

THE GENERATING HYPOTHESIS IN THE DERIVED CATEGORY OF R -MODULES

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ABSTRACT. In this paper, we prove a version of Freyd's generating hypothesis for triangulated categories: if \mathcal{D} is a cocomplete triangulated category and $S \in \mathcal{D}$ is an object whose endomorphism ring is graded commutative and concentrated in degree zero, then S generates (in the sense of Freyd) the thick subcategory determined by S if and only if the endomorphism ring of S is von Neumann regular. As a corollary, we obtain that the generating hypothesis is true in the derived category of a commutative ring R if and only if R is von Neumann regular. We also investigate alternative formulations of the generating hypothesis in the derived category. Finally, we give a characterization of the Noetherian stable homotopy categories in which the generating hypothesis is true.

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1. INTRODUCTION

In his 1966 paper *Stable Homotopy*, Freyd introduces the following conjecture:

Conjecture 1.1 (The Generating Hypothesis). *If $f : X \rightarrow Y$ is a morphism of finite spectra and $\pi_* f = 0$, then f is trivial.*

The conjecture remains open, but progress has been made using the methodology surrounding the Nilpotence Theorem of Devinatz, Hopkins, and Smith ([4], [8]). Let $L_1(-)$

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denote $E(1)$ -localization. In [3], Devinatz proves the $E(1)$ -approximate generating hypothesis when the target spectrum is the sphere: if $\pi_* f = 0$, then $L_1 f \simeq 0$. An axiomatic approach to stable homotopy theory has led to the study of other triangulated categories from a homotopy theoretic point of view ([10]), and a natural extension of this study is to try to formulate and prove the generating hypothesis in these structurally similar settings. Examples of general stable homotopy categories include localizations of the usual stable category, the derived category of a ring, the stable module category (arising in representation theory), the homotopy category of complexes of injective comodules over a Hopf algebra (e.g., the Steenrod algebra), among others.

It is not completely clear how the generating hypothesis should be stated in general stable categories; one can leave the statement as it is and get something that makes perfect sense, but there are several ways of characterizing the finite spectra from an axiomatic point of view that do not always coincide in general. Difficulties arise, for example, when the sphere is not a weak generator; i.e., $\pi_* X = 0$ does not necessarily imply that X is trivial. If such a finite X exists, then \mathcal{GH} would clearly be false (an example will be given later: the derived category of quasi-coherent sheaves over \mathbb{P}^1). One might insist that $[X, f] = 0$ for each weak generator X .

We now give two relevant definitions and formulate the version of \mathcal{GH} with which we will work. Let \mathcal{D} be a triangulated category, and let $S \in \mathcal{D}$ be a distinguished object. The *thick subcategory generated by S* , $\mathbf{thick}\langle S \rangle$, is the smallest class of objects in \mathcal{D} that contains S and is closed under suspension, retraction, and cofiber sequences. In the usual stable category, this is exactly the finite spectra; in general, it may not describe the class of small objects. An object Y is *small* if, whenever the coproduct $\coprod_{\alpha} X_{\alpha}$ exists, the natural map

$$\bigoplus_{\alpha} [Y, X_{\alpha}] \longrightarrow [Y, \coprod_{\alpha} X_{\alpha}]$$

is an isomorphism. Note that 'small' and 'finite' have the same meaning in the usual stable category.

Definition 1.2. Let \mathcal{D} be a triangulated category and let $S \in \mathcal{D}$ be a distinguished object. Write $\pi_*(-)$ for the functor $\mathrm{Hom}_{\mathcal{D}}^*(S, -)$. The *generating hypothesis* is the statement: If $f : X \rightarrow Y$ is a morphism of objects in $\mathbf{thick}\langle S \rangle$ and $\pi_* f = 0$, then $f \simeq 0$.

In a stable homotopy category, S will be the sphere object. In the derived category of a ring, for example, the sphere is the chain complex with R in degree zero and zero elsewhere. The following theorem is our main result.

Theorem A. *Let \mathcal{D} be a triangulated category with arbitrary coproducts, and let S be an object in \mathcal{D} such that $R = \pi_* S$ is commutative and concentrated in degree zero. \mathcal{GH} is true in \mathcal{D} if and only if R is von Neumann Regular.*

A ring R is *von Neumann regular* if, for every element $x \in R$, there exists an element $y \in R$ such that $xyx = x$. This class of rings, in the context of noncommutative ring theory, was

originally introduced by von Neumann to study operator algebras in functional analysis. Theorem A has the following corollary.

Corollary 1.3. *Let R be a commutative ring. \mathcal{GH} is true in the derived category of R -modules if and only if R is von Neumann regular.*

The main result actually applies to any triangulated category where $\pi_* S$ is graded commutative and concentrated in even degrees. If $\pi_* S$ is graded commutative and *not* concentrated in degree zero, we prove that if $\pi_* S$ is nonzero in only finitely many degrees or if $\pi_* S$ is connective and concentrated in even degrees, then the generating hypothesis is necessarily false. We also give an application of Theorem A to the derived category of quasicoherent sheaves over certain schemes in §3.

There is a more abstract way to view the content of Theorem A. An analysis of the proof shows that, for categories covered by the main result, the generating hypothesis is true if and only if the category is trivial in the following sense: $\mathbf{thick}\langle S \rangle$ must be exactly the collection of retracts of finite wedges of suspensions of S . In ordinary stable homotopy, this condition would imply that every finite spectrum is a wedge of suspensions of the sphere (the main theorem, of course, does not apply in this case). We conjecture that this characterization is valid for the derived category of any Grothendieck abelian category. Note that \mathcal{S} is not of this form - for any object X in the derived category of an abelian category, the cofiber Y of $2 : X \rightarrow X$ has the property that $2 : Y \rightarrow Y$ is trivial. However, $2 : M(2) \rightarrow M(2)$ is nontrivial in \mathcal{S} , where $M(2)$ is the mod 2 Moore spectrum (I learned this from Neil Strickland).

Unless otherwise indicated, for the remainder of the paper we consider a triangulated category \mathcal{D} and an object $S \in \mathcal{D}$ which has the property that its endomorphism ring $R = \pi_* S = [S, S]_*$ is commutative and concentrated in degree zero. We further assume that \mathcal{D} has arbitrary coproducts; this guarantees that idempotents split in \mathcal{D} ([2, 3.2]).

This paper is organized as follows. In §2, we develop two important necessary conditions for \mathcal{GH} to hold: the Nilpotence Criterion (2.5) and the Annihilator Criterion (2.7). In §3, we introduce (von Neumann) regular rings and show that the two criteria from §2 are equivalent to regularity. We then show that regularity is also sufficient. In §4, we explore variants of \mathcal{GH} by changing its domain of definition. In $\mathcal{D}(R)$, for example, we study \mathcal{GH} when stated for maps of objects in $\mathbf{thick}\langle S/I \rangle$, where I is a finitely generated ideal of R . In §5, we show that \mathcal{GH} holds in a Noetherian stable homotopy category if and only if $\mathbf{thick}\langle S \rangle$ is exactly the collection of retracts of finite wedges of suspensions of S .

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2. GENERALITIES AND CRITERIA

By way of motivation, let us consider the derived category of a commutative ring $\mathcal{D}(R)$, where S is the chain complex with R concentrated in degree zero. $\mathcal{D}(R)$ is a monogenic stable homotopy category ([10, 9.3.1]). If R is a field, then every object in $\mathcal{D}(R)$ is equivalent to a wedge of suspensions of the sphere; therefore, \mathcal{GH} is trivially true. The following proposition gives an apposite connection between \mathcal{GH} and direct products.

Proposition 2.1. *Suppose $R \cong R_1 \times R_2$. \mathcal{GH} is true in $\mathcal{D}(R)$ if and only if it is true in $\mathcal{D}(R_1)$ and $\mathcal{D}(R_2)$.*

Proof. This is essentially a consequence of the fact that every R -module is the direct sum of an R_1 -module and an R_2 -module and every R -module map decomposes similarly. ■

Remark 2.2. Hence, \mathcal{GH} is true in $\mathcal{D}(R)$ whenever R is a finite product of fields. In particular, \mathcal{GH} is true for $\mathbb{Z}/(n)$, provided n is square free. This condition is also necessary. In [11] we prove that if \mathcal{GH} is true in a category where S is connective and $\pi_0 S$ is projective-free (meaning that every projective $\pi_0 S$ module is free), then $\pi_* S$ is either a field or totally non-coherent. A graded ring is totally non-coherent if no proper, nonzero, finitely generated ideal is finitely presented. Hence, \mathcal{GH} is false for $\mathbb{Z}/(p^n)$ when $n > 1$ since $\mathbb{Z}/(p^n)$ is local and therefore projective-free but obviously neither a field nor totally non-coherent. We also give a specific counterexample in Example 2.4.

The next proposition is based upon Devinatz' approach to \mathcal{GH} in [3], where he proves the $E(1)$ -approximate version when the target is the sphere. The Spanier-Whitehead duality pairing $DX \wedge X \longrightarrow S$ induces a map

$$(1) \quad \pi_{-i}(DX) \otimes_R \pi_i(X) \longrightarrow R.$$

Let M be an R -module. After tensoring with M , we may examine the adjoint map

$$(2) \quad \pi_{-i}(DX) \otimes_R M \longrightarrow \text{Hom}_R(\pi_i(X), M).$$

If M is both flat and injective, then this is a natural transformation of cohomology theories. Since it is an isomorphism when $X = S$, it is an isomorphism for all $X \in \mathbf{thick}\langle S \rangle$. Now suppose there is an injection $R \longrightarrow M$ and M/R is flat. Then

$$\pi_{-i}(DX) \longrightarrow \pi_{-i}(DX) \otimes_R M$$

is also injective since the failure of this map to be injective is measured by

$$\text{Tor}_R^1(\pi_{-i}(DX), M/R),$$

which is trivial. Now consider a degree zero map $f : X \longrightarrow S$ where $X \in \mathbf{thick}\langle S \rangle$. The map f corresponds to a map $f' : S \longrightarrow DX$. If f is nontrivial, then so is f' ; f' is also nontrivial in $\pi_0(DX) \otimes_R M$. Since (2) is an isomorphism and the pairing (1) corresponds to composition, there is a map $g : S \longrightarrow X$ such that fg is nontrivial. Hence, $\pi_* f \neq 0$. We have proved:

Proposition 2.3. *If there is an R -module M that is flat and injective and if there is an injective map $R \longrightarrow M$ such that M/R is flat, then \mathcal{GH} for the target spectrum S is true in $\mathcal{D}(R)$.*

For example, if R is a von Neumann regular ring (see §3), then every R -module is flat ([13, 4.2.9]); taking M to be the injective hull of R , we obtain that \mathcal{GH} for the target S is true in $\mathcal{D}(R)$. Also, if R is self injective, then \mathcal{GH} for the target S is true in $\mathcal{D}(R)$. As we will see in the next example, however, there are self-injective rings R for which the general form of \mathcal{GH} is false in $\mathcal{D}(R)$.

Example 2.4. $\mathbb{Z}/(n)$ is self injective for all n . Hence, \mathcal{GH} for the target S is true in $\mathcal{D}(\mathbb{Z}/(p^2))$. We have observed, however, that \mathcal{GH} is false in $\mathcal{D}(\mathbb{Z}/(p^2))$ in general. Here is a counterexample: consider the map of chain complexes

$$\begin{array}{ccc} 0 & & 0 \\ \uparrow & & \uparrow \\ \mathbb{Z}/(p^2) & \xrightarrow{p} & \mathbb{Z}/(p^2) \\ \uparrow p & & \uparrow p \\ \mathbb{Z}/(p^2) & \xrightarrow{0} & \mathbb{Z}/(p^2) \\ \uparrow & & \uparrow \\ 0 & & 0 \end{array}$$

$$X \xrightarrow{f} X.$$

Though $\pi_* f = 0$, f is nontrivial; a null-chain-homotopy of f would correspond to an element $s \in \mathbb{Z}/(p^2)$ such that $p = ps$ and $ps = 0$, implying $p = 0$.

Our next proposition gives a general reason for the failure of \mathcal{GH} in the above example. In this proposition, we return to the general situation, where \mathcal{D} is a triangulated category with distinguished object S such that $\pi_* S$ is commutative and concentrated in degree zero.

Proposition 2.5 (Nilpotence Criterion). *If \mathcal{GH} is true in \mathcal{D} , then R contains no nonzero nilpotent elements.*

Proof. For now, we do not assume that R is commutative. Let $g \in R$ be nonzero nilpotent; replacing g with some power if necessary, we may assume $g^2 = 0$. Consider the following commutative diagram, where the rows are cofiber sequences. The map h is any map that

makes the diagram commute.

$$\begin{array}{ccccccc}
S & \xrightarrow{g} & S & \xrightarrow{\phi} & Y & \xrightarrow{\delta} & \Sigma S \\
\downarrow 0 & & \downarrow g & & \downarrow h & & \downarrow 0 \\
S & \xrightarrow{g} & S & \xrightarrow{\phi} & Y & \xrightarrow{\delta} & \Sigma S
\end{array}$$

Notice that h factors through both S and ΣS ; since $\pi_* S$ is concentrated in one degree, we must have $\pi_* h = 0$. Hence, any h that fills in the above diagram of cofiber sequences must be trivial by \mathcal{GH} . Now, since $g^2 = 0$, $g = f\phi$. One may take h to be ϕf , and consequently $\phi f \simeq 0$. Hence, $f = gk$. Let $x = k\phi$; then, $g = f\phi = gx$ and $xg = k\phi g = 0$. If g lies in the center of R , then we have a contradiction. In particular, if R is commutative, then it contains no nontrivial nilpotent elements. \blacksquare

Remark 2.6. For this remark, assume $\pi_* S$ is graded commutative but not necessarily concentrated in degree zero. If the map g in the proof has degree k , then one obtains that h factors through both $\Sigma^{-k} S$ and $\Sigma^{k+1} S$. Hence, the Nilpotence Criterion holds whenever $\pi_* S$ is concentrated in even degrees. Notice also that if \mathcal{GH} is true, then a nonzero element g of degree k cannot be nilpotent if $\pi_* S$ is zero outside a range of $2k+1$ consecutive degrees. In particular, if $\pi_i S$ is zero for all but finitely many i and is not concentrated in degree zero, then \mathcal{GH} is false.

In [5], Freyd proves that if the generating hypothesis is true in the stable category of spectra, then $\text{ann ann}(x) = (x)$ for all $x \in \pi_* S$. We now give a generalization of this result for \mathcal{D} . We have so far assumed that $\pi_* S$ is commutative and concentrated in degree zero; the following proposition is true without these assumptions. We write $(x)_L$ for the left ideal generated by x and $(x)_R$ for the right ideal generated by x . We also define the right annihilator of a left ideal I_L and the left annihilator of a right ideal I_R by $\text{Rann } I_L = \{x \in R \mid ix = 0 \text{ for all } i \in I_L\}$, and $\text{Lann } I_R = \{x \in R \mid xi = 0 \text{ for all } i \in I_R\}$. $\text{Rann } I_L$ is a right ideal and $\text{Lann } I_R$ is a left ideal.

Proposition 2.7 (Annihilator Criterion). *If \mathcal{GH} is true in \mathcal{D} , then for any $f \in R$, $\text{Lann Rann}(f)_L = (f)_L$.*

Proof. Let k be the degree of f . The containment $(f)_L \subseteq \text{Lann Rann}(f)_L$ is always true. Take any $g \in \text{Lann Rann}(f)_L$. Consider the cofiber sequence

$$\Sigma^k S \xrightarrow{f} S \xrightarrow{\phi} Y \xrightarrow{\delta} \Sigma^{k+1} S.$$

Since $\text{im } \pi_* \delta = \ker \pi_* f = \text{Rann}(f)_L$, we have that $\pi_*(g\delta) = 0$. By \mathcal{GH} , $g\delta \simeq 0$. Hence, $g = \psi f$ and so

$$(f)_L = \text{Lann Rann}(f)_L$$

as desired. \blacksquare

Remark 2.8. One could just as easily prove that $\text{Rann Lann}(f)_R = (f)_R$.

Example 2.9. Using the Annihilator Criterion, we give a family of examples of totally non-coherent local (and hence projective-free) rings for which \mathcal{GH} in $\mathcal{D}(R)$ is still false. Let k be a field and consider $R = k[x_1, x_2, \dots]/(x_i x_j, i \neq j, x_i^n, i \geq 1)$ or $k[x_1, x_2, \dots]/(x_i x_j, i \neq j)$. R is totally non-coherent (exercise). However, $(x_1) \not\subseteq (x_1 + x_2)$ though $\text{ann}(x_1 + x_2) \subseteq \text{ann}(x_1)$.

The above two necessary conditions, as we shall see, are also sufficient.

3. VON NEUMANN REGULAR RINGS

There is a general term for a ring that satisfies both the Annihilator Criterion and the Nilpotence Criterion. A ring (not necessarily commutative) R is (*von Neumann*) *regular* if, for every $x \in R$, there exists $y \in R$ such that $xyx = x$. For commutative rings, this is the same as the requirement that $(x^2) = (x)$ for all $x \in R$, which is in turn equivalent to the condition that every principal ideal be generated by an idempotent: if there exists an element $s \in R$ such that $sx^2 = x$, then $(x) = (sx)$ and sx is idempotent. [6] is a general reference for rings of this type. One can say something concrete about the class of commutative regular rings. The following two propositions must be well known; we include proofs for the reader's convenience.

Proposition 3.1. *If R is commutative von Neumann regular, then it is a subring of a direct product of fields.*

Proof. First, we show that every prime ideal of R is maximal. Let $P \subseteq R$ be a prime ideal. For any $x \in R$, there is some $y \in R$ such that $x^2y = x$ by regularity. Hence, $x(xy - 1) = 0$. Since R/P is an integral domain, either x is zero mod P or $xy \equiv 1 \pmod{P}$. Therefore R/P is a field and P is a maximal ideal. Next, we observe that the nilradical $\mathcal{N}(R)$ of R is trivial: $\mathcal{N}(R)$ cannot contain nonzero idempotent elements. If $e \in \mathcal{N}(R)$ is idempotent, then since e is contained in every prime ideal, $1 - e$ is contained in no prime ideal; therefore, $1 - e$ is a unit. But since $e(1 - e) = 0$, e must be zero. Now consider the map $\phi : R \rightarrow \prod_{\mathfrak{m}} R/\mathfrak{m}$ defined by $\pi_{\mathfrak{m}}\phi(x) = x \pmod{\mathfrak{m}}$. This is an injective ring homomorphism. ■

Any direct product of fields is regular, but the converse of the above Proposition is easily seen to be false; for example, $\mathbb{Z} \subseteq \mathbb{Q}$ is not a regular ring (and, of course, \mathcal{GH} is false in $\mathcal{D}(\mathbb{Z})$).

Proposition 3.2. *A commutative ring R is regular if and only if it satisfies both the Annihilator Criterion and the Nilpotence Criterion.*

Proof. Suppose R is commutative regular. In general, if e is idempotent, then $\text{ann}(e) = (1 - e)$. Since each principal ideal (x) is generated by an idempotent, we obtain

$$\text{ann ann}(x) = (x)$$

for all $x \in R$. Further, since $(x) = (x^2)$, it is clear that there are no nonzero nilpotent elements of R .

Conversely, suppose the two criteria hold. Every element of $\text{ann}(x) \cap (x)$ has the property that its square is zero; hence, by the Nilpotence criterion, this intersection is trivial. If $x^2y = 0$, then $xy \in \text{ann}(x)$, forcing $xy = 0$. Thus $\text{ann}(x^2) \subseteq \text{ann}(x)$, and by the Annihilator Criterion, $(x) \subseteq (x^2)$. Hence R is regular. \blacksquare

We have proved:

Theorem 3.3. *If \mathcal{GH} is true in \mathcal{D} , then R is von Neumann regular.*

Remark 3.4. In Remark 2.6, we noted that if \mathcal{GH} is true, then the Nilpotence Criterion holds when π_*S is concentrated in even degrees. Since the Annihilator Criterion also holds in this case, we can conclude, as above, that $(x^2) = (x)$ for all $x \in \pi_*S$. Therefore, \mathcal{GH} is false in \mathcal{D} if S is connective and π_*S is concentrated in even degrees but not concentrated in degree zero.

In order to prove the converse of Theorem 3.3, we first give circumstances under which $\mathbf{thick}\langle S \rangle$ is exactly the collection of retracts of finite wedges of suspensions of S . In this situation, \mathcal{GH} is trivially true. For the next proposition, it is not necessary to assume that π_*S is commutative or concentrated in degree zero.

Proposition 3.5. *If every finitely generated submodule of a free R -module is a summand, then every element of $\mathbf{thick}\langle S \rangle$ is a retract of a finite wedge of suspensions of S .*

Proof. Let \mathcal{C} be the collection of objects in \mathcal{D} which satisfy the conclusion of the Proposition. \mathcal{C} is contained in $\mathbf{thick}\langle S \rangle$ and contains S ; hence, it suffices to show that \mathcal{C} is thick. It is clearly closed under suspension and retraction. To show closure under cofiber sequences, we first consider maps between wedges of spheres. Let S_n be an n -fold wedge of suspensions of S and let S_m be an m -fold such wedge. Consider the cofiber sequence

$$\Sigma^{-1}C \xrightarrow{\delta} S_n \xrightarrow{f} S_m \xrightarrow{\phi} C.$$

By hypothesis, $\text{im } f_*$ is a summand of π_*S_m , so we get a decomposition $\pi_*S_m \cong \text{im } f_* \oplus \text{coker } f_*$. Since idempotents split in \mathcal{D} (we assume \mathcal{D} has arbitrary coproducts), we obtain an associated splitting $S_m \simeq K \vee L$ such that $\pi_*K \cong \text{im } f_*$ and $\pi_*L \cong \text{coker } f_*$. Since $\text{im } f_*$ is projective, $\ker f_*$ is a summand of π_*S_n . Hence we have a decomposition $\pi_*S_n \cong \ker f_* \oplus \text{coim } f_*$ and an associated splitting $S_n \simeq M \vee N$ with $\pi_*M \cong \text{coim } f_*$ and $\pi_*N \cong \ker f_*$.

Now, $f : M \vee N \rightarrow K \vee L$ has matrix form

$$\begin{pmatrix} \pi_K f \iota_M & \pi_K f \iota_N \\ \pi_L f \iota_M & \pi_L f \iota_N \end{pmatrix} = \begin{pmatrix} \pi_K f \iota_M & 0 \\ 0 & 0 \end{pmatrix}.$$

All entries but the upper left are zero since they are zero on homotopy groups and have domains that are retracts of wedges of spheres. We now need to show that $\pi_K f \iota_M$ is an

equivalence; if so, then $C \simeq \Sigma N \vee L$, making C a retract of a wedge of suspensions of the sphere. Certainly, $\pi_K f \iota_M$ induces an isomorphism of homotopy groups. This suffices, as our next claim demonstrates.

Claim. *Let $q : A \rightarrow B$ be a morphism in $\mathbf{thick}\langle S \rangle$. If q induces an isomorphism of homotopy groups, then it is an equivalence in \mathcal{D} .*

Proof. First, suppose $M \in \mathbf{thick}\langle S \rangle$ has the property $\pi_* M = [S, M]_* = 0$. Then the functor $[-, M]_*$ must vanish on all of $\mathbf{thick}\langle S \rangle$. In particular, $[M, M]_* = 0$, forcing the identity map of M to be trivial. Hence M is trivial. Since the cofiber of q satisfies this property, it must be trivial, forcing q to be an equivalence. \blacksquare

Now we consider the general case. Consider the following commutative diagram, where the rows are cofiber sequences, W_k is a wedge of suspensions of S , and the vertical compositions are the identity.

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & Y & \longrightarrow & C \\
 i_1 \downarrow & & \downarrow i_2 & & \downarrow \psi \\
 W_1 & \xrightarrow{i_2 f r_1} & W_2 & \longrightarrow & D \\
 r_1 \downarrow & & \downarrow r_2 & & \downarrow \phi \\
 X & \xrightarrow{f} & Y & \longrightarrow & C.
 \end{array}$$

Let $\theta = \phi\psi$. By the five-lemma, the induced map $\theta^* : [C, C]_* \rightarrow [C, C]_*$ is an isomorphism; hence, C is a retract of D , which is a retract of a wedge of suspensions of S by the above special case. This completes the proof. \blacksquare

Next we show that regularity guarantees that the hypothesis of Proposition 3.5 is satisfied. The following proposition follows from [6, 2.7]. We include a simple proof for the reader's convenience.

Proposition 3.6. *If T is a von Neumann regular ring, then every finitely generated submodule of a projective T -module is a summand.*

Proof. It suffices to consider a finitely generated submodule K of T^n . We have a short exact sequence

$$0 \longrightarrow K \longrightarrow T^n \longrightarrow Q \longrightarrow 0.$$

This is a finite presentation of Q . Since T is von Neumann regular, all modules are flat ([13, 4.2.9]). The module Q is therefore projective by [13, 3.2.7], so the sequence splits and K is a summand of T^n . \blacksquare

Combining this proposition with Theorem 3.3, we have proved:

Theorem 3.7. *Let \mathcal{D} be a triangulated category with arbitrary coproducts, and let $S \in \mathcal{D}$ have the property that $R = \pi_* S$ is commutative and concentrated in degree zero. \mathcal{GH} is true in \mathcal{D} if and only if R is von Neumann regular.*

Corollary 3.8. *For commutative rings, \mathcal{GH} is true in $\mathcal{D}(R)$ if and only if R is von Neumann regular.*

Remark 3.9. Observe that since Proposition 3.6 applies to noncommutative rings, we may drop the commutativity assumption for one direction of this theorem: \mathcal{GH} is true in the derived category of (right) modules over a von Neumann regular ring.

Remark 3.10. Notice that \mathcal{GH} is true in \mathcal{D} if and only if every object of $\mathbf{thick}\langle S \rangle$ is a retract of a wedge of suspensions of S . This is a very trivializing set of circumstances. In $\mathcal{D}(R)$, for example, this says that every finite object must be a wedge of suspensions of finitely generated projective modules. For any connective monogenic stable homotopy category where $\pi_0(S)$ is projective-free, this trivializing condition is equivalent to the requirement that all finite objects be finite wedges of suspensions of the sphere.

Remark 3.11. Continuing the discussion of Remark 3.4, notice that Theorem 3.7 is true when $\pi_* S$ is concentrated in even degrees; all that must be changed is the definition of regularity, in the obvious way: a graded ring R_* is regular if for every element $x \in R_k$ there exists an element $y \in R_{-k}$ such that $xyx = x$. One would also replace ‘field’ with ‘graded field’ in Proposition 3.1.

Example 3.12. As a final example, we consider the derived category $\mathcal{D}(\mathcal{Q}\text{coh}(X))$ of quasi-coherent sheaves of \mathcal{O}_X -modules over a scheme X , where $\mathcal{O} = \mathcal{O}_X$ is the structure sheaf of X . Since the categories $R\text{-Mod}$ and $\mathcal{Q}\text{coh}(X)$ are equivalent as abelian categories for the affine scheme $X = \text{Spec } R$, we have already addressed \mathcal{GH} in the affine case. According to [9, 2.6], whenever X is a finite-dimensional Noetherian scheme with enough locally frees, $\mathcal{D}(\mathcal{Q}\text{coh}(X))$ is a unital algebraic stable homotopy category with weak generators the locally free sheaves of finite rank. The sphere S is the chain complex with \mathcal{O} concentrated in degree zero. By [7, III.6.3], $\pi_* S = \text{Ext}_{\mathcal{Q}\text{coh}(X)}(\mathcal{O}, \mathcal{O}) = H^*(X; \mathcal{O})$. If $X = \mathbb{P}_R^n$, n dimensional projective space over a commutative Noetherian ring R , then $\pi_* S = R$ concentrated in degree zero ([7, III.2.7, III.5.1]). Hence, by Corollary 5.5, we conclude that \mathcal{GH} is true if and only if R is a finite product of fields. More generally, \mathcal{GH} is true in $\mathcal{D}(\mathcal{Q}\text{coh}(X))$ for any smooth Fano variety X over a field of characteristic zero; such X have no higher cohomology by the Kodaira Vanishing Theorem ([7, III.7.15]).

We are now able to give an example where \mathcal{GH} is true for $\mathbf{thick}\langle S \rangle$ but not true if stated for morphisms between any two small objects. Over \mathbb{P}_R^1 , for example, every locally free sheaf of finite rank is a direct sum of invertible sheaves ([7, V.2.6]), and the invertible sheaves over \mathbb{P}_R^1 are exactly the twists of the structure sheaf, $\mathcal{O}(k)$ for $k \in \mathbb{Z}$ ([7, II.6.17]). Hence, the collection of small objects in $\mathcal{D}(\mathcal{Q}\text{coh}(\mathbb{P}_R^1))$ is exactly the thick subcategory generated by the $\mathcal{O}(k)$. Now, $[\mathcal{O}, \mathcal{O}(-1)]_* = \text{Ext}_{\mathcal{Q}\text{coh}(\mathbb{P}_R^1)}(\mathcal{O}, \mathcal{O}(-1)) = H^*(\mathbb{P}_R^1, \mathcal{O}(-1)) = 0$ (again, use [7, III.5.1, III.6.3]). But the identity map on $\mathcal{O}(-1)$ is nontrivial.

4. VARIATIONS

In this section, we change the domain of definition of \mathcal{GH} . To distinguish among the variants, we will write $\mathcal{GH}_{\mathcal{C}}$ for the statement: If $f : X \rightarrow Y$ is a morphism of objects in \mathcal{C} and $\pi_* f = 0$, then $f \simeq 0$. Note that $\mathcal{GH} = \mathcal{GH}_{\mathbf{thick}\langle S \rangle}$. We continue to assume that $R = \pi_* S$ is commutative and concentrated in degree zero.

Consider, for example, $\mathcal{C} = \mathcal{D} = \mathcal{D}(R)$. Suppose $\mathcal{GH}_{\mathcal{D}(R)}$ is true. Then \mathcal{GH} is also true, so R is regular. Let A and B be R -modules with projective resolutions P_{\bullet} and Q_{\bullet} , respectively. Now, $\pi_* P_{\bullet} = A$ and $\pi_* Q_{\bullet} = B$ (both concentrated in degree zero), and $[P_{\bullet}, Q_{\bullet}]_* = \mathrm{Ext}_R^*(A, B)$. If $k \geq 0$, then any map $f \in [P_{\bullet}, Q_{\bullet}]_k$ must induce the zero map of homotopy groups. It must therefore be trivial by $\mathcal{GH}_{\mathcal{D}(R)}$. Hence, $\mathrm{Ext}_R^k(A, B) = 0$ for all $k \geq 0$ and all R -modules A and B . This forces every R -module to be projective, making R Noetherian. By Proposition 5.5, R must be a finite product of fields. Conversely, if R is a finite product of fields, then $\mathcal{GH}_{\mathcal{D}(R)}$ is true (this follows from the fact that Proposition 2.1 is also true for $\mathcal{GH}_{\mathcal{D}(R)}$ (the same proof works) and $\mathcal{GH}_{\mathcal{D}(R)}$ is true when R is a field).

For the next variant, we require that \mathcal{D} be a closed symmetric monoidal category (see [10, §A.2] for all relevant definitions); let S be the unit for the smash product. We will make use of the product structure and the existence of function objects (and hence a notion of duality). Corresponding to every finitely generated ideal $I = (x_1, \dots, x_n)$ of R , there is a spectrum $S/I = S/x_1 \wedge \dots \wedge S/x_n$, where S/x_i is the cofiber of x_i as a self map of S . The definition of S/I depends upon the choice of generators for I , though $\mathbf{thick}\langle S/I \rangle$ is independent of this choice ([10, 6.0.9]). Motivated by the fact that thick subcategories of this form are involved in the classification of all thick subcategories of $\mathbf{thick}\langle S \rangle$ in $\mathcal{D}(R)$, we let $\mathcal{C} = \mathbf{thick}\langle S/I \rangle$ and consider $\mathcal{GH}_{\mathcal{C}}$. First, we prove a series of Propositions establishing the relevant homotopy theoretic properties of S/I .

Let $I_k = (x_1, \dots, x_k)$. S/I_k possesses a 'unit' map $\eta_k : S \rightarrow S/I_k$ defined to be the smash product of the obvious maps $S \rightarrow S/x_i$. Write η for η_n .

Proposition 4.1. $\pi_0 \eta_k$ is the quotient map $\pi_0 S = R \rightarrow R/I_k = \pi_0 S/I_k$, and $\pi_l S/I_k = 0$ for $l < 0$.

Proof. An examination of the cofiber sequence

$$S \xrightarrow{x_1} S \xrightarrow{\eta_1} S/x_1 \longrightarrow \Sigma S,$$

coupled with the fact that $\pi_* S$ is concentrated in degree zero, yields the desired result for $k = 1$. Now proceed by induction, using the diagram

$$\begin{array}{ccccc} S/I_k & \xrightarrow{x_{k+1}} & S/I_k & \longrightarrow & S/I_{k+1} \longrightarrow \Sigma S/I_k, \\ & & \uparrow \eta_k & \nearrow \eta_{k+1} & \\ & & S & & \end{array}$$

where the triangle commutes and the top row is a cofiber sequence. ■

Corollary 4.2. $z \in I$ if and only if $\eta z = 0$.

Dually, there are maps $\delta_k : S/I_k \rightarrow \Sigma^k S$ defined to be the smash product of the obvious maps $S/x_i \rightarrow \Sigma S$. Write δ for δ_n . The proof of the following Proposition is dual to the proof of 4.1.

Proposition 4.3. $\pi_k \delta_k$ is the inclusion $\pi_k S/I_k = \text{ann } I_k \rightarrow R = \pi_0 S$, and $\pi_l S/I_k = 0$ for $l > k$.

Corollary 4.4. $z \in \text{ann } I$ if and only if $z = \delta z'$ for some $z' \in \pi_k S/I$.

Proposition 4.5. If $\mathcal{GH}_{\text{thick}\langle S/I \rangle}$ is true, then $\text{ann ann } I = I$.

Proof. Consider $z \in \text{ann ann } I$. Since $\text{im } \pi_n \delta = \text{ann } I$, $\pi_* \eta z \delta = 0$. Hence, $\eta z \delta \simeq 0$ by $\mathcal{GH}_{\text{thick}\langle S/I \rangle}$. η fits into a cofiber sequence

$$F \xrightarrow{\theta} S \xrightarrow{\eta} S/I \xrightarrow{\rho} \Sigma F.$$

Therefore $z \delta = \theta q$ for some $q : S/I \rightarrow F$. At this point, we need to make use of the duality functor $D(-) = F(-, S)$, where the objects $F(X, Y)$ are the function objects in the closed symmetric monoidal structure on \mathcal{D} . S is equivalent to its dual, and self maps of S are self dual up to unit. Further, using the definitions, one can check that η and δ are dual. Thus $z \delta = \theta q$ implies that $\eta z = \tilde{q} D \theta$, $\tilde{q} : DF \rightarrow S/I$. Consider the map $\rho \tilde{q} : DF \rightarrow \Sigma F \in \pi_0 F \wedge \Sigma F$. We will prove that $\rho \tilde{q} = 0$ by showing that $\pi_0 F \wedge \Sigma F = 0$. For each k , we have cofiber sequences

$$F_k \longrightarrow S \longrightarrow S/I_k \longrightarrow \Sigma F_k$$

which are related via the diagram of cofiber sequences

$$\begin{array}{ccccc} S & \xrightarrow{\eta_k} & S/I_k & \longrightarrow & \Sigma F_k \\ \parallel & & \downarrow & & \downarrow \\ S & \xrightarrow{\eta_{k+1}} & S/I_{k+1} & \longrightarrow & \Sigma F_{k+1} \\ \downarrow & & \downarrow & & \downarrow \\ \cdot & \longrightarrow & \Sigma S/I_k & \equiv & \Sigma S/I_k. \end{array}$$

After smashing this diagram with F , one can prove, by induction on k , that

$$\pi_l \Sigma F_k \wedge F = \pi_l \Sigma S/I_k = 0 \text{ for } l \leq 0.$$

In particular, $\pi_0 F \wedge \Sigma F = 0$. Now, since $\rho \tilde{q} = 0$, we have $\tilde{q} = \eta h$ and $\eta z = \eta h D \theta$. Since $h D \theta$ and $\theta D h$ differ by a unit, we have $h D \theta = \theta \tilde{h}$. Hence $\eta z = \eta \theta \tilde{h} = 0$, and $z \in I$ by Corollary 4.2. This completes the proof. \blacksquare

Corollary 4.6. If $\mathcal{GH}_{\text{thick}\langle S/I \rangle}$ is true, then $\text{ann ann } J = J$ whenever $S/J \in \text{thick}\langle S/I \rangle$.

Proof. If $\mathcal{GH}_{\mathbf{thick}\langle S/I \rangle}$ is true and $S/J \in \mathbf{thick}\langle S/I \rangle$, then $\mathcal{GH}_{\mathbf{thick}\langle S/J \rangle}$ is true. \blacksquare

Our next goal is to prove that this Corollary applies to I^2 . The following proposition follows from a straightforward manipulation of cofiber sequences.

Proposition 4.7. $S/I^2 \in \mathbf{thick}\langle S/I \rangle$.

We now prove the main result for this subsection.

Theorem 4.8. *Let I be a finitely generated ideal of R such that $I \cap \text{ann } I = 0$. Then, $\mathcal{GH}_{\mathbf{thick}\langle S/I \rangle}$ is true if and only if $R = I \times R/I$ as a direct product of rings and R/I is regular.*

Proof. We prove the ‘only if’ direction first. Assume $\mathcal{GH}_{\mathbf{thick}\langle S/I \rangle}$ is true. Then, by Corollary 4.6 and the previous Proposition, $\text{ann ann } I = I$ and $\text{ann ann } I^2 = I^2$. Now take $x \in \text{ann } I^2$ and fix an element $a \in I$. For all $b \in I$, $xab = 0$ since $ab \in I^2$. Thus $xa \in I \cap \text{ann } I = 0$; i.e., $x \in \text{ann } I$. This shows that $\text{ann } I^2 \subseteq \text{ann } I$, which in turn implies that $I \subseteq I^2$. Therefore $I = I^2$. By Nakayama’s lemma (as stated, for example, in [12, §2.8, Corollary 1]), there exists an element r in R such that $rI = 0$ and $1 - r \in I$. For $i \in I$, $i = i(1 - r)$; hence, I is a principal idempotent ideal and therefore a ring direct factor of R . The ring direct product decomposition $R \cong I \times T$ corresponds to a splitting $S \simeq A \vee B$ with $\pi_* A = I$ and $\pi_* B = T$. Further, $S/I \simeq B \vee \Sigma B$. Therefore $\mathbf{thick}\langle S/I \rangle = \mathbf{thick}\langle B \rangle$. Since there are no maps from A to B , $\pi_*(-)$ and $[B, -]_*$ are identical on $\mathbf{thick}\langle B \rangle$. T is therefore von Neumann regular by Theorem 3.7.

Conversely, if R/I is regular and a ring direct factor of R , then we obtain, as above, a decomposition $S \simeq A \vee B$ with $\mathbf{thick}\langle S/I \rangle = \mathbf{thick}\langle B \rangle$ and $[B, B]_* = \pi_* B = R/I$. By Theorem 3.7, $\mathcal{GH}_{\mathbf{thick}\langle S/I \rangle}$ is true. \blacksquare

The next proposition helps us identify situations where the condition $I \cap \text{ann } I = 0$ holds.

Proposition 4.9. *If $\mathcal{GH}_{\mathbf{thick}\langle S/I \rangle}$ is true and $x \wedge S/I \simeq 0$ for all $x \in I$, then $I \cap \text{ann } I = 0$.*

Proof. Let $z \in I \cap \text{ann } I$. Consider the diagram of cofiber sequences

$$\begin{array}{ccccccc} F & \xrightarrow{\theta} & S & \xrightarrow{\eta} & S/I & \xrightarrow{\rho} & \Sigma F \\ \downarrow 0 & & \downarrow z & & \downarrow h & & \downarrow 0 \\ F & \xrightarrow{\theta} & S & \xrightarrow{\eta} & S/I & \xrightarrow{\rho} & \Sigma F, \end{array}$$

where h is any map that makes the diagram commute. Note that h must factor through both S and ΣF ($\eta z \simeq 0$ since $z \in I$). Since $\pi_0 \Sigma F = 0$, $\pi_* h = 0$. By $\mathcal{GH}_{\mathbf{thick}\langle S/I \rangle}$, $h \simeq 0$. Since $z \in \text{ann } I$, $z = \delta z'$. By duality, $z = k\eta$ for some $k \in [S/I, S]_0$. Since we may take $h = \eta k$ in the above diagram, we have $\eta k \simeq 0$. Hence z is of the form $\theta l \eta$ for some

$l \in [S/I, F]_0$. We now prove $\theta_*[S/I, F]_0 = 0$, forcing z to be trivial. By naturality of duality, we have the following commutative diagram, where the vertical maps are duality isomorphisms and the bottom sequence is exact:

$$\begin{array}{ccccc} [S/I, F]_0 & \xrightarrow{\theta_*} & [S/I, S]_0 & & \\ \parallel & & \parallel & & \\ [S, DS/I \wedge F]_0 & \xrightarrow{(DS/I \wedge \theta)_*} & [S, DS/I \wedge S]_0 & \xrightarrow{(DS/I \wedge \eta)_*} & [S, DS/I \wedge S/I]_0. \end{array}$$

Since $x \wedge S/I \simeq 0$ for all $x \in I$ (and therefore the same is true of the maps $x \wedge DS/I$), the map $DS/I \wedge \eta$ splits. Hence $\theta_* = 0$, as desired. \blacksquare

This Proposition applies to the derived category of a ring; it is straightforward to check that $x \wedge S/x \simeq 0$ for all $x \in R$. We therefore have

Corollary 4.10. *Let $\mathcal{D} = \mathcal{D}(R)$, and let I be a finitely generated ideal of $R = \pi_*S$. $\mathcal{GH}_{\mathbf{thick}\langle S/I \rangle}$ is true if and only if I is a ring summand of R with R/I von Neumann regular.*

5. NOETHERIAN STABLE HOMOTOPY

In this section, we consider the case where π_*S is a Noetherian ring; we do not require that π_*S be commutative or concentrated in degree zero. To begin, we use the following generalization of a result of Freyd's ([11]). All π_*S -modules are right modules.

Proposition 5.1. *Let \mathcal{D} be a triangulated category with distinguished object S . Suppose \mathcal{GH} is true in \mathcal{D} . For every $X \in \mathbf{thick}\langle S \rangle$, if π_*X is finitely generated as a graded π_*S -module, then X is a retract of a finite wedge of suspensions of S .*

We now observe that when π_*S is Noetherian, \mathcal{GH} can only be true under trivial circumstances.

Proposition 5.2. *Let \mathcal{D} be a triangulated category and let S be an object in \mathcal{D} such that π_*S is Noetherian. Then, \mathcal{GH} is true in \mathcal{D} if and only if $\mathbf{thick}\langle S \rangle$ is exactly the collection of retracts of finite wedges of suspensions of S .*

Proof. The ‘if’ direction is trivial. For the ‘only if’ direction, we show that the collection \mathcal{C} of retracts of finite wedges of suspensions of S is thick. Since it contains S , this gives the desired conclusion. \mathcal{C} is trivially closed under suspension and retraction. If X and Y are in \mathcal{C} , then π_*X and π_*Y are finitely generated right modules over a Noetherian ring; hence, the homotopy of any cofiber C of a map from X to Y is also a finitely generated right π_*S -module. By Proposition 5.1, this implies that C is a retract of a finite wedge of suspensions of S . \blacksquare

Further, we record the following related observation concerning the derived category of a ring.

Proposition 5.3. *Let R be a ring, and let $\mathcal{D}(R)$ be the derived category of right R -modules. If $\mathbf{thick}\langle S \rangle$ is the collection of retracts of finite wedges of suspensions of S , then R is von Neumann regular.*

Proof. Let x be an element of R , and let S/x be the cofiber of $S \xrightarrow{x} S$. By hypothesis, $\pi_* S/x$ is projective; hence, $\pi_0 S/x = R/(x)_R$ is projective. Consequently, there is a projection $R \longrightarrow (x)_R$ mapping 1 to xk for some $k \in R$, and $xkx = x$. R is therefore von Neumann regular. ■

We conjecture that the collection of graded Noetherian rings for which \mathcal{GH} holds is exactly the collection of semisimple rings. A ring R is *semisimple* if, as a right module over itself, it decomposes as a finite direct sum of simple modules. It is a fact ([1, 9.4]) that every finitely generated submodule of a free module over a semisimple ring is a summand. Hence, by Proposition 3.5, \mathcal{GH} is true if $\pi_* S$ is semisimple, establishing part of our conjecture. We will prove the remaining half of the conjecture for the derived category of a ring using the following well-known result ([6, 2.16]).

Proposition 5.4. *If a ring R is right Noetherian regular, then it is semisimple.*

Proof. A ring is semisimple if and only if every ideal is a direct summand ([1, 9.6]). Since R is Noetherian, every ideal I of R is finitely generated. Since R is regular, every finitely generated ideal of R is a summand ([13, 4.2.8]). Hence, R is semisimple. ■

Corollary 5.5. *Let R be a Noetherian ring. Then, \mathcal{GH} is true in $\mathcal{D}(R)$ if and only if R is semisimple. If R is also commutative, then \mathcal{GH} is true in $\mathcal{D}(R)$ if and only if R is a finite product of fields.*

Proof. We observed that the 'if' direction is true earlier in this section. For the 'only if' direction, simply combine Propositions 5.2, 5.3, and 5.4. The final observation follows from the Artin-Wedderburn theorem, which classifies semisimple rings. ■

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