NILPOTENCE IN GROUP COHOMOLOGY

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Abstract. We study bounds on nilpotence in $H^*(BG)$, the mod $p$ cohomology of the classifying space of a compact Lie group $G$. Part of this is a report of our previous work on this problem, updated to reflect the consequences of Peter Symonds recent verification of Dave Benson’s Regularity Conjecture. New results are given for finite $p$–groups, leading to good bounds on nilpotence in $H^*(BP)$ determined by the subgroup structure of the $p$–group $P$.

1. Introduction

Fixing a prime $p$, let $H^*(BG)$ denote the mod $p$ cohomology ring of the classifying space of a compact Lie group $G$. This is a graded commutative $\mathbb{F}_p$–algebra of great interest as it is the home for mod $p$ characteristic classes of principal $G$ bundles. Furthermore, when $G$ is finite, this ring identifies with $\text{Ext}^*_{\mathbb{F}_p[G]}(\mathbb{F}_p,\mathbb{F}_p)$, and so contains much detailed module theoretic information.

As the mod $p$ cohomology of a topological space, $H^*(BG)$ is in the category $\mathcal{U}$, the category of modules over the mod $p$ Steenrod algebra $A_p$ which satisfy the unstable condition. Following Hans-Werner Henn, Jean Lannes, and Lionel Schwartz in [HLS], one can define a ‘topological’ nilpotence degree as follows. Let $\Sigma^d M$ denote the $d$th suspension (upward shift) of a graded module $M$.

**Definition 1.1.** Define $d^{\text{alg}}(G)$ to be the maximal $d$ such that $\text{Rad}(G)^d \neq 0$.

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**Definition 1.2.** Define $d^{\text{tf}}(G)$ to be the maximal $d$ such that $H^*(BG)$ contains a nonzero submodule of the form $\Sigma^d M$, with $M \in \mathcal{U}$.

This definition is clearly just dependent on the $A_p$ module structure of $H^*(BG)$, but a result in [HLS] allows for comparison with $d^{\text{alg}}(G)$. As will be reviewed in §2, $d^{\text{tf}}(G)$ is also the minimal $d$ such that the algebra homomorphism

$$q_d : H^*(BG) \to \prod_V H^*(BV) \otimes H^{\leq d}(BC_G(V))$$

is monic. Here the product is over the elementary abelian $p$–subgroups of $G$, the component maps are induced by the the group homomorphisms $V \times C_G(V) \to G$,
and $M^{d}$ denotes the quotient of a graded module $M$ by all elements of degree more than $d$.

One easily concludes (see Corollary 2.6) that

$$d^{alg}(G) \leq \begin{cases} d^L(G) & \text{if } p = 2 \\ d^L(G) + r(G) & \text{if } p \text{ is odd.} \end{cases}$$

Here $r(G)$ is the maximal rank of an elementary abelian $p$–subgroup of $G$.

Our goal here is to describe how to calculate $d^U(G)$, and, in particular, to give good group theoretic upper bounds. We note that $d^U(G/p) = 0$ and $d^U(G \times H) = d^U(G) + d^U(H)$. Furthermore, by transfer arguments, $d^U(G) \leq d^U(P)$, if $P$ is a $p$–Sylow subgroup of a finite group $G$, and a similar inequality holds for a general compact Lie group $G$, with $P$ now the evident extension of a maximal torus $T$ by a $p$–Sylow subgroup of $N_G(T)/T$.

1.1. A general bound on $d^U(G)$. We recall some terminology from [K3].

Notation 1.3. Let $C(G) < G$ be the maximal central elementary abelian $p$–subgroup, and let $c(G)$ denote its rank. Via restriction, $H^*(BC(G))$ is a finitely generated $H^*(BG)$–module, and we let $e(G)$ denote the top degree of a generator.

Finally, here and throughout the paper, we let $V$ denote an elementary abelian $p$–group, i.e. a group isomorphic to $(\mathbb{Z}/p)^r$ for some $r$.

Theorem 1.4. If $G$ is compact Lie, then

$$\max_{V < G} \{e(C_G(V)) - \dim(C_G(V))\} \leq d^U(G) \leq \max_{V < G} \{e(C_G(V)) - \dim(C_G(V))\}.$$ 

Here $\dim(G)$ denotes the dimension of a Lie group $G$ as a manifold, and so is 0 if $G$ is finite. Note that the lower bound equals the upper bound when $G$ is $p$–central – a group in which every element of order $p$ is central – and, in that case, $d^U(G) = e(G) - \dim G$.

The proof of this theorem is given in §2. Most of this is a review and slight reorganization of work in [K3], with results extended to all compact Lie groups. Some of our results were previously conditional on the verification of Dave Benson’s Regularity Conjecture [B2] which conjectured the vanishing of certain local cohomology groups. Happily, this is now a theorem of Peter Symonds [Sy], and we make very precise how the vanishing of local cohomology groups allows for improvement on Theorem 1.4.

1.2. Bounds for finite $p$–groups. Further investigations when $G$ is a finite $p$–group lead to some very nice general statements.

The first is that the upper bound given in Theorem 1.4 simplifies.

Theorem 1.5. If $P$ is a $p$–group, then $d^U(P) \leq e(P)$.

We then have two new estimates of $e(P)$.

Theorem 1.6. Suppose a $p$–group $P$ acts faithfully on a set $S$ with no fixed points. Then

$$e(P) \leq \begin{cases} |S|/2 - |S/P| & \text{if } p = 2 \\ 2|S|/p - |S/P| & \text{if } p \text{ is odd.} \end{cases}$$

Here $|S|$ is the cardinality of $S$. 
**Theorem 1.7.** Let $A < P$ be an abelian subgroup of maximal order in a $p$–group $P$. Then $e(P) \leq c(P)(2|P|/|A| - 1)$.

**Example 1.8.** Both of these last two theorems are nicely illustrated by the following example. Let $P$ be a 2–Sylow subgroup of the finite group $SU(3,4)$. $P$ is a 2–central group of order 64, of exponent 4, with $C(P) = [P, P] \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$; see [K2, §6.3] for a useful description of this group. Both theorems give us the estimate $e(P) \leq 14$, which, in fact, computation shows equals $e(P)$, and thus $d^G(P)$.

To use Theorem 1.6, let $a, b \in P$ be elements of order 4 with $a^2 \neq b^2$. Then $P$ acts faithfully on $S = P/(a) \prod P/(b)$ with no fixed points, so $e(P) \leq 32/2 - 2 = 14$.

To use Theorem 1.7, the centralizer of any element of order 4 is isomorphic to $\mathbb{Z}/4 \times \mathbb{Z}/4$, thus $e(P) \leq 2[2(64/16) - 1] = 14$.

The proofs of both Theorem 1.5 and Theorem 1.6 depend on the following monotonicity result, which at first surprised us, as it is false for arbitrary finite groups.

**Theorem 1.9.** Let $Q$ be a subgroup of a $p$–group $P$. Then $e(Q) \leq e(P)$.

Theorem 1.7 is proved using Chern classes of representations, and would be a special case of the next conjecture, where we let $n(G)$ denote the minimal dimension (over $\mathbb{C}$) of a faithful complex representation of $G$.

**Conjecture 1.10.** If $G$ is compact Lie, then $e(G) \leq 2n(G) - c(G)$.

Assuming this, one easily deduces the general estimate $d^G(G) \leq 2n(G) - c(G)$; see Remark 3.9. This should be compared to the estimate in [HLS]: $d^G(G) \leq n(G)^2$.

§3 contains the proofs of all of these theorems and a discussion of the conjecture. The most subtle argument is the proof of Theorem 1.9. Proved by induction on the order of $P$, it reduced to a problem about invariants of arbitrary $\mathbb{Z}/p$ actions on subHopf algebras of polynomial algebras over $\mathbb{F}_p$: see Problem 3.11. This we then deal with in §4, proving results which may be of independent interest.

**Remark 1.11.** We note that our paper [K3] has tables of examples made using the Jon Carlson’s cohomology website [Ca2]. Thousands more examples are now similarly accessible using the cohomology website of David Green and Simon King [GK]. Their implementation includes the calculation of the restriction of $H^*(BP)$ to $H^*(BC(P))$, so that $e(P)$ can be immediately read off of their data. For example, one see that if $P$ is the 2–Sylow subgroup of the third Conway group, so $P$ has order 1024, then $e(P) = 7$ and so, combining Theorem 1.5 with the fine points of Theorem 2.22, we see that $d^G(P) \leq 6$.

1.3. **Acknowledgements.** The organization of §2 follows the presentation I gave at the Conference on Algebraic Topology, Group Theory, and Representation Theory held on the Isle of Skye, Scotland in June, 2009. I have tried to keep the audience I had there in mind. Conversations with Benson, Symonds, and invariant theorists David Wehlau, Jim Shank, and Eddy Campbell have been helpful.

2. Old results revisited

In this section, we prove the bounds for $d^G(G)$ given in Theorem 1.4. The main steps are as follows, where terminology and notation will be defined in due course:

- $d^G(G) = \max \{d_0(Cess^*(BC_G(V))) \mid V < G\}$. See Proposition 2.8.
\[ d_0(\text{Cess}^*(BG)) = e_{\text{prim}}(G). \] See Corollary 2.14.

\[ e_{\text{prim}}(G) \leq e_{\text{indec}}(G). \] See Corollary 2.20.

\[ e_{\text{indec}}(G) \leq e(G) - \dim G. \] See Theorem 2.22 for this and a bit more.

This last inequality refines using local cohomology as follows:

\[ e_{\text{indec}}(G) = e(G) + \max\{e \mid H_m^{c(G)} - c(G) + e(H^*(BG)) \neq 0\}. \]
See Theorem 2.29.

\[ H_m^{s,t}(H^*(BG)) = 0 \text{ if } s + t > -\dim G. \] This is Symonds’ theorem [Sy].

2.1. **The basic ring structure of** \( H^*(BG) \). We begin by recalling a fundamental example. If \( V = (\mathbb{Z}/p)^r \), then

\[
H^*(BV) \simeq \begin{cases} 
\mathbb{F}_2[x_1, \ldots, x_r] & \text{if } p = 2 \\
\Lambda(x_1, \ldots, x_r) \otimes \mathbb{F}_p[y_1, \ldots, y_r] & \text{if } p \text{ is odd},
\end{cases}
\]

where \(|x_i| = 1\) and \(y_i = \beta(x_i)\). (\(\beta\) is the Bockstein homomorphism.) Furthermore, addition \(V \times V \to V\) induces a primitively generated Hopf algebra structure on \(H^*(BV)\).

More generally, \(H^*(BG)\) can be difficult to compute explicitly, particularly when \(G\) is a more interesting finite \(p\)-group. For example, if \(P\) is the 2–Sylow subgroup of \(SU_3(4)\), as in Example 1.8, a minimal presentation of the algebra \(H^*(BP)\) has 26 generators (in degrees up to 11) and 270 relations (in degrees up to 22). (See [CTVZ, group #187], or [GK, group #145].)

In spite of this, some basic ring structure has been known for a long time. In the late 1960’s [Q] D.Quillen showed that \(H^*(BG)\) is Noetherian of Krull dimension \(r(G)\); equivalently, \(H^*(BG)\) is a finitely generated module over a subpolynomial algebra on \(r(G)\) generators. A decade later J.Duflot [D] showed that its depth is at least \(c(G)\); equivalently, \(H^*(BG)\) is a free module over a subpolynomial algebra on \(c(G)\) generators.

**Remark 2.1.** The extreme situation, when \(c(G) = r(G)\), happens precisely when \(G\) is \(p\)–central. Then \(H^*(BG)\) will be Cohen–MacCauley: the depth of \(H^*(BG)\) will equal its Krull dimension. In general, there is no group theoretic criterion characterizing either groups \(G\) such that the depth of \(H^*(BG)\) equals the lower bound \(c(G)\), or groups \(G\) such that the depth of \(H^*(BG)\) equals the upper bound \(r(G)\).

Quillen’s idea was to probe \(H^*(BG)\) by its restrictions to its elementary abelian \(p\)-subgroups. The product over all such restrictions gives a ring homomorphism

\[ q_0 : H^*(BG) \to \prod_{V < G} H^*(BV). \]

Recall that, given \(K < G\), the restriction map \(H^*(BG) \to H^*(BK)\) makes \(H^*(BK)\) into a finitely generated \(H^*(BG)\)–module. Thus the codomain of \(q_0\), a ring whose Krull dimension is clearly \(r(G)\), is finitely generated over \(H^*(BG)\). Quillen then
shows that $\ker(q_0)$ is nilpotent, which then immediately implies the result about Krull dimension.

2.2. The nilpotent filtration of $\mathcal{U}$. As the mod $p$ cohomology of a topological space, $H^*(BG)$ is an unstable algebra over the mod $p$ Steenrod algebra $A_p$. We recall that an $A_p$-module $M$ is unstable, if, when $p = 2$, $Sq^k x = 0$ if $k > |x|$, and, when $p$ is odd, $\beta^n P^k x = 0$ if $2k + e > |x|$. $M$ is an unstable algebra if in addition, it is a graded commutative algebra satisfying both the Cartan and Restriction formulae.

The 1980's featured much remarkable work on $K$ and $\mathcal{U}$, the categories of unstable algebras and modules, with the algebras $H^*(BV)$ playing a special role. (See [S2] for entry into the extensive literature.)

In the 1995 paper [HLS], H.-W. Henn, J. Lannes, and L. Schwartz revisited Quillen's work from this new perspective. Following [HLS], we have the following definition.

**Definition 2.2.** If $M$ is an unstable $A_p$-module, let $d_0(M)$ be the maximal $d$ such that $M$ contains a nonzero submodule of the form $\Sigma^d N$, with $N$ unstable. If no such maximum exists, let $d_0(M) = \infty$, and let $d_0(0) = -\infty$.

Thus the invariant $d^d(G)$ of the introduction is $d_0(H^*(BG))$.

An alternate definition, easily shown equivalent to the one above, is that $d_0(M)$ is the length of the nilpotent filtration $[S1]$ of $M$,

$$\cdots \subset \text{nil}_d M \subset \text{nil}_{d-1} M \subset \text{nil}_1 M \subset \text{nil}_0 M = M,$$

where $\text{nil}_d M$ is the large submodule in the localizing subcategory of $\mathcal{U}$ generated by the $d$-fold suspensions.

Three elementary properties of $d_0(M)$ are stated in the next lemma.

**Lemma 2.3.** (a) If $M$ is nonzero in degree $d$, but zero in all higher degrees, then $d_0(M) = d$.

(b) If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence in $\mathcal{U}$, then $d_0(M_1) \leq d_0(M_2)$, and $d_0(M_2) \leq \max\{d_0(M_1), d_0(M_3)\}$.

(c) $d_0(H^*(B\mathbb{Z}/p)) = 0$.

The next properties are considerably deeper. References for (a) are [K1, Prop.2.5] or [HLS, Prop.1.3.6]. Property (b) concerns Lannes' functor $[L2] \mathcal{T}_V : \mathcal{U} \to \mathcal{U}$, the left adjoint to the functor $M \mapsto H^*(BV) \otimes M$, and a reference is [K3, Prop.3.12]. Property (c) is due to Henn [H].

**Proposition 2.4.** (a) $d_0(M \otimes N) = d_0(M) + d_0(N)$.

(b) $d_0(\mathcal{T}_V M) = d_0(M)$.

(c) $d_0(M) < \infty$ if $M$ is a finitely generated module over an Noetherian unstable algebra $K$ with structure map $K \otimes M \to M$ in $\mathcal{U}$.

2.3. The comparison between $d^{alg}(G)$ and $d^d(G)$. Note that property (c) of the last proposition implies that $d^d(G) < \infty$, so that the nilpotent filtration of $H^*(BG)$ has finite length.

In [HLS], the authors show how to generalize Quillen's map $q_0$ to realize the nilpotent filtration of $H^*(BG)$. For each $d \geq 0$, let

$$q_d : H^*(BG) \to \prod_V H^*(BV) \otimes H^{\leq d}(BC_G(V))$$

be the map of unstable algebras as defined in the introduction. They observe that $\ker q_d = \text{nil}_{d+1} H^*(BG)$, and so we have the following.
Proposition 2.5. \( d^d(G) \) is the minimal \( d \) such that \( q_d \) is monic.

If \( I \) is a nilpotent ideal in a graded Noetherian ring, let \( d^{alg}(I) \) be the maximal \( d \) such that \( I^d \neq 0 \). Thus the invariant \( d^{alg}(G) \) of the introduction is \( d^{alg}(\text{Rad}(G)) \). Note that

\[
\begin{align*}
d^{alg}(H^*(BV) \otimes \widehat{H}^\leq d(BC_G(V))) &= \begin{cases} d^{alg}(\widehat{H}^\leq d(BC_G(V))) & \text{if } p = 2 \\ d^{alg}(\Lambda(V^\#) \otimes \widehat{H}^\leq d(BC_G(V))) & \text{if } p \text{ is odd.} \end{cases}
\end{align*}
\]

Corollary 2.6. With \( d = d^d(G) \),

\[
d^{alg}(G) \leq \begin{cases} \max_V d^{alg}(\widehat{H}^\leq d(BC_G(V))) & \text{if } p = 2 \\ \max_V \{d^{alg}(\widehat{H}^\leq d(BC_G(V))) + r(V)\} & \text{if } p \text{ is odd.} \end{cases}
\]

2.4. Central essential cohomology. The following definition from [K3] is a variant of Carlson’s Depth Essential Cohomology [CTVZ].

Definition 2.7. Let \( \text{Cess}^*(BG) \) be the kernel of the map

\[
H^*(BG) \to \prod_{C(G) \subseteq V} H^*(BC_G(V)).
\]

This is an unstable \( \mathcal{A} \)-module. \( \text{Cess}^*(BG) = H^*(BG) \) exactly when the product is over the empty set, i.e. \( G \) is \( p \)-central. \( \text{Cess}^*(BG) \) can also be zero: as we will see, \( \text{Cess}^*(BG) \neq 0 \) if and only if the depth of \( H^*(BG) = c(G) \).

Proposition 2.8. \( d^d(G) = \max \{d_0(\text{Cess}^*(BC_G(V))) \mid V < G\} \).

To prove this, we first need the following consequence of the calculation of \( T_V H^*(BG) \) due to Lannes [L1].

Proposition 2.9. For all \( V < G \), \( H^*(BC_G(V)) \) is a summand of \( T_V H^*(BG) \), and thus \( d^d(C_G(V)) \leq d^d(G) \).

Proof of Proposition 2.8. This follows by downward induction on the rank of \( C(G) \). From the exact sequence

\[
0 \to \text{Cess}^*(BG) \to H^*(BG) \to \prod_{C(G) \subseteq V} H^*(BC_G(V)),
\]

one sees that

\[
d^d(G) \leq \max \{d_0(\text{Cess}^*(BG)), d^d(C_G(V)) \mid C(G) \subseteq V < G\}.
\]

But this inequality is an equality by the last proposition. \( \square \)

2.5. Primitives in central essential cohomology. For the rest of this section, we fix a compact Lie group \( G \), and let \( C = C(G) \).

By an unstable \( H^*(BC) \)-comodule, we will mean an unstable module \( M \) having an \( H^*(BC) \)-comodule structure map \( \Delta : M \to H^*(BC) \otimes M \) that is in \( \mathcal{U} \). Examples of interest to us include \( H^*(BG), H^*(BC_G(V)) \) for all \( V < G \), and \( \text{Cess}^*(BG) \), where the comodule structures are all induced by the group homomorphism \( C \times G \to G \).
Definitions 2.10. If $M$ is an unstable $H^*(BC)$–comodule, we define its associated module of primitives to be

\[ P_C M = \{ x \in M \mid \Delta(x) = 1 \otimes x \} = \operatorname{Eq} \{ M \xrightarrow{\Delta} H^*(C) \otimes M \}. \]

If $P_C M$ is finite dimensional, we let $e_{\text{prim}}(M)$ be its largest nonzero degree, or $-\infty$ if $M = 0$.

Note that $P_C M$ is again an unstable module.

Lemma 2.11. If $M$ is an unstable $H^*(BC)$–comodule, and $P_C M$ is finite dimensional, then $d_0(M) = e_{\text{prim}}(M)$.

Proof. Assume $P_C M$ is finite dimensional with largest nonzero degree $e = e_{\text{prim}}(M)$. Then $e = d_0(P_C M)$. Since $P_C M$ is an unstable submodule of $M$, $d_0(P_C M) \leq d_0(M)$. Finally, the composite

\[ M \xrightarrow{\Delta} H^*(BC) \otimes M \to H^*(BC) \otimes M^{\leq e} \]

will be monic, so that

\[ d_0(M) \leq d_0(H^*(BC) \otimes M^{\leq e}) = d_0(M^{\leq e}) = e. \]

\[ \square \]

Proposition 2.12. $P_C \text{Cess}^*(BG)$ is finite dimensional.

Proof. [K3, Thm.8.5] implies that if $P_C \text{Cess}^d(BG) \neq 0$, then $d \leq d^f(G)$. \[ \square \]

Remark 2.13. The careful reader will discover that [K3, Thm.8.5] has a rather delicate proof, using related results in [K2], all based on careful analysis of formulae in [HLS]. It would be nice to have a simpler proof of the proposition. In the next subsection we will see (Corollary 2.19) that $P_C \text{Cess}^*(BG)$ is finite dimensional if and only if $\text{Cess}^*(BG)$ has Krull dimension equal to $c(G)$. When $G$ is finite, this Krull dimension calculation is verified [K3, Prop.8.2] using a result of J. Carlson [Ca1].

We let $e_{\text{prim}}(G)$ denote $e_{\text{prim}}(\text{Cess}^*(BG))$.

Corollary 2.14. $d_0(\text{Cess}^*(BG)) = e_{\text{prim}}(G)$.

2.6. Duflot algebras. Let $c = c(G)$, the rank of $C = C(G)$, so that

\[ H^*(BC) \simeq \begin{cases} F_2[x_1, \ldots, x_c] & \text{if } p = 2 \\ \Lambda(x_1, \ldots, x_c) \otimes F_p[y_1, \ldots, y_c] & \text{if } p \text{ is odd}. \end{cases} \]

The image of the restriction homomorphism $i^* : H^*(BG) \to H^*(BC)$ will be a sub Hopf algebra of $H^*(BC)$. Thus, after a change of basis for $H^1(BC)$, it will have the form

\[ \operatorname{im}(i^*) = \begin{cases} F_2[x_1^{2^{j_1}}, \ldots, x_c^{2^{j_c}}] & \text{if } p = 2 \\ F_p[y_1^{p^{b_1}}, \ldots, y_c^{p^{b_c}}, y_{b+1}, \ldots, y_c] \otimes \Lambda(x_{b+1}, \ldots, x_c) & \text{if } p \text{ is odd}, \end{cases} \]

with the $j_i$ forming a sequence of nonincreasing nonnegative integers. (See [BrH, Rem.1.3] and [AS].) In the odd prime case, $c - b$ has group theoretic meaning as the rank of the largest subgroup of $C$ splitting off $G$ as a direct summand.
As in [K3], we will say that $G$ has type $[a_1, \ldots, a_c]$ where

$$
(a_1, \ldots, a_c) = \begin{cases} 
(2^{j_1}, \ldots, 2^{j_c}) & \text{if } p = 2 \\
(2p^{i_1}, \ldots, 2p^{i_b}, 1, \ldots, 1) & \text{if } p \text{ is odd}.
\end{cases}
$$

Recall that $e(G)$ is defined to be the largest degree of a $H^*(BG)$–module generator of $H^*(BC)$, i.e. the top degree of the finite dimensional Hopf algebra $H^*(BC) \otimes F_p$. Note that this number is determined by the type of $G$:

$$
e(G) = \sum_{i=1}^c (a_i - 1).
$$

Since $\text{im}(i^*)$ is a free commutative algebra, one can split the epimorphism of rings $i^* : H^*(BG) \to \text{im}(i^*)$, and make the next definition.

**Definition 2.15.** A Duflot algebra of $H^*(BG)$ is a subalgebra $A \subseteq H^*(BG)$, such that $i^* : A \to \text{im}(i^*)$ is an isomorphism.

**Remark 2.16.** It seems unclear that a Duflot algebra can always be chosen to also be closed under Steenrod operations. Nor does it seem that it can be always chosen to be a sub–$H^*(BC)$–comodule of $H^*(BG)$.

### 2.7. Indecomposables in central essential cohomology

For the rest of the section, now also fix a Duflot algebra $A \subseteq H^*(BG)$.

**Definitions 2.17.** If $M$ is an $A$–module, we define the $A$–indecomposables to be $Q_A M = M \otimes_A F_p$. If $Q_A M$ is finite dimensional, we let $e_{\text{indec}}(M)$ be its largest nonzero degree, or $-\infty$ if $M = 0$.

Observe that everything in the exact sequence

$$0 \to Cess^*(BG) \to H^*(BG) \to \prod_{C \not\subseteq V} H^*(BC_G(V))$$

is both an $A$–module and a $H^*(BC)$–comodule. These structures are sufficiently compatible ‘up to filtration’ that one can prove the following.

**Proposition 2.18.** The following hold.

(a) $Cess^*(BG)$ is a free $A$–module.

(b) The composite $P_C Cess^*(G) \hookrightarrow Cess^*(G) \to Q_A Cess^*(G)$ is monic.

(c) The sequence $0 \to Q_A Cess^*(G) \to Q_A H^*(G) \to \prod_{C(G) < U} Q_A H^*(C_G(U))$ is exact.

See [K3, Prop.8.1].

**Corollary 2.19.** $Q_A Cess^*(BG)$ is finite dimensional if and only if $P_C Cess^*(BG)$ is finite dimensional. In this case, $e_{\text{prim}}(Cess^*(BG)) \leq e_{\text{indec}}(Cess^*(BG))$.

**Proof.** For notational simplicity, let $M = Cess^*(BG)$. The proposition immediately implies that if $Q_A M$ is finite dimensional so is $P_C M$, and the stated inequality will hold. Conversely, suppose $P_C M$ is finite dimensional. Recall that the composite (of $A$–modules)

$$M \xrightarrow{\Delta} H^*(BC) \otimes M \to H^*(BC) \otimes M \leq e_{\text{prim}}(M)$$
is monic. As $H^*(BC) \otimes M_{\leq e_{prim}(M)}$ is certainly a finitely generated $A$–module, so is $M$.

We let $e_{indec}(G)$ denote $e_{indec}(Cess^*(BG))$.

**Corollary 2.20.** Cess$^*(BG)$ is a finitely generated free $A$–module, and $e_{prim}(G) \leq e_{indec}(G)$.

**Remark 2.21.** As we observed computationally in [K3, Appendix A], $e_{prim}(G) = e_{indec}(G)$ for all finite 2–groups $G$ of order dividing 32. We suspect that this pattern will not continue, but it would be nice to have an explicit example for which the inequality of the corollary is strict.

### 2.8. Local cohomology and Symond’s theorem.

The last step in our proof of Theorem 1.4 is the verification of the next bound.

**Theorem 2.22.** For all $G$, $e_{indec}(G) \leq e(G) - \dim G$. The inequality is strict unless $G$ is $p$–central. If $G$ is $p$–central, then $d_{lt}(G) = e_{prim}(G) = e_{indec}(G) = e(G) - \dim G$.

We note that, even when $p$ is odd, it suffices to prove this when the Duflot algebra $A$ is a polynomial algebra, i.e. when $G$ has no $\mathbb{Z}/p$ direct summands, as $d_{lt}(G \times V) = d_{lt}(G)$, $e_{prim}(G \times V) = e_{prim}(G)$, $e_{indec}(G \times V) = e_{indec}(G)$, and $e(G \times V) = e(G)$.

We need to begin with a quick summary of definitions and properties of local cohomology. A general reference for this is [BS].

Let $m$ be a maximal ideal in a graded Noetherian ring $R$. For $M$ an $R$–module,

$$M \mapsto H^s_m(M)$$

is defined to be the $s$th right derived functor of

$$M \mapsto H^0_m(M) = \text{ the } m\text{-torsion part of } M.$$

**Proposition 2.23.** $H^s_m(M) \neq 0$ only if $\text{depth}_m M \leq s \leq \dim M$. Furthermore, If $s = \text{depth}_m M$ or $s = \dim M$, then $H^s_m(M) \neq 0$.

This is the content of [BS, Cor.6.2.8].

We need some related results about how local cohomology interacts with regular $M$–sequences. Let $|z|$ denote the degree of $z \in R$.

**Lemma 2.24.** Fix $(s, t)$, and suppose that $H^{s', t'}_m(M) = 0$ for $s' < s$ and for $(s, t')$ with $t' > t$. If $z \in R$ is an $M$–regular element, then $H^{s', t'}_m(M/(z)) = 0$ for $s' < s - 1$ and for $(s - 1, t')$ with $t' > t + |z|$, and, furthermore

$$H^{s-1, t+|z|}_m(M/(z)) \simeq H^{s, t}_m(M).$$

**Proof.** By assumption, $z$ is not a zero divisor of $M$, so there is a short exact sequence of $R$–modules

$$0 \rightarrow \Sigma^{|z|} M \rightarrow M \rightarrow M/(z) \rightarrow 0.$$

The lemma then follows from the associated long exact sequence, which has the form

$$\ldots \rightarrow H^{s'-1, t'+|z|}_m(M) \rightarrow H^{s'-1, t'+|z|}_m(M/(z)) \rightarrow H^{s', t'}_m(M) \rightarrow H^{s', t'+|z|}_m(M) \rightarrow \ldots$$

$\square$
By induction on the length of a regular sequence, the lemma has the following corollary.

**Corollary 2.25.** With assumptions on \((s, t)\) and \(M\) as in the lemma, if \(z_1, \ldots, z_s\) is an \(M\)-regular sequence, then \(H_{m}^{0,t}(M/(z_1, \ldots, z_s)) = 0\) for \(t' > t + |z_1| + \cdots + |z_s|\), and

\[
H_{m}^{0,t+|z_1|+\cdots+|z_s|}(M/(z_1, \ldots, z_s)) \simeq H_{m}^{0,t}(M).
\]

We now apply this in the case when \(R = M = H^*(BG)\) and \(m = \bar{H}^*(BG)\). Let \(c = c(G), r = r(G)\). If \(z_1, \ldots, z_c\) are algebra generators for the Duflot algebra \(A\), then \(|z_1| + \cdots + |z_s| = c + e(G)\), and \(M/(z_1, \ldots, z_c) = Q_A M\), and the corollary tells us the following.

**Proposition 2.26.** Suppose \(H_{m}^{c(G),-c(G)+e'}(H^*(BG)) = 0\) for all \(e' > e\). Then \(H_{m}^{0,c(G)+e'}(Q_A H^*(BG)) = 0\) for all \(e' > e\), and

\[
H_{m}^{0,c(G)+e}(Q_A H^*(BG)) = H_{m}^{c,-c+e}(H^*(BG)).
\]

Now we note

**Proposition 2.27.** \(Q_A C_{ess}^*(BG) = H_{m}^{0,*}(Q_A C_{ess}^*(BG)) = H_{m}^{0,*}(Q_A H^*(BG))\).

Our argument is similar to that proving [K3, Prop.8.9]. We need

**Lemma 2.28** ([K3, Lem.8.8]). Assume \(c < r\). Given any sequence \(z_1, \ldots, z_c \in H^*(G)\) that generates the polynomial algebra \(A\), there exists \(z \in H^*(BG)\) such that, for all proper inclusions \(C < V, z_1, \ldots, z_c, z\) restricts to a regular sequence in \(H^*(BC_G(V))\).

**Proof of Proposition 2.27.** As \(Q_A C_{ess}^*(BG)\) is finite dimensional, we clearly have

\(Q_A C_{ess}^*(BG) = H_{m}^{0,*}(Q_A C_{ess}^*(BG))\).

By Proposition 2.18, we have an exact sequence

\[
0 \to Q_A C_{ess}^*(BG) \to Q_A H^*(BG) \to \prod_{C(G) \leq V} Q_A H^*(BC_G(V)),
\]

and this induces an exact sequence

\[
0 \to H_{m}^{0,*}(Q_A C_{ess}^*(BG)) \to H_{m}^{0,*}(Q_A H^*(BG)) \to \prod_{C(G) \leq V} H_{m}^{0,*}(Q_A H^*(BC_G(V))).
\]

But the last term here is 0, because if \(z \in H^*(BG)\) is chosen as in the lemma, then \(z\) will act regularly on each \(Q_A H^*(BC_G(V))\) with \(C(G) \leq V\).

The last two propositions combine to prove the next theorem.

**Theorem 2.29.** \(e_{indec}(G) = e(G) + \max\{e \mid H_{m}^{c(G),-c(G)+e'}(H^*(BG)) \neq 0\}\).

**Proof of Theorem 2.22.** Symonds [Sy] has proved that

\[
H_{m}^{s,t}(H^*(BG)) = 0\text{ if } s + t > -\dim G.
\]

Combined with the last theorem, this immediately implies the first part of the theorem: for all compact Lie group \(G\),

\[
e_{indec}(G) \leq e(G) - \dim G.
\]
Furthermore, this inequality will be strict if and only if

$$H_m^{c(G), -e(G) - \dim(G)}(H^*(BG)) = 0.$$ 

To deduce more, we need to recall why Symonds’s result (in the finite group case) had been conjectured by Benson. As constructed by J.P.C. Greenlees and Benson [BG], there is a spectral sequence

$$H_{m,t}^s(H^*(BG)) = E_{2}^{s,t} \Rightarrow \tilde{H}_{-s-t}(EG_+ \wedge_G S^{Ad(G)}; \mathbb{F}_p),$$

where $S^{Ad(G)}$ is the one point compactification of the adjoint representation, so Benson was conjecturing that some evident vanishing at the level of $E_\infty$ happened already at $E_2$.

By Symonds’ theorem, the group $H_m^{c(G), -e(G) - \dim(G)}(H^*(BG))$ consists of permanent cycles, as the differentials will take values in groups that are zero. As this group is certain not in the image of nonzero boundary maps, it will thus be a quotient of

$$\tilde{H}_{\dim(G)}(EG_+ \wedge_G S^{Ad(G)}; \mathbb{F}_p) \simeq \begin{cases} \mathbb{F}_p & \text{if } Ad(G) \text{ is } \mathbb{F}_p\text{-oriented.} \\ 0 & \text{if not.} \end{cases}$$

In the oriented case, $H_m^{r(G), -r(G) - \dim(G)}(H^*(BG)) \simeq \mathbb{F}_p$, by a generalization to all compact Lie groups of Benson’s argument [B1] in the finite group case. (The generalization is straightforward, using the transfer map $H^*(BV) \to H^{*+\dim(G)}(BG)$ associated to an inclusion $V < G$.)

Thus, in either the oriented or nonoriented case, we see that

$$H_m^{c(G), -e(G) - \dim(G)}(H^*(BG)) = 0$$

unless $c(P) = r(P)$, i.e. $G$ is $p$–central. In the $p$–central case, $G$ will be oriented and $e_{indec}(G) = e(G)$. But arguing as in [K3], one can do better: the top class in $Q_A H^*(BG)$ will be represented by a $H^*(BC)$–primitive, so $e_{prim}(G) = e_{indec}(G)$.

We end this section by noting that our results above include a proof of Carlson’s Depth Conjecture in the case of minimal depth, generalizing results in [G, K3]. Note that

$$e_{indec}(G) \neq -\infty \Leftrightarrow e_{indec}(G) \geq 0 \Leftrightarrow Q_A Cess^*(G) \neq 0 \Leftrightarrow Cess^*(G) \neq 0,$$

and

$$H_m^{c(G), *}(H^*(BG)) \neq 0 \Leftrightarrow H^*(BG) \text{ has depth precisely } c(G).$$

Therefore, Theorem 2.29 tells us most of the following, and Symond’s theorem tells us the rest.

**Theorem 2.30.** For $G$ compact Lie, $H^*(BG)$ has depth precisely $c(G)$ if and only if $H^*(BG)$ is not detected by restriction to the cohomology rings $H^*(BC_G(V))$ for $V < G$ of rank greater than $c(G)$. In this case, $H_m^{c(G), t}(H^*(BG)) \neq 0$ for some $-c(G) - \dim(G) \leq t \leq -c(G) - e(G)$.

**Corollary 2.31.** If $G$ is compact Lie, and $e(G) < \dim(G)$ then $H^*(BG)$ has depth greater than $c(G)$ and is detected by restriction to the cohomology rings $H^*(BC_G(V))$ for $V < G$ of rank greater than $c(G)$. 
3. New results for finite $p$ groups

We now prove various new results about $e(P)$ when $P$ is a finite $p$–group. We begin with a proof of Theorem 1.7, with part of the discussion relevant for all compact Lie groups $G$. We will next deduce Theorem 1.5 and Theorem 1.6 assuming Theorem 1.9. Finally we will prove Theorem 1.9, which will involve an excursion into invariant theory.

3.1. Upper bounds for $e(P)$ coming from Chern classes. We use Chern classes of representation to get group theoretic upper bounds for $e(P)$ when $P$ is a finite $p$–group. With $C = C(P)$, we need to get a lower bound on $\text{im}(i^*)$, the image of restriction

$$ i^* : H^*(BP) \to H^*(BC). $$

To set up notation and unify exposition, let $c = c(P)$ and let

$$ H^*(BC) \cong \begin{cases} \mathbb{F}_2[x_1, \ldots, x_c] & \text{if } p = 2 \\ \Lambda(x_1, \ldots, x_c) \otimes \mathbb{F}_p[y_1, \ldots, y_c] & \text{if } p \text{ is odd,} \end{cases} $$

where $y_i \in H^2(BC)$ denotes $\beta(x_i)$ for all primes (so that $y_i = x_i^2$ when $p = 2$). Note that each element $y_i$ is the Chern class of a unique one dimensional complex representation $\omega_i$ of $C$.

Now let $A < P$ be a maximal abelian subgroup, so that $A$ certainly contains $C$. Each $\omega_i$ extends, possibly nonuniquely, to a one dimensional representation $\tilde{\omega}_i$ of $A$. Now let $\rho_i = \text{Ind}_A^P(\tilde{\omega}_i)$, a representation of $P$ of dimension $|P : A| = |P|/|A|$.

By construction, the restriction of $\rho_i$ to $C$ will be $|P|/|A|\omega_i$, which has top Chern class $y_i^{[P]/[A]}$. We have proved the next theorem, a precise form of Theorem 1.7.

**Theorem 3.1.** The Hopf algebra $\text{im}(i^*)$ contains $\mathbb{F}_p[y_1^{[P]/[A]}, \ldots, y_c^{[P]/[A]}]$. Thus $e(P) \leq c(2|P|/|A| - 1)$.

**Remark 3.2.** Let $e_{\text{grp}}(P) = c(P)(2|P|/|A| - 1)$, where $A < P$ is an abelian subgroup of maximal order; thus the theorem says that $e(P) \leq e_{\text{grp}}(P)$. With arguments similar, but simpler, to ones we will use in the proof of Theorem 1.9, it is not hard to prove that this invariant of $p$–groups has the following monotonicity property:

if $Q < P$, then $e_{\text{grp}}(Q) \leq e_{\text{grp}}(P)$.

This property suffices to deduce that if $P$ is a finite $p$–group, then $d^H(P) \leq e_{\text{grp}}(P)$:

$$ d^H(P) \leq \max_{V \subset G}\{e(C_G(V))\} \leq \max_{V \subset G}\{e_{\text{grp}}(C_G(V))\} \leq e_{\text{grp}}(P). $$

3.2. Conjectural upper bounds for $e(G)$ coming from Chern classes. We continue in the spirit of the last subsection, and discuss how one might use Chern classes to prove Conjecture 1.10. This says that, if $n(G)$ is the minimal dimension of a faithful representation of a compact Lie group $G$, then $e(G) \leq 2n(G) - c(G)$.

With $C = C(G)$, the calculation of $e(G)$ requires understanding of the Hopf algebra $\text{im}(i^*)$, the image of the restriction

$$ i^* : H^*(BG) \to H^*(BC). $$

**Definitions 3.3.** (a) If $\rho$ is a representation of $C$, let $\mathcal{H}(\rho) \subset H^*(BC)$ be the smallest Hopf algebra containing its Chern classes. When $\rho$ is faithful, $H^*(BC)$ will be a finitely generated $\mathcal{H}(\rho)$ module, and, in this case, let $e(\rho)$ be the top degree of $Q_{\mathcal{H}(\rho)}H^*(BC)$. 

(b) If $G$ is a compact Lie group with $C = C(G)$, let $\mathcal{H}(G)$ be the smallest Hopf algebra containing all of the $\mathcal{H}(%(\rho))$, where $\rho$ ranges over all representations of $G$, restricted to $C$, and let $e_{rep}(G)$ be the top degree of $Q_{\mathcal{H}(G)}H^*(BC)$.

It is clear that for any representation $\rho$ of $G$,
$$\mathcal{H}(\rho) \subseteq \mathcal{H}(G) \subseteq \text{im}(i^*)$$
so we learn the following.

**Proposition 3.4.** $e(G) \leq e_{rep}(G) \leq e(\rho)$.

Thus Conjecture 1.10 would follow immediately from the next conjecture, which just concerns Chern classes of representations of elementary abelian groups.

**Conjecture 3.5.** Let $C$ be an elementary abelian $p$–group of rank $c$. If $\rho$ is a faithful $n$ dimensional complex representation of $C$, then $e(\rho) \leq 2n - c$.

In turn, this conjecture would be consequence of a conjectural identification of the Hopf algebra $\mathcal{H}(\rho)$. To describe this, it is convenient to use more basis free notation.

Let $V = \langle y_1, \ldots, y_c \rangle$, so that $\mathcal{H}(\rho) \subseteq S^*(V) = \mathbb{F}_p[y_1, \ldots, y_c] \subseteq H^*(BC)$. Given a subspace $W < V$, we let $\Phi^p(W) \subseteq S^p(V)$ be the span of the $p$th powers of the elements in $W$. General sub Hopf algebras of $S^*(V)$ then correspond to filtrations of $V$ as follows.

**Definition 3.6.** Let $\mathcal{F}$ be a finite length filtration of the $\mathbb{F}_p$–vector space $V$:

$$V(0) \subseteq V(1) \subseteq \cdots \subseteq V(k) = V.$$

The corresponding Hopf algebra $\mathcal{H}(\mathcal{F})$ is then defined to be

$$\mathcal{H}(\mathcal{F}) = S^*(V(0)) + \Phi(V(1)) + \cdots + \Phi^k(V(k)).$$

Note that $e(\mathcal{F})$, the top degree of $Q_{\mathcal{H}(\rho)}(H^*(BC))$, can be computed as follows. Let $c_i(\mathcal{F})$ be the rank of $V(i)/V(i - 1)$, so that $c_0(\mathcal{F}) + \cdots + c_k(\mathcal{F}) = c$. Then

$$e(\mathcal{F}) = 2 \left( \sum_{i=0}^k c_i(\mathcal{F})p^i \right) - c.$$

Now suppose $\rho$ is a faithful $n$ dimensional complex representation of $C$. This will be a sum of line bundles, possibly with multiplicities, and so will correspond to the following data:

- A finite set of distinct elements $v_1, \ldots, v_m \in V$ which span $V$.
- Multiplicities $n_1, \ldots, n_m \in \mathbb{N}$ such that $n_1 + \cdots + n_m = n$.

From this data, we define a filtration $\mathcal{F}_{\rho}$ of $V$ by letting $V(i)$ be the span of the $v_j$ such that $p^i$ does not divide $n_j$.

**Lemma 3.7.** $\mathcal{H}(\rho) \subseteq \mathcal{H}(\mathcal{F}_{\rho})$.

**Proof.** Let $ch(\rho)$ denote the total Chern class. We will have

$$ch(\rho) = \prod_{j=1}^m (1 + v_j)^{n_j} = \prod_{i} \prod_{v_j \in V(i) - V(i-1)} (1 + v_j^{p^i})^{n_j/p^i}.$$

As $v_j^{p^i} \in \mathcal{H}(\mathcal{F}_{\rho})$ for $v_j \in V(i) - V(i-1)$, we see that all the homogenous components of $ch(\rho)$ are in $\mathcal{H}(\mathcal{F}_{\rho})$ as well. $\square$
We conjecture equality in the last lemma.

**Conjecture 3.8.** $\mathcal{H}(\rho) = \mathcal{H}(\mathcal{F}_\rho)$.

As the estimate $e(\mathcal{F}_\rho) \leq 2n - c$ is easily checked, this conjecture implies Conjecture 3.5, and thus Conjecture 1.10.

**Remark 3.9.** Note that, for any $V < G$, $n(C_G(V)) \leq n(G)$ and $c(C_G(V)) \geq c(G)$. Thus, if Conjecture 1.10 were true, we could deduce

$$d^H(G) \leq \max_{V < G} \{e(C_G(V))\} \leq \max_{V < G} \{2n(C_G(V)) - c(C_G(V))\} \leq 2n(G) - c(G).$$

### 3.3. Proofs of Theorem 1.5 and Theorem 1.6 assuming Theorem 1.9.

Here we assume Theorem 1.9, which says that if $P$ is a $p$-group, and $Q < P$, then $e(Q) \leq e(P)$, and deduce Theorem 1.5 and Theorem 1.6.

**Proof of Theorem 1.5.** This is immediate: $d^H(P) \leq \max_{V < P} \{e(C_P(V))\} \leq e(P)$. □

**Proof of Theorem 1.6.** Suppose a $p$-group $P$ acts faithfully on a set $S$ with no fixed points. We wish to show that $e(P) \leq |S|/2 - |S/P|$ when $p = 2$, and $e(P) \leq 2|S|/p - |S/P|$ when $p$ is odd.

Note that $S/P$ is the set of orbits of $S$, so $S$ has a decomposition into orbits

$$S = \prod_{i=1}^{n} S_i,$$

with $|S_i| = p^{r_i}$, and each $r_i \geq 1$. Then $P$ admits an embedding

$$P \leq \prod_{i=1}^{n} W(r_i),$$

where $W(r)$ denotes the Sylow subgroup of the symmetric group $\Sigma_{p^r}$.

Assuming Theorem 1.9, we would then have the bound

$$e(P) \leq \sum_{i=1}^{n} e(W(r_i)).$$

The next proposition will thus complete the proof of Theorem 1.6 □

**Proposition 3.10.** When $p = 2$, $e(W(r)) = 2^{r-1} - 1$. When $p$ is odd, $e(W(1)) = 0$, and, for $r \geq 1$, then $e(W(r)) = 2^{p^r-1} - 1$.

**Proof.** We begin by identifying $C(r) = C(W(r))$. We claim that $C(r) \simeq \mathbb{Z}/p$. This is easily proved by induction on $r$, as $W(r+1)$ is the semidirect product

$$W(r+1) = W(r)^p \rtimes \mathbb{Z}/p,$$

so that

$$C(r+1) = (C(r)^p)_{\mathbb{Z}/p},$$

the diagonal copy of $C(r)$ in $C(r)^p$.

Now we determine $\text{im}(i(r)^*) \subset H^*(BC(r))$, where $i(r) : C(r) \to W(r)$ is the inclusion.

The case when $r = 1$ is elementary: $C(1) = W(1) = \mathbb{Z}/p$, so $\text{im}(i(1)^*) = H^*(B\mathbb{Z}/p)$ and $e(W(1)) = 0$ for all primes $p$. 


To proceed by induction, we observe that the inclusions
\[ C(r + 1) \to C(r)^p \to W(r)^p \to W(r + 1) \]
induce a factorization of \( i(r + 1)^* \) as
\[ H^*(BW(r + 1)) \to H^*(BW(r)^p)^{\mathbb{Z}/p} \xrightarrow{i(r)^*} H^*(BC(r)^p)^{\mathbb{Z}/p} \to H^*(BC(r + 1)), \]
with the first map epic as indicated.

Now let \( p \) be odd. Identifying \( H^*(BC(r)) \) with \( \Lambda(x) \otimes \mathbb{F}_p[y] \), we prove by induction that, for \( r \geq 2 \), \( \text{im}(i(r)^*) = \mathbb{F}_p[y^p^r - 1] \) so that \( e(W(r)) = 2p^r - 1 \).

The case when \( r = 2 \) is slightly special: \( \text{im}(2)^* \) will be the image of
\[ (\Lambda(x_1, \ldots, x_p) \otimes \mathbb{F}_p[y_1, \ldots, y_p])^{\mathbb{Z}/p} \to \Lambda(x) \otimes \mathbb{F}_p[y] \]
under the map induced by sending each \( x_i \) to \( x \) and \( y_i \) to \( y \). Recall also that this image will be a Hopf algebra. As \( y^p \) is the image of the invariant \( y_1 \cdots y_p \), while \( x \) and \( y \) are easily checked to not be in this image, we see that \( \text{im}(2)^* = \mathbb{F}_p[y^p] \).

Assume by induction that \( \text{im}(i(r)^*) = \mathbb{F}_p[y^p^r - 1] \). Then, reasoning as above,
\[ \text{im}(i(r + 1)^*) = \{ \mathbb{F}_p[y_1^p^r - 1, \ldots, y_p^p^r - 1]^{\mathbb{Z}/p} \to \mathbb{F}_p[y] \} = \mathbb{F}_p[y^p^r]. \]

The case when \( p = 2 \) is similar. Identifying \( H^*(BC(r)) \) with \( \mathbb{F}_2[x] \), one proves by induction that, for \( r \geq 1 \), \( \text{im}(i(r)^*) = \mathbb{F}_2[x^{2^r - 1}] \) so that \( e(W(r)) = 2^{r - 1} - 1 \).

\( \square \)

3.4. Reduction of Theorem 1.9 to invariant theory. We begin the proof of Theorem 1.9. Our goal is to show that, if \( Q \) is a subgroup of a \( p \)-group \( P \), then \( e(Q) \leq e(P) \). Thus we need to somehow compare the image of the restriction
\[ H^*(BP) \to H^*(BC(P)) \]
to the image of the restriction
\[ H^*(BQ) \to H^*(BC(Q)). \]

We make some first reductions.

First of all, by induction of the index of \( Q \) in \( P \), we can assume that \( Q \) has index \( p \), and thus will be normal in \( P \). Then \( \mathbb{Z}/p \simeq P/Q \) will act on \( H^*(BQ) \) and also on \( H^*(C(Q)) \), with \( C(Q)^{\mathbb{Z}/p} = C(P) \cap Q \).

Next, suppose that \( C(P) \) is not contained in \( Q \). Then there would exist a central element \( \sigma \in P \) of order \( p \), not in \( Q \). It follows easily that then \( \langle \sigma \rangle \times Q = P \), and we conclude that \( e(P) = e(Q) \).

Thus we will assume that \( C(P) \) is contained in \( Q \). Suppose \( P \) admits a direct product decomposition \( P = \langle \sigma \rangle \times P_1 \), with \( \sigma \) of order \( p \). Then \( \sigma \) would be contained in \( C(P) \) and thus \( Q = \langle \sigma \rangle \times Q_1 \) with \( Q_1 = P_1 \cap Q \). Then \( e(P) = e(P_1) \) and \( e(Q) = e(Q_1) \).

We are reduced to needing to prove that \( e(Q) \leq e(P) \) under the following assumptions:

- \( Q \) is normal of index \( p \), so \( \mathbb{Z}/p \simeq P/Q \) acts on both \( H^*(BQ) \) and \( C = C(Q) \).
- \( C(P) = C^{\mathbb{Z}/p} \).
- \( P \) has no nontrivial elementary abelian direct summands.
In this situation, the restriction map $H^*(BP) \to H^*(C(P))$ factors
\[
H^*(BP) \to H^*(BQ)^{\mathbb{Z}/p} \to H^*(BC)^{\mathbb{Z}/p} \hookrightarrow H^*(BC) \to H^*(BC^{\mathbb{Z}/p}),
\]
and the last assumption tell us that the image lands in the part of $H^*(BC^{\mathbb{Z}/p})$ generated by $\beta(H^1(BC^{\mathbb{Z}/p}))$.

Let $V$ denote $\beta(H^1(BC)) \subseteq H^2(BC)$, so $V$ is just the dual of the vector space $C$. Let $V_{\mathbb{Z}/p}$ denote the $\mathbb{Z}/p$-coinvariants $V/(x - \sigma x : x \in V)$, where $\sigma$ generates $\mathbb{Z}/p$, so that the part of $H^*(BC^{\mathbb{Z}/p})$ generated by $\beta(H^1(BC^{\mathbb{Z}/p}))$ identifies with $S^*(V_{\mathbb{Z}/p})$.

The image of $H^*(BP) \to H^*(BC) \to S^*(V)$ will be the Hopf algebra $\mathcal{H}(\mathcal{F})$ associated to a filtration $\mathcal{F}$ of $V$:
\[
V(0) \subseteq V(1) \subseteq \cdots \subseteq V(n) = V.
\]

Here we are using notation as in §3.2: given $W < V$, $\Phi^k(W) \subset S^k(V)$ denotes the span of the $p^k$th powers of the elements in $W$, and then
\[
\mathcal{H}(\mathcal{F}) = S^*(V(0)) + \Phi(V(1)) + \cdots + \Phi^n(V(n)).
\]

It is important to note that the filtration $\mathcal{F}$ will be preserved by the $\mathbb{Z}/p$ action on $V$. One way to see this is to note that $\mathcal{F}$ is a natural invariant of $Q$: as described in [K3, §6], this filtration of $V$ records the history of the 0–line in the spectral sequence associated to the central extension
\[
C \to Q \to Q/C.
\]

From our observations above, the image of $H^*(BP) \to H^*(C(P))$ will be contained in the image of
\[
\mathcal{H}(\mathcal{F})^{\mathbb{Z}/p} \hookrightarrow S^*(V)^{\mathbb{Z}/p} \hookrightarrow S^*(V) \to S^*(V_{\mathbb{Z}/p}).
\]

Recall that $e(\mathcal{F})$ is defined to be $\sum_{k=0}^n c_k(\mathcal{F})(2p^k - 1)$ where $c_k(\mathcal{F})$ is the rank of $V(k)/V(k - 1)$. As $e(Q) \leq e(\mathcal{F})$, we will be able to deduce that $e(Q) \leq e(P)$ if we can solve the following problem in invariant theory.

**Problem 3.11.** Given a filtration $\mathcal{F}$ of a $\mathbb{Z}/p$–module $V$, find a filtration $\mathcal{F}_{\mathbb{Z}/p}$ of $V_{\mathbb{Z}/p}$ such that
- The image of $\mathcal{H}(\mathcal{F})^{\mathbb{Z}/p} \to S^*(V_{\mathbb{Z}/p})$ is contained in $\mathcal{H}(\mathcal{F}_{\mathbb{Z}/p})$, and
- $e(\mathcal{F}) \leq e(\mathcal{F}_{\mathbb{Z}/p})$.

In the next section we find such a filtration $\mathcal{F}_{\mathbb{Z}/p}$: see Theorem 4.6.

## 4. New results in invariant theory

In this section $\mathcal{F}$ is a filtration of an $\mathbb{F}_p[\mathbb{Z}/p]$–module $V$,
\[
V(0) \subseteq V(1) \subseteq \cdots \subseteq V(n) = V,
\]
and we wish to understand the image of the composite
\[
\mathcal{H}(\mathcal{F})^{\mathbb{Z}/p} \hookrightarrow S^*(V)^{\mathbb{Z}/p} \hookrightarrow S^*(V) \to S^*(V_{\mathbb{Z}/p}),
\]
with our goal to solve Problem 3.11. Throughout we let $\sigma$ be a generator for $\mathbb{Z}/p$. 
4.1. \(\mathbb{Z}/p\)-modules. The modular representation theory of \(\mathbb{Z}/p\) is quite tame. There are \(p\) indecomposable \(\mathbb{F}_p[\mathbb{Z}/p]\)-modules, \(V_1, \ldots, V_p\), where \(V_i\) has dimension \(i\). An explicit model for \(V_i\) is the vector space with basis \(x_1, \ldots, x_i\) with

\[
\sigma x_j = \begin{cases} 
  x_j + x_{j-1} & \text{if } 1 < j \leq i \\
  x_1 & \text{if } j = 1.
\end{cases}
\]

A general \(\mathbb{F}_p[\mathbb{Z}/p]\)-module \(V\) decomposes as a direct sum

\[V \simeq m_1V_1 \oplus m_2V_2 \oplus \cdots \oplus m_pV_p.\]

We say that \(V\) is reduced if \(m_1 = 0\).

We let \(\text{rad}(V)\) and \(\text{soc}(V)\) be the radical and socle of a module \(V\). Thus \(\text{soc}(V) = V^\mathbb{Z}/p\) and \(V/\text{rad}(V) = V_{\mathbb{Z}/p}\). In the usual way, we define \(\text{soc}(V) \subset \text{soc}^2(V) \subset \ldots\) and \(\text{rad}(V) \supset \text{rad}^2(V) \supset \ldots\).

The submodule \(m_1V_1\) in a decomposition of \(V\) can be regarded as the image of a section of the quotient map \(\text{soc}(V) \to \text{soc}(V)/\text{soc}(V) \cap \text{rad}(V)\). Thus \(V\) is reduced precisely when \(\text{soc}(V) \subset \text{rad}(V)\), or equivalently, when the composite \(V^\mathbb{Z}/p \to V \to V_{\mathbb{Z}/p}\) is zero.

4.2. The case when the filtration is trivial. Given a \(\mathbb{Z}/p\)-module \(V\), a special case of our general problem is to understand the image of

\[S^*(V)^{\mathbb{Z}/p} \hookrightarrow S^*(V) \to S^*(V_{\mathbb{Z}/p}).\]

We remark that, in spite of the simple classification of modules \(V\), a complete calculation of \(S^*(V)^{\mathbb{Z}/p}\) is not known in all cases, and is the subject of much research. Even so, we prove the following theorem.

**Theorem 4.1.** If \(V = W \oplus U\), where \(W\) is trivial and \(U\) is reduced, the image of \(S^*(V)^{\mathbb{Z}/p} \to S^*(V_{\mathbb{Z}/p})\) is \(S^*(W \oplus \Phi(U_{\mathbb{Z}/p}))\).

Here is a more invariant way of stating this. Given \(V\), let \(W_{\mathbb{Z}/p}\) be the image of the composite \(V^\mathbb{Z}/p \hookrightarrow V \to V_{\mathbb{Z}/p}\). Then the image of

\[S^*(V)^{\mathbb{Z}/p} \hookrightarrow S^*(V) \to S^*(V_{\mathbb{Z}/p})\]

will be

\[S^*(W_{\mathbb{Z}/p} + \Phi(V_{\mathbb{Z}/p})).\]

The next example both illustrates the theorem and will be used in its proof.

**Example 4.2.** Suppose \(V = mV_2\), where the \(i\)th copy of \(V_2\) has basis \(\{x_i, y_i\}\) with \(\sigma y_i = y_i + x_i\) and \(\sigma x_i = x_i\). The kernel of the quotient \(V \to V_{\mathbb{Z}/p}\) is the span of the \(x_i\)'s, so we can view \(V_{\mathbb{Z}/p}\) as having basis given by the \(y_i\)'s. The theorem in this case is asserting that the image of the composite

\[\mathbb{F}_p[x_1, \ldots, x_m, y_1, \ldots, y_m]^{\mathbb{Z}/p} \hookrightarrow \mathbb{F}_p[x_1, \ldots, x_m, y_1, \ldots, y_m] \to \mathbb{F}_p[y_1, \ldots, y_m]\]

is \(\mathbb{F}_p[y_1^p, \ldots, y_m^p]\). The main theorem of [CH] is a description of generators of \(S^*(mV_2)^{\mathbb{Z}/p}\) as polynomials in the \(x_i\)'s and \(y_j\)'s; see also [W]. One sees that all of these are sent to 0 modulo the ideal \((x_1, \ldots, x_m)\) except for the ‘norm’ generators \(\prod_{j=0}^{p-1} \sigma^j y_i = y_i^p - x_i^{p-1} y_i\), which map to \(y_i^p\). So the assertion of the theorem is true in this case.
According to Theorem 4.1, when $V = W \oplus U$ with $W$ trivial, then $S^*(V)^\mathbb{Z}/p = S^*(W) \otimes S^*(U)^\mathbb{Z}/p$ and $S^*(W \oplus \Phi(U_{\mathbb{Z}/p})) = S^*(W) \otimes S^*(\Phi(U_{\mathbb{Z}/p}))$. Therefore, it suffices to prove that, when $V$ is reduced, there is an equality

$$I(V) = S^*(\Phi(V_{\mathbb{Z}/p})),$$

where $I(V) = \text{im}\{S^*(V)^\mathbb{Z}/p \hookrightarrow S^*(V) \rightarrow S^*(V_{\mathbb{Z}/p})\}$.

The previous example showed that this holds when $V = mV_2$. We use this to show that the equality holds for a general reduced $V$. Recall that $V_{\mathbb{Z}/p} = V/\text{rad}(V)$. If we let $\bar{V} = V/\text{rad}^2(V)$, and let $\bar{V}$ be the projective cover of $V$, then $\bar{V} = mV_2$ and $V = mV_1$. The surjections $\bar{V} \rightarrow V \rightarrow V$ will induce isomorphisms $\bar{V}_{\mathbb{Z}/p} = V_{\mathbb{Z}/p} = \bar{V}_{\mathbb{Z}/p}$, and then inclusions

$$I(\bar{V}) \subseteq I(V) \subseteq I(\bar{V}) = S^*(\Phi(V_{\mathbb{Z}/p})).$$

Finally, to see that all of these inclusions are, in fact, equalities, we note $I(\bar{V})$ is easily seen to contain $S^*(\Phi(V_{\mathbb{Z}/p}))$: our proof of Proposition 3.10 showed that $I(V_p) = S^*(\Phi(V_1))$, and so $I(\bar{V})$ certainly contains $S^*(\Phi(mV_1))$. □

4.3. The case when $V = W \oplus U$ with $W$ trivial and $U$ reduced. Now suppose that there exists a decomposition of filtered $\mathbb{Z}/p$–modules $V = W \oplus U$, with $W$ trivial and $U$ reduced. Define a filtration $\mathcal{F}_{\mathbb{Z}/p}$ of $V_{\mathbb{Z}/p}$ by letting

$$V_{\mathbb{Z}/p}(k) = (W(k) + U(k - 1) + \text{rad}(V))/\text{rad}(V).$$

Proposition 4.3. The image of $\mathcal{H}(\mathcal{F})^\mathbb{Z}/p \rightarrow S^*(V_{\mathbb{Z}/p})$ is contained in $\mathcal{H}(\mathcal{F}_{\mathbb{Z}/p})$.

Proof. Just as in the proof of Theorem 4.1, it suffices to prove this when $V$ is reduced, so that $\mathcal{F}_{\mathbb{Z}/p}$ is defined by letting $V_{\mathbb{Z}/p}(k) = (V(k - 1) + \text{rad}(V))/\text{rad}(V)$. Also, similar to the proof of Theorem 4.1, we let $\bar{V} = V/\text{rad}^2(V)$, with filtration $\bar{F}$ defined by $\bar{V}(k) = (V(k) + \text{rad}^2(V))/\text{rad}^2(V)$. Then

$$\text{im}\{\mathcal{H}(\mathcal{F})^\mathbb{Z}/p \rightarrow S^*(V_{\mathbb{Z}/p})\} \subseteq \text{im}\{\mathcal{H}(\bar{F})^\mathbb{Z}/p \rightarrow S^*(V_{\mathbb{Z}/p})\},$$

and the filtrations $\mathcal{F}_{\mathbb{Z}/p}$ and $\bar{F}_{\mathbb{Z}/p}$ of $V_{\mathbb{Z}/p}$ agree.

Thus it suffices to also assume that $V$ satisfies $\text{rad}^2(V) = 0$. In this case, let elements $y_1, \ldots, y_t \in V$, of filtration $k_1, \ldots, k_t$, project to a filtered basis of $V_{\mathbb{Z}/p}$, and let $x_j = \sigma y_j$. Then

$$\mathcal{H}(\mathcal{F}) \subseteq \mathbb{F}_p[x_1, \ldots, x_t, y^{k_{x_1}}_1, \ldots, y^{k_{x_t}}_t]$$

as algebras with $\mathbb{Z}/p$ action, and so the image of $\mathcal{H}(\mathcal{F})^\mathbb{Z}/p \rightarrow S^*(V_{\mathbb{Z}/p})$ is contained in $\mathcal{H}(\mathcal{F}_{\mathbb{Z}/p}) = \mathbb{F}_p[y^{k_{x_1}+1}_1, \ldots, y^{k_{x_t}+1}_t]$, as this is the image of

$$\mathbb{F}_p[x_1, \ldots, x_t, y^{k_{x_1}}_1, \ldots, y^{k_{x_t}}_t]^\mathbb{Z}/p \rightarrow \mathbb{F}_p[y_1, \ldots, y_t].$$

□

4.4. The general case. Unfortunately, at least when $p \geq 3$, a general filtered $\mathbb{Z}/p$–module $V$ need not admit a direct sum decomposition as filtered modules of the form $V = W \oplus U$, with $W$ trivial and $U$ reduced.

Example 4.4. With $p \geq 3$, let $V(0) = V_2$ embedding ‘diagonally’ in $V_1 \oplus V_3 = V(1) = V$. Then the image of $\text{soc}(V) \rightarrow V/\text{rad}(V)$ is $V_1$, generated by an element
of \(V(0)\), but not of \(\text{soc}(V(0))\), and we see that there is no isomorphism \(V \simeq V_1 \oplus V_3\) as filtered modules\(^1\).

This phenomenon goes away if we assume that \(\text{rad}(V) \subseteq V(0)\).

**Lemma 4.5.** If \(\text{rad}(V) \subseteq V(0)\), then there exists a decomposition of filtered \(\mathbb{Z}/p\)-modules \(V = W \oplus U\), with \(W\) trivial and \(U\) reduced.

We temporarily postpone the proof.

Now let \(F\) be an arbitrary filtration of a \(\mathbb{Z}/p\)-module \(V\). Define a filtration \(F_{\mathbb{Z}/p}\) of \(V_{\mathbb{Z}/p}\) by letting

\[
V_{\mathbb{Z}/p}(k) = (\text{soc}(V(k) + \text{rad}(V)) + V(k - 1) + \text{rad}(V))/\text{rad}(V).
\]

Note that, if \(V = W \oplus U\) with \(W\) trivial and \(U\) reduced, then the filtration \(F_{\mathbb{Z}/p}\) agrees with the filtration of the same name in the last subsection.

**Theorem 4.6.** The image of \(\mathcal{H}(F)_{\mathbb{Z}/p} \to S^*(V_{\mathbb{Z}/p})\) is contained in \(\mathcal{H}(F_{\mathbb{Z}/p})\).

**Proposition 4.7.** \(e(F) \leq e(F_{\mathbb{Z}/p})\).

**Proof of Theorem 4.6.** Let \(F'\) be the filtration of \(V\) defined by letting

\[
V'(k) = V(k) + \text{rad}(V).
\]

Then \(\mathcal{H}(F) \subseteq \mathcal{H}(F')\), so that

\[
\text{im}\{\mathcal{H}(F)_{\mathbb{Z}/p} \to S^*(V_{\mathbb{Z}/p})\} \subseteq \text{im}\{\mathcal{H}(F'_{{\mathbb{Z}/p}}) \to S^*(V_{\mathbb{Z}/p})\}.
\]

Moreover, \(F'_{\mathbb{Z}/p} = F_{\mathbb{Z}/p}\). Thus the proposition follows from Proposition 4.3. \(\square\)

**Proof of Proposition 4.7.** One can choose a filtered basis for \(V\) consisting of elements \(x_\alpha, y_\beta, z_\gamma\) such that the set of elements \(z_\gamma = z_\gamma + \text{rad}(V)\) span \((\text{soc}(V) + \text{rad}(V))/\text{rad}(V)\), and the set of elements \(x_\alpha\) span \(\text{rad}(V)\). Each \(y_\beta\) will generate a \(\mathbb{Z}/p\)-submodule of dimension at most \(p\), and with radical of filtration at most equal to \(|y_\beta|\), the filtration of \(y_\beta\), and such modules, together with the \(z_\gamma\), span \(V\). In \(V_{\mathbb{Z}/p}\), the element \(\bar{y}_\beta = y_\beta + \text{rad}(V)\) will have filtration \(|y_\beta| + 1\), while \(|\bar{z}_\beta| \geq |z_\beta|\), with the possibility of \(>\) due to the phenomenon illustrated in Example 4.4.

It follows that

\[
e(F) = \sum_\alpha 2^p|\alpha| + \sum_\beta 2^p|y_\beta| + \sum_\gamma 2^p|z_\gamma| - r(V)
\leq \sum_\beta 2^p|y_\beta| + \sum_\gamma 2^p|z_\gamma| - r(V)
\leq \sum_\beta 2^p|y_\beta| + \sum_\gamma 2^p|z_\gamma| - r(V_{\mathbb{Z}/p})
= e(F_{\mathbb{Z}/p}).
\]

\(\square\)

**Proof of Lemma 4.5.** Filter \(V_{\mathbb{Z}/p}\) by letting \(F_kV_{\mathbb{Z}/p} = (V(k) + \text{rad}(V))/\text{rad}(V)\).

Then let \(W_{\mathbb{Z}/p} = (\text{soc}(V) + \text{rad}(V))/\text{rad}(V) \subseteq V_{\mathbb{Z}/p}\) be filtered by letting \(F_kW_{\mathbb{Z}/p} = W_{\mathbb{Z}/p} \cap F_kV_{\mathbb{Z}/p}\). It is easy to choose a filtered complement \(U_{\mathbb{Z}/p}\) so that \(F_kV_{\mathbb{Z}/p} = F_kW_{\mathbb{Z}/p} \oplus F_kU_{\mathbb{Z}/p}\) as filtered \(\mathbb{Z}/p\)-vector spaces.

\(^1\)We thank Dave Benson for showing us this example.
The point is now that, as $\text{rad}(V) \subseteq V(0)$, one can choose a lifting

$$
\xymatrix{
W_{\mathbb{Z}/p} \ar[r] & V_{\mathbb{Z}/p} & V \\
U_{\mathbb{Z}/p} \ar[r] & V_{\mathbb{Z}/p} & V
}
$$
as filtered vector spaces so that the image is contained in $\text{soc}(V)$, and thus can be viewed as a lifting of filtered $\mathbb{Z}/p$–modules. For if $x + \text{rad}(V) = y + \text{rad}(V)$ with $x \in V(k)$ and $y \in \text{soc}(V)$, then $y \in V(k) \cap \text{soc}(V) = \text{soc}(V(k))$. The conclusion of the lemma follows if we let $W$ be the image of such a lifting, and $U$ equal to the filtered $\mathbb{Z}/p$–module generated by any lifting

$$
\xymatrix{
U_{\mathbb{Z}/p} \ar[r] & V_{\mathbb{Z}/p} & V
}
$$

\[\square\]

**Example 4.8.** We illustrate how Theorem 4.6 and Proposition 4.7 work when our filtered module $V$ is as in Example 4.4. Thus let $p \geq 3$, and let $\mathcal{F}$ be the filtration given by having $V(0) = V_2$ ‘diagonally’ embedded in $V = V(1) = V_1 \oplus V_3$. Then $\mathcal{F}_{\mathbb{Z}/p}$ is the filtration having $V_{\mathbb{Z}/p}(0) = V_1$ embedded as the first factor of $V_{\mathbb{Z}/p} = V_{\mathbb{Z}/p}(1) = V_1 \oplus V_1$.

Explicitly, $V = V(1) = \langle x_1, x_2, y_1, y_2 \rangle$, and $V(0) = \langle x_1, x_2 \rangle$, with action: $\sigma(y_2) = y_2 + y_1 + x_2, \sigma(y_1) = y_1 + x_1, \sigma(x_2) = x_2 + x_1, \sigma(x_1) = x_1$. The quotient map $V \to V_{\mathbb{Z}/p}$ identifies with the map $V \to \langle \bar{x}_2, \bar{y}_2 \rangle$ sending $x_1$ to 0, $x_2$ and $-y_1$ to $\bar{x}_2$, and $y_2$ to $\bar{y}_2$. Then Theorem 4.6 says that the image of $\mathbb{F}_p[x_1, x_2, y_1^p, y_2^p]_{\mathbb{Z}/p}$ in $\mathbb{F}_p[\bar{x}_2, \bar{y}_2^2]$ will be contained in $\mathbb{F}_p[\bar{x}_2, \bar{y}_2^2]$. Note that $e(\mathcal{F}) = 4p + 2 \leq 2p^2 = e(\mathcal{F}_{\mathbb{Z}/p})$, as predicted by Proposition 4.7.

**References**


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