MODULI OF SUSPENSION SPECTRA

JOHN R. KLEIN

Abstract. For a 1-connected spectrum E, we study the moduli space of suspension spectra which come equipped with a weak equivalence to E. We construct a spectral sequence converging to the homotopy of the moduli space in positive degrees. In the metastable range, we get a complete homotopical classification of the path components of the moduli space. Our main tool is Goodwillie's calculus of homotopy functors.

1. Introduction

It's well known that the singular cohomology of a space has the structure of a commutative graded ring. The corresponding statement for spectra fails to hold. The problem arises from the observation that a spectrum E might not have a "diagonal map" $E \to E \wedge E$. Of course, a diagonal map exists when E is a suspension spectrum. This motivates

Question. Let E be a spectrum. When can we find a based space X and a weak equivalence of spectra $\Sigma^\infty X \simeq E$? In how many ways?

Our first result gives a criterion for deciding the existence part of this question in the metastable range. Recall that E is *r-connected* if its homotopy groups $\pi_*(E)$ vanish when $* \le r$. Write $\dim E \le n$ if E is, up to homotopy, a CW spectrum with cells in dimensions $\le n$. Recall that the *k-th extended power* $D_k(E)$ is the homotopy orbit spectrum of Σ_k acting on the k-fold smash product $E^{\wedge k}$.

Theorem A (Existence). *There is an obstruction*

$$\delta_E \in [E, \Sigma D_2(E)]\,,$$

which is trivial when E has the homotopy type of a suspension spectrum.

Conversely, if $\delta_E = 0$, E is r-connected, $r \ge 1$ and $\dim E \le 3r+2$, then E has the homotopy type of a suspension spectrum.

Before stating our second result, we comment on the relation between Theorem A and one of the early results of Kuhn ([Ku, Th. 1.2])

Date: October 17, 2002.
2000 MSC. Primary: 55P42, 55P43. Secondary: 55P40, 55P65.
The author is partially supported by NSF Grant DMS-0201695.

which says that a connected spectrum E is a *retract* of a suspension spectrum if and only if there is a weak equivalence

$$\Sigma^\infty \Omega^\infty E \;\simeq\; \bigvee_{k\geq 1} D_k(E)\,.$$

Theorem A implies one can remove the word "retract" from Kuhn's result in the metastable range.

Next consider the collection of pairs (X, h) in which X is a based space and $h\colon \Sigma^\infty X \to E$ is a weak equivalence. Equate two such pairs (X, h) and (Y, g) if and only if there is a map of spaces $f\colon X \to Y$ such that $g \circ \Sigma^\infty f$ is homotopic to h (in particular, f is a homology isomorphism). This generates an equivalence relation. Let Θ_E denote the associated set of equivalence classes.

Our second result identifies Θ_E in the metastable range.

Theorem B (Enumeration)**.** *Assume Θ_E is nonempty and is equipped with basepoint. Then there is a basepoint preserving function*

$$\phi\colon \Theta_E \to [E, D_2(E)]\,.$$

If E is r-connected, $r \geq 1$ and $\dim E \leq 3r + 2$, then ϕ is a surjection. If in addition $\dim E \leq 3r + 1$, ϕ is a bijection. In particular, Θ_E has the structure of an abelian group when $\dim E \leq 3r + 1$.

Theorem B leads to the possibility of calculating Θ_E in a number of simple cases. For example, table 4.1 appearing in Mahowald's memoir [Mah] provides extensive calculations of Θ_E at the prime 2 in the case of spectra with two cells (see §8 below).

In fact, Θ_E is the set of path components of a space $\mathfrak{M}_E$ which is defined as follows: let C_E be the category whose objects are pairs (Y, h) with Y a based space and $h\colon \Sigma^\infty Y \to E$ a weak equivalence. A morphism $(Y, h) \to (Z, g)$ consists of a map of based spaces $f\colon Y \to Z$ such that $g \circ \Sigma^\infty f = h$. Then define

$$\mathfrak{M}_E \;:=\; |C_E|\,,$$

i.e., the geometric realization of (the nerve of) C_E.[1] We call $\mathfrak{M}_E$ the *moduli space* of suspension structures on E.

Our third result gives a spectral sequence converging to the homotopy of $\mathfrak{M}_E$ in positive degrees. Formulating it requires some preparation.

[1]There are set-theoretic difficulties presented by this definition, since C_E isn't a small category. This problem can be avoided in several ways (compare [Wa, p. 379]).

For $q \geq 2$, let W_q be the spectrum with Σ_q-action which classifies the q-th layer of the Goodwillie tower of the identity functor from based spaces to based spaces. Unequivariantly, W_q is a wedge of $(q-1)$ copies of the $(1-q)$-sphere spectrum (see Johnson [Jo]). If $q \leq 1$, we take W_q to be the trivial spectrum.

If E is a spectrum, then its q-fold smash power $E^{\wedge q}$ has the structure of a (naive) spectrum with Σ_q-action. Give the smash product $W_q \wedge E^{\wedge q}$ the diagonal Σ_q-action. We are then entitled to form

$$W_q \wedge_{h\Sigma_q} E^{\wedge q}\,,$$

i.e, the homotopy orbit spectrum of Σ_q acting on $W_q \wedge E^{\wedge q}$. Let $F(E, W_q \wedge_{h\Sigma_q} E^{\wedge q})$ denote the function space of spectrum maps from E to $W_q \wedge_{h\Sigma_q} E^{\wedge q}$. With these definitions we are ready to state our third result.

Theorem C (Spectral Sequence)**.** *Let E be 1-connected. Assume $\mathfrak{M}_E$ is non-empty and equipped with basepoint. Then there is a first quadrant spectral sequence converging to $\pi_{p+1}(\mathfrak{M}_E)$ with*

$$E^1_{p,q} := \pi_p(F(E, W_q \wedge_{h\Sigma_q} E^{\wedge q}))$$

and d^1-differential of bi-degree $(-1,1)$.

Note when $\dim E \leq n$, we have $E^1_{p,q} = 0$ for $p \leq q+1-n$. Thus for fixed p, $E^1_{p,q}$ is non-zero for finitely many q.

Our last result says that the obstruction δ_E is, in some sense, 2-local.

Theorem D (Localization)**.** *For any connected spectrum E, the homotopy class δ_E becomes trivial after inverting the prime 2.*

Furthermore, if $f\colon E \to E'$ is a map of spectra which is a 2-local weak equivalence, then $\delta_E = 0$ if and only if $\delta_{E'} = 0$.

Consequently, if E and E' satisfy the connectivity assumptions of Theorem A, then E is weak equivalent to a suspension spectrum if and only E' is weak equivalent to a suspension spectrum.

Remarks. Starting with the work with Berstein and Hilton [B-Hi], the problem of deciding when a space has the homotopy type of a (single or iterated) suspension has been intensively studied.

To the best of my knowledge, the literature contains much less information about the desuspension problem for spectra, aside from the trivial stable range case (Freudenthal's theorem) and a p-local version considered by Gray [Gr].

My original interest in the desuspension question for spectra came from embedding theory. In a future paper, we intend to use the above results to attack certain embedding questions.

Acknowledgements. This paper would not have come into being had I not had the fortune of discussing mathematics with Greg Arone, Randy McCarthy and Tom Goodwillie in the middle to late 1990s. I am indebted to them for their help. I am also grateful to Bob Bruner for explaining to me how to compute the stable homotopy of stunted projective space in low degrees. I am also grateful to Bruner for drawing my attention to the early work of Nick Kuhn.

Outline. §2 consists of preliminary material. The proof of Theorem A is contained in §3. In §4 we prove Theorem B. In §5 we express the connected components of the moduli space $\mathfrak{M}_E$ as the classifying space of a suitable monoid. §6 contains the proof of Theorem C. Theorem D is proved in §7. In §8 we provide examples in connection with Theorem B. In §9 we discuss some loose ends.

2. Preliminaries

In this section we give the conventions and tools used in the rest of the paper. We make no claim to completeness.

Spaces. All spaces below will be compactly generated, and **Top** will denote the category of compactly generated spaces. In particular, we make the convention that products are to be retopologized with respect to the compactly generated topology. Let $\mathbf{Top}_*$ denote the category of based spaces. A *weak equivalence* of spaces is shorthand for (a chain of) weak homotopy equivalence(s).

We use the usual connectivity terminology for spaces: A non-empty space is *r-connected* if its homotopy vanishes in degrees $\le r$ for every choice of basepoint (in particular, every non-empty space is (-1)-connected). A map $A \to B$ of spaces, with B nonempty, is *r-connected* if for any choice of basepoint in B, the homotopy fiber with respect to this choice of basepoint is an $(r-1)$-connected space.

A commutative square of spaces

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array}$$

is said to be *k-cocartesian* if the map $\mathrm{hocolim}(B \leftarrow A \to C) \to D$ is k-connected. It is ∞-cartesian if it is k-cartesian for all k. Dually, the square is *k-cartesian* if the map $A \to \mathrm{holim}\,(B \to D \leftarrow C)$ is k-connected. The square is ∞-(co)cartesian if it is k-(co)cartesian for all k.

Spectra. A *spectrum* E will be taken to mean a collection of based spaces $\{E_i\}_{i\in\mathbb{N}}$ together with based maps $\Sigma E_i \to E_{i+1}$ where ΣE_i is the reduced suspension of E_i. A *morphism* of spectra $E \to E'$ consists of maps $E_i \to E'_i$ that are compatible with the structure maps. We denote the category of spectra by $\mathbf{Sp}$.

A map of spectra is *r-connected* if it induces a surjection on homotopy up through degree r and an isomorphism in degrees less than r. A spectrum is r-connected if the map to the trivial spectrum (consisting of the one point space in each degree) is $(r+1)$-connected.

A map of spectra is a *weak equivalence* if it is r-connected for all integers r. This notion of weak equivalence comes from a Quillen model structure on $\mathbf{Sp}$. In this model structure, the fibrant objects are the Ω-spectra (those spectra E such that $E_i \to \Omega E_{i+1}$ is a weak equivalence for all i. A cofibrant object is (a retract of) a spectrum which is built up from the zero object by attaching cells. The model structure on $\mathbf{Sp}$ comes equipped with functorial factorizations; in particular, fibrant and cofibrant approximation is functorial. For details, see for example Schwede [Sc].

We typically apply fibrant and/or cofibrant approximation functors to maintain homotopy invariance. To avoid clutter, we usually suppress the application of these approximations in the notation.

For example, if E is a fibrant spectrum we write $\Omega^\infty E$ for the zero space E_0 of E. If E isn't fibrant, to get a good construction we first replace E by its associated Ω-spectrum $E^\sharp$ and then define $\Omega^\infty E$ as $E_0^\sharp$.

If X is a based space, its *suspension spectrum* $\Sigma^\infty X$ has j-th space $\Sigma^j X$, the j-fold reduced suspension of X (for this to have the correct homotopy type, we assume that X is a cofibrant space; i.e., the retract of a cell complex). In particular, $Q(X) := \Omega^\infty\Sigma^\infty X$ is the reduced stable homotopy functor. A map of spaces $Y \to \Omega^\infty E$ is adjoint to a map of spectra $\Sigma^\infty Y \to E$.

As in the introduction, we write $\dim E \le n$ if and only if E is, up to homotopy, obtained from the zero object by attaching cells of dimension $\le n$. A spectrum is *finite* if it is built up from the zero object by attaching a finite number of cells. A spectrum is *homotopy finite* if it is weak equivalent to a finite spectrum.

In this paper, we can get away with a notion of smash product which is associative, commutative and unital up to homotopy (see e.g. Lewis, May and Steinberger [L-M-S, Chap. 2]). We also need to know that a construction of the extended power spectrum

$$D_k(E) = (E^{\wedge k})_{h\Sigma_k} := (E\Sigma_k)_+ \wedge_{\Sigma_k} E^{\wedge k}$$

exists, is functorial, homotopy invariant and coincides with the usual one in the case of spaces. For details, see [L-M-S, Chap. 6].

Truncation. Let Y be a based space, W be a 1-connected spectrum and $f\colon \Sigma^\infty Y \to W$ a map of spectra. Assume $\dim W \le n$ and that f is n-connected.

Lemma 2.1. *If $n \ge 0$, there exists a space Z, and an $(n-1)$-connected map $g\colon Z \to Y$ such that the composite*

$$\Sigma^\infty Z \overset{\Sigma^\infty g}{\to} \Sigma^\infty Y \overset{f}{\to} W$$

is a weak equivalence.

Proof. If $n \le 1$, then W is weak equivalent to the zero object, and the proof is trivial in this case.

Assume then that $n \ge 2$. Let $h\mathbb{Z}$ be the Eilenberg-Mac Lane spectrum. Then the induced map

$$h\mathbb{Z} \wedge \Sigma^\infty Y \to h\mathbb{Z} \wedge W$$

can be thought of as a map of chain complexes (via the Dold-Kan correspondence). Applying the *truncation lemma* [Kl, 4.1], We obtain a space Z and an $(n-1)$-connected map $Z \to Y$ such that the composite

$$h\mathbb{Z} \wedge \Sigma^\infty Z \to h\mathbb{Z} \wedge \Sigma^\infty Y \to h\mathbb{Z} \wedge W$$

is a weak equivalence. But this composite is obtained by smashing the composite

$$\Sigma^\infty Z \to \Sigma^\infty Y \to W$$

with $h\mathbb{Z}$. The result then follows by application of Whitehead's theorem. □

Corollary 2.2 (Freudenthal)**.** *Let E be a an r-connected spectrum, $r \ge 1$. Assume $\dim E \le 2r+2$. Then E is weak equivalent to a suspension spectrum.*

Proof. This follows from 2.1 because the map $\Sigma^\infty \Omega^\infty E \to E$ is $(2r+2)$-connected (see 3.2 below). □

Another description of the moduli space. We can use Lemma 2.1 to give an alternative description of the moduli space when E is 1-connected. Let C'_E be the (not full) subcategory of C_E having the same objects, but with the property that morphisms $(Y, h) \to (Y', h')$ satisfy the condition that $Y \to Y'$ is a *weak homotopy equivalence.* Let $\mathfrak{M}'_E$ denote the realization of C'_E.

Proposition 2.3. *Assume that E is fibrant, cofibrant and 1-connected. Then the inclusion $\mathfrak{M}'_E \subset \mathfrak{M}_E$ is a homotopy equivalence.*

Proof. Let $Y \mapsto Y^+$ denote the plus construction. Applying functorial factorization, we can arrange it so that the natural map $Y \to Y^+$ is a cofibration.

Observe that if (Y, h) is an object of C_E, then the 1-connectedness of E implies Y has trivial first homology. Therefore, Y^+ is a 1-connected space. So if $(Y, h) \to (Z, h')$ is a morphism, the induced map of plus constructions $Y^+ \to Z^+$ is a weak equivalence by Whitehead's theorem.

Since E is fibrant and cofibrant, the space $\Omega^\infty E$ is cofibrant and 1-connected. The natural map $\Omega^\infty E \to (\Omega^\infty E)^+$ is a cofibration which is a weak equivalence. Consequently, there is a retraction $r\colon (\Omega^\infty E)^+ \to \Omega^\infty E$.

Let $i\colon C'_E \to C_E$ denote the inclusion. Define a functor

$$\pi\colon C_E \to C'_E$$

by the rule $\pi(Y, h) = (Y^+, h^\flat)$, where $h^\flat\colon \Sigma^\infty(Y^+) \to E$ is adjoint to the composite

$$Y^+ \xrightarrow{(\hat{h})^+} (\Omega^\infty E)^+ \xrightarrow{r} \Omega^\infty E\,.$$

Then there is an evident natural transformation from the identity functor of C_E to the composite functor $i \circ \pi$. There is a similar evident natural transformation from the identity functor of C'_E to $\pi \circ i$. It follows that i induces a homotopy equivalence on realizations. $\square$

3. Proof of Theorem A

After defining the obstruction δ_E, the idea of the remainder of the proof will be to construct a highly connected map from a suspension spectrum to E. The proof is then completed by applying Lemma 2.1.

Definition of the obstruction. We describe below a certain fibration sequence of spectra

$$D_2(E) \to S_2(E) \to E$$

in which $S_2\colon \mathbf{Sp} \to \mathbf{Sp}$ is a certain homotopy functor.

Assuming this construction has been specified, we have:

Definition 3.1. The class

$$\delta_E \in [E, \Sigma D_2(E)]$$

is the obstruction to splitting the above fibration sequence, i.e., the homotopy class of its connecting map to the right.

Construction of the fibration. Consider the homotopy functor

$$\Sigma^\infty\Omega^\infty\colon \mathbf{Sp} \to \mathbf{Sp}$$

which assigns to a spectrum the suspension spectrum of its zero space. Let $S_k(E)$ denote the k-th stage of the Goodwillie tower of this functor and let $F_k(E) := \text{fiber}(S_k(E) \to S_{k-1}(E))$ denote the k-th layer.

The following result has been noted by several people, including Goodwillie, Arone, McCarthy, and Ahearn and Kuhn [A-K, Cor. 1.3].

Lemma 3.2. *Assume E is r-connected. Then the map*

$$\Sigma^\infty\Omega^\infty E \to S_k(E)$$

is $((k{+}1)r + k{+}1)$-connected. Consequently, the Goodwillie tower of $\Sigma^\infty\Omega^\infty E$ is convergent if E is 0-connected. Furthermore, there is a natural weak equivalence of functors

$$F_k(E) \;\simeq\; D_k(E)\,.$$

Applying 3.2, we see that the bottom of the tower yields a fibration sequence of spectra

$$D_2(E) \to S_2(E) \to E$$

together with a $(3r{+}3)$-connected map $\Sigma^\infty\Omega^\infty E \to S_2(E)$.

Corollary 3.3. *Assume E is r-connected and $\dim E \le 3r + 3$. Then $\delta_E = 0$ if and only if the map $\Sigma^\infty\Omega^\infty E \to E$ admits a section up to homotopy.*

Proof. The class δ_E is trivial if and only if the $S_2(E) \to E$ has a section. The dimension constraint on E and the lemma show that $S_2(E) \to E$ admits a section if and only if $\Sigma^\infty\Omega^\infty E \to E$ admits a section up to homotopy. □

Now assume E is r-connected, $r \ge 1$, $\dim E \le 3r + 3$ and $\delta_E = 0$. By the above remarks, we are entitled to choose a homotopy section $\sigma\colon E \to \Sigma^\infty\Omega^\infty E$. Let

$$\Omega^\infty\sigma\colon \Omega^\infty E \to \Omega^\infty\Sigma^\infty\Omega^\infty E$$

be the corresponding map of zero spaces.

There is another map $c\colon \Omega^\infty E \to \Omega^\infty\Sigma^\infty\Omega^\infty E$ (*not* an infinite loop map) which is defined by taking the adjoint to the identity map $\Sigma^\infty\Omega^\infty E \to \Sigma^\infty\Omega^\infty E$.

Let Y be the homotopy pullback of the maps $\Omega^\infty\sigma$ and c. Thus we have an ∞-cartesian square of spaces

$$(1)\qquad \begin{array}{ccc} Y & \xrightarrow{j} & \Omega^\infty E \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \Omega^\infty\sigma} \\ \Omega^\infty E & \xrightarrow[c]{} & \Omega^\infty\Sigma^\infty\Omega^\infty E \end{array}$$

(commutative up to preferred homotopy).

Lemma 3.4. *With respect to the above assumptions, let* $\hat{j}\colon \Sigma^\infty Y \to E$ *be the adjoint to the map labeled* j *in diagram (1). Then* $\hat{j}$ *is* $(3r+2)$-*connected.*

Proof. Consider the diagram of spectra

$$(2)\qquad \begin{array}{ccccc} \Sigma^\infty Y & \xrightarrow{\Sigma^\infty j} & \Sigma^\infty\Omega^\infty E & \xrightarrow{q_E} & E \\ {\scriptstyle \Sigma^\infty i}\downarrow & & \downarrow{\scriptstyle \Sigma^\infty\Omega^\infty\sigma} & & \downarrow{\scriptstyle\sigma} \\ \Sigma^\infty\Omega^\infty E & \xrightarrow[\Sigma^\infty c]{} & \Sigma^\infty\Omega^\infty\Sigma^\infty\Omega^\infty E & \xrightarrow[q_{\Sigma^\infty\Omega^\infty E}]{} & \Sigma^\infty\Omega^\infty E \end{array}$$

in which the maps labeled with q are the counits to the adjunction for E and $\Sigma^\infty\Omega^\infty E$. The map $\hat{j}$ is therefore given by the composite $q_E \circ \Sigma^\infty j$, and the composite along the bottom, $q_{\Sigma^\infty\Omega^\infty E} \circ \Sigma^\infty c$ is clearly the identity. The left square commutes up to a preferred homotopy and is ∞-cocartesian. The right square commutes on the nose.

The maps $\Omega^\infty\sigma$ and c in diagram (1) are both $(2r+1)$-connected. Consequently, by the dual Blakers-Massey theorem (see e.g., [Go2]) diagram (1) is a $(4r+2)$-cocartesian square of spaces. In particular, the left square in diagram (2) is a $(4r+2)$-cocartesian square of spectra.

Regarding the right square in diagram (2), If C denotes the homotopy cofiber of σ, then the evident map $C \to D_2(E)$ is $(3r+3)$-connected. Similarly, if C' denotes the homotopy cofiber of $\Sigma^\infty\Omega^\infty\sigma$, an argument using the Blakers-Massey theorem, which we omit, shows that the evident map $C' \to \Sigma^\infty\Omega^\infty D_2(E)$ is also $(3r+3)$-connected.

Lemma 3.2 shows that the map $\Sigma^\infty\Omega^\infty D_2(E) \to D_2(E)$ is $(4r+4)$-connected (because $D_2(E)$ is $(2r+1)$-connected). We infer that the map $C' \to C$ is also $(3r+3)$-connected. Therefore the right square in diagram (2) is $(3r+3)$-cocartesian.

Putting both squares together, it follows that diagram (2) is $(3r+3)$-cocartesian. Since the bottom composite is the identity map, we infer that the top composite $\hat{j}$ is $(3r+2)$-connected, as asserted. ∎

Completion of the proof. Assume in addition to the above that $\dim E \leq 3r{+}2$. Since the map $\hat{j}\colon \Sigma^\infty Y \to E$ is $(3r{+}2)$-connected we can apply 2.1. This gives a based space Z and a based map $Z \to Y$ such that the composite $\Sigma^\infty Z \to \Sigma^\infty Y \to E$ is a weak equivalence. This completes the proof of Theorem A.

4. Proof of Theorem B

Step 1. Let $\mathcal{S}_E$ denote the homotopy classes of sections of the fibration $S_2(E) \to E$. Note that the abelian group $[E, D_2(E)]$ acts freely and transitively on $\mathcal{S}_E$ (cf. Lemma 6.1 below). Thus if $\mathcal{S}_E$ is given a basepoint, it follows that $[E, D_2(E)]$ and $\mathcal{S}_E$ are isomorphic (the isomorphism is dependent on the choice of basepoint).

There is a function

$$\phi\colon \Theta_E \to \mathcal{S}_E$$

defined by sending (Y, h) to the class represented by

$$E \simeq \Sigma^\infty Y \to S_2(\Sigma^\infty Y) \simeq S_2(E)$$

(the map in the middle is the preferred section of $S_2(\Sigma^\infty Y) \to \Sigma^\infty Y$).

If $E \to S_2(E)$ is a representative of $\mathcal{S}_E$, and $\dim E \leq 3r + 2$, we can perform the constructions in the previous section to get an element of Θ_E (this element is not necessarily unique). It is straightforward to check that this element of Θ_E maps to the given element of $\mathcal{S}_E$. Thus, when $\dim E \leq 3r{+}2$, the function $\phi\colon \Theta_E \to \mathcal{S}_E$ is onto.

To complete the proof of Theorem B, it will be sufficient to show that ϕ is one-to-one when $\dim E \leq 3r + 1$. Choose a basepoint (Y, h) for Θ_E, then $\mathcal{S}_E$ inherits a basepoint and $\mathcal{S}_E$ becomes identified with $[E, D_2(E)]$. Thus we may rewrite ϕ as a basepoint preserving function

$$\Theta_E \to [E, D_2(E)]\,,$$

where the basepoint of the codomain is the zero element. Since Y was chosen arbitrarily, it suffices to show that ϕ is one-to-one at the inverse image of the basepoint.

Step 2. We digress to develop a relative version of Theorem A. Suppose that $A \rightarrowtail E$ is a cofibration in the category of spectra. We write $\dim(E, A) \leq n$ if E is obtained from A up to homotopy by attaching cells of dimension $\leq n$. Assume that $A = \Sigma^\infty Z$ is a suspension spectrum.

Theorem 4.1. *There is an obstruction*

$$\delta_{(E,A)} \in [E/A, \Sigma D_2(E)]$$

whose triviality is necessary to finding a cofibration $Z \to W$ and a weak equivalence $\Sigma^\infty W \simeq E$ extending the identity on $\Sigma^\infty Z = A$. If E is r-connected ($r \geq 1$) and $\dim(E, A) \leq 3r+2$, then the vanishing of this obstruction is also sufficient.

The obstruction $\delta_{(E,A)}$ is defined in the same way as δ_E taking care to notice that the restriction of δ_E to A has a preferred trivialization. The proof of 4.1 is virtually the same as the proof of Theorem A, so we omit it.

Step 3. We now return to the proof of Theorem B. We shall apply 4.1 in the following situation: choose a representative (Y, h) for an element in Θ_E (call this the basepoint). Suppose that (Y', h') represents an another element in Θ_E. Both elements combine to assemble to give a weak equivalence

$$\Sigma^\infty(Y \vee Y') \xrightarrow{\sim} E \vee E\,.$$

Applying 4.1 to the pair $(E \wedge I_+, E \vee E)$ we get a necessary obstruction

$$\delta_{(E\wedge I_+, E\vee E)} \in [\Sigma E, \Sigma D_2(E)] = [E, D_2(E)]$$

to finding a cofibration $Y \vee Y' \rightarrowtail W$ of based spaces and a weak equivalence

$$(\Sigma^\infty W, \Sigma^\infty(Y \vee Y')) \simeq (E \wedge I_+, E \vee E)\,.$$

It follows from this that the zig-zag $Y \to W \leftarrow Y'$ equates (Y, h) with (Y', h') in Θ_E.

Thus, fixing (Y, h) and allowing (Y', h') to vary defines a function

$$\Theta_E \to [E, D_2(E)]\,,$$

which is just another description of the function ϕ (we omit the details).

Assume that E is r-connected and $\dim E \leq 3r+1$. Then 4.1 shows that (Y', h') maps to zero under ϕ if and only if (Y', h') is the basepoint of Θ_E. This completes the proof of Theorem B.

5. $\mathfrak{M}_E$ AS A CLASSIFYING SPACE

Fix a cofibrant based space Z. Let $\mathbf{Top}_{*/Z}$ denote the category of based spaces over Z. An *object* y of this category consists of a based space Y together with a based map $p_Y : Y \to Z$. A *morphism* $(Y, p_Y) \to (Y', p_{Y'})$ is given by a map of based spaces $f : Y \to Y'$ such that $p_{Y'} \circ f = p_Y$.

Since $\mathbf{Top}_{*/Z}$ is an over category of the Quillen model category $\mathbf{Top}_*$, it follows that it too has the structure of a Quillen model category (see [Qu]). A *weak equivalence* is a morphism $y \to y'$ whose underlying map of spaces is a weak homotopy equivalence. We say $y \to y'$ is a *fibration* if its underlying map of spaces is. We say that $y \to y'$ is a *cofibration* if it satisfies the left lifting property with respect to the acyclic fibrations.

Let $w\mathbf{Top}_{*/Z}$ denote the subcategory consisting of the weak equivalences. Let $w_{(y)}\mathbf{Top}_{*/Z}$ denote the full subcategory of $w\mathbf{Top}_{*/Z}$ consisting of those objects connected to y by a chain of weak equivalences.

The proof of the following proposition, which we attribute to Waldhausen, is proved by the same method as [Wa, 2.2.5]. We omit the details.

Proposition 5.1. *There is a weak equivalence of spaces*

$$|w_{(y)}\mathbf{Top}_{*/Z}| \simeq BG(y)\,,$$

where $G(y)$ denotes the topological monoid of self homotopy equivalences of y in $\mathbf{Top}_{*/Z}$, and $BG(y)$ is its classifying space.

We apply this in the following special case: let E be a 1-connected fibrant and cofibrant spectrum. The category C'_E, whose realization is the moduli space $\mathfrak{M}'_E$ (cf. §2), is just the full subcategory of $w\mathbf{Top}_{*/\Omega^\infty E}$ whose objects Y are such that the map $p_Y\colon Y \to \Omega^\infty E$ is adjoint to a weak equivalence of spectra.

If we combine Proposition 5.1 with Proposition 2.3, we obtain

Corollary 5.2. *Let y be an object of C_E which is fibrant and cofibrant as an object of $\mathbf{Top}_{*/\Omega^\infty E}$. Let $\mathfrak{M}_{E,(y)}$ be the connected component of $\mathfrak{M}_E$ which contains y. Then there is a homotopy equivalence*

$$\mathfrak{M}_{E,(y)} \simeq BG(y)\,.$$

6. Proof of Theorem C

Outline of the proof. The basepoint $y = (Y, h)$ of $\mathfrak{M}_E$ can be taken as an object of C_E. Taking the plus construction if necessary, we can assume without loss in generality that Y is 1-connected (see the argument in the proof of Proposition 2.3). We may also assume that y is fibrant and cofibrant when considered as an object of $\mathbf{Top}_{*/\Omega^\infty E}$.

We will construct a tower of fibrations of based spaces

$$\cdots \to T_3(y) \to T_2(y) \to T_1(y)$$

such that

- $T_1(y)$ is contractible;
- for $k > 1$, there is a weak equivalence

  $$\mathrm{fiber}(T_k(y) \to T_{k-1}(y)) \;\simeq\; \Omega^\infty(W_k \wedge_{h\Sigma_k} Y^{\wedge k})\,,$$

  where W_k denotes the spectrum that classifies the k-th layer of the Goodwillie tower of the identity functor on based spaces.

- there is a weak equivalence

$$\Omega\mathfrak{M}_{E,(y)} \simeq \lim_k T_y(y)\,.$$

Assuming this has been done, we can define the spectral sequence $\{E^r_{p,q}\}$ as the homotopy spectral sequence of the tower $\{T_k(y)\}_k$.

We now digress to discuss generalities about section spaces and the basic properties of the Goodwillie tower of the identity functor.

Digression. Suppose $p\colon E \to Z$ is a fibration of based spaces. We say that p is *induced* if there exists a commutative ∞-cartesian square of based spaces

$$\begin{array}{ccc} E & \longrightarrow & P \\ {\scriptstyle p}\downarrow & & \downarrow \\ Z & \longrightarrow & B \end{array}$$

with the property that P is contractible.

Let

$$\mathrm{Sec}(p)$$

denote the space of sections of p. Suppose that Z is connected. If $p\colon E \to Z$ is induced, there is an "action" $\Omega B \times E \to E$. If there is a section $Z \to E$ one can combine it with this action to produce a map of fibrations $\Omega B \times Z \to E$ covering the identity map of Z. This implies that p is weak fiber homotopically trivial. In particular,

Lemma 6.1. *Assume $p\colon E \to Z$ is induced. Assume that* $\mathrm{Sec}(p)$ *is non-empty and comes equipped with basepoint. Then there is a weak equivalence of based spaces*

$$\mathrm{Sec}(p) \;\simeq\; F(Z, \Omega B)\,.$$

We next recall for the reader the basic properties of the Goodwillie tower of the identity functor on based spaces (cf. Goodwillie [Go1], [Go2], [Go3], [Go4], Johnson [Jo] and Arone [Ar]).

Theorem 6.2. *There is a tower of fibrations of homotopy functors on based spaces*

$$\cdots \to P_2(X) \to P_1(X)$$

and compatible natural transformations $X \to P_k(X)$ such that

- *$P_1(X) = Q(X)$ is the stable homotopy functor;*
- *the k-the layer $L_k(X)$ is naturally weak equivalent to the functor*

$$X \mapsto \Omega^\infty(W_k \wedge_{h\Sigma_k} X^{\wedge k})\,;$$

- *if X is 1-connected, then the natural map*

$$X \to \lim_k P_k(X)$$

is a weak equivalence.

An additional fact we will need, but which to my knowledge is not stated in the literature, is

Addendum 6.3. *For each $k \geq 2$, the fibration $P_k(Y) \to P_{k-1}(Y)$ is induced.*

This is due to Goodwillie (private communication). We will assume 6.3 without providing a proof.

Completion of the proof. We are now in a position to define the tower $\{T_k(y)\}_k$.

Definition 6.4. Let $T_k(y)$ be the space of lifts

$$\begin{array}{ccc} & & P_k(Y) \\ & \nearrow & \downarrow \\ Y & \longrightarrow & P_1(Y) \end{array}$$

where $Y \to P_1(Y)$ is the natural map. Note that $T_k(y)$ comes equipped with a basepoint defined by the natural map $Y \to P_k(Y)$.

From the definition of $T_k(y)$, there is an evident tower of fibrations of based spaces

$$\cdots \to T_2(y) \to T_1(y)$$

with $T_1(y) = *$. Furthermore, the fiber of the map $T_k(y) \to T_{k-1}(y)$ at the basepoint is identified with the space of lifts

$$\begin{array}{ccc} & & P_k(Y) \\ & \nearrow & \downarrow \\ Y & \longrightarrow & P_{k-1}(Y)\,. \end{array}$$

Note that this is just the space of sections of the pulled back fibration $Y \times_{P_{k-1}(Y)} P_k(Y) \to Y$. The latter fibration is induced and comes equipped with a preferred section.

Applying 6.1, 6.2 and 6.3, we obtain a weak equivalence of based spaces

$$\mathrm{fiber}(T_k(y) \to T_{k-1}(y)) \;\simeq\; F(Y, \Omega^\infty(W_k \wedge_{h\Sigma_k} Y^{\wedge k}))\,.$$

Taking adjunctions, we see that $F(Y, \Omega^\infty(W_k \wedge_{h\Sigma_k} Y^{\wedge k}))$ is weak equivalent to the space $F(E, W_k \wedge_{h\Sigma_k} E^{\wedge k})$ (where we are using the identification $\Sigma^\infty Y \simeq E$).

To complete the proof of Theorem C, it suffices to identify the inverse limit of the tower $\{T_k(y)\}_k$. It is clear that this inverse limit is just the space of lifts

$$\begin{array}{ccc} & & \lim_k P_k(Y) \\ & \nearrow & \downarrow \\ Y & \longrightarrow & P_1(Y)\,. \end{array}$$

Recall that $P_1(Y) = Q(Y)$ and since Y is 1-connected, $\lim_k P_k(Y) \simeq Y$. With respect to these identifications, we infer that this space of lifts is weak equivalent to the underlying space of the topological monoid $G(y)$. Applying 5.2 and 2.3 completes the proof of Theorem C.

7. Proof of Theorem D

Let $\mathrm{tr}\colon D_2(E) \to E^{\wedge 2}$ be the transfer and let $\pi\colon E^{\wedge 2} \to D_2(E)$ be the projection ([L-M-S, Chap. 4]). It is well known that the composite

$$p \circ \mathrm{tr}\colon D_2(E) \to D_2(E)$$

is a weak equivalence after inverting 2. To prove the first part of Theorem D, it is clearly enough to show

Lemma 7.1. *For a connected spectrum E, the class*

$$(\Sigma \mathrm{tr}) \circ \delta_E \in [E, \Sigma E \wedge E]$$

is trivial.

Proof. Consider the functor from spectra to spectra defined by $E \mapsto E \wedge E$. This is a homogeneous homotopy functor of degree 2.

We will describe a natural transformation

$$\Phi\colon \Sigma^\infty \Omega^\infty E \to E \wedge E\,.$$

By taking adjoints it is enough to describe a natural transformation of spaces $\Omega^\infty E \to \Omega^\infty(E \wedge E)$. It will suffice to do this when E is a coordinate free spectrum, i.e., a spectrum indexed over finite dimensional subspaces of $\mathbb{R}^\infty$ (with induced inner product). If $V \subset \mathbb{R}^\infty$ is a finite dimensional subspace, there is a map

$$\Omega^V E_V \to \Omega^{V \oplus V}(E_V \wedge E_V)$$

which assigns to a map $f\colon S^V \to E_V$ the map

$$f \wedge f\colon S^V \wedge S^V \to E_V \wedge E_V\,.$$

Taking the colimit over all V, the domain becomes $\Omega^\infty E$, and the codomain becomes identified with $\Omega^\infty(E \wedge E)$ once we choose an isometry $\mathbb{R}^\infty \oplus \mathbb{R}^\infty \to \mathbb{R}^\infty$. This describes the natural transformation Φ.

Since $E \mapsto E^{\wedge 2}$ is a homogeneous quadratic functor, it coincides with the second stage of its Goodwillie tower and its first stage is trivial. So the map of towers in degrees ≤ 2 may be displayed as

$$\begin{array}{ccccc} D_2(E) & \longrightarrow & E \wedge E & & \\ & \searrow & & \searrow\!\!\!\!= & \\ & & S_2(E) & \longrightarrow & E \wedge E \\ & & \downarrow & & \downarrow \\ & & E & \longrightarrow & *\,. \end{array}$$

We assert that the induced natural transformation $D_2(E) \to E \wedge E$ of second layers is in fact homotopic to the transfer.[2]

Here is a sketch proof: the natural transformation $D_2(E) \to E \wedge E$ is determined by what it does on the category of sets of cardinality ≤ 2 (by identifying a based finite set T with its suspension spectrum). This observation is due to McCarthy [Mc, Th. 4.6], and independently to Arone, Dwyer and Rezk (unpublished). I first learned of the result from Dwyer.

In particular, it is enough to check that the map is the transfer when E is a suspension spectrum. But for suspension spectra $\Sigma^\infty X$, it is not difficult to exhibit a natural weak equivalence

$$S_2(\Sigma^\infty X) \simeq (\Sigma^\infty_{\mathbb{Z}_2}(X \wedge X))^{\mathbb{Z}_2}\,,$$

where the right side is the $\mathbb{Z}_2$-fixed points of the equivariant stable homotopy of $X \wedge X$. With respect to this identification, the map $D_2(\Sigma^\infty X) \to S_2(\Sigma^\infty X)$ corresponds to the norm map. But the norm map followed by the forgetful map $(\Sigma^\infty_{\mathbb{Z}_2}(X \wedge X))^{\mathbb{Z}_2} \to (\Sigma^\infty(X \wedge X))$ is the transfer. This completes the sketch proof of the assertion.

Finally, if we take horizontal homotopy cofibers of the maps from stage two to stage one, we obtain a homotopy commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & * \\ {\scriptstyle \delta_E}\downarrow & & \downarrow \\ \Sigma D_2(E) & \underset{\Sigma \mathrm{tr}}{\longrightarrow} & \Sigma E \wedge E \end{array}$$

which shows that $(\Sigma \mathrm{tr}) \circ \delta_E$ is null homotopic. □

We now prove the second part of Theorem D. Let $f\colon E \to E'$ be a map which is a 2-local weak equivalence. Assume that E and E' satisfy

[2] Randy McCarthy has informed me that the proof of this is implicit in the recent Ph.D. thesis of K. Baxter Bauer [Ba].

the connectivity hypotheses of Theorem B. Then the naturality of the obstruction shows that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\delta_E} & \Sigma D_2(E) \\ f\big\downarrow & & \big\downarrow \Sigma D_2(f) \\ E' & \xrightarrow[\delta_{E'}]{} & \Sigma D_2(E') \end{array}$$

is homotopy commutative. If we invert 2, then by the first part, the obstructions δ_E and δ'_E vanish. It we localize at 2, then f and $\Sigma D_2(f)$ become weak equivalences and the δ_E and $\delta_{E'}$ coincide. Thus, integrally, $\delta_E = 0$ if and only $\delta'_E = 0$.

8. Illustrations of Theorem B

We give two examples.

Two cells. For $p > 1$ and $q \leq 3p-3$, consider the suspension spectrum $E = \Sigma^\infty(S^p \cup_f e^{q+1})$ where $f\colon S^q \to S^p$ is some map. According to Theorem B, we have

$$\Theta_E \;\cong\; [E, D_2(E)] \;\cong\; \pi_{q+1}(D_2(S^p))\,,$$

where the second of these isomorphisms comes from that fact that a homotopy class $E \to D_2(E)$ maps the top cell of E into $D_2(S^p) \subset D_2(E)$.

The first non-trivial group occurs when $q = 2p-1$. The group $\pi_{2p}(D_2(S^p))$ is isomorphic to $\mathbb{Z}$ if p is even and $\mathbb{Z}_2$ if p is odd. The distinct elements of Θ_E are represented by the suspension spectra of $S^p \cup_g e^{2p}$, with

$$g = f + k[\iota, \iota]\,,$$

where k is an integer (mod 2 if p is odd). The group structure on Θ_E is given by adding these integers.

In fact, at the prime 2, Mahowald has computed $\pi_{q+1}(D_2(S^p))$ for $q \leq \min(3p-3, 2p+29)$ (see [Mah, table 4.1], see also Milgram [Mi, table 13.5]). For example, assuming in addition $p \equiv 1 \bmod 16$, the first few groups are

j	0	1	2	3	4	5
$\pi_{2p+j}(D_2(S^p))$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$

.

Tori. Let $E = \Sigma^\infty(S^p \times S^p)$, with $p > 1$. Then

$$\Theta_E \;\cong\; [E, D_2(E)] \;\cong\; \pi_{2p}(D_2(S^p \vee S^p))\,.$$

When p is even, the group $\pi_{2p}(D_2(S^p \vee S^p))$ is isomorphic to $\mathbb{Z}^{\oplus 3}$. When p is odd, it is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$. The elements of Θ_E can be represented by the suspension spectra of the complexes

$$(S^p \vee S^p) \cup_g e^{2p}$$

with attaching map

$$g = k[\iota, \iota] + \ell[\iota, \iota] + m\omega\,,$$

where ω is the attaching map for the top cell of $S^p \times S^p$, and k, ℓ and m are integers (take k and ℓ modulo 2 if p is odd). Hence the elements of Θ_E are specified by triple (k, ℓ, m). The identity element of Θ_E is $(0, 0, 1)$ and addition is given by

$$(k, \ell, m) + (k', \ell', m') = (k + k', \ell + \ell', m + m' - 1)\,.$$

9. Loose Ends

Relation to the James-Hopf invariant. Another approach to desuspension questions is to inductively desuspend cell-by-cell (using a suitable cell decomposition for the spectrum E). We outline here how this relates to our approach.

The idea is this: when E can be written as a homotopy cofiber of a map of suspension spectra $f\colon \Sigma^\infty A \to \Sigma^\infty B$, it turns out that δ_E is closely related to the James-Hopf invariant

$$H_2(f) \in [\Sigma^\infty A, D_2(\Sigma^\infty B)]\,.$$

There is a homomorphism

$$\psi\colon [\Sigma^\infty A, D_2(\Sigma^\infty B)] \to [E, \Sigma D_2(E)]$$

given by suspending, precomposing with the connecting map $E \to \Sigma\Sigma^\infty A$ and postcomposing with the inclusion $\Sigma D_2(\Sigma^\infty B) \to \Sigma D_2(E)$. We assert that ψ maps $H_2(f)$ to δ_E (we defer the proof to another paper).

If we also assume that B is r-connected ($r \geq 1$), $\dim B \leq 2r{+}1$, A is $2r$-connected and $\dim A \leq 3r{+}1$, then we infer that E is r-connected and $\dim E \leq 3r{+}2$. Furthermore, obstruction theory implies that ψ is an isomorphism. We conclude that E desuspends if and only if $H_2(f) = 0$ (the 'if' part of this statement is well known).

Musings on the spectral sequence. It would be desirable to have a version of the spectral sequence in Theorem C which converges to $\pi_*(\mathfrak{M}_E)$ in degree zero.

The spectral sequence still lacks a geometric interpretation. Conjecturally, the spectral sequence should be a packaging machine for the obstructions to equipping a spectrum with the structure of an

"E_∞-coalgebra over the sphere spectrum."

What I have in mind here comes from an observation that $S_2(E)$ is a model for $(E \wedge E)^{\mathbb{Z}_2}$ = the categorical fixed points of $\mathbb{Z}_2$ acting on $E \wedge E$, and a choice of homotopy section to the map $S_2(E) \to E$ can be thought of as commutative (but not necessarily associative) "diagonal" for E.

It is tempting to conjecture that the spectral sequence arises from a tower whose k-stage encodes, in some sense, the moduli space of spectra equipped with choice of commutative diagonal that is coherently homotopy associative up to order $k-1$.

References

[A-K] Ahearn, S. T., Kuhn, N. J.: Product and other fine structure in polynomial resolutions of mapping spaces. *Algebr. Geom. Topol.* **2**, 1123–1150 (2002)

[Ar] Arone, G.: A generalization of the Snaith-type filtration. *Trans. Amer. Soc.* **351**, 591–647 (1999)

[B-Hi] Berstein, I., Hilton, P. J.: On suspensions and comultiplications. *Topology* **2**, 73–82 (1963)

[Ba] Bauer, K. B.: On Hopf algebra type and rational calculus decompositions. Ph.D. thesis, University of Illinois Urbana-Champaign 2001

[Go1] Goodwillie, T. G.: Calculus I, the derivative of a homotopy functor. *K-theory* **4**, 1–27 (1990)

[Go2] Goodwillie, T. G.: Calculus II: analytic functors. *K-theory* **5**, 295–332 (1992)

[Go3] Goodwillie, T. G.: Calculus III: the Taylor series of a functor. unpublished

[Go4] Goodwillie, T. G.: The differential calculus of homotopy functors. In: Proceedings of the International Congress of Mathematicians, (Kyoto, 1990), pp. 621–630. Math. Soc. Japan 1991

[Gr] Gray, B.: Desuspension at an odd prime. In: Algebraic topology, Aarhus 1982, Springer LNM 1051, pp. 360–370. Springer 1984

[Jo] Johnson, B.: The derivatives of homotopy theory. *Trans. Amer. Math. Soc.* **347**, 1295–1321 (1995)

[Kl] Klein, J. R.: Poincaré embeddings and fiberwise homotopy theory. *Topology* **38**, 597–620 (1999)

[Ku] Kuhn, N. J.: Suspension spectra and homology equivalences. *Trans. Amer. Math. Soc.* **283**, 303–313 (1984).

[L-M-S] Lewis, L. G., May, J. P., Steinberger, M., McClure, J. E.: Equivariant Stable Homotopy Theory. (LNM, Vol. 1213). Springer 1986

[Mah] Mahowald, M.: The metastable homotopy of S^n. (Memoirs of the Amer. Math. Soc, No. 72). Amer. Math. Soc. 1967

[Mc] McCarthy, R.: On n-excisive functors of module categories. to appear in *Math. Proc. Cambridge Philos. Soc.*

[Mi] Milgram, R. J.: Unstable Homotopy from the Stable Point of View. (LNM, Vol. 368). Springer 1974

[Qu] Quillen, D.: Homotopical Algebra. (LNM, Vol. 43). Springer 1967

[Sc] Schwede, S.: Spectra in model categories and applications to the algebraic cotangent complex. *J. Pure Appl. Algebra* **120**, 77–104 (1997)

[Wa] Waldhausen, F.: Algebraic K-theory of spaces. In: Algebraic and geometric topology, Proceedings Rutgers 1983, LNM **1126**, pp. 318–419. Springer 1985

Wayne State University, Detroit, MI 48202
E-mail address: klein@math.wayne.edu