BOUNDS FOR THE CUP-LENGTH OF POINCARÉ SPACES AND THEIR APPLICATIONS

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ABSTRACT. Our main result offers a new (quite systematic) way of deriving bounds for the cup-length of Poincaré spaces over fields; we outline a general research program based on this result. For the oriented Grassmann manifolds, already a limited realization of the program leads, in many cases, to the exact values of the cup-length and to interesting information on the Lyusternik-Shnirel'man category.

1. INTRODUCTION AND STATEMENT OF RESULTS

The cup-length over a field R, $\operatorname{cup}_R(X)$, of a path connected topological space X is the supremum of all numbers c such that there exist positive dimensional cohomology classes $a_1, \dots, a_c \in H^*(X; R)$ such that their cup product $a_1 \cup \cdots \cup a_c$ is nonzero. Instead of the standard notation $a \cup b$, we shall mostly write $a \cdot b$ or just ab. Although the main topic of this paper is the cup-length, we shall also keep in mind its relation to another important invariant: we have $\operatorname{cat}(X) \geq 1 + \operatorname{cup}_R(X)$, where $\operatorname{cat}(X)$ is the L-S category (the Lyusternik-Shnirel'man category). We recall that $\operatorname{cat}(X)$ is defined to be the least positive integer k such that X can be covered by k open subsets each of which is contractible in X. If no such integer exists, then one puts $\operatorname{cat}(X) = \infty$. Note that some authors (see e.g. [5]) prefer to modify the definition by subtracting 1 from our value of $\operatorname{cat}(X)$.

The L-S category is very hard to compute; a longstanding problem in topology (cf. Ganea's list [7]) is the task to find its value for familiar manifolds. In general, the cup-length is difficult to calculate, and (a formula for) its value remains unknown for many commonly used spaces, e.g. for the great majority of the Grassmann manifolds $O(n)/O(k) \times O(n-k)$; explicit formulae for $\operatorname{cup}_{\mathbb{Z}_2}(O(n)/O(k) \times O(n-k))$ are known for $k \leq 5$, with some exceptions (cf. R. Stong's [12]).

1.1 Statement of the main result. When trying to find the cup-length of a space, one might start from calculations of the height for some nonzero elements in its cohomology. Recall that if $x \in H^i(X; R)$ $(i > 0)$ is a nonzero cohomology class, then the height of x over R, denoted by $\operatorname{ht}_R(x)$, is the supremum of all the numbers k such that $x^k \neq 0$; of course, one has $\operatorname{cup}_R(X) \geq \operatorname{ht}_R(x)$. Our main result is the following general theorem which we shall prove in Section 2. Its most interesting part, (b), shows that if X is an R-Poincaré space, then height-related information may lead

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also to upper bounds for $\operatorname{cup}_R(X)$; part (a) (in particular (a2) and (a3)) is almost obvious, but we include it in order to have the theorem as a comfortable reference tool.

Theorem A. Let R be a field, and $X \neq \emptyset$ be a path-connected R-Poincaré space of formal dimension n. Let the first two nonzero reduced R-cohomology groups of X, in dimensions less than n, be $\tilde{H}^r(X;R)$ and $\tilde{H}^q(X;R)$, $r \leq q < n$. Then the cup-length satisfies the following.

(a) One always has

$$\operatorname{cup}_R(X) \leq \frac{n}{r}. \tag{a1}$$

If there is a cohomology class $x \in H^r(X;R) \setminus \{0\}$ such that $r \cdot \operatorname{ht}_R(x) = n$, then

$$\operatorname{cup}_R(X) = \frac{n}{r}. \tag{a2}$$

If one finds $a_1, \ldots, a_s \in H^r(X;R) \setminus \{0\}$ such that $a_1^{t_1} \cdots a_s^{t_s} \neq 0$ for some non-negative integers $t_1, \ldots, t_s$ such that $t_1 + \cdots + t_s > 0$ and $r(t_1 + \cdots + t_s) < n$, then one has

$$\operatorname{cup}_R(X) \geq 1 + t_1 + \cdots + t_s. \tag{a3}$$

(b) Suppose that $r < q$ and there exists a basis $a(1)_r, \ldots, a(t)_r$ for the R-vector space $H^r(X;R)$ such that

$$a(1)_r^{k_1+1} = 0, \ldots, a(t)_r^{k_t+1} = 0$$

for some positive integers $k_1, \ldots, k_t$ satisfying the condition $r(k_1 + \cdots + k_t) < n$.

Then the upper bound given in (a) improves to

$$\operatorname{cup}_R(X) \leq k_1 + \cdots + k_t + \left[\frac{n - r(k_1 + \cdots + k_t)}{q}\right] < \frac{n}{r}. \tag{b1}$$

We remark that if X has the homotopy type of an $(r-1)$-connected $(r \geq 1)$ finite CW-complex, then, as proved by Grossman in [8], $\operatorname{cat}(X) \leq 1 + \dfrac{\dim(X)}{r}$; as a consequence, one then also has $\operatorname{cup}_R(X) \leq \dfrac{\dim(X)}{r}$. Note that Grossman's inequality and our (a1) in Theorem A coincide for X having the homotopy type of an $(r-1)$-connected $(r \geq 1)$ finite CW-complex if X is an R-Poincaré space; but our (a1) requires just R-homological $(r-1)$-connectedness, while for Grossman's upper bound the standard $(r-1)$-connectedness is required.

Theorem A can serve as a basis for the following research program.

Research Program. Let X be an R-Poincaré space such as we suppose in the theorem, such that $r < q$ (if $r = q$, then one readily sees that $\operatorname{cup}_R(X) = 2$), and such that $H^r(X;R)$ is finitely generated as a vector space over R. Then

(1) use (a1) to calculate the initial upper bound for $\operatorname{cup}_R(X)$;
(2) study vanishing of products of elements in $H^*(X;R)$ and find (possibly using (a2) or (a3)) a lower bound, as high as you can, for $\operatorname{cup}_R(X)$;
(3) if the best upper and lower bounds you have obtained do not coincide, then find bases (as many as you can) in $H^r(X;R)$, study vanishing of powers of their elements and, when possible, apply (b1) to obtain a better upper bound

Note that this Research Program can be applied, in particular, to any R-orientable closed n-dimensional manifold with R-cohomology different from the R-cohomology of the n-sphere S^n.

1.2 Statement of some applications of Theorem A. To illustrate its usefulness, we shall use Theorem A, with $r = 2$, $R = \mathbb{Z}_2$ or $r = 4$, $R = \mathbb{Q}$, for deriving several new results on the cup-length of the oriented Grassmann manifolds; at the same time, we shall pay attention to their L-S category. In many cases, in spite of knowing the cup-length, one might be very far from knowing the exact value of the L-S category (see e.g. [5]). But we shall show that for the oriented Grassmann manifolds $\tilde{G}_{n,k} = SO(n)/SO(k) \times SO(n-k)$, with $n \geq 2k \geq 6$, the cup-length yields, at least in some cases, a good amount of information about the L-S category (we shall take $k \geq 3$, because for $\tilde{G}_{n,1} = S^{n-1}$ and for $\tilde{G}_{n,2}$ (complex quadrics) the cup-length and L-S category are known). More precisely, already our limited (in other words: just illustrative) realization of the Research Program brings new non-trivial estimates for the cup-length; in particular, we find the exact values of $\mathrm{cup}_{\mathbb{Z}_2}(\tilde{G}_{6,3})$ and also of $\mathrm{cup}_{\mathbb{Q}}(\tilde{G}_{n,k})$ for infinitely many (n, k) with $k \geq 4$. At the same time, it turns out that for $\tilde{G}_{n,3}$ the cup-length lower bound (that is, 1 plus the cup-length) and the L-S category can be very close to each other. In two cases, for $\tilde{G}_{6,3}$ and $\tilde{G}_{7,3}$, our cup-length lower bound and the Grossman upper bound differ by just 1, for $\tilde{G}_{8,3}$ the difference is 2.

Now we pass to detailed statements of the applications which we have roughly outlined. In order to simplify the notation, we shall write $\mathrm{cup}(X)$ instead of $\mathrm{cup}_{\mathbb{Z}_2}(X)$, and $\mathrm{ht}(x)$ instead of $\mathrm{ht}_{\mathbb{Z}_2}(x)$.

Using Theorem A(a) (mainly (a3); cf. (2) in the Research Program), we prove the following new lower bounds for the cup-length of the oriented Grassmann manifolds in Section 3.

Proposition B. For the oriented Grassmann manifolds $\tilde{G}_{n,k}$, $n \geq 2k \geq 6$, we have

(a) $\mathrm{cup}(\tilde{G}_{6,3}) \geq 3$;

(b) $$\mathrm{cup}(\tilde{G}_{n,3}) \geq \begin{cases} \dfrac{n+3}{2} & \text{if } n \geq 7 \text{ is odd, } n \notin \{9, 11\}, \\ \dfrac{n+2}{2} & \text{if } n \geq 8 \text{ is even, } n \notin \{10, 12\}. \end{cases}$$

(c) Each of $\mathrm{cup}(\tilde{G}_{9,3})$, $\mathrm{cup}(\tilde{G}_{10,3})$, $\mathrm{cup}(\tilde{G}_{11,3})$, $\mathrm{cup}(\tilde{G}_{12,3})$ is at least 5.

(d) If $k \geq 4$, then

$$\mathrm{cup}(\tilde{G}_{n,k}) \geq \begin{cases} \dfrac{n-k+6}{2} & \text{if } n-k+3 \geq 7 \text{ is odd, } n-k+3 \notin \{9, 11\}, \\ \dfrac{n-k+5}{2} & \text{if } n-k+3 \geq 8 \text{ is even, } n-k+3 \notin \{10, 12\}, \\ 5 & \text{if } n-k+3 \in \{9, 10, 11, 12\}. \end{cases}$$

(e) If $k \geq 4$, then

$$\mathrm{cup}_{\mathbb{Q}}(\tilde{G}_{n,k}) \geq \begin{cases} 1 + \left[\dfrac{k}{2}\right]\left[\dfrac{n-k}{2}\right] & \text{if } 4\left[\dfrac{k}{2}\right]\left[\dfrac{n-k}{2}\right] < k(n-k), \\ \left[\dfrac{k}{2}\right]\left[\dfrac{n-k}{2}\right] & \text{if } 4\left[\dfrac{k}{2}\right]\left[\dfrac{n-k}{2}\right] = k(n-k). \end{cases}$$

The author acknowledges that Proposition B(e) was suggested by Parameswaran Sankaran. Proposition B obviously implies the following (the upper estimates are the Grossman upper bounds).

Corollary C. For the oriented Grassmann manifolds $\tilde{G}_{n,k}$, $n \geq 2k \geq 6$, we have:

(a) $4 \leq \mathrm{cat}(\tilde{G}_{6,3}) \leq 5$;
(b) $\frac{n+5}{2} \leq \mathrm{cat}(\tilde{G}_{n,3}) \leq \frac{3n-7}{2}$ if n is odd, $n \notin \{9, 11\}$, and $\frac{n+4}{2} \leq \mathrm{cat}(\tilde{G}_{n,3}) \leq \frac{3n-7}{2}$ if n is even, $n \notin \{6, 10, 12\}$. In addition to this, we have $6 \leq \mathrm{cat}(\tilde{G}_{9,3}) \leq 10$, $6 \leq \mathrm{cat}(\tilde{G}_{10,3}) \leq 11$, $6 \leq \mathrm{cat}(\tilde{G}_{11,3}) \leq 13$, $6 \leq \mathrm{cat}(\tilde{G}_{12,3}) \leq 14$.
(c) If $k \geq 4$, then

$$\mathrm{cat}(\tilde{G}_{n,k}) \geq \begin{cases} 6 & \text{if } (n,k) = (8,4), \\ 2 + \left[\frac{k}{2}\right]\left[\frac{n-k}{2}\right] & \text{if } 4\left[\frac{k}{2}\right]\left[\frac{n-k}{2}\right] < k(n-k), \\ 1 + \left[\frac{k}{2}\right]\left[\frac{n-k}{2}\right] & \text{if } (n,k) \neq (8,4) \text{ and } 4\left[\frac{k}{2}\right]\left[\frac{n-k}{2}\right] = k(n-k). \end{cases}$$

In Section 4, we shall prove the following upper bounds for $\mathrm{cup}(\tilde{G}_{n,k})$ $(n \neq 6)$ and $\mathrm{cup}_{\mathbb{Q}}(\tilde{G}_{n,k})$ $(k \geq 4)$; the proofs can be seen as a realization of the third step of our Research Program. The manifold $\tilde{G}_{6,3}$ will be treated in a special way; note that for this manifold the upper bound coincides with the lower bound, so that we obtain $\mathrm{cup}(\tilde{G}_{6,3}) = 3$. Also note that the upper bound for $\mathrm{cup}_{\mathbb{Q}}(\tilde{G}_{n,k})$ $(k \geq 4)$ given below coincides with the lower bound given in Proposition B(e) in infinitely many cases.

Proposition D. For the oriented Grassmann manifolds $\tilde{G}_{n,k}$, $n \geq 2k \geq 6$, we have

(a) $\mathrm{cup}(\tilde{G}_{6,3}) \leq 3$. As a consequence of this and Proposition B(a), we have $\mathrm{cup}(\tilde{G}_{6,3}) = 3$.
(b) We have

$$\mathrm{cup}(\tilde{G}_{n,3}) \leq \begin{cases} \left[\frac{2^{s+2}-7}{3}\right] & \text{if } n = 2^s+1, s \geq 3, \\ \left[\frac{2^{s+2}-3}{3}\right] & \text{if } n = 2^s+2, s \geq 3, \\ \left[\frac{2^{s+2}+5\cdot 2^p-8}{3}\right] & \text{if } n = 2^s+2^p+1, s > p \geq 1, \\ \left[\frac{2^{s+2}+5\cdot 2^p+3t-7}{3}\right] & \text{if } n = 2^s+2^p+t+1, s > p \geq 1, \\ & 1 \leq t \leq 2^p - 1 \end{cases}$$

$$\mathrm{cup}(\tilde{G}_{n,4}) \leq \begin{cases} \left[\dfrac{5\cdot 2^s - 13}{3}\right] & \text{if } n = 2^s+1, s \geq 3, \\ 2^{s+1} - 4 & \text{if } n = 2^s+2, s \geq 3, \\ 2^{s+1} - 3 & \text{if } n = 2^s+3, s \geq 3, \\ \left[\dfrac{2^{s+1}+4n-17}{3}\right] & \text{if } 2^s+4 \leq n \leq 2^{s+1}; \end{cases}$$

$$\mathrm{cup}(\tilde{G}_{n,k}) \leq \begin{cases} \left[\dfrac{(k+1)\cdot 2^s + k - k^2 - 1}{3}\right] & \text{if } n = 2^s+1, s \geq 3, k \geq 5 \\ \left[\dfrac{2^{s+1}+kn-k^2-1}{3}\right] & \text{if } 2^s+2 \leq n \leq 2^{s+1}, k \geq 5; \end{cases}$$

(c) $\mathrm{cup}_{\mathbb{Q}}(\tilde{G}_{n,k}) \leq \dfrac{k(n-k)}{4}$ if $k \geq 4$. As a consequence of this and Proposition B(e), we have $\mathrm{cup}_{\mathbb{Q}}(\tilde{G}_{n,k}) = \dfrac{k(n-k)}{4}$ for n even and k (≥ 4) even, as well as for $n = 4t+9$ ($t \geq 1$) and $k = 4$.

Note that Proposition B(d) gives $\mathrm{cup}(\tilde{G}_{8,4}) \geq 5$, while Proposition B(e) yields $\mathrm{cup}_{\mathbb{Q}}(\tilde{G}_{8,4}) \geq 4$ and, from Proposition D(c), we can see that $\mathrm{cup}_{\mathbb{Q}}(\tilde{G}_{8,4}) = 4$. This indicates that, for a given $\tilde{G}_{n,k}$, the value of $\mathrm{cup}(\tilde{G}_{n,k})$ may be higher than $\mathrm{cup}_{\mathbb{Q}}(\tilde{G}_{n,k})$. But the lower bounds presently known for the $\mathbb{Z}_2$-cup-length of $\tilde{G}_{n,k}$ with $k \geq 4$ and $(n,k) \neq (8,4)$, given in Proposition B(d), do not exceed the corresponding lower bounds for the rational cup-length, given in Proposition B(e).

We hope that our Research Program based on Theorem A can lead to further interesting results on the cup-length and L-S category not only for the oriented Grassmann manifolds but also for other manifolds.

2. ON THE CUP-LENGTH OF R-POINCARÉ SPACES / PROOF OF THE MAIN RESULT

In the spirit of W. Browder's [4], by an R-Poincaré space of formal dimension n we understand a path connected topological space X for which there is an element $\mu \in H_n(X) \cong R$ such that the $\cap$-product homomorphism $\cap\mu : H^k(X;R) \to H_{n-k}(X;R)$, $x \mapsto x \cap \mu$, is an isomorphism for each k. By saying that an R-Poincaré space X is R-homologically t-connected ($t \geq 0$) we understand that its reduced cohomology groups $\tilde{H}^i(X;R)$ vanish for all $i \leq t$. For example, any t-connected (in the standard sense) closed n-dimensional manifold orientable over R is an R-homologically t-connected R-Poincaré space of formal dimension n.

2.1 Proof of Theorem A. If X is an R-Poincaré space of formal dimension n, then the cup product pairing $H^k(X;R) \times H^{n-k}(X;R) \to R$ is nonsingular (see e.g. [4] or [9]); as a consequence, for any nonzero $x \in H^k(X;R)$ there exists some $y \in H^{n-k}(X;R)$ such that $x \cup y$ is nonzero in $H^n(X;R)$. In particular, this immediately implies (a3).

For the rest of the proof, first note that the hypothesis of Theorem A implies that the space X is R-homologically $(r-1)$-connected and it has at least three nontrivial unreduced R-cohomology groups. If there are just three, then Proposition A(a) is verified in an obvious way. Indeed, in such a case we have $q = r = \dfrac{n}{2}$, and one readily sees that $\mathrm{cup}_R(X) = 2$.

So suppose now that X has at least four nontrivial unreduced R-cohomology groups, so that we have $q > r$. Of course (see the beginning of this proof), any cup product of the maximum length, that is, of the length $\mathrm{cup}_R(X)$, must belong to $H^n(X;R) \cong R$. So the cup-length of X is realized by a cup product

$$x(1)_r^{p_1} \cdots x(s)_r^{p_s} y_q^v z_{q+i_1}^{j_1} \cdots z_{q+i_m}^{j_m} \in H^n(X;R) \setminus \{0\}, \tag{$\bullet$}$$

where $x(1)_r, \dots, x(s)_r \in H^r(X;R)$, $y_q \in H^q(X;R)$, $z_{q+i_l} \in H^{q+i_l}(X;R)$ are nonzero cohomology classes and $p_1, \dots, p_s$, v, $j_1, \dots, j_m$, $i_1, \dots, i_m$ are non-negative integers. Denote $p = p_1 + \cdots + p_s$. Then, of course, $\mathrm{cup}_R(X) = p + v + j_1 + \cdots + j_m$.

From this it is clear that

$$pr + vq + j_1(q + i_1) + \cdots + j_m(q + i_m) = n,$$

therefore

$$n \geq r(p + v + j_1 + \cdots + j_m).$$

In other words, we obtain

$$n \geq r \cdot \mathrm{cup}_R(X),$$

which proves (a1). If there is a cohomology class $x \in H^r(X;R) \setminus \{0\}$ such that $r \cdot \mathrm{ht}_R(x) = n$, then obviously $\mathrm{cup}_R(X) \geq \dfrac{n}{r}$; this together with (a1) proves (a2). The rest of Theorem A(a) is clear in view of what we have said in the beginning of this proof.

We pass to the proof of Theorem A(b). Now we suppose that $r < q$, and we have a basis $a(1)_r, \dots, a(t)_r$ for $H^r(X;R)$ such that

$$a(1)_r^{k_1+1} = 0, \dots, a(t)_r^{k_t+1} = 0$$

for some positive integers $k_1, \dots, k_t$ such that $r(k_1 + \cdots + k_t) < n$. As above, take an element realizing the cup-length of X, hence some

$$c := x(1)_r^{\pi_1} \cdots x(s)_r^{\pi_s} y_q^v z_{q+i_1}^{j_1} \cdots z_{q+i_m}^{j_m} \in H^n(X;R) \setminus \{0\}, \tag{$\bullet$}$$

where $x(1)_r, \dots, x(s)_r \in H^r(X;R) \setminus \{0\}$, $y_q \in H^q(X;R) \setminus \{0\}$, $z_{q+i_l} \in H^{q+i_l}(X;R) \setminus \{0\}$, and $\pi_1, \dots, \pi_s$, v, $j_1, \dots, j_m$, $i_1, \dots, i_m$ are non-negative integers. We denote $\pi = \pi_1 + \cdots + \pi_s$. But now $x(i)_r = \sum_{j=1}^t \alpha_{i,j} a(j)_r$ for some uniquely determined coefficients $\alpha_{i,1}, \dots, \alpha_{i,t} \in R$. So c is a linear combination of cup products of the form

$$a(1)_r^{p_1} \cdots a(t)_r^{p_t} y_q^v z_{q+i_1}^{j_1} \cdots z_{q+i_m}^{j_m}, \tag{$\bullet\bullet$}$$

where $p_1, \dots, p_t$ are non-negative integers such that $p_1 + \cdots + p_t = \pi$. Since $c \neq 0$, at least one of the products ($\bullet\bullet$) must be nonzero; of course, in such a nonzero cup product, the exponent of $a(i)_r$ must be less than or equal to k_i for all $i = 1, \dots, t$. We conclude that the cup-length is realized by a nonzero element

$$a(1)_r^{p_1} \cdots a(t)_r^{p_t} y_q^v z_{q+i_1}^{j_1} \cdots z_{q+i_m}^{j_m} \in H^n(X;R) \setminus \{0\} \tag{$\bullet\bullet\bullet$}$$

where $p_1, \dots, p_t, v, j_1, \dots, j_m, i_1, \dots, i_m$ are non-negative integers, $p_1 \le k_1, \dots, p_t \le k_t$, $\mathrm{cup}_R(X) = p_1 + \cdots + p_t + v + j_1 + \cdots + j_m$, and $y_q \in H^q(X;R) \setminus \{0\}$, $z_{q+i_l} \in H^{q+i_l}(X;R) \setminus \{0\}$. Using the fact that $q - r > 0$, we obtain

$$\begin{aligned} n &= r(p_1 + \cdots + p_t) + q(v + j_1 + \cdots + j_m) + i_1 j_1 + \cdots + i_m j_m \\ &= q(p_1 + \cdots + p_t + v + j_1 + \cdots + j_m) + (r - q)(p_1 + \cdots + p_t) + i_1 j_1 + \cdots + i_m j_m \\ &\ge q(p_1 + \cdots + p_t + v + j_1 + \cdots + j_m) + (r - q)(p_1 + \cdots + p_t) \\ &= q \,\mathrm{cup}_R(X) - (q - r)(p_1 + \cdots + p_t) \\ &\ge q \,\mathrm{cup}_R(X) - (q - r)(k_1 + \cdots + k_t). \end{aligned}$$

The proof of Theorem A is finished.

3. APPLICATIONS: LOWER BOUNDS FOR $\tilde{G}_{n,k}$ / PROOF OF PROPOSITION B

3.1 Selected facts about the oriented Grassmann manifolds. The oriented Grassmann manifold $\tilde{G}_{n,k}$ consists of oriented k-dimensional vector subspaces in $\mathbb{R}^n$, similarly the Grassmann manifold $G_{n,k}$ consists of all k-dimensional vector subspaces in $\mathbb{R}^n$. For obvious reasons, we shall suppose that $2k \le n$ and, for the reason explained in the Introduction, we restrict ourselves to $k \ge 3$.

Let $\gamma_{n,k}$ (briefly γ) denote the canonical k-plane bundle over $G_{n,k}$, $\tilde{\gamma}_{n,k}$ (briefly $\tilde{\gamma}$) be the canonical k-plane bundle over $\tilde{G}_{n,k}$, and ξ be the canonical line bundle associated with the double covering $p : \tilde{G}_{n,k} \to G_{n,k}$. We denote $w_i = w_i(\gamma)$ in $H^i(G_{n,k}; \mathbb{Z}_2)$ and $\tilde{w}_i = w_i(\tilde{\gamma})$ in $H^i(\tilde{G}_{n,k}; \mathbb{Z}_2)$ the corresponding Stiefel-Whitney classes. Of course, $\tilde{w}_1 = 0$, and w_1 is easily seen to coincide with the first Stiefel-Whitney class of ξ.

By Borel [2], the $\mathbb{Z}_2$-cohomology ring $H^*(G_{n,k}; \mathbb{Z}_2)$ can be identified with a quotient ring,

$$H^*(G_{n,k}; \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \dots, w_k]/I_{n,k},$$

where the ideal $I_{n,k}$ is generated by the homogeneous components of

$$\frac{1}{1 + w_1 + \cdots + w_k}$$

in dimensions $n - k + 1, \dots, n$.

Calculations in the ring $H^*(\tilde{G}_{n,k}; \mathbb{Z}_2)$ ($k \ge 3$) may be very difficult. Fortunately, they can be made a little easier thanks to the Gysin exact sequence associated with the double covering $p : \tilde{G}_{n,k} \to G_{n,k}$. Indeed, for showing that $\tilde{w}_2^{i_2} \cdots \tilde{w}_k^{i_k} \ne 0$, it is enough to verify that $w_2^{i_2} \cdots w_k^{i_k} \in H^{2i_2 + \cdots + k i_k}(G_{n,k}; \mathbb{Z}_2)$ is not a multiple of the class w_1.

In view of Theorem A (and in view of the step (3) of our Research Program), it is good to keep in mind an explicit description of the first nontrivial reduced $\mathbb{Z}_2$-cohomology group for those oriented Grassmann manifolds we are interested in.

Lemma E. For the oriented Grassmann manifolds $\tilde{G}_{n,k}$ ($n \ge 2k \ge 6$), we have $H^2(\tilde{G}_{n,k}; \mathbb{Z}_2) = \{0, \tilde{w}_2\} \cong \mathbb{Z}_2$.

Proof. Let $f : V_{n,k} \to \tilde{G}_{n,k}$ (where $V_{n,k}$ is the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$) be the natural fiber bundle with fiber $SO(k)$. The Lemma is readily implied by the corresponding Serre exact sequence.

3.2 Preparations for the proof of Propositions B(a)-(c). To any integer $n \geq 6$, we find s as the unique integer such that $2^s < n \leq 2^{s+1}$. Since $\dim(G_{n,3}) < 4n \leq 2^{s+3}$, we have

$$(1 + w_2 + w_3)^{2^{s+3}} = 1. \qquad (*)$$

To show that

$$w_2^{i_2} w_3^{i_3} \in H^{2i_2+3i_3}(G_{n,3}; \mathbb{Z}_2)$$

cannot be expressed as a multiple of the class w_1, it is enough to show that $w_2^{i_2}w_3^{i_3}$ is not in the ideal $J_{n,3}$ of $\mathbb{Z}_2[w_2, w_3]$ generated by the homogeneous components of

$$\frac{1}{1 + w_2 + w_3} = (1 + w_2 + w_3)^{2^{s+3}-1}$$

(we have used (*)) in dimensions $n-2$, $n-1$, n. Therefore the ideal $J_{n,3}$ is generated by homogeneous polynomials g_{n-2}, g_{n-1}, g_n, obtained as the homogeneous components in dimensions $n-2$, $n-1$ and n, respectively, of

$$\sum_{i=0}^{2^{s+3}-1} \sum_{j=0}^{i} \binom{i}{j} w_2^{i-j} w_3^{j}. \qquad (**)$$

Note that the binomial coefficient $\binom{2^{s+3}-1}{i}$ is odd for every $i = 0, 1, \ldots, 2^{s+3}-1$. For $\kappa = n-2,\ n-1,\ n$ one calculates from (**) that

$$g_\kappa = \sum_{\frac{\kappa}{3} \leq i \leq \frac{\kappa}{2}} \binom{i}{3i-\kappa} w_2^{3i-\kappa} w_3^{\kappa-2i}. \qquad (***)$$

3.3 Proof of Proposition B(a). From (***) we obtain that the ideal $J_{6,3}$ in $\mathbb{Z}_2[w_2, w_3]$ is generated by $g_4 = w_2^2$ and $g_6 = w_3^2 + w_2^3$. So w_2w_3 is not in $J_{6,3}$. Consequently, $\tilde{w}_2\tilde{w}_3 \neq 0$, and (cf. (a3) in Theorem A) $\operatorname{cup}(\tilde{G}_{6,3}) \geq 3$, as we have claimed.

3.4 Proof of Proposition B(b). We say that a homogeneous polynomial $h_a \in \mathbb{Z}_2[w_2, w_3]$, in dimension a, is w_2-monomial ("monomial" is an adjective here) if $h_a = w_2^{\frac{a}{2}}$. As a realization of the step (2) of our Research Program, we should find a as high as we can (this value of a will be called an available target dimension) such that no element in the a-dimensional homogeneous component of $J_{n,3}$ is w_2-monomial. This then gives that $\tilde{w}_2^{\frac{a}{2}} \neq 0$, and Proposition B(b) is readily implied by (a2) or (a3) of Theorem A.

Odd values of n. Let us first suppose that n is odd, $n \geq 7$, $n \notin \{9, 11\}$. We shall show that $n+1$ is an available target dimension; so our aim now is to verify that none of the elements w_3g_{n-2}, w_2g_{n-1}, $w_3g_{n-2} + w_2g_{n-1}$ is w_2-monomial.

If $n = 6t+1$, then using (***) we calculate that

$$w_2g_{n-1} = w_2w_3^{2t} + \cdots + w_2^{3t+1},$$

$$w_3g_{n-2} = 0 \cdot w_2w_3^{2t} + \cdots + (3t-1)w_2^{3t-2}w_3^2,$$

and

$$w_2g_{n-1} + w_3g_{n-2} = w_2w_3^{2t} + \cdots + w_2^{3t+1}$$

Hence none of $w_3 g_{n-2}$, $w_2 g_{n-1}$, $w_3 g_{n-2} + w_2 g_{n-1}$ is w_2-monomial, indeed.

If $n = 6t+3$, we shall suppose that $t > 1$; the case $n = 9$ has a separate treatment. Write $n = 3 \cdot 2^{k+1}(2l+1) + 3$. Using (***), one readily verifies that none of the polynomials $w_2 g_{n-1}$, $w_3 g_{n-2}$ is w_2-monomial. Further, one calculates that

$$w_2 g_{n-1} + w_3 g_{n-2} = \sum_{i=2t+1}^{3t} \binom{i+1}{3i-6t-1} w_2^{3i-6t-1} w_3^{6t+2-2i} + 1 \cdot w_2^{3t+2}.$$

Therefore,

(a) if $k \geq 1$ and l is arbitrary, then
$$w_2 g_{n-1} + w_3 g_{n-2} = \cdots + w_2^2 w_3^{2^{k+1}(2l+1)} + \cdots + w_2^{3\cdot 2^k(2l+1)+2},$$

(b) if $k = 0$ and l is at least 2 and even, then
$$w_2 g_{n-1} + w_3 g_{n-2} = \cdots + w_2^5 w_3^{4l} + \cdots + w_2^{6l+5},$$

(c) if $k = 0$ and l is at least 1 and odd, then
$$w_2 g_{n-1} + w_3 g_{n-2} = \cdots + w_2^{6l+2} w_3^2 + w_2^{6l+5}.$$

We conclude that for $n = 6t+3$ with $t > 1$, the polynomial $w_2 g_{n-1} + w_3 g_{n-2}$ is not w_2-monomial.

For n odd, we are left with the case $n = 6t+5$; we suppose that $t > 1$ ($n = 11$ is treated separately). We write now $n = 3 \cdot 2^{k+1} \cdot (2l+1) - 1$.

If $l \geq 1$, then we obtain from (***) that

$$w_2 g_{n-1} = w_2^{3\cdot 2^k} w_3^{3\cdot 2^{k+2}\cdot l} + \cdots + w_2^{\frac{n+1}{2}},$$
$$w_3 g_{n-2} = w_3^{\frac{n+1}{3}} + \cdots + (3\cdot 2^k(2l+1) - 2) w_2^{3\cdot 2^k(2l+1)-3} w_3^2.$$

Hence for $l \geq 1$ none of the polynomials $w_2 g_{n-1}$, $w_3 g_{n-2}$, $w_2 g_{n-1} + w_3 g_{n-2} = w_3^{\frac{n+1}{3}} + \cdots + w_2^{\frac{n+1}{2}}$ is w_2-monomial.

We are left with $n = 2^{k+1} \cdot 3 \cdot (2l+1) - 1$ for $l = 0$; now the argument used for $l \geq 1$ does not work. Note that we take $k > 1$, because we have $n > 11$.

Now one calculates from (***) that

$$g_{n-2} = w_3^{\frac{n-2}{3}} + \cdots + 0 \cdot w_2^{\frac{n-5}{2}} w_3.$$

Further, if k is odd, then

$$w_2 g_{n-1} = \cdots + w_2^{\frac{n+7}{3}} w_3^{\frac{n-11}{9}} + \cdots + w_2^{\frac{n+1}{2}},$$

and if k is even, then

$$w_2 g_{n-1} = \cdots + w_2^{\frac{n+4}{3}} w_3^{\frac{n-5}{9}} + \cdots + w_2^{\frac{n+1}{2}}.$$

We conclude that none of the polynomials $w_2 g_{n-1}$, $w_3 g_{n-2}$, $w_2 g_{n-1} + w_3 g_{n-2} = w_3^{\frac{n+1}{3}} + \cdots + w_2^{\frac{n+1}{2}}$ is w_2-monomial.

Even values of n. Let us suppose that n is even, $n \geq 8$, $n \notin \{10, 12\}$. By a similar analysis as above, we could show that n is an available target dimension. But the hard work can be avoided. Indeed, we have the standard "inclusion" $i : \tilde{G}_{n-1,3} \to \tilde{G}_{n,3}$ such that $i^*(\tilde{\gamma}_{n,3}) = \tilde{\gamma}_{n-1,3}$. Since $n-1$ is odd, we know, by what we have computed above, that n is an available target dimension for $\tilde{G}_{n-1,3}$. This implies that $\tilde{w}_2^{\frac{n}{2}} \in H^n(\tilde{G}_{n-1,3}; \mathbb{Z}_2)$ does not vanish, and therefore $\tilde{w}_2^{\frac{n}{2}} \in H^n(\tilde{G}_{n,3}; \mathbb{Z}_2)$ cannot be zero. Since n is now less than $\dim(\tilde{G}_{n,3})$, (a3) of Theorem A applies. Proposition B(b) is proved.

3.5 Proof of Proposition B(c). To finish the proof of Proposition B, we are left with the special cases, $n = 9$, 10, 11, 12. For $n = 9$, we obtain that $g_7 = w_2^2 w_3$, $g_8 = w_2 w_3^2 + w_2^4$, $g_9 = w_3^3$. So $\tilde{w}_2^4 \neq 0$, while $\tilde{w}_2^5 = 0$ (indeed: we have $w_2^5 = w_3 g_7 + w_2 g_8$). In the three remaining cases, it is then readily seen that $\tilde{w}_2^4$ is not zero (apply the "inclusions" $i : \tilde{G}_{9,3} \to \tilde{G}_{9+s,3}$ such that $i^*(\tilde{\gamma}_{9+s,3}) = \tilde{\gamma}_{9,3}$, $s = 1, 2, 3$). Proposition B(c) is proved.

3.6 Proof of Proposition B(d). Let $j : \tilde{G}_{a,b} \to \tilde{G}_{a+1,b+1}$ be the "inclusion" such that $j^*(\tilde{\gamma}_{a+1,b+1}) = \tilde{\gamma}_{a,b} \oplus \varepsilon^1$, where ε^1 is the trivial line bundle. Then we have $j^*(w_2(\tilde{\gamma}_{a+1,b+1})) = w_2(\tilde{\gamma}_{a,b})$. By an obvious iteration, we obtain the "inclusion" $i : \tilde{G}_{n-k+3,3} \to \tilde{G}_{n,k}$ such that $i^*(w_2(\tilde{\gamma}_{n,k})) = w_2(\tilde{\gamma}_{n-k+3,3})$. In the proofs of Propositions B(b), B(c), we have shown that certain powers of $w_2(\tilde{\gamma}_{n-k+3,3})$ do not vanish; of course, then the same powers of $w_2(\tilde{\gamma}_{n,k})$ do not vanish, and the rest is clear: one applies (a3) from Theorem A. Proposition B(d) is proved.

3.7 Proof of Proposition B(e). Now $k \geq 4$ and, using the known description of the cohomology algebra $H^*(\tilde{G}_{n,k}; \mathbb{Q})$ (cf. for instance [11; Proposition 5]), one readily verifies that we have $r = 4$ in Theorem A. Let $p_i(\tilde{\gamma}_{n,k}) \in H^{4i}(\tilde{G}_{n,k}; \mathbb{Q})$ be the ith rational Pontrjagin class. The height of $p_1(\tilde{\gamma}_{n,k})$ is $\left[\frac{k}{2}\right]\left[\frac{n-k}{2}\right]$ (see e.g. the proof of [11; Theorem 1]). The lower bounds stated in Proposition B(e) are then implied by (a2) or (a3) of Theorem A.

The proof of Proposition B is finished.

4. APPLICATIONS: UPPER BOUNDS FOR $\tilde{G}_{n,k}$ / PROOF OF PROPOSITION D

4.1 Proof of Proposition D(a). From the proof of Proposition B(a), we know that $\tilde{w}_2 \tilde{w}_3 \neq 0$. Hence there exists a nonzero cohomology class $a \in H^4(\tilde{G}_{6,3}; \mathbb{Z}_2)$ such that $\tilde{w}_2 \cdot \tilde{w}_3 \cdot a \neq 0$. It is clear that $\operatorname{cup}(\tilde{G}_{6,3}) \geq 3$, and that $\operatorname{cup}(\tilde{G}_{6,3})$ could be more than 3 only if a could be decomposed as a product of two cohomology classes (in view of Lemma E in 3.1, the only decomposition, not excluded a priori, would be $a = \tilde{w}_2^2$) or if $\tilde{w}_2^4$ would be nonzero. But the element a is indecomposable, because, in addition to the fact that $H^1(\tilde{G}_{6,3}; \mathbb{Z}_2) = 0$, we have (Lemma E) $H^2(\tilde{G}_{6,3}; \mathbb{Z}_2) = \{0, \tilde{w}_2\}$, and $\tilde{w}_2^2 = 0$ (cf. 3.3). So we conclude that $\operatorname{cup}(\tilde{G}_{6,3}) \leq 3$. Proposition D(a) is proved.

4.2 Proof of Proposition D(b). We apply Theorem A(b), with $r = 2$, $R = \mathbb{Z}_2$. In view of Lemma E, we have $\tilde{w}_2$ as the only choice of basis in $H^2(\tilde{G}_{n,k}; \mathbb{Z}_2)$ ($n \geq 2k \geq 6$). Dutta and Khare [6] calculated the height of w_2 in $H^*(G_{n,k}; \mathbb{Z}_2)$ as follows (we quote just the results we need, hence for $n \geq 2k \geq 6$).

Lemma F. (S. Dutta, S. Khare [6]) For the Grassmann manifolds $G_{n,k}$ ($n \geq 2k \geq 6$), one has

$$\mathrm{ht}(w_2(\gamma_{n,3})) = \begin{cases} 2^s - 1 & \text{if } n = 2^s + 1, \\ 2^s & \text{if } n = 2^s + 2, \\ 2^s + 2^{p+1} - 2 & \text{if } n = 2^s + 2^p + 1, s > p \geq 1, \\ 2^s + 2^{p+1} - 1 & \text{if } n = 2^s + 2^p + t + 1, s > p \geq 1, \\ & 1 \leq t \leq 2^p - 1; \end{cases}$$

$$\mathrm{ht}(w_2(\gamma_{n,4})) = \begin{cases} 2^s - 1 & \text{if } n = 2^s + 1, \\ 2^{s+1} - 4 & \text{if } n = 2^s + 2, \\ 2^{s+1} - 4 & \text{if } n = 2^s + 3, \\ 2^{s+1} - 1 & \text{if } 2^s + 4 \leq n \leq 2^{s+1}; \end{cases}$$

$$\mathrm{ht}(w_2(\gamma_{n,k})) = \begin{cases} 2^s - 1 & \text{if } n = 2^s + 1, k \geq 5 \\ 2^{s+1} - 1 & \text{if } 2^s + 2 \leq n \leq 2^{s+1}, k \geq 5. \end{cases}$$

It is clear that $\mathrm{ht}(w_2(\gamma_{n,k}))$, briefly $\mathrm{ht}(w_2)$, cannot exceed half of the dimension of the corresponding Grassmann manifold. But, as we see from Lemma F, it is mostly smaller. In an obvious way, the above quoted results on $\mathrm{ht}(w_2)$ enable us to find, for each pair (n,k) under consideration, some c, mostly smaller than half of the dimension, such that $\tilde{w}_2^{c+1} = 0$. For instance, for $G_{2^s+1,3}$ we know by Lemma F that $\mathrm{ht}(w_2) = 2^s - 1$. Therefore $w_2^{2^s} = 0$, and of course for $\tilde{G}_{2^s+1,3}$ we have $\tilde{w}_2^{2^s} = 0$. Realizing the step (3) of our Research Program, using Theorem A(b), we obtain the upper bound stated in Proposition D(b) for this case. The remaining cases are similar: when $\mathrm{ht}(w_2)$ is less than half of the dimension, then we obtain the upper bound stated in Proposition D(b) by applying Theorem A(b), and in the cases where $\mathrm{ht}(w_2)$ is precisely half of the dimension, we have just half of the dimension as an upper bound for $\mathrm{cup}(\tilde{G}_{n,k})$ from Theorem A(a).

4.3 Proof of Proposition D(c). The upper bound given in Proposition D(c) is obtained from (a1) of Theorem A (with $r = 4$).

Proposition D is proved.

5. REMARKS

5.1 Remark. In a special case, for the real flag manifolds $F(n_1+\cdots+n_s) = O(n_1+\cdots+n_s)/O(n_1)\times\cdots\times O(n_s)$, we derived an upper bound of the same type as (b1) of Theorem A, with $r = 1$ and $R = \mathbb{Z}_2$, in [10; Proposition 3.2.2]. We based it there on specific properties of the $\mathbb{Z}_2$-cohomology of the flag manifolds, but we did not recognize the full potential, now expressed in Theorem A(b). In [10], we also illustrated the strength of that special case of (b1) by calculating the exact value of the $\mathbb{Z}_2$-cup-length of $F(1,2,n_3)$ for all $n_3 \geq 3$.

5.2 Remark. As pointed out by Akira Kono after having seen an early version of the author's calculations for $\tilde{G}_{n,3}$, one can deal with $G_2/SO(4)$ in a similar way (note that $G_2/SO(4)$ also is a simply-connected irreducible compact Riemannian symmetric space). Using Borel and Hirzebruch's [3; 17.3], one calculates that $\mathrm{cup}(G_2/SO(4)) = 4$. As a consequence, $\mathrm{cat}(G_2/SO(4)) \geq 5$. On the other hand, the Grossman upper bound yields $\mathrm{cat}(G_2/SO(4)) \leq 5$, and so $\mathrm{cat}(G_2/SO(4)) = 5$.

5.3 Remark. In view of Theorem A(b), it would be interesting to know the exact values of $\mathrm{ht}(\tilde{w}_2)$.

5.4 Remark. For the oriented Grassmann manifolds $\tilde{G}_{n,k}$ we know that $2\,\mathrm{ht}(\tilde{w}_2) < k(n-k)$ whenever n is odd, independently of Dutta-Khare's Lemma F. Indeed, if n is odd, then (see e.g. [1; Theorem 1.1]) $w_2(\tilde{G}_{n,k}) = \tilde{w}_2$. Hence the value of $\tilde{w}_2^{\frac{k(n-k)}{2}}$ on the fundamental class of the manifold $\tilde{G}_{n,k}$ is one of its Stiefel-Whitney numbers. But, as is well known, all the Stiefel-Whitney numbers of $\tilde{G}_{n,k}$ vanish.

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