

THE SECOND REAL JOHNSON-WILSON THEORY AND NON-IMMERSIONS OF RP^n , PART II

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ABSTRACT. This paper is a continuation of the study begun in [KW]. We analyze $ER(2)^{16*+8}(RP^{2n})$ and compute $ER(2)^*(RP^{16K+1})$ and use these to prove more non-immersion theorems for RP^n including many in fairly low dimensions. In particular, we get 12 new non-immersion results for RP^n where $n \leq 192$, the range included in the tables [Dav]. These complement the 10 already found in [KW].

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1. INTRODUCTION

This paper is a continuation of [KW], which we refer to as Part I. We make free use of the notation and results of Part I.

The main theorem of [Dav84] states that for

$$n = m + \alpha(m) - 1 \quad k = 2m - \alpha(m)$$

there does not exist an axial map:

$$RP^{2^k - 2k - 2} \times RP^{2n} \longrightarrow RP^{2^k - 2n - 2}$$

and so, by [Jam63], $RP^{2n} \not\subseteq \mathbb{R}^{2^k}$ for these n and k .

This is proven using the equivalent of $E(2)^*(-)$ by showing that the $u^{2^k - 1 - n} = 0$ on the right would have to go to a non-zero element on the left. That prevents the existence of the axial map. In Part I we constructed a purely algebraic surjection

$$ER(2)^{16*}(RP^{2^k - 2k - 4}) \longrightarrow E(2)^{16*}(RP^{2^k - 2k - 2})$$

that allowed us to show the axial map

$$RP^{2^k - 2k - 4} \times RP^{2n} \longrightarrow RP^{2^k - 2n - 2}$$

did not exist if we added the restrictions to k and n that $n = 7$ or $0 \pmod 8$ and $-k - 2 = 1, 2, 5$ or $6 \pmod 8$. This improved some non-immersions results by 2.

In this paper we are able to include the $n = 3$ and $4 \pmod 8$ cases by analyzing $ER(2)^{16*+8}(RP^{2n})$ and constructing a similar algebraic map

$$ER(2)^{16*+8}(RP^{2^k - 2k - 4}) \longrightarrow E(2)^{16*+8}(RP^{2^k - 2k - 2})$$

with the previous restrictions on k .

In order to describe $ER(2)^{16*+8}(RP^{2n})$ properly we need to define and study an element $y \in ER(2)^8(RP^\infty)$. The simple version of our answer, similar to our understanding of $ER(2)^{16*}(RP^{2n})$ in Theorem 1.6 of Part I, is:

Theorem 1.1. *A 2-adic basis for $ER(2)^{16*+8}(RP^{2n})$ consists of the elements $\alpha_2\alpha^k u^j$, with $0 \leq k$ and $0 < j < n$, yu^j , with $0 \leq j \leq n-4$, and, when*

$n = 3$ or 4 modulo 8 , no other elements,

$n = 2$ or 5 modulo 8 , $\alpha^k yu^{n-3}$,

$n = 1$ or 6 modulo 8 , $\alpha^k yu^{n-3}$, and yu^{n-2} ,

$n = 7$ or 0 modulo 8 , yu^{n-3} , yu^{n-2} , and yu^{n-1} ,

and no others.

In addition to this we need some information about $ER(2)^*(-)$ of products and then we are able to prove:

Theorem 1.2. *When the pair $(m, \alpha(m))$ is, modulo 8 , $(0, 3), (5, 6), (4, 7)$ or $(1, 2)$, then*

$$RP^{2(m+\alpha(m))} \not\subseteq \mathbb{R}^{2(2m-\alpha(m)+1)}.$$

The most interesting pair to us is $(m, \alpha(m)) = (0, 3)$. Let $m = 8 + 16 + 2^i$, then $2(m + \alpha(m)) = 54 + 2^{i+1}$ and $2(2m - \alpha(m) + 1) = 92 + 2^{i+2}$. So, we get

$$RP^{54+2^{i+1}} \not\subseteq \mathbb{R}^{92+2^{i+2}}.$$

The lowest dimensional cases are

$$RP^{118} \not\subseteq \mathbb{R}^{220} \quad RP^{182} \not\subseteq \mathbb{R}^{348}$$

However, the importance to us is that it gets on Don Davis's tables, [Dav]. Notice also that for these cases there is now only a knowledge gap of 1 between best known non-immersions and best known immersions.

Next we move on to compute $ER(2)^*(RP^{16K+1})$, analyze $ER(2)^{8*}(RP^{16K+1})$, and construct an algebraic map

$$ER(2)^{8*}(RP^{16K+1}) \longrightarrow E(2)^{8*}(RP^{16K+2})$$

that allows us to do similar things for non-immersions when, in our axial maps, $-k-1 = 1 \pmod{8}$. The theory $E(2)^*(-)$ cannot make use of the odd spaces because the top cell is not connected algebraically, but for $ER(2)^*(-)$ the connection is strong for $16K+1$ and $16K+9$. We have not done the computation for $16K+9$ because, although there are surely more non-immersions there, they are not of low enough dimension to inspire us to do the work, whereas the $16K+1$ case gives lots of nice new low dimensional results.

Theorem 1.3. *A 2-adic basis for $ER(2)^{16*}(RP^{16K+1})$ is given by the elements $\alpha^k u^j$, $0 \leq k$, $0 < j \leq 8K+1$.*

A 2-adic basis for $ER(2)^{16+8}(RP^{16K+1})$ is given by the elements*

$$\begin{aligned} \alpha_2\alpha^k u^j & \quad 0 \leq k \quad 0 < j < 8K \\ \alpha_2\alpha^k u^{8K} & = \alpha^{k+1} yu^{8K-3} \\ & w\alpha^k u \\ yu^j & \quad 0 \leq j < 8K. \end{aligned}$$

Using this we get:

Theorem 1.4. *For the mod 8 pairs $(m, \alpha(m)) = (6, 6), (1, 4), (2, 6), (5, 4)$ we have:*

$$RP^{2(m+\alpha(m)-1)} \not\subseteq \mathbb{R}^{2(2m-\alpha(m)+1)}.$$

Let's look at the numbers. First, the pair $(1, 4)$. The lowest possible non-immersions we get from this are

$$RP^{56+2^{i+1}} \not\subseteq \mathbb{R}^{93+2^{i+2}}$$

which also implies another new result:

$$RP^{57+2^{i+1}} \not\subseteq \mathbb{R}^{93+2^{i+2}}.$$

The lowest dimensional examples are:

$$RP^{120} \not\subseteq \mathbb{R}^{221} \quad RP^{121} \not\subseteq \mathbb{R}^{221} \quad RP^{184} \not\subseteq \mathbb{R}^{349} \quad RP^{185} \not\subseteq \mathbb{R}^{349}$$

The next pair to look at is $(5, 4)$. From this we get

$$RP^{16+2^{i+1}+2^{j+1}} \not\subseteq \mathbb{R}^{13+2^{i+2}+2^{j+2}}.$$

When $i = 3$ this is:

$$RP^{32+2^{j+1}} \not\subseteq \mathbb{R}^{45+2^{j+2}}.$$

The lowest dimensional examples are:

$$RP^{96} \not\subseteq \mathbb{R}^{173} \quad RP^{160} \not\subseteq \mathbb{R}^{301}$$

When $i = 4$ this is:

$$RP^{48+2^{j+1}} \not\subseteq \mathbb{R}^{77+2^{j+2}}.$$

which also implies that

$$RP^{49+2^{j+1}} \not\subseteq \mathbb{R}^{77+2^{j+2}}.$$

The lowest dimensional examples are:

$$RP^{112} \not\subseteq \mathbb{R}^{205} \quad RP^{113} \not\subseteq \mathbb{R}^{205} \quad RP^{176} \not\subseteq \mathbb{R}^{333} \quad RP^{177} \not\subseteq \mathbb{R}^{333}$$

In the tables, [Dav], the best known results for non-immersions for RP^n for $n \leq 192$ are listed. 95 of these are solved completely because it is known that RP^n immerses in the next higher dimension. Of the remaining 97 cases we improve on 12 in this paper, $n = 96, 112, 113, 118, 120, 121, 160, 176, 177, 182, 184,$ and 185, making for a total of 22 when combined with 10 from Part I, $n = 48, 62, 80, 94, 110, 126, 144, 158, 174,$ and 190.

The tables also list what is known for $n = d + 2^i$ for $0 \leq d < 64$. Of these 64 cases, 24 are known completely. Of the remaining 40 we improve on 10, 6 from this paper, $d = 32, 48, 49, 54, 56,$ and 57, and 4 from Part I, $d = 16, 30, 46,$ and 62.

We are fairly confident that $ER(2)^*(-)$ will not give any more results in these low dimensions. Before attacking the present cases in this paper, computer computations were made on all of the cases we believed we could approach below 192 and we have now proven all of the results that seemed to be there.

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2. INJECTIONS

$ER(2)^*(X)$ has already been computed for $X = RP^\infty$, RP^{2n} , and $RP^\infty \wedge RP^\infty$. However, we need a more detailed analysis.

Theorem 2.1. *The map $ER(2)^{16*+8}(RP^\infty) \rightarrow E(2)^{16*+8}(RP^\infty)$ is an injection with cokernel given by $v_2^4 u^{\{1-3\}}$.*

Proof. Recall that a 2-adic basis for $E(2)^*(RP^\infty)$ is given by

$$v_2^i \alpha^k u^j \quad 0 \leq i < 8 \quad 0 \leq k \quad 1 \leq j.$$

The elements of degree $8 \bmod 16$ are

$$v_2^4 \alpha^k u^j \quad 0 \leq k \quad 1 \leq j.$$

From Theorem 8.1 of Part 1 we can read off the elements of $ER(2)^{16*+8}(RP^\infty)$. From the x^1 -torsion we have

$$\alpha_2 \alpha^k u^j \quad 0 \leq k \quad 1 \leq j.$$

From the x^3 -torsion we have

$$w \alpha^k u, \quad k \geq 0, \text{ and } w u^j \quad 1 < j.$$

Mapping these elements to $E(2)^*(RP^\infty)$ we have

$$\alpha_2 \alpha^k u^j \longrightarrow 2v_2^4 \alpha^k u^j = v_2^4 \alpha^{k+1} u^{j+1}$$

plus higher filtration terms,

$$w \alpha^k u \longrightarrow v_2^4 \alpha^{k+1} u$$

and

$$w u^j \longrightarrow v_2^4 \alpha u^j = v_2^4 u^{j+2}, \quad j > 1$$

modulo higher terms. From this we can see the injection and that the only terms missed are $v_2^4 u^{\{1-3\}}$. □

Theorem 2.2. *The map $ER(2)^{16*+8}(RP^\infty \wedge RP^\infty) \rightarrow E(2)^{16*+8}(RP^\infty \wedge RP^\infty)$ is an injection with cokernel given by $v_2^4 u_1^{\{1-3\}} u_2^{\{1-3\}}$.*

Proof. We have both $E(2)^*(RP^\infty \wedge RP^\infty)$ and $ER(2)^*(RP^\infty \wedge RP^\infty)$ written down in Theorem 17.1 of Part I. $E(2)^*(RP^\infty \wedge RP^\infty)$ is

$$\begin{aligned} v_2^s \alpha^k u_1^i u_2 & \quad 0 \leq s < 8 \quad 0 \leq k \quad 0 < i \\ v_2^s u_1^i u_2^j & \quad 0 \leq s < 8 \quad 0 < i \quad 1 < j. \end{aligned}$$

The elements in degree $16 * + 8$ are those with $s = 4$. We can also write down $ER(2)^*(RP^\infty \wedge RP^\infty)$ and the map to $E(2)^*(RP^\infty \wedge RP^\infty)$ in degrees $16 * + 8$: From the x^1 -torsion we get

$$\begin{aligned} \alpha_2 \alpha^k u_1^i u_2 & \longrightarrow 2v_2^4 \alpha^k u_1^i u_2 = v_2^4 \alpha^{k+1} u_1^{i+1} u_2 \\ \alpha_2 u_1^i u_2^j & \longrightarrow 2v_2^4 u_1^i u_2^j = v_2^4 u_1^{i+1} u_2^{j+2} \quad 0 < i \quad 1 < j, \end{aligned}$$

all modulo higher filtrations. From the x^3 -torsion we get:

$$\begin{aligned} w \alpha^k u_1 u_2 & \longrightarrow v_2^4 \alpha^{k+1} u_1 u_2 \quad 0 \leq k \\ w u_1^i u_2^{\{1,2,3\}} & \longrightarrow v_2^4 u_1^{i+2} u_2^{\{1,2,3\}} \quad 1 < i \\ w u_1 u_2^j & \longrightarrow v_2^4 u_1 u_2^{j+2} \quad 1 < j. \end{aligned}$$

From this we see the injection and that the only elements missed are those stated. □

Note that there are no elements in degrees $16 * + 8$ divisible by x .

Corollaries 8.3 and 17.3 of Part I give us an isomorphism for these spaces in degrees $16*$ so we get:

Corollary 2.3. *The map $ER(2)^{8*}(X) \rightarrow E(2)^{8*}(X)$ is an injection for $X = RP^\infty$ and $RP^\infty \wedge RP^\infty$.*

Not much more work is required to prove:

Proposition 2.4. *The map $ER(2)^{16*+i}(X) \rightarrow E(2)^{16*+i}(X)$ is an injection for $X = RP^\infty$ and $RP^\infty \wedge RP^\infty$ when $i = 0, 1, 2, 3, 4, 5$ and 8.*

From this we know that there are no elements divisible by x in any of these degrees.

3. y , A NEW ELEMENT

We need to introduce a new element that we have good control over. We know we have an isomorphism of $ER(2)^{16*}(RP^\infty)$ and $E(2)^{16*}(RP^\infty)$ and that we have the same relation, $0 = 2u +_F \alpha u^2 +_F u^4$, in both. We can use this to solve for u^4 as:

$$u^4 = -_F(2u) -_F(\alpha u^2) = 2ug + \alpha u^2 h$$

where g and h are invertible power series. As it stands, g and h are not uniquely determined, but if we insist that none of the terms of h be divisible by 2 (we can move such terms to g) then we can make our choice of g and h unique.

Recall that in $E(2)^*(-)$ we have set $v_2^8 = 1$. We now multiply this relation, when viewed only as being in $E(2)^*(RP^\infty)$, by v_2^4 , which is a unit in $E(2)^*(-)$, to get a relation

$$v_2^4 u^4 = v_2^4(2ug + \alpha u^2 h) = (2v_2^4)ug + (v_2^4 \alpha)u^2 h.$$

The image of the element α_2 from $ER(2)^*$ in $E(2)^*$ is $2v_2^4$ and the image of w is $v_2^4 \alpha$ so, in this relation, all of the terms on the right hand side are in the image from $ER(2)^*(RP^\infty)$ and we can use them to define a new element

$$y = \alpha_2 u g + w u^2 h$$

that reduces to $v_2^4 u^4 \in E(2)^8(RP^\infty)$. (The lack of uniqueness of g and h would not affect anything here. It does not matter whether we convert $2v_2^4$ to α_2 or $v_2^4 \alpha$ to w if we have a $v_2^4 2\alpha$ that could be factored either way because $\alpha_2 \alpha = 2w \in ER(2)^*$, [KW07].)

Although we struggled with this element a great deal in our original computations and then managed to eliminate it for our work in Part I, it was only with the work of Bruner, Davis and Mahowald in [BDM02, DM] that we realized its importance for our work with non-immersion theorems.

The element y has many interesting properties. We collect a few here.

Theorem 3.1. *There is an element $y \in ER(2)^8(RP^\infty)$ that maps to $v_2^4 u^4 \in E(2)^8(RP^\infty)$. We have relations:*

$$\begin{aligned} y^2 &= u^8 & \alpha y &= w u^4 & w y &= \alpha u^4 & 2y &= \alpha_2 u^4 & \alpha_2 y &= 2u^4 \\ \alpha_3 y &= \alpha_1 u^4 & \alpha_1 y &= \alpha_3 u^4 & x y &= x w u^2 h & x^3 y &= 0. \end{aligned}$$

Proof. We have already constructed y with the property that it reduces to $v_2^4 u^4$. The first five relations take place in degrees $8*$ where we have an injection of $ER(2)^{8*}(RP^\infty) \rightarrow E(2)^{8*}(RP^\infty)$ so we can prove the relations by substituting $v_2^4 u^4$ for y , $2v_2^4$ for α_2 and $v_2^4 \alpha$ for w . They all follow quickly then. The next relation is in degree 4 modulo 16 and we have an injection here too as well from Proposition 2.4 so it also follows by replacing α_i with $2v_2^i$. Only the next relation

requires anything else. It is in degree -4 modulo 16 and we do not have an injection in this degree. We have to resort to the definition (which we could also have used for the other relations).

$$\begin{aligned}\alpha_1 y &= \alpha_1(\alpha_2 u g + w u^2 h) = (\alpha_1 \alpha_2) u g + (\alpha_1 w) u^2 h = \\ &= (2\alpha_3) u g + (\alpha \alpha_3) u^2 h = \alpha_3(2u g + \alpha u^2 h) = \alpha_3 u^4.\end{aligned}$$

This uses the relations in the coefficient ring: $2\alpha_3 = \alpha_1 \alpha_2$ and $\alpha \alpha_3 = \alpha_1 w$, from [KW07].

$$y = \alpha_2 u g + w u^2 h,$$

so, since $x\alpha_2 = 0$, we get the next relation. Since $x^3 w = 0$ the last one follows. \square

We know $E(2)^*(-)$ and $ER(2)^*(-)$ for RP^∞ and $RP^\infty \wedge RP^\infty$. From Theorem 3.4 of Part I we know that we have a Künneth theorem for $RP^\infty \wedge RP^\infty$. The standard map $RP^\infty \times RP^\infty \rightarrow RP^\infty$ induces a coproduct that can be computed from the formal group law, i.e. $u \rightarrow u_1 +_F u_2$. However, things are much nicer than that:

Theorem 3.2. *The coproduct of u , up to a unit, is $u_1 - u_2$. The coproduct of y , up to a unit, is*

$$y_1 - 2\alpha_2 u_1^3 u_2 + 3\alpha_2 u_1^2 u_2^2 - 2\alpha_2 u_1 u_2^3 + y_2.$$

Proof. Because we have injections in degrees $8*$ it is enough to prove this for the image in $E(2)^*(-)$. The first statement is well-known and comes from the fact that $0 = [2](X) = X +_F X$. This implies that $X +_F Y$ is divisible by $X - Y$ (because plugging in $Y = X$ gives zero) and so $X +_F Y = (X - Y)g$ where g is a power series in X and Y that is invertible, i.e. a unit. Since the coproduct is given by the formal group law we have $u \rightarrow u_1 +_F u_2 = (u_1 - u_2)$ up to a unit.

To compute the coproduct of y up to a unit we can just compute for u^4 . Up to a unit this is $u_1^4 - 4u_1^3 u_2 + 6u_1^2 u_2^2 - 4u_1 u_2^3 + u_2^4$. Multiply this by v_2^4 and replace $v_2^4 u_i^4$ with y_i and $2v_2^4$ with α_2 . \square

4. REWRITING $ER(2)^*(RP^\infty)$

We would like to rewrite our answer for $ER(2)^{16*+8}(RP^\infty)$ using our new element y . Recall, from Theorem 8.1 of Part I, our description of $ER(2)^*(RP^\infty)$.

The x^1 -torsion generators are given by:

$$\alpha_i \alpha^k u^j \quad 0 \leq i < 4 \quad 0 \leq k \quad 1 \leq j$$

where $\alpha_0 = 2$.

The x^3 -torsion generators are given by:

$$w^\epsilon \alpha^k u, \quad \epsilon + k > 0, \quad w u^j \quad 1 < j, \quad \text{and} \quad u^j \quad 3 < j.$$

The only x^7 -torsion generators are

$$u^{\{1-3\}}.$$

From $y = \alpha_2 u g + w u^2 h$ it is easy to see that we can replace the x^3 -torsion generators, $w u^j$, $1 < j$, using $y u^{j-2}$.

We would also like to replace some of the x^1 -torsion generators,

$$\alpha_2 \alpha^k u^j$$

with $\alpha^{k+1}yu^{j-3}$. This last element is not x^1 -torsion though. When we are in $ER(2)^*(RP^{2n})$ and j is big enough, this can be x^1 torsion.

$$\begin{aligned} y &= \alpha_2 u g + w u^2 h \\ \alpha_2 u g &= y - w u^2 h \\ \alpha_2 u &= y g^{-1} - w u^2 h g^{-1} \\ \alpha_2 u^3 &= y u^2 g^{-1} - w u^4 h g^{-1} \end{aligned}$$

We know that $wu^4 = \alpha y$ so this is:

$$\alpha_2 u^3 = y u^2 g^{-1} - \alpha y h g^{-1}.$$

The lead term (i.e. the term with lowest filtration) here is αy and the whole right hand side must be x^1 -torsion even if the lead term isn't. When the higher filtration terms are all zero we can replace $\alpha_2 \alpha^k u^j$ with $\alpha^{k+1} y u^{j-3}$.

This is enough to give us what we want.

5. $ER(2)^{16*+8}(RP^{2n})$

We have, for $ER(2)^{16*+8}(-)$ a theorem similar to Theorem 13.4 of Part I for $ER(2)^{16*}(-)$.

Theorem 5.1. *For all $n > 3$ there is a short exact sequence:*

$$(5.2) \quad 0 \leftarrow ER(2)^{16*+8}(RP^{2n-2}) \leftarrow ER(2)^{16*+8}(RP^{2n}) \\ \leftarrow ER(2)^{16*+8}(RP^{2n}/RP^{2n-2}) \leftarrow 0.$$

We have elements $\alpha_2 \alpha^k u^j \in ER(2)^{16*+8}(RP^{2n})$, $0 \leq k$, $0 < j < n$. We also have elements yu^j for $0 \leq j \leq n-4$.

Depending on n modulo 8 there are other elements in $ER(2)^{16*+8}(RP^{2n})$.

For $n = 8K + 4$ and $8K + 3$ there are no other elements.

For $n = 8K + 2$ there is an x^5 -torsion element, z_{16K-38} , that reduces to $v_2 u^{8K+2}$ in the Bockstein spectral sequence such that

$$x^2 \alpha^k z_{16K-38} = \alpha^k y u^{8K-1}.$$

For $n = 8K + 1$ there is an x^5 -torsion element, z_{16K-22} , that reduces to $v_2 u^{8K+1}$ in the Bockstein spectral sequence such that

$$x^2 \alpha^k z_{16K-22} = \alpha^k y u^{8K-2}$$

and an x^7 -torsion element, z_{16K-4} that reduces to $v_2^6 u^{8K+1}$ in the Bockstein spectral sequence such that

$$x^2 u z_{16K-22} = x^4 z_{16K-4} = y u^{8K-1}.$$

For $n = 8K$ there are x^7 -torsion elements, z_{16K-20} , and z_{16K-18} that reduce to $v_2^6 u^{8K-1}$ and $v_2^3 u^{8K}$ respectively in the Bockstein spectral sequence such that

$$\begin{aligned} x^4 z_{16K-20} &= y u^{8K-3} \\ x^4 u z_{16K-20} &= y u^{8K-2} \end{aligned}$$

and

$$x^4 u^2 z_{16K-20} = x^6 z_{16K-18} = y u^{8K-1}.$$

For $n = 8K + 7$ there are x^7 -torsion elements, z_{16K-36} , and z_{16K-34} that reduce to $v_2^6 u^{8K+6}$ and $v_2^3 u^{8K+7}$ respectively in the Bockstein spectral sequence such that

$$x^4 z_{16K-36} = y u^{8K+4}$$

$$x^4 u z_{16K-36} = y u^{8K+5}$$

and

$$x^4 u^2 z_{16K-36} = x^6 z_{16K-34} = y u^{8K+6}.$$

For $n = 8K + 6$ there is an x^5 -torsion element, z_{16K-8} , that reduces to $v_2 u^{8K+6}$ in the Bockstein spectral sequence such that

$$x^2 \alpha^k z_{16K-8} = \alpha^k y u^{8K+3}$$

and an x^7 -torsion element, z_{16K-36} that reduces to $v_2^6 u^{8K+6}$ in the Bockstein spectral sequence such that

$$x^2 u z_{16K-8} = x^4 z_{16K-36} = y u^{8K+4}.$$

For $n = 8K + 5$ there is an x^5 -torsion element, z_{16K+10} , that reduces to $v_2 u^{8K+5}$ in the Bockstein spectral sequence such that

$$x^2 \alpha^k z_{16K+10} = \alpha^k y u^{8K+2}.$$

This gives us our Theorem 1.1.

Proof. We have computed the Bockstein spectral sequence for all of the spaces RP^{2n-2} , RP^{2n} , and RP^{2n}/RP^{2n-2} . From this we can just read off the elements in degree $16* + 8$. In every case the x^1 -torsion elements $\alpha_2 \alpha^k u^j$ for $j < n - 1$ correspond using the map induced by $RP^{2n-2} \rightarrow RP^{2n}$. Likewise for the elements $w \alpha^k u$, and $y u^j$, $0 \leq j \leq n - 5$ so we will ignore these elements. In the proof we are constantly using the fact that we already know all of the groups. We also make use of the fact that the map $ER(2)^*(RP^{2n}/RP^{2n-2}) \rightarrow ER(2)^*(RP^{2n})$ was computed explicitly in Equation 13.1 of Part I.

First note that $\alpha_0 \alpha^k u^{n-1} = 2 \alpha^k u^{n-1} = \alpha^{k+1} u^n$.

For $n = 4 \pmod 8$, there is nothing else in $ER(2)^{16*+8}(RP^{2n-2})$. All that is left of 5.2 is $\alpha^k z_{2n} \in ER(2)^{16*+8}(RP^{2n}/RP^{2n-2})$ and $\alpha_2 \alpha^k u^{n-1}$ and $y u^{n-4}$ in $ER(2)^{16*+8}(RP^{2n})$. Since there are no elements of higher filtration we can use Section 4 to replace $\alpha_2 \alpha^k u^{n-1}$ with $\alpha^{k+1} y u^{n-4}$. Note that $\alpha^k y u^{n-4}$ is represented by $v_2^4 \alpha^k u^n$ in the Bockstein spectral sequence. The long exact sequence forces $\alpha^k z_{2n} \rightarrow \alpha^k y u^{n-4}$ but so does our direct computation using Equation 13.1 of Part I.

For $n = 3 \pmod 8$, $ER(2)^{16*+8}(RP^{2n}/RP^{2n-2}) = 0$. We must have $\alpha^k y u^{n-4} \rightarrow x^2 \alpha^k u^{n-1} v_2$. [Technically, we need to worry that perhaps $y u^{n-4}$ goes to $x^2 \alpha^{3k} u^{n-1} v_2$ for some k . If this is the case then the boundary homomorphism on $x^2 u^{n-1} v_2$ must be non-trivial but we can check that there is nowhere for it to go. Consequently we will ignore this kind of possibility in the rest of this proof.]

For $n = 2 \pmod 8$ things are a little more complicated. The only elements in $ER(2)^{16*+8}(RP^{2n}/RP^{2n-2})$ are $x^2 w \alpha^k z_{2n-18}$ and we can compute directly that they go to $x^2 \alpha^{k+1} u^n v_2$. $\alpha^k y u^{n-4}$ must go to $x^2 \alpha^k u^{n-1} v_2$. The only possibility left is for $x^2 u^n v_2$ to go to $x^4 u^{n-1} v_2^6$. Recall from above that this last element is $y u^{n-3}$.

For $n = 1 \pmod 8$, we compute the map to $ER(2)^{16*+8}(RP^{2n})$ directly and we have

$$\begin{aligned} w \alpha^k z_{2n-18} &\longrightarrow \alpha^{k+1} y u^{n-4} \\ x^2 w \alpha^k z_{2n} &\longrightarrow x^2 \alpha^{k+1} u^n v_2 \end{aligned}$$

Keep in mind that this last represents $\alpha^{k+1} y u^{n-3}$. $y u^{n-4}$ must map to $x^4 v_2^6 u^{n-2}$, $x^2 v_2 u^n = y u^{n-3}$ to $x^4 v_2^6 u^{n-1}$, and $x^4 v_2^6 u^n = y u^{n-2}$ to $x^6 v_2^3 u^{n-1}$.

For $n = 0 \pmod 8$ we compute $x^6 z_{2n-18} \rightarrow x^6 u^n v_2^3 = yu^{n-1}$ and $w\alpha^k z_{2n} \rightarrow \alpha^{k+1} yu^{n-4}$. That leaves $yu^{n-4} \rightarrow x^4 u^{n-2} v_2^6$, $x^4 u^{n-1} v_2^6 = yu^{n-3} \rightarrow x^4 u^{n-1} v_2^6$, and $x^4 u^n v_2^2 = yu^{n-2} \rightarrow x^6 u^{n-1} v_2^3$.

For $n = 7 \pmod 8$ we compute $x^4 z_{2n-18} \rightarrow x^4 u^n v_2^6 = yu^{n-2}$ and $x^6 z_{2n} \rightarrow x^6 u^n v_2^3 = yu^{n-1}$. That leaves $x^4 u^{n-1} v_2^2 = yu^{n-3} \rightarrow x^4 u^{n-1} v_2^6$ and $\alpha^k yu^{n-2} \rightarrow x^2 \alpha^k u^{n-1} v_2$.

For $n = 6 \pmod 8$ we compute $x^2 \alpha^k z_{2n-18} \rightarrow x^2 \alpha^k u^n v_2 = \alpha^k yu^{n-3}$ and $x^4 z_{2n} \rightarrow x^4 u^n v_2^6 = yu^{n-2}$. All that is left is $\alpha^k yu^{n-4} \rightarrow x^2 \alpha^k u^{n-1} v_2$.

The $n = 5 \pmod 8$ case is simple again with $\alpha^k z_{2n-18} \rightarrow \alpha^k yu^{n-4}$ and $x^2 \alpha^k z_{2n} \rightarrow x^2 \alpha^k u^n v_2 = \alpha^k yu^{n-3}$.

□

6. ALGEBRAIC MAPS

We can now see, from Theorem 5.1 and Part I, that we have purely algebraic maps, no topology used or implied, of

$$ER(2)^{8*}(RP^{2n}) \longrightarrow E(2)^{8*}(RP^{2n+2}) \quad n = 1, 2, 5, 6 \pmod 8.$$

These maps are neither injective nor surjective. However, they are close enough to surjective for our purposes since the only elements they miss are the $v_2^4 u^{\{1-3\}}$. These low powers of u are never involved with our non-immersion results.

7. LAST OF THE EVEN SPACES

The goal of this section is to prove Theorem 1.2.

We begin again with the main theorem of [Dav84], for

$$n = m + \alpha(m) - 1 \quad k = 2m - \alpha(m)$$

there does not exist an axial map:

$$RP^{2n} \times RP^{2^k - 2k - 2} \longrightarrow RP^{2^k - 2n - 2}$$

and so $RP^{2n} \not\subseteq \mathbb{R}^{2^k}$. This is proven by using the equivalent of $E(2)^*(-)$ and showing that the $u^{2^{k-1}-n} = 0$ on the right would have to go to a non-zero element on the left. That same element would prevent the existence of an axial map,

$$RP^{2n+2} \times RP^{2^k - 2k - 2} \longrightarrow RP^{2^k - 2n - 2}$$

and likewise

$$RP^{2n+2} \times RP^{2^k - 2k - 2} \longrightarrow RP^{2^k - 2n - 4}.$$

Furthermore, if $u^{2^{k-1}-n}$ went to non-zero then we must also have $u^{2^{k-1}-n-1} = 0$ going to a non-zero element. If $n+1 = 3 \pmod 8$ then, from Theorem 1.1, we know $yu^{2^{k-1}-n-5} = 0$ in $ER(2)^*(RP^{2^k - 2n - 4})$ and, if $-k-2 = \{1, 2, 5, 6\} \pmod 8$ we have a commuting diagram:

(7.1)

$$\begin{array}{ccc} E(2)^*(RP^{2n+2}) \otimes_{E(2)^*} E(2)^*(RP^{2^k - 2k - 2}) & \longleftarrow & E(2)^*(RP^{2^k - 2n - 4}) \\ \uparrow & & \uparrow \\ ER(2)^*(RP^{2n+2}) \otimes_{ER(2)^*} ER(2)^*(RP^{2^k - 2k - 4}) & \longleftarrow & ER(2)^*(RP^{2^k - 2n - 4}). \end{array}$$

Our element $yu^{2^{k-1}-n-5}$ in the lower right hand corner maps to $v_2^4 u^{2^{k-1}-n-1}$ in the upper right hand corner. v_2^4 is a unit so the result of Davis shows that

this element maps non-trivially to the upper left hand side of our diagram and so $yu^{2^{K-1}-n-5}$ must map non-trivially to the lower left hand side.

If we show that the corresponding element in the tensor product maps non-trivially to $ER(2)^*(-)$ of the product, then we will have the following proposition:

Proposition 7.2. *When $n = m + \alpha(m) - 1$ and $k = 2m - \alpha(m)$, with $n + 1 = 3 \pmod 8$ and $-k - 2 = \{1, 2, 5, 6\} \pmod 8$, there is no axial map*

$$RP^{2n+2} \times RP^{2^K-2k-4} \longrightarrow RP^{2^K-2n-4}.$$

and so $RP^{2n+2} \not\subseteq \mathbb{R}^{2k+2}$.

To derive the proof of Theorem 1.2 from this we have to untangle some equations to get our $(m, \alpha(m))$ pairs. We have

$$n + 1 = m + \alpha(m) = 3 \pmod 8$$

and

$$-k - 2 = -2m + \alpha(m) - 2 = \{1, 2, 5, 6\} \pmod 8.$$

The equation for k gives

$$2m - \alpha(m) = \{5, 4, 1, 0\} \pmod 8.$$

The equation for n gives

$$m + \alpha(m) = 3 \pmod 8.$$

Adding, we have

$$3m = \{0, 7, 4, 3\} \pmod 8.$$

Multiply by 3 to get

$$m = \{0, 5, 4, 1\} \pmod 8.$$

Substituting into

$$\alpha(m) = -m + 3 \pmod 8$$

we get

$$\alpha(m) = \{3, 6, 7, 2\}$$

and this gives us $(m, \alpha(m))$ pairs $(0, 3), (5, 6), (4, 7)$ and $(1, 2) \pmod 8$.

To complete our proof of Theorem 1.2 we have to show that the obstruction which we know is non-zero in our tensor product actually maps non-trivially to the $ER(2)^*(-)$ of the product.

We do not need to know much about the actual obstruction. It is enough to know that it must be a linear combination of our 2-adic basis for the tensor product in the upper left hand corner of our diagram. The basis here, in degrees $16*+8$, is given by $v_2^4 \alpha^k u_1^i u_2$ with $i \leq n+1$ and $v_2^4 u_1^i u_2^j$ with $i \leq n+1$ and $1 < j \leq 2^{K-1} - k - 1$.

We need to discuss the obstruction just a little in order to be careful with our comparison. Recall from Theorem 3.2 that computing the coproduct of $v_2^4 u^{2^{K-1}-n-1}$ is, up to a unit, $(u_1 - u_2)^{2^{K-1}-n-1}$. The algorithm, Remark 15.3 of Part I, never lowers the powers of the u 's so the obstruction must be a linear combination of the 2-adic basis elements $v_2^4 u_1^i u_2^j$ with $i \leq n+1$, $1 < j \leq 2^{K-1} - k - 1$, and $i + j \geq 2^{K-1} - n - 1$. (The missing $v_2^4 u_1^{\{1,2,3\}} u_2^{\{1,2,3\}}$ never figure in here.)

We need to show that the elements which hit these from $ER(2)^*(-)$ map non-trivially from the tensor product:

$$(7.3) \quad ER(2)^*(RP^{2n+2}) \otimes_{ER(2)^*} ER(2)^*(RP^{2^K-2k-4})$$

$$\longrightarrow ER(2)^*(RP^{2n+2} \times RP^{2^K-2k-4}).$$

This is easy for most of these elements. Consider the elements $u_1^i u_2^j y_2$ with $3 < i \leq n+1$ and for $3 < j \leq 2^{K-1} - k - 6$. These reduce non-trivially (and independently) to $v_2^4 u_1^i u_2^{j+4}$ in $E(2)^{8*}(-)$. That's not quite enough though. We also need the elements $u_1^i u_2^{2^{K-1}-k-5} y_2$ for $3 < i \leq n+1$.

The following result finishes off what we need.

Proposition 7.4. *When $q = 1, 2, 5$ or 6 modulo 8 and $m \leq 8K$ and $8K + 8 < q$, the element $u_1^m u_2^{q-3} y_2 \in ER(2)^*(RP^{2m} \wedge RP^{2q})$ is non-zero.*

Proof. The element yu^{q-3} is represented in the spectral sequence for $ER(2)^*(RP^{2q})$ by $x^2 v_2 u^q$ (from Theorem 5.1) so the element $u_1^m u_2^{q-3} y_2$ is represented by $x^2 u_1^m v_2 u_2^q$. Thus it is enough to show that the element $v_2 u_1^m u_2^q$ survives in the spectral sequence to E^3 . In Theorem 19.2 of Part I we have computed the entire E^2 term of the spectral sequence and this term is there with no restrictions on q . There can be no differentials on this element since it is the product of two honest elements (this does use the restriction on q).

All we need to do now is show this element is not in the image of d^2 . The differential d^2 has degree $35 = -13$. Our element $u_1^m u_2^q v_2$ has degree $-16(m+q) - 6$ so the source that would have to hit it would have to have degree $-16(m+q) - 41$, in particular, it must be odd degree. From Theorem 19.2 of Part I, the odd degree elements in the E^2 term of our Bockstein spectral sequence are:

$$\begin{aligned} & v_2^{2s} \alpha^k z_{-16q-17} \\ & v_2^{2s} u_1^i z_{-16q-17} \quad 0 < i < m \\ \text{and} \quad & v_2^{2s+1} \alpha^k u_1^{m-1} z_{-16q-17} \end{aligned}$$

The only elements with degree equal to -9 modulo 16 are:

$$\begin{aligned} & v_2^4 \alpha^k z_{-16q-17} \\ \text{and} \quad & v_2^4 u_1^i z_{-16q-17} \quad 0 < i < m. \end{aligned}$$

In the proof of Proposition 19.3 of Part I we showed that d^2 must be trivial on $z_{-16q-17}$ when $m \leq 8K$. d^2 also commutes with multiplication by α and u_1 . We also know that since $d^2(v_2^4) = 0$, d^2 commutes with multiplication by v_2^4 . Consequently, d^2 is trivial on all of the elements listed above when $m \leq 8K$. \square

This completes the proof of Theorem 1.2.

8. $ER(2)^*(RP^{16K+1})$

We begin our computation by setting up the Bockstein spectral sequence.

E^1 is just $E(2)^*(RP^{16K+1})$, which is nothing more than

$$E(2)^*(RP^{16K}) \oplus E(2)^*(S^{16K+1}).$$

So, we have a 2-adic basis, (it isn't really necessary to use this notation for the torsion free part and so it isn't necessary to go to the 2-adic completion of $ER(2)$, it is just convenient notation now), letting $2 = \alpha_0$,

E^1

$$\begin{aligned} & v_2^i \alpha^k u^j \quad 0 \leq i < 8 \quad 0 \leq k < 8 \quad 0 < j \leq 8K. \\ & v_2^i \alpha_0^q \alpha^k \iota_{16K+1} \quad 0 \leq i < 8 \quad 0 \leq q < 8 \quad 0 \leq k < 8K. \end{aligned}$$

All of the first part is even degree and all of the second part is odd degree. d^1 is even degree so it is induced by the maps

$$RP^{16K} \longrightarrow RP^{16K+1} \longrightarrow S^{16K+1}$$

where we already know it. Thus we can just read off our d^1 from Theorem 13.2 Part I, for the RP^{16K} part and Section 5 Part I for the S^{16K+1} part.

$$d^1(v_2^{2s-5}\alpha^k u^j) = 2v_2^{2s}\alpha^k u^j = v_2^{2s}\alpha^{k+1}u^{j+1} \quad j < 8K$$

(modulo higher powers of u).

$$d^1(\alpha_0^q v_2^{2s+1}\alpha^k \iota_{16K+1}) = \alpha_0^{q+1} v_2^{2s-2}\alpha^k \iota_{16K+1}$$

E^2 is given by:

$$v_2^{2s}\alpha^k u \quad 0 \leq k, \quad v_2^{2s}u^j \quad 1 < j \leq 8K, \quad v_2^{2s+1}\alpha^k u^{8K} \quad 0 \leq k$$

$$v_2^{2s}\alpha^k \iota_{16K+1}.$$

We confront a new problem now. d^2 has degree 35 and we have both odd and even degree elements so it could be non-zero. If so, by naturality it must have its source in the RP^{16K} part and its target in the S^{16K+1} part. Furthermore the source cannot be something in the image from the E^2 for $ER(2)^*(RP^\infty)$ because we know that d^2 is zero on all of those elements. All we are left with for possible sources is

$$v_2^{2s+1}\alpha^k u^{8K} \quad 0 \leq k$$

The differential d^2 is trivial on v_2^2 and α so it commutes with multiplication by these elements. Since v_2^2 is a unit, if there is a d^2 then it must be non-zero on $v_2 u^{8K}$ which has degree $-6 - 16(8K) = 16K - 6$. The degree of the target must be this plus 35, or $16K + 29 = 16K - 19$. The possible targets have degrees $-12s - 32k + 16K + 1$. Working modulo 16 we need $-3 = -12s + 1$ so we see that $s = 3$ and we have $16K - 19 = -36 - 32k + 16K + 1$, which gives $-19 = -36 - 32k + 1$ modulo 48. This is $16 = -32k$ which suggests the solution of $k = 1$ (other alternatives are $k = 3q + 1$).

If there is a d^2 we would conjecture that it starts with

$$d^2(v_2 u^{8K}) = v_2^6 \alpha \iota_{16K+1}$$

and this would lead to

$$d^2(v_2^{2s+1}\alpha^k u^{8K}) = v_2^{6+2s}\alpha^{k+1}\iota_{16K+1}.$$

We now know what to look for. If we can show that the element $\alpha \iota_{16K+1}$ is in the image from S^{16K+1} and that x^2 times it must be zero, then our conjectured d^2 is correct.

If we look carefully at $ER(2)^*(S^{16K+1})$ we see that there are a number of known 2-torsion free elements, namely, all of $w^\epsilon \alpha^k \iota_{16K+1}$, $\alpha_{\{1,3\}} \alpha^k \iota_{16K+1}$ and $\alpha_2 \iota_{16K+1}$. If we look at

$$RP^{16K} \longrightarrow RP^{16K+1} \longrightarrow S^{16K+1}$$

we know that $ER(2)^*(RP^{16K})$ is all torsion so our torsion free elements must all inject into $ER(2)^*(RP^{16K+1})$. From this we know that the element $\alpha \iota_{16K+1}$ is in $ER(2)^*(RP^{16K+1})$, but we don't yet know if x^2 kills it.

For that we need the diagram:

$$(8.1) \quad \begin{array}{ccccc} RP^{16K} & \xrightarrow{=} & RP^{16K} & & \\ \downarrow & & \downarrow & & \\ RP^{16K+1} & \xrightarrow{\quad} & RP^{16K+2} & \xrightarrow{\quad} & S^{16K+2} \\ \downarrow & & \downarrow & & \downarrow = \\ S^{16K+1} & \xrightarrow{\quad} & RP^{16K+2}/RP^{16K} & \xrightarrow{\quad} & S^{16K+2} \end{array}$$

Each row and column is a cofibration giving rise to a long exact sequence. Our goal is to show that $x^2\alpha\iota_{16K+1}$ in $ER(2)^*(RP^{16K+1})$ is zero. It is the image of the same named element in $ER(2)^*(S^{16K+1})$. From Corollary 9.3 of Part I we know that the element

$$z_{16K-16} \in ER(2)^*(RP^{16K+2}/RP^{16K})$$

maps to $x\iota_{16K+1}$ in $ER(2)^*(S^{16K+1})$. Consequently, $x\alpha z_{16K-16}$ must map to our element of interest, $x^2\alpha\iota_{16K+1}$.

Rather than go through S^{16K+1} on our way to RP^{16K+1} we can now go through RP^{16K+2} . The element z_{16K-16} maps to u^{8K+1} (Equation 13.1 Part I) and so $x\alpha z_{16K-16}$ maps to $x\alpha u^{8K+1}$. Since $2x = 0$ we can use the relation (Equation 1.3 Part I) on αu^2 and we will get xu^{8K+3} plus even higher terms, but we know that u^{8K+3} is zero in $ER(2)^*(RP^{16K+2})$ (Theorem 1.6 Part I). So, it follows that we have $x^2\alpha\iota_{16K+1} = 0$ in $ER(2)^*(RP^{16K+1})$ and we can compute our d^2 as we conjectured.

We get more information out of that computation. It shows us that u^{8K+1} is represented in the spectral sequence by $x\iota_{16K+1}$ because both come from z_{16K-16} . Multiply u^{8K} by α_2 and in $E(2)^*(RP^{16K+2})$ this is $v_2^4\alpha u^{8K+1}$. Consequently, $\alpha_2\alpha^k u^{8K}$ is represented by $v_2^4\alpha^{k+1}u^{8K+1}$ and in RP^{16K+1} this is represented by $v_2^4\alpha^{k+1}x\iota_{16K+1}$. From our discussion in Section 4 we can replace this $\alpha_2\alpha^k u^{8K}$ with $\alpha^{k+1}yu^{8K-3}$.

We now have our E^3 :

$$v_2^{2s}\alpha^k u \quad 0 \leq k, \quad v_2^{2s}u^j \quad 1 < j \leq 8K, \quad v_2^{2s}\iota_{16K+1}.$$

d^3 is even degree again so the even and odd degree parts don't mix. In S^{16K+1} d^3 takes v_2^2 to αv_2^4 but this element is not there, so there is no d^3 on the odd part. On the even part we already know the d^3 differentials:

$$\begin{aligned} d^3(v_2^{\{6,2\}}\alpha^k u) &= v_2^{\{0,4\}}\alpha^{k+1}u \\ d^3(v_2^{\{6,2\}}u^j) &= v_2^{\{0,4\}}\alpha u^j = v_2^{\{0,4\}}u^{j+2} \quad 1 < j \leq 8K - 2 \end{aligned}$$

We get: E^4 :

$$v_2^{\{0,4\}}u^{\{1-3\}}, \quad v_2^{\{6,2\}}u^{\{8K-1,8K\}}, \quad v_2^{2s}\iota_{16K+1}.$$

Our d^4 is odd degree again and so must go from the RP^{16K} part to the S^{16K+1} part if at all. d^4 has degree 21 and must be zero on things in the image from RP^∞ so a non-zero differential must start out on

$$v_2^{\{6,2\}} u^{8K-1}$$

and hit one of

$$v_2^{2s} \iota_{16K+1}.$$

The source degrees are $-36 - 16(8K - 1) = 16K - 20$ and $16K + 4$. Adding 21 to see what degree our target would have to be we get $16K + 1$ and $16K + 25$. Since v_2^4 commutes with d^4 , if we have a d^4 it must be

$$d^4(v_2^6 u^{8K-1}) = \iota_{16K+1} \quad d^4(v_2^2 u^{8K-1}) = v_2^4 \iota_{16K+1}.$$

If $x^4 \iota_{16K+1} = 0$ then this must be the case.

The argument here is the same as before using the diagram 8.1. z_{8K-16} maps to $x \iota_{16K+1}$ in S^{16K+1} so we want to study $x^3 z_{8K-16}$ and we will again get to RP^{16K+1} by way of RP^{16K+2} where $x^3 z_{16K-16}$ maps to $x^3 u^{8K+1} = 0$ (because $x^3 u^4 = 0$).

This computes our d^4 and we have left for our

E^5 :

$$v_2^{\{0,4\}} u^{\{1-3\}}, \quad v_2^{\{6,2\}} u^{8K}, \quad v_2^{\{2,6\}} \iota_{16K+1}.$$

d^5 is once again even degree and the dimensions don't work for the odd part so it is zero there and RP^{8K} determines it is zero on the even part.

We have to consider the possibility of a d^6 which is of degree 7. Again, it must go from even to odd by naturality. It would commute with v_2^4 so if d^6 is non-zero on $v_2^2 u^{8K}$ it will be non-zero on the other element. The degree here is $-12 + 16K$. Add 7 to look for the target to get $-5 + 16K$. Elements that are in odd degrees are in degrees $16K + 1 - 12$ and $16K + 1 - 36$ so there can be no d^6 .

All we have left is d^7 and we know, for starters, that $d^7(v_2^4 u^{1-3}) = u^{1-3}$. We can also read off from RP^{8K} that $d^7(v_2^2 u^{8K}) = v_2^6 u^{8K}$.

The only issue remaining is how d^7 works on

$$v_2^{\{2,6\}} \iota_{16K+1}.$$

The element yu^{8K-1} is represented by $x^4 v_2^6 u^{8K+1}$ in the Bockstein spectral sequence for $ER(2)^*(RP^{16K+2})$ and by $x^6 v_2^3 u^{8K}$ in the Bockstein spectral sequence for $ER(2)^*(RP^{16K})$ (Theorem 5.1).

It must pass through $ER(2)^*(RP^{16K+1})$ non-trivially and it must be divisible by x^4 in here. The only even degree candidates for such an element in the Bockstein spectral sequence for $ER(2)^*(RP^{16K+1})$ are $x^{\{4,6\}} v_2^6 u^{8K}$ and $x^5 v_2^{\{2,6\}} \iota_{16K+1}$. The degree of yu^{8K-1} is $16K \pm 24$ which is 8 mod 16. Checking $x^{\{4,6\}} v_2^6 u^{8K}$ and $x^5 v_2^{\{2,6\}} \iota_{16K+1}$ we see that, modulo 16, their degrees are 8, -10 , 0, and 8 respectively so the only possibilities are $x^4 v_2^6 u^{8K}$ and $x^5 v_2^6 \iota_{16K+1}$. However, modulo 48, these are $-20 - 36 + 16K$ and $-36 - 36 + 16K$, or, $16K - 8$ and $16K - 24$ so it must be represented by $x^5 v_2^6 \iota_{16K+1}$ so our last undecided differential must be $d^7(v_2^2 \iota_{16K+1}) = v_2^6 \iota_{16K+1}$.

This concludes our computation of $ER(2)^*(RP^{16K+1})$ using the Bockstein spectral sequence and we collect our results here.

Theorem 8.2. *The Bockstein spectral sequence for $ER(2)^*(RP^{16K+1})$ is as follows.*

E^1

$$\begin{aligned} v_2^i \alpha^k u^j & \quad 0 \leq i < 8 \quad 0 \leq k \quad 0 < j \leq 8K. \\ v_2^i \alpha_0^q \alpha^k \iota_{16K+1} & \quad 0 \leq i < 8 \quad 0 \leq q \quad 0 \leq k. \end{aligned}$$

$$d^1(v_2^{2s-5} \alpha^k u^j) = 2v_2^{2s} \alpha^k u^j = v_2^{2s} \alpha^{k+1} u^{j+1} \quad j < 8K$$

(modulo higher powers of u).

$$d^1(\alpha_0^q v_2^{2s+1} \alpha^k \iota_{16K+1}) = \alpha_0^{q+1} v_2^{2s-2} \alpha^k \iota_{16K+1}$$

 E^2

$$\begin{aligned} v_2^{2s} \alpha^k u \quad 0 \leq k, \quad v_2^{2s} u^j \quad 1 < j \leq 8K, \quad v_2^{2s+1} \alpha^k u^{8K} \quad 0 \leq k \\ v_2^{2s} \alpha^k \iota_{16K+1}. \end{aligned}$$

$$d^2(v_2^{2s+1} \alpha^k u^{8K}) = v_2^{6+2s} \alpha^{k+1} \iota_{16K+1}.$$

 E^3 :

$$v_2^{2s} \alpha^k u \quad 0 \leq k, \quad v_2^{2s} u^j \quad 1 < j \leq 8K, \quad v_2^{2s} \iota_{16K+1}.$$

$$\begin{aligned} d^3(v_2^{\{6,2\}} \alpha^k u) &= v_2^{\{0,4\}} \alpha^{k+1} u \\ d^3(v_2^{\{6,2\}} u^j) &= v_2^{\{0,4\}} \alpha u^j = v_2^{\{0,4\}} u^{j+2} \quad 1 < j \leq 8K - 2 \end{aligned}$$

 E^4 :

$$v_2^{\{0,4\}} u^{\{1-3\}}, \quad v_2^{\{6,2\}} u^{\{8K-1, 8K\}}, \quad v_2^{2s} \iota_{16K+1}.$$

$$d^4(v_2^{\{6,2\}} u^{8K-1}) = v_2^{\{0,4\}} \iota_{16K+1}.$$

 $E^5 = E^6 = E^7$:

$$v_2^{\{0,4\}} u^{\{1-3\}}, \quad v_2^{\{6,2\}} u^{8K}, \quad v_2^{\{2,6\}} \iota_{16K+1}.$$

$$d^7(v_2^4 u^{\{1-3\}}) = u^{\{1-3\}}$$

$$d^7(v_2^2 u^{8K}) = v_2^6 u^{8K}$$

$$d^7(v_2^2 \iota_{16K+1}) = v_2^6 \iota_{16K+1}$$

We identify all of the elements in degree $8*$. This completes the proof of Theorem 1.3.

Theorem 8.3. *A 2-adic basis for the elements in $ER(2)^{8*}(RP^{16K+1})$ is given by the following.*

From the x^1 -torsion elements we have

$$\alpha^k u^j \quad 0 < k \quad 1 < j \leq 8K$$

represents elements with the same name. Also

$$\alpha_2 \alpha^k u^j \quad 0 \leq k \quad 0 < j < 8K$$

is represented by

$$2v_2^4 \alpha^k u^j = v_2^4 \alpha^{k+1} u^{j+1}$$

modulo higher powers of u .

From the x^2 -torsion we have

$$\alpha^k u^{8K+1} \quad k > 0$$

is represented by

$$\alpha^k x \iota_{16K+1}.$$

and

$$\alpha_2 \alpha^k u^{8K} = \alpha^{k+1} y u^{8K-3}$$

is represented by

$$v_2^4 \alpha^{k+1} x \iota_{16K+1}.$$

From the x^3 -torsion we have

$$\alpha^{k+1} u$$

represents the element with the same name and

$$v_2^4 \alpha^{k+1} u$$

represents

$$w \alpha^k u.$$

$$u^j \quad 3 < j \leq 8K$$

represents the element with the same name and

$$v_2^4 u^j \quad 3 < j \leq 8K$$

represents

$$y u^{j-4}.$$

From the x^4 -torsion we have

$$x \iota_{16K+1}$$

represents

$$u^{8K+1}$$

and

$$v_2^4 x \iota_{16K+1}$$

represents

$$y u^{8K-3}.$$

From the x^7 -torsion we have

$$u^{\{1-3\}}$$

represent elements of the same name.

$$x^4 v_2^6 u^{8K}$$

represents

$$y u^{8K-2}.$$

and

$$x^5 v_2^6 \iota_{16K+1}$$

represents

$$y u^{8K-1}.$$

Proof. We can find all of the elements in degrees $8*$ by looking at the Bockstein spectral sequence. If we check the elements that are x^1 -torsion, i.e. the image of d^1 , the elements in degrees $8*$ are just $2v_2^{\{0,4\}}\alpha^k u^j$, and these are, modulo higher filtrations, $v_2^{\{0,4\}}\alpha^{k+1}u^{j+1}$, $0 < j < 8K$. These can be written as $\alpha_i\alpha^k u^j$ with $i = 0$ and 2 , $0 < j < 8K$.

Elements in degree $8*$ coming from the x^2 torsion are represented by x times $v_2^{\{0,4\}}\alpha^{k+1}t_{16K+1}$. In our computation of d^2 in the Bockstein spectral sequence we showed that z_{16K-16} mapped to xt_{16K+1} . We also showed it mapped to u^{8K+1} . This was only x^2 torsion if we multiplied by α^{k+1} . This gives us our $\alpha^{k+1}u^{8K+1}$ which is $2\alpha^k u^{8K}$.

From the proof of Theorem 5.1 we know that $w\alpha^k z_{16K-16}$ maps to $\alpha^{k+1}yu^{8K-3}$ and is represented by $v_2^4\alpha^{k+1}u^{8K+1}$, or, x times $v_2^4\alpha^{k+1}t_{16K+1}$. This identifies our remaining $8*$ degree elements coming from the x^2 torsion.

The only elements that come from the x^3 -torsion are again standard elements, $w^\epsilon\alpha^k u$ with $\epsilon + k > 0$, u^j with $3 < j \leq 8K$, and yu^j with $0 \leq j \leq 8K - 4$.

From the x^4 -torsion we get $xv_2^{\{0,4\}}t_{16K+1}$ in the expected degrees. We have already identified each of these as u^{8K+1} and yu^{8K-3} respectively.

Of course our x^7 -torsion $u^{\{1-3\}}$ is standard.

We have only the x^7 -torsion elements $v_2^6 u^{8K}$ and $v_2^6 t_{16K+1}$ remaining to consider. The only elements in the appropriate degrees are x^4 times the first and x^5 times the second. We have already identified the last one as yu^{8K-1} . We have not identified the necessary element yu^{8K-2} which we now see must be $x^4 v_2^6 u^{8K}$. □

We have a Corollary:

Corollary 8.4. *There is a purely algebraic map*

$$ER(2)^{8*}(RP^{16K+1}) \longrightarrow E(2)^{8*}(RP^{16K+2})$$

which only misses the elements $v_2^4 u^{\{1-3\}}$.

9. AXIAL MAPS AND ODD SPACES

Recall that Don Davis uses $E(2)^*(-)$ to show that the axial map

$$RP^{2^K-2k-2} \times RP^{2n} \longrightarrow RP^{2^K-2n-2}$$

does not exist when $n = 2(m + \alpha(m) - 1)$ and $k = 2(2m - \alpha(m))$ giving him

$$RP^{2(m+\alpha(m)-1)} \not\subseteq \mathbb{R}^{2(2m-\alpha(m))}.$$

From the previous section we know that there is an algebraic map, which, for our purposes, is surjective enough, when $-2k - 2 = 2 \pmod{16}$:

$$ER(2)^*(RP^{2^K-2k-3}) \longrightarrow ER(2)^*(RP^{2^K-2k-2})$$

We will be able to use our standard tricks to show that

$$RP^{2(m+\alpha(m)-1)} \not\subseteq \mathbb{R}^{2(2m-\alpha(m))+1}.$$

when

$$-k - 1 = 1 = -2m + \alpha(m) - 1 \pmod{8}$$

and

$$m + \alpha(m) - 1 = n = \{3, 4, 7, 0\}.$$

Presumably one could compute $ER(2)^*(RP^{16K+9})$ and prove a similar theorem with $-k-1=5$. Although the results are probably new they are not of sufficiently low dimensions as to interest us.

Before we proceed let's check out the numbers here. We have

$$\begin{aligned} 2m - \alpha(m) &= -2 \pmod{8} \\ m + \alpha(m) &= \{4, 5, 0, 1\} \end{aligned}$$

Adding we get

$$3m = \{2, 3, 6, 7\}.$$

Multiply by 3 (always mod 8)

$$m = \{6, 1, 2, 5\}.$$

Then

$$\alpha(m) = \{6, 4, 6, 4\}.$$

This is Theorem 1.4 in the introduction.

The rest of the paper is dedicated to the proof.

10. THE 16^* CASES.

We now have the diagram in degrees 16^* :

(10.1)

$$\begin{array}{ccc} E(2)^*(RP^{2n}) \otimes_{E(2)^*} E(2)^*(RP^{2^K-2k-2}) & \longleftarrow & E(2)^*(RP^{2^K-2n-2}) \\ \uparrow & & \uparrow \\ ER(2)^*(RP^{2n}) \otimes_{ER(2)^*} ER(2)^*(RP^{2^K-2k-3}) & \longleftarrow & ER(2)^*(RP^{2^K-2n-2}). \end{array}$$

We assume throughout that $2^K - 2k - 3$ is equal to 1 mod 16.

In the case of $n = 7$ or $0 \pmod{8}$, we have isomorphisms in degrees 16^* on the far left term and the right term, and, in the relevant degrees, a surjection in the right side of the tensor product.

We use the fact that $u^{2^{K-1}-n}$ on the right is zero.

Consequently, with the Don Davis's restrictions on n and k we know our obstruction is non-zero in the tensor product. All we have to do now is show that the elements corresponding to that tensor product also exist in

$$ER(2)^{16^*}(RP^{2n} \wedge RP^{2^K-2k-3}).$$

As in previous cases, because the algorithm always increases the number of u 's and because the coproduct of u can be computed as $u_1 - u_2$ up to a unit, we just need the elements $u_1^i u_2^j$ to be non-zero when $i + j$ is big. (We don't really have to worry about the $\alpha u_1^i u_2$ terms because they don't have enough u 's in them.)

Most of the $u_1^i u_2^j$ are obviously non-zero and independent because they reduce to $E(2)^*(-)$. The only elements this doesn't work for are taken care of by the following theorem.

Theorem 10.2. *When $n \leq 8M < 8M + 8 < 8K$, in*

$$ER(2)^{16^*}(RP^{2n} \wedge RP^{16K+1}).$$

the element $u_1^n u_2^{8K+1}$ is non-zero.

If this element is non-zero, since the elements $u_1^i u_2^{8K+1}$ are defined and u_1^{n-i} times them is non-zero, they too are all non-zero.

This result will complete the proof of the theorem in the previous section for the $n = 7, 0$ cases.

11. PRODUCTS WITH AN ODD SPACE

We study the Bockstein spectral sequence for

$$ER(2)^*(RP^{2n} \wedge RP^{16K+1}).$$

where $2n < 16K + 1$. The E^1 term is, as usual, just

$$E(2)^*(RP^{2n} \wedge RP^{16K+1}).$$

This has a few more pieces than we are used to. Because

$$E(2)^*(RP^{16K+1}) \simeq E(2)^*(RP^{16K}) \oplus E(2)^*(S^{16K+1}).$$

Since $E(2)^*(S^{16K+1})$ is free it doesn't affect the Tor term, only the tensor product term. So our E^1 is

$$\begin{aligned} E(2)^*(RP^{2n}) \otimes E(2)^*(RP^{16K}) \oplus E(2)^*(RP^{2n}) \otimes E(2)^*(S^{16K+1}) \\ \oplus \Sigma^{-16(8K)-1} E(2)^*(RP^{2n}) \end{aligned}$$

Keep in mind that

$$E(2)^*(RP^{2n}) \otimes E(2)^*(S^{16K+1}) \simeq \Sigma^{16K+1} E(2)^*(RP^{2n})$$

The $-16(8K) - 1$ looks silly and can be replaced with $16K - 1$ since we are working modulo 48.

We need our 2-adic basis for our E^1 term.

$$v_2^s \alpha^k u_1^i u_2 \quad 0 \leq k \quad 0 < i \leq n \quad s < 8$$

$$v_2^s u_1^i u_2^j \quad 0 < i \leq m \quad 1 < j \leq 8K \quad s < 8$$

and

$$v_2^s \alpha^k u_1^i \iota_{16K+1} \quad 0 \leq k \quad 0 < i \leq n \quad s < 8$$

$$v_2^s \alpha^k u_1^i z_{16K-17} \quad 0 \leq k \quad 0 \leq i < n \quad s < 8$$

We know that $x \iota_{16K+1}$ represents u_2^{8K+1} so $x u_1^n \iota_{16K+1}$ represents $u_1^n u_2^{8K+1}$. There can be no differential on $u_1^n \iota_{16K+1}$ because it is a product of elements. All we have to do is show that it is not in the image of d^1 . Since d^1 is even degree we only have to worry about it on the odd degree elements since $u_1^n \iota_{16K+1}$ is odd degree.

d^1 has degree 18 so if d^1 is to hit $u_1^n \iota_{16K+1}$ it must start on some $\alpha^k u_1^i z_{16K-17}$ because they are the only elements in the appropriate degree mod 16. Since d^1 commutes with α we would have to have $u_1^i z_{16K-17}$ hitting $u_1^n \iota_{16K+1}$ and since d^1 commutes with multiplication by u_1 we would have to have d^1 be non-trivial on z_{16K-17} .

In the Bockstein spectral sequence for $ER(2)^*(RP^{16M+16} \times RP^{16K+2})$, $8M+8 < 8K$ we have, from Theorem 19.2 of Part I, that $d^1(z_{16K-33}) = 0$. From Theorem 1.2 of [GW], z_{16K-33} maps to $u_1 z_{16K-17}$ in the spectral sequence for $RP^{16M+16} \times RP^{16K}$. Since this passes through the spectral sequence for $RP^{16M+16} \times RP^{16K+1}$, z_{16K-33} maps to $u_1 z_{16K-17}$ here as well so $d^1(u_1 z_{16K-17}) = 0$, i.e. u_1 multiplied times $d^1(z_{16K-17})$ is zero. All elements killed by multiplication by u_1 go to zero

under the map to $RP^{8M} \times RP^{16K+1}$, and so, in here, our $d^1(z_{16K-17}) = 0$ and our result follows by naturality.

This concludes our proof for the $n = 7, 0$ cases.

12. THE $16 * + 8$ CASES.

Again, we have the diagram when each factor is restricted to degrees $8*$:

$$(12.1) \quad \begin{array}{ccc} E(2)^*(RP^{2n}) \otimes_{E(2)^*} E(2)^*(RP^{2^K-2k-2}) & \longleftarrow & E(2)^*(RP^{2^K-2n-2}) \\ \uparrow & & \uparrow \\ ER(2)^*(RP^{2n}) \otimes_{ER(2)^*} ER(2)^*(RP^{2^K-2k-3}) & \longleftarrow & ER(2)^*(RP^{2^K-2n-2}). \end{array}$$

We continue to assume that $2^K - 2k - 3$ is equal to 1 mod 16.

We now switch to the cases of $n = 3, 4 \pmod 8$. We now have $yu^{2^{K-1}-n-4} = 0$ on the lower right which maps to $v_2^4 u^{2^{K-1}-n} = 0$ in the upper right. [Dav84] shows this goes to non-zero in the upper left hand corner.

In the tensor product we have surjections for degrees $8*$ in both factors for high powers of u_i .

The discussion now is nearly identical to that for the even products we studied first in this paper. The end result that we need to complete the work is:

Theorem 12.2. *When $m \leq 8M$ and $8M + 8 < 8K$ the element*

$$u_1^m u_2^{8K-3} y_2 \in ER(2)^*(RP^{2m} \wedge RP^{16K+1})$$

is non-zero.

We have already written down the E^1 term for this. The element that represents $u_1^m u_2^{8K-3} y_2$ is $v_2^4 x u_1^m \iota_{16K+1}$. All we have to do is show that $v_2^4 u_1^m \iota_{16K+1}$ is not the target of a d^1 . If there is such a differential for $8M + 8$, just like in the last case, $d^1(z_{-16n-17})$ must be non-zero and u_1 times the target must be zero. All such target elements go to zero when we map down to $m \leq 8M < 8M + 8$ and so the differential is trivial there.

This concludes the proof of the final cases.

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