

# RATIONALIZED EVALUATION SUBGROUPS OF A MAP AND THE RATIONALIZED $G$ -SEQUENCE

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ABSTRACT. Let  $f: X \rightarrow Y$  be a based map of simply connected spaces. The corresponding evaluation map  $\omega: \text{map}(X, Y; f) \rightarrow Y$  induces a homomorphism of homotopy groups whose image in  $\pi_n(Y)$  is called the  $n$ th evaluation subgroup of  $f$ . The  $n$ th Gottlieb group of  $X$  occur as the special case in which  $X = Y$  and  $f = 1_X$ . We identify the homomorphism induced on rational homotopy groups by this evaluation map, in terms of a map of complexes of derivations constructed using Sullivan minimal models. Our identification allows for the characterization of the rationalization of the  $n$ th evaluation subgroup of  $f$ . It also allows for the identification of several long exact sequences of rational homotopy groups, including the long exact sequence induced on rational homotopy groups by the evaluation fibration. As a consequence, we obtain an identification of the rationalization of the so-called  $G$ -sequence of the map  $f$ . This is a sequence—in general not exact—of groups and homomorphisms that includes the Gottlieb groups of  $X$  and the evaluation subgroups of  $f$ . We use these results to study the  $G$ -sequence in the context of rational homotopy theory. We give new examples of non-exact  $G$ -sequences and uncover a relationship between the homology of the rational  $G$ -sequence and negative derivations of rational cohomology. We also relate the splitting of the rational  $G$ -sequence of a fibre inclusion to a well-known conjecture in rational homotopy theory.

## 1. INTRODUCTION

Suppose given a based map  $f: X \rightarrow Y$  of simply connected CW complexes. Denote by  $\text{map}(X, Y; f)$  the path component of the space of (unbased) maps  $X \rightarrow Y$  consisting of those maps that are homotopic to  $f$ . Then evaluation at the basepoint of  $X$  gives a map  $\omega: \text{map}(X, Y; f) \rightarrow Y$ . We refer to this map as the *evaluation map*. We define the  *$n$ th evaluation subgroup of  $f$*  to be the subgroup  $G_n(Y, X; f) = \omega_{\#}(\pi_n(\text{map}(X, Y; f)))$  of  $\pi_n(Y)$ . The famous Gottlieb groups  $G_*(X)$  occur as the special case when  $X = Y$  and  $f = 1_X$  [Got69]. The Gottlieb groups of a space have been much studied by homotopy theorists (see [Opr95] for a survey of results and references). While many general results are known, explicit computation of  $G_*(X)$  appears difficult and such results are limited to a small number of sporadic examples. One reason that accounts in part for this difficulty is the fact that a map of spaces  $f: X \rightarrow Y$  does not necessarily induce a corresponding homomorphism of Gottlieb groups, since in general  $f_{\#}(G_n(X)) \not\subseteq G_n(Y)$ . In particular, attempts to study  $G_*(X)$  via a cell decomposition of  $X$  are frustrated, since it is not clear what effect a cell attachment may have on the Gottlieb group. One tool for studying

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*Date:* June 16, 2003.

*2000 Mathematics Subject Classification.* 55P62, 55D23.

*Key words and phrases.* Evaluation map, evaluation subgroup, Gottlieb group,  $G$ -sequence,  $\omega$ -homology, function spaces, rational homotopy, minimal models, derivations.

$G_*(X)$  that attempts to circumvent this problem is the so-called *G-sequence of a map*, as developed and studied by Lee and Woo in a series of papers [LW88a, LW88b, LW90, LW93].

The *G-sequence* of a map  $f: X \rightarrow Y$  is a sequence

$$\cdots \rightarrow G_n(X) \rightarrow G_n(Y, X; f) \rightarrow G_n^{rel}(Y, X; f) \rightarrow G_{n-1}(X) \rightarrow \cdots$$

of groups and homomorphisms that derives from the long exact homotopy sequence of the map  $f$ . In this sequence,  $G_n(X)$  is the Gottlieb group of  $X$ ,  $G_n(Y, X; f)$  is the evaluation subgroup of  $f$  introduced above, and the third term  $G_n^{rel}(Y, X; f)$  is a suitably defined “relative” term. The *G-sequence* arises as follows. One has the following commutative diagram of spaces:

$$\begin{array}{ccc} \text{map}(X, X; 1) & \xrightarrow{f_*} & \text{map}(X, Y; f) \\ \omega \downarrow & & \downarrow \omega \\ X & \xrightarrow{f} & Y \end{array}$$

Now pass to the corresponding induced homomorphisms of homotopy groups. In a standard way, the induced homomorphisms of homotopy groups  $(f_*)_{\#}$  and  $f_{\#}$  can be fitted into the long exact homotopy sequences of the maps  $f_*$  and  $f$  respectively. Then the evaluation maps induce maps of each term in the long exact sequences, resulting in a homotopy ladder. The *G-sequence* of the map  $f$  is then the image of the top long exact homotopy sequence in that of the bottom. A portion of the *G-sequence* is shown here:

$$\begin{array}{ccccc} \pi_{n+1}(f_*) & \longrightarrow & \pi_n(\text{map}(X, X; 1)) & \xrightarrow{(f_*)_{\#}} & \pi_n(\text{map}(X, Y; f)) \\ \downarrow \omega_{\#} & & \downarrow \omega_{\#} & & \downarrow \omega_{\#} \\ G_{n+1}^{rel}(Y, X; f) & \longrightarrow & G_n(X) & \xrightarrow{f_{\#}} & G_n(Y, X; f) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_{n+1}(f) & \longrightarrow & \pi_n(X) & \xrightarrow{f_{\#}} & \pi_n(Y) \end{array}$$

The homomorphisms in the *G-sequence* are the same as those in the long exact homotopy sequence of the map  $f$ . Therefore, the *G-sequence* forms a chain complex (consecutive compositions are trivial). However, it is not necessarily exact. Consequently, it determines graded homology groups called the  *$\omega$ -homology* of  $f: X \rightarrow Y$ . The determination of this invariant of the homotopy type of the map  $f$  is an extremely difficult problem even under restricted circumstances. Examples of non-exactness, that is, of nontrivial  $\omega$ -homology, are given in [PW97, PSW98]. Some general conditions are known under which the *G-sequence* is exact. For instance, it is exact when  $f$  is a null-homotopic cellular inclusion [LW93], or when  $f$  is a homotopy monomorphism [PW01].

In this paper is to bring the techniques of rational homotopy theory to bear on problems and questions concerning evaluation subgroups (of a map) in general, and the *G-sequence* in particular. Our main goal is to expand the range of application of these techniques in this area. We are primarily concerned with establishing a suitable framework for considering such questions. At the same time, we obtain a number of results of interest in their own right. We now outline the results of

the paper. Our main results are established in Section 2 and Section 3. The basic result in Section 2 is Theorem 2.1, in which we identify the map induced on rational homotopy groups by the evaluation map  $\omega: \text{map}(X, Y; f) \rightarrow Y$ . We describe this induced homomorphism as the map induced on homology by a map of complexes of derivations of the Sullivan minimal models of  $X$  and  $Y$ . Theorem 2.1 has a number of immediate corollaries. For instance, we obtain a characterization of the rationalized evaluation subgroups of a map (Corollary 2.7) that extends a well known characterization, due to Félix and Halperin, of the (rationalized) Gottlieb groups of a space in terms of derivations of its minimal model.

In Section 3, we extend and amplify the basic result of Theorem 2.1. We show that the relevant long exact sequences of rational homotopy groups of function spaces are naturally expressed as homology exact sequences of derivations complexes of Sullivan minimal models. In particular, we obtain a description, within the framework of derivation spaces, of the  $G$ -sequence of a map after rationalization.

In Section 4, we use the framework established in Section 2 and Section 3 to study the exactness and non-exactness of the rationalized  $G$ -sequence. Just as in the integral setting, the rationalized  $G$ -sequence is not exact in general. Example 4.1 gives a simple example for which the rationalized  $G$ -sequence fails to be exact at each of the three types of term that occur. Since nonexactness rationally implies nonexactness integrally, this example provides a new, complete example of the failure of exactness of the  $G$ -sequence. This example also shows that the framework established in Section 2 and Section 3 provides a fairly effective setting in which to carry out explicit computations. By way of contrast, in Theorem 4.3 we give one set of conditions under which the rationalized  $G$ -sequence is exact. In Theorem 4.5 we show that under certain circumstances, the rationalized  $G$ -sequence may be exact at all occurrences of one type of term, while failing to be exact at the other types of term. This same result establishes a relationship, albeit under rather restrictive circumstances, between the (vanishing of a certain type of the)  $\omega$ -homology of a map  $f: X \rightarrow Y$  and the (vanishing of) negative-degree derivations of the rational cohomology of  $X$ . The last development that we present in Section 4 is a connection between the  $G$ -sequence of certain fibre inclusion maps and a well-known conjecture in rational homotopy theory. Let  $X \rightarrow E \rightarrow S^{2r+1}$  be a fibration with base an odd-dimensional sphere. For certain types of fibre space  $X$ , a conjecture of Halperin asserts that the fibration should be rationally TNCZ (see Conjecture 2.10 below). In Theorem 4.9, we show that this is the case exactly when the rationalized  $G$ -sequence of the fibre inclusion  $X \rightarrow E$  reduces to a certain short exact sequence. In this way, we obtain an equivalent phrasing of the conjecture of Halperin, in terms of the ideas studied in this paper.

We now discuss existing results on rational homotopy, function spaces, and the Gottlieb groups of a space that relate to this paper. Our aim in this discussion is to indicate the basic results that exist in the area. At the same time, we identify how these results relate to our work, and our points of departure from them. From results in rational homotopy theory, we can identify three “tributary streams” that flow into this work precisely at Theorem 2.1—our first main result—at varying levels of generality. There is also a fourth tributary flowing into our work, from outside rational homotopy theory. This consists of the work on the  $G$ -sequence, already indicated above. We say a little bit more on this below.

The first tributary of antecedent results in rational homotopy theory concerns the rationalized Gottlieb groups of a space. In [FH82], Félix and Halperin gave a characterization of the rationalized Gottlieb groups of a space, in terms of derivations of the Sullivan minimal model. At its most specialized level, Theorem 2.1 retrieves this characterization (Corollary 2.6) and extends it to a similar characterization of the rationalized evaluation subgroups of a general map (Corollary 2.7). Félix and Halperin went on to prove a remarkable result concerning the rationalized Gottlieb groups of a finite complex [FH82, Th.III]. Their result significantly extends results of Gottlieb from [Got69], and relates the rationalized Gottlieb groups with the rational Lusternik-Schnirelmann category. Unfortunately, no analogous result seems forthcoming for the rationalized evaluation subgroups of a map. Nonetheless, our characterization of the rationalized evaluation subgroups of a map is as effective for concrete computations as is the earlier characterization of the rationalized Gottlieb groups.

The second tributary concerns the rational homotopy type of  $B \operatorname{aut}_1(X)$ —the classifying space for fibrations with fibre  $X$ . Although we are not concerned with this classifying space as such, the connection arises because we have an isomorphism of homotopy groups  $\pi_{i+1}(B \operatorname{aut}_1(X)) \cong \pi_i(\operatorname{map}(X, X; 1))$ , and also the Gottlieb groups of  $X$  are obtained as the image in homotopy groups of the connecting homomorphism of the corresponding classifying fibration [Got69, Th.2.6]. At a higher level of generality than the rationalized Gottlieb groups, Theorem 2.1 includes an identification of the map induced on rational homotopy groups by the evaluation map  $\omega: \operatorname{map}(X, X; 1) \rightarrow X$ , and in particular it identifies the rational homotopy groups of  $\operatorname{map}(X, X; 1)$ . Whilst the identification of these groups we give is familiar in rational homotopy theory, our proof for this—obtained by restricting that of Theorem 2.1—is the first direct and detailed one that has actually appeared in the literature. We support this assertion as follows: In [Sul78, Sec.11], Sullivan sketched (with no proof) a model for the rational homotopy type of  $B \operatorname{aut}_1(X)$ , from which the description of the rational homotopy groups of  $\operatorname{map}(X, X; 1)$  contained in Theorem 2.1 follows. A justification of Sullivan’s model for  $B \operatorname{aut}_1(X)$  may be gleaned by collating results from a number of sources spread through the literature (e.g. [SS, Tan84, Tan83, Gat97]). But to date, no direct proof of Sullivan’s model has been given. Even amongst those articles that focus specifically on the rational homotopy *groups*, either of  $B \operatorname{aut}_1(X)$  or of  $\operatorname{map}(X, X; 1)$ —and that therefore avoid the technical problems of dealing with the rational homotopy *type*—we still do not find complete details. Meier [Mei82, (1.4), (2.6)] outlines the basic idea, but is actually focussed on a special kind of situation in which the minimal model can be replaced by its cohomology. Grivel [Gri94] focusses on the same special case as Meier, and quotes Sullivan’s model directly. Félix and Thomas [FT94, Sec.2.3] give exactly the description of the rational homotopy groups of  $\operatorname{map}(X, X; 1)$  contained in Theorem 2.1, but no details of the proof are given. As well as including a direct and detailed proof for the case of  $\operatorname{map}(X, X; 1)$ , Theorem 2.1 extends this identification of the rational homotopy groups to the general case of  $\operatorname{map}(X, Y; f)$ . It thus provides a natural framework for the study of rational homotopy groups of function spaces.

Finally, the third tributary from rational homotopy theory consists of a model, due to Sullivan and Haefliger, for the rational homotopy type of  $\operatorname{map}(X, Y; f)$ . At

its full level of generality, Theorem 2.1 identifies the map induced on rational homotopy groups by  $\omega: \text{map}(X, Y; f) \rightarrow Y$ , a general evaluation map. The precursor to the identification we give, and to our general line of proof, is the approach of Thom in [Tho57], although obviously there is no reference to minimal models in his work. Pursuing the river analogy a little further, we may think of Thom’s result, if not as the source, then at least as somewhere in the headwaters. We emphasize the connection between Theorem 2.1 and Thom’s approach by retrieving a basic result of his in Corollary 2.9. Coming further downstream, Sullivan also described in [Sul78, Sec.11] a model for the space of sections of a fibration homotopic to a given section. By specializing to the trivial fibration  $X \times Y \rightarrow X$ , this yields a model for the function space  $\text{map}(X, Y; f)$ —and more generally a model for the rational homotopy *type* of the general evaluation map  $\omega$ . A detailed proof for Sullivan’s model in this case was given by Haefliger [Hae82]. Now this model should in principle determine the rational homotopy groups of the function space (see [FT94] and [MR85], where it is used quite effectively). However, the model in question is a (non-minimal) DG algebra model. Therefore, the homomorphism induced by  $\omega$  on rational homotopy groups—which is exactly the information we require to proceed with our development—is available only indirectly, at best. By focussing on the rational homotopy *groups*—as opposed to the rational homotopy *type*, we have arrived in Theorem 2.1 at an entirely new characterization of the map induced on rational homotopy groups by  $\omega: \text{map}(X, Y; f) \rightarrow Y$ . Furthermore, we have been able to give a direct proof that avoids many of the technical complexities of Haefliger’s work and is completely independent of it.

The remaining tributary flowing into our work is from outside rational homotopy theory, and concerns classical results on the Gottlieb groups of a space, and more recent results on evaluation subgroups of a map and the  $G$ -sequence. We have already mentioned some of the results in this area. In his original work on evaluation subgroups, Gottlieb observed that a map of spaces does not necessarily induce a map of Gottlieb groups, and gave conditions under which it does [Got69, Sec.1]. Gottlieb briefly mentions the evaluation subgroups of a map in [Got69], but did not study them as such. Kim and Woo established a number of basic properties of the evaluation subgroup of a map in [WK84]. As we mentioned earlier, Woo and Lee, and other authors, introduced and studied the  $G$ -sequence of a map. The results basically fall into one of three areas: conditions under which the  $G$ -sequence is exact (e.g. [LW88b, Th.12] and [LW90]), examples of non-exactness (e.g. [LW93] and [PW97]), and extensions and generalizations of evaluation subgroups and the  $G$ -sequence (e.g. [LW98, LW01]). These results give the stepping-off point for our work in Sections 3 and 4. Using our description of the rationalized  $G$ -sequence (Theorem 3.7), we extend the known exactness results to several new cases. We also give new instances of non-exactness. More significantly, by focussing on the rational setting, our methods make the production of such examples straightforward. On the other hand, our results and examples are not restricted to amplifying previous results in this area. In Corollary 4.10, we suggest a different kind of result that relates properties of the  $G$ -sequence to the triviality of a fibration. Furthermore, some results in Section 3—including Theorem 3.2 and Theorem 3.10—are of interest independently of any relation to the  $G$ -sequence.

We finish this introduction by setting some notation and terminology. Throughout this paper,  $X$  and  $Y$  will denote simply connected CW complexes of finite type.

By *vector space* we mean a rational graded vector space. By *algebra*, we mean the kind of commutative graded algebras over the rationals that arise in rational homotopy. That is, they are non-negatively graded, connected ( $H^0 = \mathbb{Q}$ ) and usually simply connected ( $H^1 = 0$ ), with cohomology of finite type. For a vector space  $V$ , we denote the free commutative graded algebra generated by  $V$  by  $\Lambda V$ . We use the acronym DG to denote differential graded: Thus, DG vector space, DG algebra, and so-forth. For a DG algebra, the differential is of degree  $+1$ . In other situations, however, particularly when we consider the complex of derivations of a DG algebra, the differential is of degree  $-1$ . We will generally refer to a DG vector space whose differential is of degree  $-1$  as a chain complex. If  $f: A \rightarrow B$  is a map, either topological or algebraic, then  $f^*$  denotes pre-composition by  $f$  and  $f_*$  denotes post-composition by  $f$ . In any setting in which it is appropriate, we use  $H(f)$  to denote the map induced on homology (or cohomology) by  $f$ , and  $f_\#$  to denote the map induced on homotopy groups by the map of spaces  $f$ . We use  $\omega$  in a generic way to denote an evaluation map, and we denote the identity map of a topological space or the identity homomorphism of an algebra by  $1$ . We denote the rationalization of a space  $X$  by  $X_{\mathbb{Q}}$  and of a map  $f$  by  $f_{\mathbb{Q}}$  (cf. [HMR75]).

We assume that the reader is familiar with the basics of rational homotopy. In particular, a space  $X$  has a *minimal model*  $\mathcal{M}_X$ , which is a certain type of DG algebra. Namely,  $\mathcal{M}_X$  is a free algebra  $\Lambda V$  with a decomposable differential, that is,  $d(V) \subseteq \Lambda^{\geq 2}V$ . Furthermore, a map of spaces  $f: X \rightarrow Y$  induces a map of minimal models  $\mathcal{M}_f: \mathcal{M}_Y \rightarrow \mathcal{M}_X$ . We refer to this induced map as the *Sullivan minimal model* of the map  $f$ . It is a complete rational homotopy invariant for a map, and in principle all rational homotopy theoretic information about  $f$  can be retrieved from it. Passing to cohomology, for example, gives  $H(\mathcal{M}_f): H^*(\mathcal{M}_Y) \rightarrow H^*(\mathcal{M}_X)$ , which corresponds to the homomorphism of rational cohomology algebras induced by  $f$ . The results of this paper illustrate how deeper information about a space or map may be retrieved from the minimal model by making correspondingly more sophisticated constructions with the model. Our basic reference for rational homotopy theory is [FHT01].

## 2. DERIVATION SPACES

Our purpose in this and the next section is to give a unified description in rational homotopy theory of all the terms involved in the definition of the  $G$ -sequence. Informally stated, we show that the homology theory of derivation complexes of Sullivan minimal models provides an algebraic model for the rational homotopy theory of function spaces at the level of homotopy groups.

We focus on the following commutative square that appears in the homotopy ladder from which the  $G$ -sequence arises:

$$(1) \quad \begin{array}{ccc} \pi_n(\mathrm{map}(X, X; id)) & \xrightarrow{(f_*)_\#} & \pi_n(\mathrm{map}(X, Y; f)) \\ \omega_\# \downarrow & & \downarrow \omega_\# \\ \pi_n(X) & \xrightarrow{f_\#} & \pi_n(Y) \end{array}$$

It turns out that identifying the rationalization of this commutative square is sufficient to arrive not only at a characterization of the rationalized evaluation subgroups of  $f$ , but also at a description of the rationalization of the  $G$ -sequence.

Furthermore, our identification of the rationalization of this square allows us to conclude several subsidiary results of interest.

We say two maps of vector spaces  $f: U \rightarrow V$  and  $g: U' \rightarrow V'$  are *equivalent* if there exists isomorphisms  $\alpha$  and  $\beta$  which make the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \alpha \downarrow \cong & & \beta \downarrow \cong \\ U' & \xrightarrow{g} & V' \end{array}$$

commutative. This notion of equivalence for vector space maps extends in the obvious way to sequences of vector space maps, commutative squares of vector space maps, and any other diagram of vector space maps.

Suppose a DG algebra  $(A, d_A)$  is isomorphic to  $\mathbb{Q}$  in degree zero, that is,  $A^0 \cong \mathbb{Q}$ . Then the map  $\varepsilon: A \rightarrow \mathbb{Q}$  that sends all elements of positive degree to zero, and is the identity in degree zero, is an augmentation. This will be the situation in all cases of interest to us here, and thus we refer to  $\varepsilon: A \rightarrow \mathbb{Q}$  as *the* augmentation. Here, as in the sequel, we regard  $\mathbb{Q}$  as the trivial DG algebra concentrated in degree zero and with trivial differential. Thus  $\varepsilon$  is a DG algebra map.

Given DG algebras  $(A, d_A)$  and  $(B, d_B)$  and a (fixed) DG algebra map  $\phi: A \rightarrow B$ , define a  $\phi$ -*derivation* of degree  $n$  to be a linear map  $\theta: A \rightarrow B$  that *reduces degree by  $n$*  and satisfies the derivation law  $\theta(xy) = \theta(x)\phi(y) - (-1)^{n|x|}\phi(x)\theta(y)$ . We will only consider derivations of positive degree, that is, those that reduce degree by some positive integer. When  $n = 1$  we require additionally that  $d_B \circ \theta = -\theta \circ d_A$ . Let  $\text{Der}_n(A, B; \phi)$  denote the vector space of  $\phi$ -derivations of degree  $n$ , for  $n > 0$ . Finally, define a linear map  $\delta: \text{Der}_n(A, B; \phi) \rightarrow \text{Der}_{n-1}(A, B; \phi)$  by  $\delta(\theta) = d_B \circ \theta - (-1)^{|\theta|}\theta \circ d_A$ . A standard check now shows that  $\delta^2 = 0$  and thus  $(\text{Der}_*(A, B; \phi), \delta)$  is a chain complex. In order to cut down on cumbersome notation, we will usually suppress the differential from our notation, and write  $H_n(\text{Der}(A, B; \phi))$  for the homology in degree  $n$  of the chain complex  $(\text{Der}_*(A, B; \phi), \delta)$ . That is,  $H_n(\text{Der}(A, B; \phi))$  denotes the homology represented by  $\delta$ -cycles of  $\text{Der}_*(A, B; \phi)$  that reduce degree by  $n$ .

A special case of the preceding that is of interest to us is the one in which  $A = B$  and  $\phi = 1_B$ . In this case, the chain complex of derivations  $\text{Der}_*(B, B; 1)$  is just the usual complex of derivations on the DG algebra  $B$ . Note once again that we restrict the derivations in degree 1 to the cycles and that the complex is zero in non-positive degrees. Pre-composition with the DG algebra map  $\phi: A \rightarrow B$  thus gives a map of chain complexes  $\phi^*: \text{Der}(B, B; 1) \rightarrow \text{Der}(A, B; \phi)$ . Furthermore, post-composition by the augmentation  $\varepsilon: B \rightarrow \mathbb{Q}$  induces DG vector space maps  $\varepsilon_*: \text{Der}_*(A, B; \phi) \rightarrow \text{Der}_*(A, \mathbb{Q}; \varepsilon)$  and  $\varepsilon_*: \text{Der}_*(B, B; 1) \rightarrow \text{Der}_*(B, \mathbb{Q}; \varepsilon)$ .

All of the above can be applied to a map of minimal models. Suppose  $f: X \rightarrow Y$  is a map of spaces, and  $\mathcal{M}_f: \mathcal{M}_Y \rightarrow \mathcal{M}_X$  is the corresponding Sullivan minimal model of the map  $f$ . Then we have a commutative square of chain complexes

$$(2) \quad \begin{array}{ccc} \text{Der}_*(\mathcal{M}_X, \mathcal{M}_X; 1) & \xrightarrow{(\mathcal{M}_f)^*} & \text{Der}_*(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f) \\ \varepsilon_* \downarrow & & \downarrow \varepsilon_* \\ \text{Der}_*(\mathcal{M}_X, \mathbb{Q}; \varepsilon) & \xrightarrow{(\widehat{\mathcal{M}_f})^*} & \text{Der}_*(\mathcal{M}_Y, \mathbb{Q}; \varepsilon) \end{array}$$

In this square,  $\varepsilon: \mathcal{M}_X \rightarrow \mathbb{Q}$  is the augmentation. Both horizontal maps are obtained by pre-composing with the same map  $\mathcal{M}_f$ , but in different contexts. Since we will need to distinguish between these two maps notationally in the sequel, we have used an extra decoration on the bottom one.

The main result of this section is the following:

**Theorem 2.1.** *Let  $X$  and  $Y$  be simply connected CW complexes of finite type, with  $X$  finite. For  $n \geq 2$ , the commutative square obtained by rationalizing (1) is equivalent to the square obtained by passing to homology in degree  $n$  from (2). That is, the commutative squares*

$$\begin{array}{ccc} \pi_n(\text{map}(X, X; 1)) \otimes \mathbb{Q} & \xrightarrow{(f_*)_{\#} \otimes \mathbb{Q}} & \pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q} \\ \omega_{\#} \otimes \mathbb{Q} \downarrow & & \downarrow \omega_{\#} \otimes \mathbb{Q} \\ \pi_n(X) \otimes \mathbb{Q} & \xrightarrow{f_{\#} \otimes \mathbb{Q}} & \pi_n(Y) \otimes \mathbb{Q} \end{array}$$

and

$$\begin{array}{ccc} H_n(\text{Der}(\mathcal{M}_X, \mathcal{M}_X; 1)) & \xrightarrow{H((\mathcal{M}_f)^*)} & H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)) \\ H(\varepsilon_*) \downarrow & & \downarrow H(\varepsilon_*) \\ H_n(\text{Der}(\mathcal{M}_X, \mathbb{Q}; \varepsilon)) & \xrightarrow{H(\widehat{(\mathcal{M}_f)^*})} & H_n(\text{Der}(\mathcal{M}_Y, \mathbb{Q}; \varepsilon)) \end{array}$$

are equivalent for each  $n \geq 2$ .

We prove this result below. First, we comment on some ingredients of the statement and proof, and give some immediate consequences.

*Remark 2.2.* In rational homotopy theory there is a standard way to identify the rational homotopy groups of a space  $X$ , and more generally the homomorphism of rational homotopy groups induced by a map of spaces. Namely, the rational homotopy groups are identified with the dual of the vector space of indecomposables of the minimal model, thus  $\pi_*(X) \otimes \mathbb{Q} \cong \text{Hom}(Q(\mathcal{M}_X), \mathbb{Q})$ . As for maps,  $f_{\#} \otimes \mathbb{Q}$  is identified with the dual of the map of vector spaces of indecomposables  $Q(\mathcal{M}_f): Q(\mathcal{M}_Y) \rightarrow Q(\mathcal{M}_X)$  induced by the Sullivan minimal model  $\mathcal{M}_f: \mathcal{M}_Y \rightarrow \mathcal{M}_X$  of a map  $f: X \rightarrow Y$  (see [FHT01, Sec.15(d)] for details). From the bottom maps in the two squares of Theorem 2.1, we obtain a superficially different description of  $f_{\#} \otimes \mathbb{Q}$ . But it is easy to see that this agrees with the standard one: Note that the derivation law implies  $\text{Der}_*(\mathcal{M}_X, \mathbb{Q}; \varepsilon) \cong \text{Hom}(Q(\mathcal{M}_X), \mathbb{Q})$ , while the minimality of  $\mathcal{M}_X$  implies that  $\delta = 0$  in the chain complex  $\text{Der}_*(\mathcal{M}_X, \mathbb{Q}; \varepsilon)$ . Thus we have  $\text{Hom}(Q(\mathcal{M}_X), \mathbb{Q}) \cong H_*(\text{Der}(\mathcal{M}_X, \mathbb{Q}; \varepsilon))$ . The agreement between our description of  $f_{\#} \otimes \mathbb{Q}$  and the standard one is obvious from this isomorphism.

Theorem 2.1 also contains and depends upon basic results concerning the rationalization of function space components and evaluation subgroups due to several authors. We consider this material here.

When  $X$  is a finite complex, the function space  $\text{map}(X, Y)$  has the homotopy type of a CW complex by the result of Milnor [Mil59]. In fact, by Hilton-Mislin-Roitberg [HMR75, Ch. II, Th. 2.5] the components  $\text{map}(X, Y; f)$  are nilpotent complexes. Moreover, given a rationalization  $e_Y: Y \rightarrow Y_{\mathbb{Q}}$  of  $Y$  the induced map  $(e_Y)_*: \text{map}(X, Y; f) \rightarrow \text{map}(X, Y_{\mathbb{Q}}; e_Y \circ f)$  is a rationalization of  $\text{map}(X, Y; f)$

[HMR75, Ch. II, Th. 3.11]. By [Smi96, Th.2.3], rationalization in the initial variable  $e_X: X \rightarrow X_{\mathbb{Q}}$  induces a weak equivalence  $(e_X)^*: \text{map}(X_{\mathbb{Q}}, Y_{\mathbb{Q}}; f_{\mathbb{Q}}) \rightarrow \text{map}(X, Y_{\mathbb{Q}}; f_{\mathbb{Q}} \circ e_X)$ . These results, together with the naturality of the various maps involved, imply the following result.

**Theorem 2.3.** *Let  $f: X \rightarrow Y$  be a map between simply connected complexes of finite type with  $X$  finite. Let  $f_{\mathbb{Q}}: X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$  denote the rationalization of  $f$ . The commutative squares*

$$\begin{array}{ccc} \pi_n(\text{map}(X, X; 1)) \otimes \mathbb{Q} & \xrightarrow{(f_*)_{\#} \otimes \mathbb{Q}} & \pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q} \\ \omega_{\#} \otimes \mathbb{Q} \downarrow & & \downarrow \omega_{\#} \otimes \mathbb{Q} \\ \pi_n(X) \otimes \mathbb{Q} & \xrightarrow{f_{\#} \otimes \mathbb{Q}} & \pi_n(Y) \otimes \mathbb{Q} \end{array}$$

and

$$\begin{array}{ccc} \pi_n(\text{map}(X_{\mathbb{Q}}, X_{\mathbb{Q}}; 1)) & \xrightarrow{((f_{\mathbb{Q}})_{*})_{\#}} & \pi_n(\text{map}(X_{\mathbb{Q}}, Y_{\mathbb{Q}}; f_{\mathbb{Q}})) \\ (\omega)_{\#} \downarrow & & \downarrow (\omega)_{\#} \\ \pi_n(X_{\mathbb{Q}}) & \xrightarrow{(f_{\mathbb{Q}})_{\#}} & \pi_n(Y_{\mathbb{Q}}) \end{array}$$

are equivalent for each  $n \geq 2$ .

The result  $G_n(X_{\mathbb{Q}}) \cong G_n(X) \otimes \mathbb{Q}$  for  $X$  a simply connected finite complex due to Lang [Lan75] is an easy consequence of Theorem 2.3, as is its generalization  $G_*(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) \cong G_*(Y, X; f) \otimes \mathbb{Q}$  (c.f. [Smi96, PW97]) for  $X, Y$  simply connected complexes of finite type with  $X$  finite. The corresponding localization result for the relative Gottlieb group can be deduced from the preceding discussion, as well.

**Theorem 2.4.** *Let  $f: X \rightarrow Y$  be a map between simply connected complexes of finite type with  $X$  finite. Then  $G_n^{\text{rel}}(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) \cong G_n^{\text{rel}}(Y, X; f) \otimes \mathbb{Q}$ , for  $n \geq 3$ .*

*Proof.* Consider, as in the introduction, the long exact homotopy sequence of the induced map  $f_*: \text{map}(X, X; 1) \rightarrow \text{map}(X, Y; f)$  and its relative homotopy group  $\pi_n(f_*)$  for  $n \geq 3$ . We also have  $(f_{\mathbb{Q}})_*: \text{map}(X_{\mathbb{Q}}, X_{\mathbb{Q}}; 1) \rightarrow \text{map}(X_{\mathbb{Q}}, Y_{\mathbb{Q}}; f_{\mathbb{Q}})$  and the relative group  $\pi_n((f_{\mathbb{Q}})_*)$ . The results cited above and the 5-Lemma, imply the maps induced on the various function spaces involved by the rationalizations  $e_X: X \rightarrow X_{\mathbb{Q}}$  and  $e_Y: Y \rightarrow Y_{\mathbb{Q}}$  induce a rationalization homomorphism  $R: \pi_n(f_*) \rightarrow \pi_n((f_{\mathbb{Q}})_*)$ . The maps  $e_X$  and  $e_Y$  also induce a rationalization  $r: \pi_n(f) \rightarrow \pi_n(f_{\mathbb{Q}})$ . By naturality, we obtain a commutative diagram

$$\begin{array}{ccc} \pi_n(f_*) & \xrightarrow{R} & \pi_n((f_{\mathbb{Q}})_*) \\ \downarrow \omega_{\#} & & \downarrow \omega_{\#} \\ \pi_n(f) & \xrightarrow{r} & \pi_n(f_{\mathbb{Q}}). \end{array}$$

Thus  $G_n(X_{\mathbb{Q}}, Y_{\mathbb{Q}}; f_{\mathbb{Q}}) = \omega_{\#}(\pi_n((f_{\mathbb{Q}})_*)) = r \circ \omega_{\#}(\pi_n(f_*)) \cong G_n^{\text{rel}}(Y, X; f) \otimes \mathbb{Q}$   $\square$

Combining the preceding, we obtain the following:

**Corollary 2.5.** *Let  $f: X \rightarrow Y$  be a map between simply connected complexes of finite type with  $X$  finite. Then the rationalization of the  $G$ -sequence of  $f$*

$$\cdots \rightarrow G_n(X) \otimes \mathbb{Q} \rightarrow G_n(Y, X; f) \otimes \mathbb{Q} \rightarrow G_n^{rel}(Y, X; f) \otimes \mathbb{Q} \rightarrow G_{n-1}(X) \otimes \mathbb{Q} \rightarrow \cdots$$

*is equivalent to the  $G$ -sequence of the rationalization of  $f$*

$$\cdots \rightarrow G_n(X_{\mathbb{Q}}) \rightarrow G_n(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) \rightarrow G_n^{rel}(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) \rightarrow G_{n-1}(X_{\mathbb{Q}}) \rightarrow \cdots$$

In the next section, we identify the rational  $G$ -sequence in the context of rational homotopy theory. From Theorem 2.1 we immediately retrieve minimal model descriptions of the rationalized Gottlieb groups of a space, and of the rationalized evaluation subgroups of a map. The first of these is well-known:

**Corollary 2.6.** *Let  $X$  be a simply connected finite complex. The rationalized  $n$ th Gottlieb group  $G_n(X_{\mathbb{Q}}) \cong G_n(X) \otimes \mathbb{Q}$  is isomorphic to the image of the induced homomorphism*

$$H(\varepsilon_*) : H_n(\mathrm{Der}(\mathcal{M}_X, \mathcal{M}_X; 1)) \rightarrow H_n(\mathrm{Der}(\mathcal{M}_X, \mathbb{Q}; \varepsilon))$$

for  $n \geq 2$ .

This is easily translated into the standard minimal model description of the rationalized Gottlieb groups given by Félix and Halperin (see [FHT01, Sec.29(c)]). Specifically, they describe a Gottlieb element of the minimal model  $\mathcal{M}_X = \Lambda(V)$  as a linear map  $\theta: V^n \rightarrow \mathbb{Q}$  that extends to a derivation of  $\mathcal{M}_X$  satisfying  $d\theta = (-1)^n \theta d$ . Such a derivation  $\theta$  is a cycle in  $\mathrm{Der}(\mathcal{M}_X, \mathcal{M}_X; 1)$ , and the class that it represents has non-zero image under  $H(\varepsilon_*)$  precisely when the original linear map  $\theta: V^n \rightarrow \mathbb{Q}$  is non-zero. On recalling that  $H_n(\mathrm{Der}(\mathcal{M}_X, \mathbb{Q}; \varepsilon)) \cong \mathrm{Der}(\mathcal{M}_X, \mathbb{Q}; \varepsilon) \cong \mathrm{Hom}(Q(\mathcal{M}_X), \mathbb{Q})$ , we see the two descriptions agree.

**Corollary 2.7.** *Let  $f: X \rightarrow Y$  be a map between simply connected complexes of finite type with  $X$  finite. The rationalized  $n$ th evaluation subgroup  $G_n(Y_{\mathbb{Q}}, X_{\mathbb{Q}}, f_{\mathbb{Q}}) \cong G_n(Y, X, f) \otimes \mathbb{Q}$  of the map  $f$  is isomorphic to the image of the induced homomorphism*

$$H(\varepsilon_*) : H_n(\mathrm{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)) \rightarrow H_n(\mathrm{Der}(\mathcal{M}_Y, \mathbb{Q}; \varepsilon)).$$

for  $n \geq 2$ .

This identification of the rationalized evaluation subgroups of the map  $f$  can be conveniently phrased in a way comparable to the Félix-Halperin description of the rationalized Gottlieb groups: An evaluation subgroup element of the minimal model  $\mathcal{M}_f: \mathcal{M}_Y \rightarrow \mathcal{M}_X$ , with  $\mathcal{M}_Y = \Lambda(W)$ , is a linear map  $\theta: W^n \rightarrow \mathbb{Q}$  that extends to an  $\mathcal{M}_f$ -derivation  $\theta: \mathcal{M}_Y \rightarrow \mathcal{M}_X$  satisfying  $d_Y \theta = (-1)^n \theta d_X$ . Whenever such a  $\theta \in \mathrm{Der}_n(\mathcal{M}_Y, \mathbb{Q}; \varepsilon)$  is non-zero, it is a non-zero element in the image of  $H(\varepsilon_*)$ .

In view of the preceding remarks, we introduce the following vocabulary and notation.

**Definition 2.8.** Suppose  $\phi: A \rightarrow B$  is a map of DG algebras. We define the *evaluation subgroup of  $\phi$*  as the image of the map

$$H(\varepsilon_*) : H_n(\mathrm{Der}(A, B; \phi)) \rightarrow H_n(\mathrm{Der}(A, \mathbb{Q}; \varepsilon)).$$

We denote it by  $G_n(A, B; \phi)$ . In the special case in which  $A = B$  and  $\phi = 1_B$ , we refer to the *Gottlieb group of  $B$* , and use the notation  $G_n(B)$ .

From the previous discussion, we see that  $G_n(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f) \cong G_n(Y, X; f) \otimes \mathbb{Q}$  and  $G_n(\mathcal{M}_X) \cong G_n(X) \otimes \mathbb{Q}$ .

*Proof of Theorem 2.1.* We will define vector space isomorphisms  $\Phi$ ,  $\Phi_f$ ,  $\Psi_X$ , and  $\Psi_Y$  to give the following equivalence of commutative squares:

$$\begin{array}{ccc}
H_n(\mathrm{Der}(\mathcal{M}_X, \mathcal{M}_X; 1)) & \xrightarrow{H((\mathcal{M}_f)^*)} & H_n(\mathrm{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)) \\
\Phi \nearrow \cong & \downarrow H(\varepsilon_*) & \Phi_f \nearrow \cong \\
\pi_n(\mathrm{map}(X, X; 1)) \otimes \mathbb{Q} & \xrightarrow{(f_*)_{\#} \otimes 1} & \pi_n(\mathrm{map}(X, Y; f)) \otimes \mathbb{Q} \\
\downarrow \omega_{\#} & \downarrow & \downarrow H(\varepsilon_*) \\
H_n(\mathrm{Der}(\mathcal{M}_X, \mathbb{Q}; \varepsilon)) & \xrightarrow{H((\mathcal{M}_f)^*)} & H_n(\mathrm{Der}(\mathcal{M}_Y, \mathbb{Q}; \varepsilon)) \\
\Psi_X \nearrow \cong & \downarrow \omega_{\#} & \Psi_Y \nearrow \cong \\
\pi_n(X) \otimes \mathbb{Q} & \xrightarrow{f_{\#} \otimes 1} & \pi_n(Y) \otimes \mathbb{Q}
\end{array}$$

We obtain  $\Phi_f$  as the rationalization of a natural homomorphism

$$\Phi'_f: \pi_n(\mathrm{map}(X, Y; f)) \rightarrow H_n(\mathrm{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)),$$

which we now define. A representative of a homotopy class  $\alpha \in \pi_n(\mathrm{map}(X, Y; f))$  determines, via the exponential correspondence, a map  $F: S^n \times X \rightarrow Y$  that satisfies  $F \circ i_2 = f$ , where  $i_2: X \rightarrow S^n \times X$  is the inclusion. The map  $F$  is often called an *affiliated map* for (a representative of)  $\alpha$ . Passing to minimal models, we obtain a map  $\mathcal{M}_F: \mathcal{M}_Y \rightarrow \mathcal{M}_{S^n} \otimes \mathcal{M}_X$  with  $\mathcal{M}_{i_2} \circ \mathcal{M}_F = \mathcal{M}_f$  (equals, not just up to DG homotopy—see Proposition A.2). Now  $S^n$  is a *formal space*, which means there is a quasi-isomorphism of DG algebras  $\psi: \mathcal{M}_{S^n} \rightarrow H^*(S^n; \mathbb{Q})$ . In turn, this gives a quasi-isomorphism  $\psi \otimes 1: \mathcal{M}_{S^n} \otimes \mathcal{M}_X \rightarrow H^*(S^n; \mathbb{Q}) \otimes \mathcal{M}_X$ . Write  $H^*(S^n; \mathbb{Q})$  as  $\Lambda(s_n)/(s_n^2)$  if  $n$  is even, or as  $\Lambda(s_n)$  if  $n$  is odd. Given  $\chi \in \mathcal{M}_Y$ , we may write

$$(\psi \otimes 1) \circ \mathcal{M}_F(\chi) = 1 \otimes \mathcal{M}_f(\chi) + s_n \otimes \theta_G(\chi),$$

thus defining a linear map  $\theta_G: \mathcal{M}_Y \rightarrow \mathcal{M}_X$  that reduces degree by  $n$ . A standard check—using the fact that  $(\psi \otimes 1) \circ \mathcal{M}_F(\chi)$  is a DG algebra map—shows that  $\theta_G$  is an  $\mathcal{M}_f$ -derivation that is a  $\delta_{\mathcal{M}_f}$ -cycle. Define  $\Phi'_f(\alpha) = [\theta_G] \in H_n(\mathrm{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f))$ .

To show  $\Phi'_f(\alpha)$  is well-defined, suppose that  $g_1 \simeq g_2: S^n \rightarrow \mathrm{map}(X, Y; f)$  are two representatives of  $\alpha$  with affiliated maps  $F, G: S^n \times X \rightarrow Y$  respectively. Then the homotopy  $K: S^n \times I \rightarrow \mathrm{map}(X, Y; f)$  from  $g_1$  to  $g_2$  gives a homotopy  $H: S^n \times X \times I \rightarrow Y$ , from  $F$  to  $G$ , by setting  $H(s, x, t) = K(s, t)(x)$ . Further, since  $K$  is a *based homotopy*, the homotopy  $H$  satisfies  $H \circ i = J$ , where  $i$  denotes the inclusion  $i(x, t) = (*, x, t)$  and  $J$  denotes the homotopy  $J(x, t) = f(x)$  that is stationary at  $f$ . Now the homotopy  $H$  gives a DG homotopy  $\mathcal{M}_H: \mathcal{M}_Y \rightarrow \mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt)$  between minimal models for  $F$  and  $G$ . Translating the restriction on  $H$  into minimal model terms allows us to assume that  $\mathcal{M}_H$  has form such that

$$(3) \quad (\psi \otimes 1 \otimes 1) \circ \mathcal{M}_H(\chi) = 1 \otimes \mathcal{M}_f(\chi) \otimes 1 + \sum_{i \geq 0} s_n \otimes \alpha_i(\chi) \otimes t^i + \sum_{i \geq 0} s_n \otimes \beta_i(\chi) \otimes t^i dt,$$

for an element  $\chi \in \mathcal{M}_Y$ . This translation is intuitively plausible, but its justification requires some technical details, which we provide in Proposition A.2. Since the DG homotopy  $\mathcal{M}_H$  is from  $\mathcal{M}_F$  to  $\mathcal{M}_G$ , then at  $t = 0$  we have  $\alpha_0(\chi) = \theta_F(\chi)$ , and from  $t = 1$ , we have  $\sum_{i \geq 0} \alpha_i(\chi) = \theta_G(\chi)$ . To establish well-definedness, we must

show these differ by a boundary in  $\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)$ . To this end, use (3) to write separate expressions for  $(\psi \otimes 1 \otimes 1) \circ \mathcal{M}_H(\chi\chi')$  and  $(\psi \otimes 1 \otimes 1) \circ \mathcal{M}_H(\chi)(\psi \otimes 1 \otimes 1) \circ \mathcal{M}_H(\chi')$ . Since a DG homotopy is an algebra map, these expressions agree. By equating them and collecting like terms we obtain equations

$$\beta_i(\chi\chi') = (-1)^{n|\chi|} \mathcal{M}_f(\chi)\beta_i(\chi') + (-1)^{|\chi'|} \beta_i(\chi)\mathcal{M}_f(\chi')$$

for each  $i \geq 0$ . By substituting  $\gamma_i(\chi) = (-1)^{|\chi|} \beta_i(\chi)$  for  $\chi \in \mathcal{M}_Y$ , we obtain derivations  $\gamma_i \in \text{Der}_{n+1}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)$ . On the other hand, use (3) to write separate expressions for  $(\psi \otimes 1 \otimes 1) \circ \mathcal{M}_H(d\chi)$  and  $d(\psi \otimes 1 \otimes 1) \circ \mathcal{M}_H(\chi)$ , with the latter obtained by applying  $d$  to both sides of (3). Since a DG homotopy respects differentials, these expressions agree. By equating them and collecting like terms we obtain equations

$$\beta_i(d\chi) = (-1)^{|\chi|} (i+1)\alpha_{i+1}(\chi) + (-1)^n d\beta_i(\chi)$$

for each  $i \geq 0$ . With the previous substitution, this gives  $d\gamma_i(\chi) - (-1)^{n+1} \gamma_i d(\chi) = (-1)^{n+1} (i+1)\alpha_i(\chi)$ , that is,

$$\alpha_{i+1}(\chi) = \delta_{\mathcal{M}_f} \left( (-1)^{n+1} \frac{1}{i+1} \gamma_i(\chi) \right)$$

for each  $i \geq 0$ . It follows that the difference of derivations  $\theta_G - \theta_F = \sum_{i \geq 1} \alpha_i$  is a  $\delta_{\mathcal{M}_f}$ -boundary in  $\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)$ . Hence  $\Phi'_f$  is well-defined.

It is not difficult to show that  $\Phi'_f$  is a homomorphism. But since the proof requires some technical notions from rational homotopy theory, we postpone it to the appendix (Proposition A.3). As we stated before, the map of vector spaces  $\Phi_f$  is now obtained as the rationalization of the (group) homomorphism  $\Phi'_f$ . The map  $\Phi$  is defined in the same way, specializing to the case in which  $Y = X$  and  $f = 1_X$ . The maps  $\Psi_X$  and  $\Psi_Y$  are the standard minimal model identification of the rational homotopy groups of a space, as discussed in Remark 2.2.

Next, we show  $\Phi_f$  is surjective. Denote by  $[S_{\mathbb{Q}}^n \times X_{\mathbb{Q}}, Y_{\mathbb{Q}}]_{f_{\mathbb{Q}}}$  the subset of the set of homotopy classes of maps  $S_{\mathbb{Q}}^n \times X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$  consisting of classes represented by a map that restricts to  $f_{\mathbb{Q}}$  on  $X_{\mathbb{Q}}$ . By Theorem 2.3, we identify  $\pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q}$  with  $\pi_n(\text{map}(X_{\mathbb{Q}}, Y_{\mathbb{Q}}; f_{\mathbb{Q}}))$ , and hence with  $[S_{\mathbb{Q}}^n \times X_{\mathbb{Q}}, Y_{\mathbb{Q}}]_{f_{\mathbb{Q}}}$ . Now suppose given  $[\theta] \in H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f))$ . Use  $\theta$  to define

$$\phi(\chi) = 1 \otimes \mathcal{M}_f(\chi) + s_n \otimes \theta(\chi)$$

for  $\chi \in \mathcal{M}_Y$ . Since  $\theta$  is an  $\mathcal{M}_f$ -derivation that is a cycle, this defines a DG algebra map  $\phi: \mathcal{M}_Y \rightarrow H^*(S^n; \mathbb{Q}) \otimes \mathcal{M}_X$ . Now lift  $\phi$  through the surjective quasi-isomorphism  $\psi \otimes 1$  as in [FHT01, Lem.12.4], to obtain a map  $\tilde{\phi}: \mathcal{M}_Y \rightarrow \mathcal{M}_{S^n} \otimes \mathcal{M}_X$  that satisfies  $(\varepsilon \cdot 1) \circ \tilde{\phi} = \mathcal{M}_f: \mathcal{M}_Y \rightarrow \mathcal{M}_X$ . By the standard correspondence between maps of minimal models and maps of rational spaces, this gives a map  $F: S_{\mathbb{Q}}^n \times X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$  that satisfies  $i_2 \circ F \sim f_{\mathbb{Q}}: X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$ . Using, for example, [FHT01, Th.9.7], we can adjust  $F$  into a homotopic map  $F': S_{\mathbb{Q}}^n \times X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$  that satisfies  $i_2 \circ F' = f_{\mathbb{Q}}: X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$ , so that  $F'$  represents a class of  $[S_{\mathbb{Q}}^n \times X_{\mathbb{Q}}, Y_{\mathbb{Q}}]_{f_{\mathbb{Q}}}$ . As described at the start of this paragraph,  $F'$  corresponds to a homotopy class  $\alpha \in \pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q}$ . Evidently, we have  $\Phi_f(\alpha) = \theta$ .

Finally, we show  $\Phi_f$  is injective. Since  $\Phi_f$  is a vector space homomorphism, it is sufficient to show that  $\alpha \in \pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q}$  is zero whenever  $\Phi_f(\alpha) = 0$ . Using the identification of the previous paragraph, let  $G: S_{\mathbb{Q}}^n \times X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$  be an

affiliated map for  $\alpha$ . Suppose that  $\theta_G = \delta(\eta)$  for  $\eta \in \text{Der}_{n+1}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)$ . Using  $\eta$ , define a map  $\mathcal{M}_H: \mathcal{M}_Y \rightarrow H^*(S^n; \mathbb{Q}) \otimes \mathcal{M}_X \otimes \Lambda(t, dt)$  by

$$\Gamma(\chi) = 1 \otimes \mathcal{M}_f(\chi) \otimes 1 + s_n \otimes \theta_G(\chi) \otimes (1 - t) + s_n \otimes \eta(\chi) \otimes dt.$$

A routine check verifies that  $\Gamma$  is a DG algebra map. Furthermore, it is a DG homotopy from  $\mathcal{M}_G$  to the map  $E: \mathcal{M}_Y \rightarrow H^*(S^n; \mathbb{Q}) \otimes \mathcal{M}_X$  given by  $E(\chi) = 1 \otimes \mathcal{M}_f(\chi)$ . Now this latter map is a Sullivan model of an affiliated map for  $0 \in \pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q}$ . It follows that  $\alpha = 0$ . We observe that, strictly speaking, we have not justified that the homotopy between the maps  $S_{\mathbb{Q}}^n \times X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$  is relative to  $X_{\mathbb{Q}}$ , which corresponds to the homotopy between the maps  $S^n \rightarrow \text{map}(X_{\mathbb{Q}}, Y_{\mathbb{Q}}; f_{\mathbb{Q}})$  being based. However, a based map from a sphere is null-homotopic if and only if it is based null-homotopic (cf. [Spa89, p.27]). It follows that  $\alpha = 0$ , and thus  $\Phi_f$  is injective.

Commutativity of the cube diagram follows from the naturality of the homomorphism  $\Phi'_f$ . By naturality, we mean the following: Suppose given maps of spaces  $f: A \rightarrow B$  and  $g: B \rightarrow C$ . Then we have induced maps of function spaces  $g_*: \text{map}(A, B; f) \rightarrow \text{map}(A, C; g \circ f)$  and  $f^*: \text{map}(B, C; g) \rightarrow \text{map}(A, C; g \circ f)$ . For either case we obtain a commutative square involving  $\Phi_{g \circ f}$ . Namely, we have  $H((\mathcal{M}_g)^*) \circ \Phi_f = \Phi_{g \circ f} \circ (g_*)_{\#}$  in the first case, and  $H((\mathcal{M}_f)_*) \circ \Phi_g = \Phi_{g \circ f} \circ (f^*)_{\#}$  in the second case, as is easily checked. Since  $\Phi_f$  is obtained from  $\Phi'_f$  by localization, the isomorphism  $\Phi_f$  has the same naturality property. This is sufficient to conclude that the top, bottom, left, and right faces of the cube commute. For the evaluation map  $\omega: \text{map}(X, Y; f) \rightarrow Y$  can be identified as  $i^*: \text{map}(X, Y; f) \rightarrow \text{map}(x_0, Y; y_0)$ , where  $x_0 \in X$  and  $y_0 \in Y$  denote basepoints, and  $i: x_0 \rightarrow X$  inclusion of the basepoint. Likewise for the evaluation map  $\omega: \text{map}(X, X; 1) \rightarrow X$ , and then  $f: X \rightarrow Y$  can be identified with  $f_*: \text{map}(x_0, X; x_0) \rightarrow \text{map}(x_0, Y; y_0)$  (cf. also Remark 2.2). Finally, the front and rear faces commute because the squares (1) and (2) are commutative.  $\square$

We finish this section with two more immediate consequences of Theorem 2.1. The first retrieves the basic result of Thom, in the rational homotopy setting.

**Corollary 2.9.** ([Tho57, Th.2]) *Let  $Y = K(V, m)$  be an Eilenberg-Mac Lane space, for  $V$  a finite dimensional (ungraded) rational vector space. If  $X$  is a finite CW complex, and  $f: X \rightarrow Y$  is any map, then  $\pi_n(\text{map}(X, Y; f)) \cong H^{m-n}(X; V)$ .*

*Proof.* The minimal model for  $Y$  is  $\Lambda V^*$  with zero differential, where  $V^*$  denotes the dual vector space of  $V$ . It follows easily that  $H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)) \cong \text{Hom}(V^*, H^{m-n}(\mathcal{M}_X)) \cong \text{Hom}(V^*, H^{m-n}(X; \mathbb{Q})) \cong H^{m-n}(X; V)$ .  $\square$

Notice that this result—with the remarks on rationalization preceding Theorem 2.3—easily extends to yield the rational homotopy type of  $\text{map}(X, Y; f)$ , in case  $Y$  is a rational  $H$ -space. That is, a space whose rationalization is an  $H$ -space. For if  $Y$  is a rational  $H$ -space, then so too is  $\text{map}(X, Y; 0)$ , which is homotopy equivalent—via translation by  $f$ —to  $\text{map}(X, Y; f)$ . Now a rational  $H$ -space is determined up to rational homotopy type by its rational homotopy groups. Furthermore, a rational  $H$ -space is a product of rational Eilenberg-Mac Lane spaces. Hence  $\text{map}(X, Y; f)$  has the rational homotopy type of a product of spaces  $\text{map}(X, Y_i; 0)$ , for  $Y_i$  a rational Eilenberg-Mac Lane space as in the corollary.

We give a further consequence of Theorem 2.1 concerning the rational homotopy groups of certain function spaces. Define an  $F_0$ -space to be a finite simply connected

complex with finite dimensional rational homotopy (a *rationaly elliptic space*) such that  $H^{odd}(X, \mathbb{Q}) = 0$ . This type of space features in the following well-known conjecture of Halperin (cf. [FHT01, p.516]):

**Conjecture 2.10.** *Suppose  $X$  is an  $F_0$ -space. Then any fibration  $X \rightarrow E \rightarrow B$  of simply connected spaces is TNCZ, that is, the fibre inclusion  $j: X \rightarrow E$  induces a surjection on rational cohomology.*

The rational homotopy of  $\text{map}(X, X; 1)$  for  $X$  an  $F_0$ -space is directly related to Conjecture 2.10 (see [Mei82] for details). The following result extends [Mei82, Prop.2.6] (also compare [Gri94, Cor.4.6] and [Hau93, Th.B]):

**Corollary 2.11.** *Let  $f: X \rightarrow Y$  be a map between  $F_0$ -spaces. Then for  $r \geq 1$ ,*

$$\pi_{2r}(\text{map}(X, Y; f)) \otimes \mathbb{Q} \cong \text{Der}_{2r}(H^*(Y, \mathbb{Q}), H^*(X, \mathbb{Q}); H(f)).$$

*Proof.* The argument given by Grivel—for the case in which  $Y = X$  and  $f = 1_X$ —can be used word for word to show

$$H_{2r}(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)) \cong \text{Der}_{2r}(H^*(Y, \mathbb{Q}), H^*(X, \mathbb{Q}); H(f)).$$

The result now follows from Theorem 2.1. □

### 3. DERIVATION SPACES AND LONG EXACT SEQUENCES

In this section, we identify the rationalized long exact homotopy sequences of the maps  $f: X \rightarrow Y$  and  $f_*: \text{map}(X, X; 1) \rightarrow \text{map}(X, Y; f)$ , and hence the rationalized  $G$ -sequence of  $f$ . Our identifications flow from the following observation: Suppose given vector space homomorphisms  $\phi_n: A_n \rightarrow B_n$  for each  $n$ . Then, up to equivalence, *there is a unique way to fit these into a three-term long exact sequence*. Since we rely on this observation for our basic results, we make a formal statement of the fact.

**Lemma 3.1.** *Suppose given equivalences of vector space homomorphisms*

$$\begin{array}{ccc} A_n & \xrightarrow{\phi_n} & B_n \\ \cong \downarrow \alpha_n & & \cong \downarrow \beta_n \\ A'_n & \xrightarrow{\phi'_n} & B'_n \end{array}$$

for each  $n$ . Then for any two long exact sequences of vector spaces containing the  $\phi_n$  and the  $\phi'_n$  thus

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & A_n & \xrightarrow{\phi_n} & B_n & \xrightarrow{p_n} & C_n & \longrightarrow & \cdots \\ & & \cong \downarrow \gamma_{n+1} & & \cong \downarrow \alpha_n & & \cong \downarrow \beta_n & & \cong \downarrow \gamma_n & & \\ \cdots & \longrightarrow & C'_{n+1} & \xrightarrow{\partial'_{n+1}} & A'_n & \xrightarrow{\phi'_n} & B'_n & \xrightarrow{p'_n} & C'_n & \longrightarrow & \cdots \end{array},$$

there exist isomorphisms  $\gamma_n: C_n \rightarrow C'_n$  that make the long exact sequences equivalent.

*Proof.* We define a map  $\gamma_n$  as follows: Decompose  $C_n$  and  $C'_n$  as  $C_n \cong \text{im}(p_n) \oplus D_n$  and  $C'_n \cong \text{im}(p'_n) \oplus D'_n$ , where  $D_n$  and  $D'_n$  are complements. For  $x = p_n(y) \in \text{im}(p_n)$ , define  $\gamma_n(x) = g'_n \circ \beta_n(y)$ . For  $x \in D_n$ , set  $\gamma_n(x) = y'$ , where  $y' \in D'_n$  is

such that  $\partial'_n(y'_n) = \alpha_{n-1} \circ \partial_n(x)$ . It is straightforward to check that  $\gamma_n$  is a well-defined isomorphism. Indeed, it restricts to give isomorphisms  $\text{im}(p_n) \cong \text{im}(p'_n)$  and  $D_n \cong D'_n$ . The required commutativity properties follow immediately.  $\square$

There is nothing remarkable in this observation. It is important, nonetheless, since it allows us to choose descriptions of the long exact homotopy sequences that we need in whatever way is most convenient for our purposes.

We begin with the long exact homotopy sequence of the map  $f_*: \text{map}(X, X; 1) \rightarrow \text{map}(X, Y; f)$ . From Theorem 2.1, we see that the map this induces on rational homotopy groups can be identified with  $H((\mathcal{M}_f)^*)$ . Since this map is a homomorphism induced on homology by a map of chain complexes, there is a standard way to fit it into a long exact sequence, which we now describe.

Given a map  $\phi: A \rightarrow B$  of chain complexes, define a *relative* chain complex  $\text{Rel}_*(\phi)$  as follows:  $\text{Rel}_n(\phi) = A_{n-1} \oplus B_n$ , with differential  $\delta_\phi$  of degree  $-1$  given by  $\delta_\phi(a, b) = (\delta_A(a), \delta_B(b) - \phi(a))$ . Now define chain maps  $J: B_n \rightarrow \text{Rel}_n(\phi)$  and  $P: \text{Rel}_n(\phi) \rightarrow A_{n-1}$  by  $J(b) = (0, b)$  and  $P(a, b) = a$ . On passing to homology, we obtain a long exact sequence of the following form:

$$\cdots \longrightarrow H_{n+1}(\text{Rel}(\phi)) \xrightarrow{H(P)} H_n(A) \xrightarrow{H(\phi)} H_n(B) \xrightarrow{H(J)} H_n(\text{Rel}(\phi)) \longrightarrow \cdots,$$

in which  $H_n(\text{Rel}(\phi))$  denotes the  $n$ th homology of the chain complex  $\text{Rel}_*(\phi)$ . We refer to this long exact sequence as the *long exact homology sequence of the map  $\phi$* . Furthermore, this construction is natural. For suppose given a commutative square of DG vector spaces

$$(4) \quad \begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \alpha \downarrow & & \downarrow \beta \\ A' & \xrightarrow{\phi'} & B' \end{array}$$

Then the obvious map  $(\alpha, \beta): \text{Rel}_*(\phi) \rightarrow \text{Rel}_*(\phi')$  is a chain map that satisfies  $(\alpha, \beta)J = J'\beta$  and  $\alpha P = P'(\alpha, \beta)$ . Thus we obtain a homology ladder

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_{n+1}(\text{Rel}(\phi)) & \xrightarrow{H(P)} & H_n(A) & \xrightarrow{H(\phi)} & H_n(B) & \xrightarrow{H(J)} & H_n(\text{Rel}(\phi)) & \longrightarrow & \cdots \\ & & \downarrow H(\alpha, \beta) & & \downarrow H(\alpha) & & \downarrow H(\beta) & & \downarrow H(\alpha, \beta) & & \\ \cdots & \longrightarrow & H_{n+1}(\text{Rel}(\phi')) & \xrightarrow{H(P')} & H_n(A') & \xrightarrow{H(\phi')} & H_n(B') & \xrightarrow{H(J')} & H_n(\text{Rel}(\phi')) & \longrightarrow & \cdots \end{array}$$

In particular, we can apply this construction to the map of chain complexes

$$(5) \quad (\mathcal{M}_f)^*: \text{Der}(\mathcal{M}_X, \mathcal{M}_X; 1) \rightarrow \text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)$$

induced by the minimal model  $\mathcal{M}_f: \mathcal{M}_Y \rightarrow \mathcal{M}_X$  of the map  $f: X \rightarrow Y$ .

To display the long exact sequence that appears in the following result, we condense some of our notation as in the proof of Theorem 2.1. We use  $\text{Der}(\mathcal{M}_X)$  to denote  $\text{Der}(\mathcal{M}_X, \mathcal{M}_X; 1)$ , and  $\text{Der} \mathcal{M}(Y, X; f)$  to denote  $\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)$ .

**Theorem 3.2.** *The long exact sequence induced by*

$$f_*: \text{map}(X, X; 1) \rightarrow \text{map}(X, Y; f)$$

on rational homotopy groups is equivalent to the long exact homology sequence of the map (5). Specifically, this is a long exact sequence

$$\begin{array}{ccc}
 & \cdots & \xrightarrow{H(J)} & H_{n+1}(\text{Rel}((\mathcal{M}_f)^*)) \\
 & \searrow & \xrightarrow{H(P)} & \\
 H_n(\text{Der}(\mathcal{M}_X)) & \xrightarrow{H((\mathcal{M}_f)^*)} & H_n(\text{Der } \mathcal{M}(Y, X; f)) & \xrightarrow{H(J)} \cdots
 \end{array}$$
  

$$\begin{array}{ccc}
 & \cdots & \xrightarrow{H(J)} & H_3(\text{Rel}((\mathcal{M}_f)^*)) \\
 & \searrow & \xrightarrow{H(P)} & \\
 H_2(\text{Der}(\mathcal{M}_X)) & \xrightarrow{H((\mathcal{M}_f)^*)} & H_2(\text{Der } \mathcal{M}(Y, X; f)) &
 \end{array}$$

in which  $\text{Rel}_*(\mathcal{M}_f)^*$  is the relative chain complex of the map (5), as described above.

*Proof.* By Theorem 2.3 we may assume  $f: X \rightarrow Y$  is map of rational spaces. Theorem 2.1 gives equivalences of vector space maps

$$\begin{array}{ccc}
 \pi_n(\text{map}(X, X; 1)) & \xrightarrow{(f_*)\#} & \pi_n(\text{map}(X, Y; f)) \\
 \Phi_1 \downarrow \cong & & \cong \downarrow \Phi_f \\
 H_n(\text{Der}(\mathcal{M}_X, \mathcal{M}_X; 1)) & \xrightarrow{H((\mathcal{M}_f)^*)} & H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)),
 \end{array}$$

for each  $n \geq 2$ . The top horizontal maps are contained in the long exact sequence induced by  $f_*: \text{map}(X, X; 1) \rightarrow \text{map}(X, Y; f)$  on rational homotopy groups. The bottom horizontal maps are contained in the long exact homology sequence of the map (5). From Lemma 3.1, these sequences are equivalent.  $\square$

*Remark 3.3.* When we refer to the long exact homotopy sequence of a map, we mean this in the sense of [Hil65, Chaps.3,4]: Recall that given a map  $f: X \rightarrow Y$ , this sequence is as follows:

$$\cdots \rightarrow \pi_n(X) \xrightarrow{f\#} \pi_n(Y) \rightarrow \pi_n(f) \rightarrow \pi_{n-1}(X) \rightarrow \cdots \rightarrow \pi_2(X) \xrightarrow{f\#} \pi_2(Y).$$

If  $f$  is the inclusion of a subspace, then the groups  $\pi_n(f)$  are just the usual homotopy groups of a pair. Generally,  $\pi_n(f)$  is defined as homotopy classes of pairs  $(g_1, g_2)$  such that the diagram

$$\begin{array}{ccc}
 S^{n-1} & \xrightarrow{g_1} & X \\
 \downarrow & & \downarrow f \\
 CS^{n-1} & \xrightarrow{g_2} & Y
 \end{array}$$

commutes. Since  $\pi_2(f)$  is not necessarily abelian and we are interested in rationalizing this sequence, we stop at  $\pi_2(Y)$ . On the other hand, one can convert  $f$  into a fibration and use the corresponding long exact sequence in homotopy. Either approach suits our purposes and indeed the same sequence of homotopy groups

results from either. From the above, we see that if  $\mathcal{F}$  denotes the homotopy fibre of the map  $f_*: \text{map}(X, X; 1) \rightarrow \text{map}(X, Y; f)$ , then for  $n \geq 2$  we have

$$\pi_{n+1}(f^*) \otimes \mathbb{Q} \cong \pi_n(\mathcal{F}) \otimes \mathbb{Q} \cong H_{n+1}(\text{Rel}((\mathcal{M}_f)^*)),$$

where  $\text{Rel}_*((\mathcal{M}_f)^*)$  is the relative chain complex of the map (5).

The preceding result specializes to give a description of the long exact sequence induced by a general map on rational homotopy groups. The minimal model  $\mathcal{M}_f: \mathcal{M}_Y \rightarrow \mathcal{M}_X$  of the map  $f: X \rightarrow Y$  also induces a map of chain complexes

$$(6) \quad (\mathcal{M}_f)^*: \text{Der}(\mathcal{M}_X, \mathbb{Q}; \varepsilon) \rightarrow \text{Der}(\mathcal{M}_Y, \mathbb{Q}; \varepsilon).$$

**Theorem 3.4.** *The long exact sequence induced by  $f: X \rightarrow Y$  on rational homotopy groups is equivalent to the long exact homology sequence of the map (6). Specifically, this is a long exact sequence*

$$\begin{array}{ccccccc} & & & \dots & \xrightarrow{H(J)} & H_{n+1}(\text{Rel}((\mathcal{M}_f)^*)) & \\ & & & & & \uparrow & \\ & & & & & \xrightarrow{H(P)} & \\ H_n(\text{Der}(\mathcal{M}_X, \mathbb{Q}; \varepsilon)) & \xrightarrow{H((\mathcal{M}_f)^*)} & H_n(\text{Der}(\mathcal{M}_Y, \mathbb{Q}; \varepsilon)) & \xrightarrow{H(J)} & \dots & & \\ & & & & & \uparrow & \\ & & & \dots & \xrightarrow{H(J)} & H_3(\text{Rel}((\mathcal{M}_f)^*)) & \\ & & & & & \uparrow & \\ & & & & & \xrightarrow{H(P)} & \\ H_2(\text{Der}(\mathcal{M}_X, \mathbb{Q}; \varepsilon)) & \xrightarrow{H((\mathcal{M}_f)^*)} & H_2(\text{Der}(\mathcal{M}_Y, \mathbb{Q}; \varepsilon)) & & & & \end{array}$$

in which  $\text{Rel}_*((\mathcal{M}_f)^*)$  is the relative chain complex of the map (6).

*Proof.* Argue exactly as in the proof of Theorem 3.2, using Theorem 2.1 and Lemma 3.1.  $\square$

*Remark 3.5.* There is already a standard way to describe the long exact sequence induced by a map on rational homotopy groups, using minimal models. This uses the notion of a so-called K-S model of the map  $\mathcal{M}_f: \mathcal{M}_Y \rightarrow \mathcal{M}_X$  [FHT01, Sec.15(d)]. The description we give above, however, is better suited to our purposes. Note that if  $F$  denotes the homotopy fibre of the map  $f: X \rightarrow Y$ , then for  $n \geq 2$  we have

$$\pi_{n+1}(f) \otimes \mathbb{Q} \cong \pi_n(F) \otimes \mathbb{Q} \cong H_{n+1}(\text{Rel}((\mathcal{M}_f)^*)),$$

where  $\text{Rel}_*((\mathcal{M}_f)^*)$  is the relative chain complex of the map (6). It is perhaps interesting to compare the description given in Theorem 3.4 to the standard description of the long exact sequence in rational homotopy groups of a fibration.

We now turn our attention to the  $G$ -sequence, and identify it within our current framework.

Suppose given a DG algebra map  $\phi: A \rightarrow B$ . Starting from this map, we can construct the following commutative square of DG vector spaces:

$$\begin{array}{ccc} \text{Der}_*(B, B; 1) & \xrightarrow{\phi^*} & \text{Der}_*(A, B; \phi) \\ \varepsilon_* \downarrow & & \downarrow \varepsilon_* \\ \text{Der}_*(B, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{\phi}^*} & \text{Der}_*(A, \mathbb{Q}; \varepsilon). \end{array}$$

In this diagram,  $\varepsilon$  denotes the augmentation of either  $A$  or  $B$ , and we have used a decoration to distinguish the lower horizontal map from the upper. On passing to homology and using the naturality of the relative chain complex construction, we obtain the following homology ladder:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{H(J)} & H_{n+1}(\mathrm{Rel}(\phi^*)) & \xrightarrow{H(P)} & H_n(\mathrm{Der}(B, B; 1)) & \xrightarrow{H(\phi^*)} & H_n(\mathrm{Der}(A, B; \phi)) \cdots \\ & & \downarrow H(\varepsilon_*, \varepsilon_*) & & \downarrow H(\varepsilon_*) & & \downarrow H(\varepsilon_*) \\ \cdots & \xrightarrow{H(J)} & H_{n+1}(\mathrm{Rel}(\widehat{\phi}^*)) & \xrightarrow{H(\widehat{P})} & H_n(\mathrm{Der}(B, \mathbb{Q}; \varepsilon)) & \xrightarrow{H(\widehat{\phi}^*)} & H_n(\mathrm{Der}(A, \mathbb{Q}; \varepsilon)) \cdots \end{array}$$

for  $n \geq 2$ . We supplement Definition 2.8 with the following vocabulary.

**Definition 3.6.** Suppose  $\phi: A \rightarrow B$  is a map of DG algebras. For  $n \geq 3$  we define the  $n$ th relative evaluation subgroup of  $\phi$  as the image of the map

$$H(\varepsilon_*, \varepsilon_*): H_n(\mathrm{Rel}(\phi^*)) \rightarrow H_n(\mathrm{Rel}(\widehat{\phi}^*)).$$

We denote it by  $G_n^{rel}(A, B; \phi)$ . Then the image of the upper long exact sequence in the lower, of the ladder above, gives a (not necessarily exact) sequence

$$\begin{array}{c} \cdots \xrightarrow{H(\widehat{J})} G_{n+1}^{rel}(A, B; \phi) \\ \downarrow H(\widehat{P}) \\ G_n(B) \xrightarrow{H(\widehat{\phi}^*)} G_n(A, B; \phi) \xrightarrow{H(\widehat{J})} G_n^{rel}(A, B; \phi) \\ \downarrow H(\widehat{P}) \\ \cdots \xrightarrow{H(\widehat{J})} G_3^{rel}(A, B; \phi) \\ \downarrow H(\widehat{P}) \\ G_2(B) \xrightarrow{H(\widehat{\phi}^*)} G_2(A, B; \phi) \end{array}$$

We refer to this sequence as *the  $G$ -sequence of the map  $\phi: A \rightarrow B$* .

All of the above can be applied to the minimal model  $\mathcal{M}_f: \mathcal{M}_Y \rightarrow \mathcal{M}_X$  of the map  $f: X \rightarrow Y$ . By doing so, and then collecting together previous results, we obtain the following result.

**Theorem 3.7.** *The rationalization of the  $G$ -sequence of the map  $f: X \rightarrow Y$ , as far as the term  $G_2(Y, X; f)$ , is equivalent to the  $G$ -sequence of the corresponding map of Sullivan models.*

*Proof.* Starting from the cube displayed in the proof of Theorem 2.1, we extend each of the four left-to-right maps into their respective long exact sequences. This is then completed into an equivalence of ladders, by defining isomorphisms  $\gamma_n$  and  $\widehat{\gamma}_n$  to give a commutative square

$$\begin{array}{ccc} \pi_n(f_*) & \xrightarrow{\gamma_n} & H_n(\mathrm{Rel}(\phi^*)) \\ \omega_\# \downarrow & & \downarrow H(\varepsilon_*, \varepsilon_*) \\ \pi_n(f) & \xrightarrow{\widehat{\gamma}_n} & H_n(\mathrm{Rel}(\widehat{\phi}^*)) \end{array}$$



$\phi: A \rightarrow B$ , let  $\tilde{\phi}: A \rightarrow \tilde{B}$  be the DG algebra map which agrees with  $\phi$  in positive degrees and vanishes in degree zero.

A DG algebra map  $\phi: A \rightarrow B$  together with the short exact augmentation sequence

$$0 \longrightarrow \tilde{B} \xrightarrow{i} B \xrightarrow{\varepsilon} \mathbb{Q} \longrightarrow 0$$

of DG algebras gives rise to the short exact sequence of DG vector spaces

$$0 \longrightarrow \mathrm{Der}_*(A, \tilde{B}; \tilde{\phi}) \xrightarrow{i_*} \mathrm{Der}_*(A, B; \phi) \xrightarrow{\varepsilon_*} \mathrm{Der}_*(A, \mathbb{Q}; \varepsilon) \longrightarrow 0.$$

This in turn gives a long exact sequence on homology, in the usual way, of the form (8)

$$\cdots \xrightarrow{\Delta_*} H_n(\mathrm{Der}(A, \tilde{B}; \tilde{\phi})) \xrightarrow{H(i_*)} H_n(\mathrm{Der}(A, B; \phi)) \xrightarrow{H(\varepsilon_*)} H_n(\mathrm{Der}(A, \mathbb{Q}; \varepsilon)) \cdots$$

for  $n \geq 2$ . Call this sequence the *long exact derivation homology sequence* of the DG algebra map  $\phi: A \rightarrow B$ .

To display the sequence that appears in the following result, we have once again condensed some of our notation. We use  $\mathrm{Der} \mathcal{M}(Y, X; f)$  and  $\mathrm{Der}(\mathcal{M}_Y, \mathbb{Q})$ , to denote  $\mathrm{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)$ , respectively  $\mathrm{Der}(\mathcal{M}_Y, \mathbb{Q}; \varepsilon)$ , as in the proof of Theorem 2.1. In addition, here we use  $\mathrm{Der} \mathcal{M}(Y, \tilde{X}; \tilde{f})$  to denote  $\mathrm{Der}(\mathcal{M}_Y, \tilde{\mathcal{M}}_X; \tilde{\mathcal{M}}_f)$ .

**Theorem 3.10.** *The long exact sequence induced by the evaluation fibration (7) on rational homotopy groups is equivalent to the long exact derivation homology sequence (8) of the map  $\mathcal{M}_f: \mathcal{M}_Y \rightarrow \mathcal{M}_X$ . Specifically, this is a long exact sequence*

$$\begin{array}{ccc} \cdots & \xrightarrow{H(\varepsilon_*)} & H_{n+1}(\mathrm{Der}(\mathcal{M}_Y, \mathbb{Q})) \\ & \searrow \Delta_* & \downarrow \\ H_n(\mathrm{Der} \mathcal{M}(Y, \tilde{X}; \tilde{f})) & \xrightarrow{H(i_*)} & H_n(\mathrm{Der} \mathcal{M}(Y, X; f)) \xrightarrow{H(\varepsilon_*)} \cdots \\ & \searrow \Delta_* & \downarrow \\ \cdots & \xrightarrow{H(\varepsilon_*)} & H_3(\mathrm{Der}(\mathcal{M}_Y, \mathbb{Q})) \\ & \searrow \Delta_* & \downarrow \\ H_2(\mathrm{Der} \mathcal{M}(Y, \tilde{X}; \tilde{f})) & \xrightarrow{H(i_*)} & H_2(\mathrm{Der} \mathcal{M}(Y, X; f)) \xrightarrow{H(\varepsilon_*)} H_2(\mathrm{Der}(\mathcal{M}_Y, \mathbb{Q})). \end{array}$$

**Corollary 3.11.** *Let  $X$  and  $Y$  be simply connected spaces with  $X$  finite. Then the rational homotopy groups of the based function space  $\mathrm{map}_*(X, Y; f)$  are given by*

$$\pi_n(\mathrm{map}_*(X, Y; f)) \otimes \mathbb{Q} \cong H_n(\mathrm{Der}(\mathcal{M}_Y, \tilde{\mathcal{M}}_X; \tilde{\mathcal{M}}_f))$$

for  $n \geq 2$ .

#### 4. EXAMPLES, COMPUTATIONS, AND FURTHER CONSEQUENCES

We illustrate the effectiveness of the framework established in the previous two sections with examples. First, we give a composite example that includes specific computation of many of the ingredients of the above. Our example is one in which the  $G$ -sequence of a map fails to be exact (after rationalization) at each of the three types of term that occur.

We begin with some notational conventions. Suppose that  $(A, d_A)$  and  $(B, d_B)$  are minimal algebras, with  $A = \Lambda(W)$  and  $B = \Lambda(V)$  for suitable graded vector spaces  $W$  and  $V$ . Let  $\phi: A \rightarrow B$  be a fixed DG algebra map. Since any linear map  $W \rightarrow B$  extends in a unique way to a  $\phi$ -derivation, we can view the space  $\text{Hom}_*(W, B)$  of negative degree linear maps as a subspace of  $\text{Der}_*(A, B; \phi)$ . (Of course, in degree 1 we must restrict to cycles!) With this view  $\text{Hom}_*(W, B) = \text{Der}_*(A, B; \phi)$  as graded spaces although the differential depends on the derivation structure. This point of view, whereby a derivation is specified on generators and then extended to the whole algebra, is one that we will invariably adopt in any practical calculation.

Now suppose given a basis  $\{w_1, w_2, w_3, \dots\}$  for  $W$  and an element  $P \in B$  with  $|P| < |w_i|$  we will write  $P\partial w_i$  for the  $\phi$ -derivation carrying  $w_i$  to  $P$  and vanishing on the other  $w_j$ . Thus any derivation can be expressed as a sum  $\sum_i P_i \partial w_i$ . When  $B = \mathbb{Q}$ , we write  $w_j^*$  rather than  $1\partial w_j$  for the derivation dual to  $w_j$ .

**Example 4.1.** Let  $f = (f_1, f_2): \mathbb{H}P^2 \rightarrow S^8 \times \mathbb{H}P^4$  be the map with coordinate functions  $f_1: \mathbb{H}P^2 \rightarrow S^8$  obtained by pinching out the bottom cell and  $f_2: \mathbb{H}P^2 \rightarrow \mathbb{H}P^4$  the inclusion. We will use our framework from above to compute various terms from the long exact sequences corresponding to Theorem 3.2 and Theorem 3.4. Denote  $\mathbb{H}P^2$  by  $X$  and  $S^8 \times \mathbb{H}P^4$  by  $Y$ , thus  $f: X \rightarrow Y$ . Our computation will show, using Theorem 3.7, that the  $G$ -sequence of  $f$  is non-exact at the terms  $G_4(Y, X)$ ,  $G_8^{rel}(Y, X)$ , and  $G_{11}(X)$ .

First,  $\mathcal{M}_X = \Lambda(x_4, x_{11})$ , with differential given on generators by  $d(x_4) = 0$ , and  $d(x_{11}) = x_4^3$ , and  $\mathcal{M}_Y = \Lambda(y_8, y_{15}, y_4, y_{19})$  with differential  $d(y_8) = 0$ ,  $d(y_{15}) = y_8^2$ ,  $d(y_4) = 0$ , and  $d(y_{19}) = y_4^5$ . In both models, subscripts denote degrees. Then the Sullivan model of  $f$ , which we denote by  $\phi: \mathcal{M}_Y \rightarrow \mathcal{M}_X$ , is given on generators by  $\phi(y_8) = x_4^2$ ,  $\phi(y_{15}) = x_4 x_{11}$ ,  $\phi(y_4) = x_4$ , and  $\phi(y_{19}) = x_4^2 x_{11}$ .

For degree reasons,  $\text{Der}_i(\mathcal{M}_X, \mathcal{M}_X; 1) = 0$  unless  $i = 3, 4, 7$  or  $11$ . Furthermore,  $\text{Der}_*(\mathcal{M}_X, \mathcal{M}_X; 1)$  is spanned by the derivations  $x_4^2 \partial x_{11}$ ,  $x_4^*$ ,  $x_4 \partial x_{11}$ , and  $x_{11}^*$  of degree 3, 4, 7, and 11 respectively. An easy computation reveals that  $\delta(x_4^*) = -3x_4^2 \partial x_{11}$ , but that  $x_4 \partial x_{11}$  and  $x_{11}^*$  are both (non-exact) cocycles. It follows from Theorem 2.1 that

$$\pi_i(\text{map}(X, X; 1)) \otimes \mathbb{Q} \cong H_i(\text{Der}(\mathcal{M}_X, \mathcal{M}_X; 1)) = \begin{cases} \mathbb{Q} & \text{if } i = 7, 11 \\ 0 & \text{otherwise} \end{cases}$$

Further, it is evident that

$$\pi_i(X) \otimes \mathbb{Q} \cong H_i(\text{Der}(\mathcal{M}_X, \mathbb{Q}; \varepsilon)) = \begin{cases} \mathbb{Q} & \text{if } i = 4, 11 \\ 0 & \text{otherwise} \end{cases}$$

with the non-zero cohomology in degrees 4 and 11 generated by cocycles  $x_4^*$  and  $x_{11}^*$ , respectively. Given these generators, we see that

$$H(\varepsilon_*): H_i(\text{Der}(\mathcal{M}_X, \mathcal{M}_X; 1)) \rightarrow H_i(\text{Der}(\mathcal{M}_X, \mathbb{Q}; \varepsilon)),$$

that is, the homomorphism  $\omega_{\#} \otimes \mathbb{Q}: \pi_i(\text{map}(X, X; 1)) \otimes \mathbb{Q} \rightarrow \pi_i(X) \otimes \mathbb{Q}$  induced by the evaluation map on rational homotopy groups, is an isomorphism in degree 11 and is zero in all other degrees. It follows from Theorem 2.1—see Corollary 2.6—that  $G_i(\mathcal{M}_X) = 0$  other than in degree 11, where we have  $G_{11}(\mathcal{M}_X) \cong \mathbb{Q}$ . Up to this point, our observations are both well-known, and also easily obtained by a number of standard methods.

We now show that the rationalized  $G$ -sequence is non-exact at the  $G_{11}(\mathcal{M}_X)$  term. Recall that this term of the  $G$ -sequence, together with its adjacent terms, is obtained from the diagram

$$\begin{array}{ccccc} \text{Rel}_{12}(\phi^*) & \xrightarrow{P} & \text{Der}_{11}(\mathcal{M}_X, \mathcal{M}_X; 1) & \xrightarrow{\phi^*} & \text{Der}_{11}(\mathcal{M}_Y, \mathcal{M}_X; \phi) \\ \downarrow (\varepsilon_*, \varepsilon_*) & & \downarrow \varepsilon_* & & \downarrow \varepsilon_* \\ \text{Rel}_{12}(\widehat{\phi}^*) & \xrightarrow{\widehat{P}} & \text{Der}_{11}(\mathcal{M}_X, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{\phi}^*} & \text{Der}_{11}(\mathcal{M}_Y, \mathbb{Q}; \varepsilon), \end{array}$$

by passing to homology and then considering the image of the top sequence in the bottom. A brute force calculation will display the result, but we opt to argue at a more general level so as to indicate some reason for non-exactness. It is evident that  $H(\widehat{\phi}^*) \circ H(\varepsilon_*)([x_{11}^*]) = 0 \in H_{11}(\text{Der}(\mathcal{M}_Y, \mathbb{Q}; \varepsilon))$ —indeed, this latter term is zero, since it is isomorphic to  $\pi_{11}(Y) \otimes \mathbb{Q} = 0$ . The key point for non-exactness here, however, is that in the top sequence we have  $H(\phi^*)([x_{11}^*]) \neq 0 \in H_{11}(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \phi))$ . In fact, a straightforward check shows that  $\phi^*(x_{11}^*) = x_4 \partial y_{15} + x_4^2 \partial y_{19}$ . Since  $\text{Der}_{12}(\mathcal{M}_Y, \mathcal{M}_X; \phi) = 0$ , there are no non-zero boundaries in degree 11 and hence  $H(\phi^*)([x_{11}^*]) \neq 0$ . Consequently,  $[x_{11}^*]$  cannot be in the image of  $H(P)$  in the top sequence. Therefore, since  $H(\varepsilon_*)$  is an isomorphism in degree 11,  $H(\varepsilon_*)([x_{11}^*]) = [x_{11}^*]$  cannot be in the image of  $H(\widehat{P}) \circ H(\varepsilon_*, \varepsilon_*)$ . It follows from these facts that  $[x_{11}^*] \in G_{11}(\mathcal{M}_X)$  is a non-zero element in the kernel of  $H(\widehat{\phi}^*)$  and yet is not in the image of  $H(\widehat{P}): G_{12}^{rel}(\mathcal{M}_Y, \mathcal{M}_X; \phi) \rightarrow G_{11}(\mathcal{M}_X)$ .

Next consider the term  $G_4(\mathcal{M}_Y, \mathcal{M}_X; \phi)$ : Before passing to homology, the relevant diagram is the following:

$$\begin{array}{ccccc} \text{Der}_4(\mathcal{M}_X, \mathcal{M}_X; 1) & \xrightarrow{\phi^*} & \text{Der}_4(\mathcal{M}_Y, \mathcal{M}_X; \phi) & \xrightarrow{J} & \text{Rel}_4(\phi^*) \\ \downarrow \varepsilon_* & & \downarrow \varepsilon_* & & \downarrow (\varepsilon_*, \varepsilon_*) \\ \text{Der}_4(\mathcal{M}_X, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{\phi}^*} & \text{Der}_4(\mathcal{M}_Y, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{J}} & \text{Rel}_4(\widehat{\phi}^*), \end{array}$$

The derivation  $\theta = y_4^* + 5x_4x_{11}\partial y_{19} \in \text{Der}_4(\mathcal{M}_Y, \mathcal{M}_X; \phi)$  is a cocycle, as is easily checked. Under  $H(\varepsilon_*): H_4(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \phi)) \rightarrow H_4(\text{Der}(\mathcal{M}_Y, \mathbb{Q}; \varepsilon))$ , we have  $H(\varepsilon_*)([\theta]) = [y_4^*] \neq 0$ . Since  $[y_4^*] = H(\widehat{\phi}^*)([x_4^*])$ , it follows that  $H(\widehat{J})([y_4^*]) = 0$ . As we noted above, however,  $G_4(\mathcal{M}_X) = 0$ . Therefore,  $[y_4^*] \in G_4(\mathcal{M}_Y, \mathcal{M}_X; \phi)$  is a non-zero element in the kernel of  $H(\widehat{J}): G_4(\mathcal{M}_Y, \mathcal{M}_X; \phi) \rightarrow G_4^{rel}(\mathcal{M}_Y, \mathcal{M}_X; \phi)$  that is not in the image of  $H(\widehat{\phi}^*): G_4(\mathcal{M}_X) \rightarrow G_4(\mathcal{M}_Y, \mathcal{M}_X; \phi)$ .

Finally, consider the term  $G_8^{rel}(\mathcal{M}_Y, \mathcal{M}_X; \phi)$ : Here, we begin with the following diagram:

$$\begin{array}{ccccc} \text{Der}_8(\mathcal{M}_Y, \mathcal{M}_X; \phi) & \xrightarrow{J} & \text{Rel}_8(\phi^*) & \xrightarrow{P} & \text{Der}_7(\mathcal{M}_X, \mathcal{M}_X; 1) \\ \downarrow \varepsilon_* & & \downarrow (\varepsilon_*, \varepsilon_*) & & \downarrow \varepsilon_* \\ \text{Der}_8(\mathcal{M}_Y, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{J}} & \text{Rel}_8(\widehat{\phi}^*) & \xrightarrow{\widehat{P}} & \text{Der}_7(\mathcal{M}_X, \mathbb{Q}; \varepsilon) \end{array}$$

We find that  $(-2x_4\partial x_{11}, y_8^* + 2x_{11}\partial y_{19}) \in \text{Rel}_8(\phi^*)$  is a cocycle that has non-zero image in  $H(\varepsilon_*, \varepsilon_*): H_8(\text{Rel}(\phi^*)) \rightarrow H_8(\text{Rel}(\widehat{\phi}^*))$ . Furthermore, it is evident that  $H(\widehat{P}) \circ H(\varepsilon_*, \varepsilon_*)((-2x_4\partial x_{11}, y_8^* + 2x_{11}\partial y_{19})) = H(\widehat{P})([(0, y_8^*)]) = 0$ —indeed,

$H_7(\text{Der}(\mathcal{M}_X, \mathbb{Q}; \varepsilon)) \cong \pi_7(X) \otimes \mathbb{Q} = 0$ . To see that  $[(0, y_8^*)]$  is not in the image of  $H(\widehat{J}): G_8(\mathcal{M}_Y, \mathcal{M}_X; \phi) \rightarrow G_8^{rel}(\mathcal{M}_Y, \mathcal{M}_X; \phi)$ , we will compute  $G_8(\mathcal{M}_Y, \mathcal{M}_X; \phi)$  to be zero.

A general derivation  $\gamma \in \text{Der}_8(\mathcal{M}_Y, \mathcal{M}_X; \phi)$  can be written as

$$\gamma = \lambda_1 y_8^* + \lambda_2 x_{11} \partial y_{19},$$

for coefficients  $\lambda_i \in \mathbb{Q}$ . To find the cocycles of this form, we first observe that  $\delta(\gamma)$  must vanish on the generators  $y_8$  and  $y_4$ , for degree reasons. On  $y_{15}$ , we compute as follows:

$$\begin{aligned} \delta(\gamma)(y_{15}) &= (d(\gamma) - (\gamma)d)(y_{15}) \\ &= 0 - \gamma(y_8^2) \\ &= -2\gamma(y_8)\phi(y_8) \\ &= -2\lambda_1 x_4^2. \end{aligned}$$

Thus, if  $\gamma$  is a cocycle, then we must have  $\lambda_1 = 0$ . A similar computation shows that  $\delta(\gamma)(y_{19}) = \lambda_2 x_4^3$ , and thus that there are no non-zero cocycles in  $\text{Der}_8(\mathcal{M}_Y, \mathcal{M}_X; \phi)$ . In summary, we have computed that

$$\pi_8(\text{map}(X, Y; f)) \otimes \mathbb{Q} \cong H_8(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \phi)) = 0.$$

This last part of our computation is easily confirmed using the result of Corollary 2.11. It follows, of course, that  $G_8(\mathcal{M}_Y, \mathcal{M}_X; \phi) = 0$  and, in particular, that  $[(0, y_8^*)]$  is not in the image of  $H(\widehat{J}): G_8(\mathcal{M}_Y, \mathcal{M}_X; \phi) \rightarrow G_8^{rel}(\mathcal{M}_Y, \mathcal{M}_X; \phi)$ .

*Remarks 4.2.* The first example of a non-exact  $G$ -sequence, given in [PW97], was in dimension one. A higher dimensional example was produced later in [PSW98]. With the approach illustrated in the above example, it is straightforward to produce higher dimensional examples of non-exact rationalized  $G$ -sequences. Observe that non-exact ordinary  $G$ -sequences are produced as a result, since if a sequence of abelian groups is not exact after tensoring with  $\mathbb{Q}$  then it was not exact to begin with. Thus, non-exactness rationally implies non-exactness integrally.

Although the  $G$ -sequence in general is non-exact, there are certain situations in which it is perfectly well behaved, at least after rationalization. We now mention one such situation. We say that a space  $X$  is an  $H_0$ -space if its rational cohomology algebra is a free graded algebra (exterior algebra on the odd-degree generators tensored with a polynomial algebra on the even-degree generators). Equivalently, we could require that  $X$  be an  $H$ -space after rationalization, whence such a space is also referred to as a *rational  $H$ -space*. Recall also the definition of an  $F_0$ -space from before Corollary 2.11.

**Theorem 4.3.** *Let  $f: X \rightarrow Y$  be any map from an  $F_0$ -space  $X$  to an  $H_0$ -space  $Y$  that induces the zero homomorphism on rational homotopy groups. Then the rationalized  $G$ -sequence splits into short exact sequences*

$$0 \longrightarrow G_{n+1}(Y, X; f) \otimes \mathbb{Q} \xrightarrow{J} G_{n+1}^{rel}(Y, X; f) \otimes \mathbb{Q} \xrightarrow{P} G_n(X) \otimes \mathbb{Q} \longrightarrow 0$$

for each  $n \geq 2$ .

*Proof.* Our assumption that  $f_{\#} \otimes \mathbb{Q} = 0$  means that the long exact sequence induced by  $f$  on rational homotopy groups splits. Furthermore, since  $Y$  is an  $H_0$ -space, we

have  $G_n(Y) \otimes \mathbb{Q} = \pi_n(Y) \otimes \mathbb{Q}$  for each  $n$ . It follows that  $G_n(Y, X) \otimes \mathbb{Q} = \pi_n(Y) \otimes \mathbb{Q}$  for each  $n$ . From these general considerations, we have short exact sequences

$$(9) \quad 0 \longrightarrow G_{n+1}(Y, X; f) \otimes \mathbb{Q} \xrightarrow{J} \pi_{n+1}(f) \otimes \mathbb{Q} \xrightarrow{P} \pi_n(X) \otimes \mathbb{Q} \longrightarrow 0$$

for  $n \geq 2$ . To sharpen this to the statement of the theorem, we work within our minimal model framework.

We use some results of Halperin [Hal77], on the rational homotopy of an  $F_0$ -space. These state that  $X$  has minimal model  $(\mathcal{M}_X, d_X)$  of the form  $\mathcal{M}_X = \Lambda(x_1, \dots, x_n) \otimes \Lambda(y_1, \dots, y_n)$  where  $|x_i|$  is even,  $|y_j|$  is odd,  $d_X(x_i) = 0$  and  $d_X(y_j) \in \Lambda(x_1, \dots, x_n)$ . Furthermore, the cohomology is evenly graded, and any cocycle in  $I(y_1, \dots, y_n)$ , the ideal of  $\mathcal{M}_X$  generated by the  $y_i$ , is exact. It is well-known that the minimal model of an  $H_0$ -space takes the form  $\mathcal{M}_Y = \Lambda(z_1, z_2, \dots)$ , with trivial differential. The map  $f: X \rightarrow Y$  has Sullivan minimal model  $\phi: \mathcal{M}_Y \rightarrow \mathcal{M}_X$  that is determined by the  $\phi(z_i)$ . Since each  $z_i$  is a cocycle, it follows that each  $\phi(z_i) \in \mathcal{M}_X$  is a cocycle. From the results of Halperin mentioned above, we can write  $\phi(z_i) = \chi_i + d(\xi_i)$ , for suitable  $\chi_i \in \Lambda(x_1, \dots, x_n)$  and  $\xi_i \in I(y_1, \dots, y_n)$ . The assumption that  $f$  induces zero on rational homotopy groups translates into the further restriction that each  $\chi_i$  is decomposable.

The short exact sequences (9), translated into our derivation setting, correspond to short exact sequences

$$0 \rightarrow G_{n+1}(\mathcal{M}_Y, \mathcal{M}_X; \phi) \xrightarrow{H(\widehat{J})} H_{n+1}(\text{Rel}(\widehat{\phi}^*)) \xrightarrow{H(\widehat{P})} H_n(\text{Der}(\mathcal{M}_X, \mathbb{Q}; \varepsilon)) \rightarrow 0$$

We first prove that each map  $H(\widehat{P})$  restricts to give a surjection

$$H(\widehat{P}): G_{n+1}^{rel}(\mathcal{M}_Y, \mathcal{M}_X; \phi) \rightarrow G_n(\mathcal{M}_X).$$

The Gottlieb elements of  $\mathcal{M}_X$  are precisely the  $y_j^*$ , dual to the odd-degree generators of  $\mathcal{M}_X$ . This can be seen from the description of the Gottlieb elements given in Corollary 2.6, together with the facts about the minimal model of  $X$  recalled earlier. Now write  $\phi(z_i) = \chi_i + d(\xi_i)$  as above and, for each  $j$ , define a derivation

$$\theta_j = - \sum_i y_j^*(\xi_i) \partial z_i \in \text{Der}_{|y_j|+1}(\mathcal{M}_Y, \mathcal{M}_X; \phi).$$

Then  $\delta(\theta_j) = - \sum_i d(y_j^*(\xi_i)) \partial z_i$ , since the differential in  $\mathcal{M}_Y$  is trivial. On the other hand, we have

$$\begin{aligned} \phi^*(y_j^*)(z_i) &= y_i^*(\phi(z_i)) = y_i^*(\chi_i + d(\xi_i)) = 0 + y_i^*(d(\xi_i)) \\ &= -d(y_i^*(\xi_i)), \end{aligned}$$

with the last step following because  $y_j^*$  is a  $\delta$ -cycle. Consequently,  $(y_j^*, \theta_j) \in \text{Rel}_{|y_j|+1}(\phi^*)$  is a  $\delta_{\phi^*}$ -cycle. Since  $H(\widehat{P}) \circ H(\varepsilon_*, \varepsilon_*)(y_j^*, \theta_j) = y_j^*$ , it follows that  $H(\widehat{P})$  does restrict to the desired surjection.

The map  $H(\widehat{J})$  is injective on rational homotopy groups, as we have already observed, and therefore restricts to an injection in the rationalized  $G$ -Sequence. So it only remains to show exactness at the  $G_{n+1}^{rel}(Y, X; f)$  terms. Suppose that

$$H(\varepsilon_*, \varepsilon_*)([(\theta, \psi)]) \in \ker(H(\widehat{P}): G_{n+1}^{rel}(\mathcal{M}_Y, \mathcal{M}_X; \phi) \rightarrow G_n(\mathcal{M}_X)),$$

for some cocycle  $(\theta, \psi) \in \text{Rel}_{n+1}(\phi^*)$ . The fact that  $H(\widehat{P})([(\varepsilon_*(\theta), \varepsilon_*(\psi))]) = 0$  implies that  $\varepsilon_*(\theta) = 0$ . Thus  $(\varepsilon_*, \varepsilon_*)(\theta, \psi) = (0, \varepsilon_*(\psi))$ . Now define a derivation

$\bar{\psi} \in \text{Der}_{n+1}(\mathcal{M}_Y, \mathcal{M}_X; \phi)$  by setting  $\bar{\psi} = \psi$  on generators of  $\mathcal{M}_Y$  of degree  $n+1$  and  $\bar{\psi} = 0$  on all other generators of  $\mathcal{M}_Y$ . It is easily seen that  $\bar{\psi}$  is a cycle. Indeed,  $d_X \bar{\psi} = 0$  on all generators of  $\mathcal{M}_Y$ , since  $\bar{\psi}$  has non-zero image only in degree zero, and  $\bar{\psi} d_Y = 0$  since  $d_Y = 0$ . Thus  $H(\varepsilon_*)([\bar{\psi}]) \in G_{n+1}(\mathcal{M}_Y, \mathcal{M}_X; \phi)$  satisfies

$$H(\hat{J}) \circ H(\varepsilon_*)([\bar{\psi}]) = [(0, \varepsilon \circ \bar{\psi})] = [(0, \varepsilon \circ \psi)] = H(\varepsilon_*, \varepsilon_*)([(\theta, \psi)]).$$

That is,  $\ker(H(\hat{P})) \cap G_{n+1}^{\text{rel}}(\mathcal{M}_Y, \mathcal{M}_X; \phi) \subseteq H(\hat{J})(G_{n+1}(\mathcal{M}_Y, \mathcal{M}_X; \phi))$  and the rationalized  $G$ -sequence is exact at each  $G_{n+1}^{\text{rel}}(Y, X; f)$  term.  $\square$

*Remark 4.4.* Various conditions are known, under which the  $G$ -sequence of a map  $f: X \rightarrow Y$  is exact. For instance, it is exact when  $f$  is null-homotopic [LW93], and when  $f$  is a homotopy monomorphism [PW01]. The hypotheses of Theorem 4.3 are well-suited for rational homotopy theory. Both types of space are well-known, and it is easy to give examples to which the theorem applies. In fact, for fixed  $X$  and  $Y$ , the maps to which it applies are classified up to rational homotopy by the decomposable rational cohomology of  $X$  that occurs in those (even) degrees in which  $Y$  has a generator of rational cohomology. We emphasize that the  $H_0$ -space  $Y$  must be allowed to have polynomial generators in rational cohomology, and hence be infinite-dimensional, otherwise the theorem reduces to the case in which the map  $f$  is rationally null-homotopic. Furthermore, the hypothesis that  $f$  be zero on rational homotopy groups is necessary. For example, the map  $f: S^4 \rightarrow \mathbb{H}P^\infty$ , given by inclusion of the bottom cell, has a non-exact rationalized  $G$ -sequence, as is easily confirmed by computations similar to those of Example 4.1

Since the  $G$ -sequence of a map  $f: X \rightarrow Y$  is a boundary sequence, but not usually an exact sequence, it is natural to consider its homology. This gives the so-called  $\omega$ -homology of  $f$  [LW93]. In general, one obtains an  $\omega$ -homology group at each of the three types of term. In the following, we restrict our attention to the  $\omega$ -homology that occurs at the Gottlieb group term  $G_*(X)$ , denoted  $H_*^{\omega}(X, Y; f)$  in [LW93]. Thus we consider the sub-quotients of the Gottlieb groups  $G_*(X)$  defined by

$$H_n^{\omega}(X, Y; f) = \frac{\ker\{f_{\#}: G_n(X) \rightarrow G_n(Y, X; f)\}}{\text{im}\{P: G_{n+1}^{\text{rel}}(Y, X; f) \rightarrow G_n(X)\}}.$$

When  $Y$  is an  $H_0$ -space, the rational  $\omega$ -homology of  $f: X \rightarrow Y$  is related to the the negative derivations on the rational cohomology of  $X$  that are induced by derivations on the minimal model. To be precise, define a linear map of degree zero

$$\varphi_X: H_*(\text{Der}(\mathcal{M}_X, \mathcal{M}_X; 1)) \rightarrow \text{Der}_*(H^*(X; \mathbb{Q}), H^*(X; \mathbb{Q}); 1)$$

by the rule  $\varphi_X([\theta])([\chi]) = [\theta(\chi)]$ , for  $\theta$  a cycle in  $\text{Der}_*(\mathcal{M}_X, \mathcal{M}_X; 1)$  and  $\chi$  a cocycle in  $\mathcal{M}_X$ . It is straightforward to check that  $\varphi_X$  is well-defined. (cf. [Gri94, Prop.1.6]. In fact,  $\varphi_X$  is a morphism of graded Lie algebras.)

In the next result, and the example that follows it, we illustrate that the rationalized  $G$ -sequence may be exact at all occurrences of one type of term, while failing to be exact at the other types of term. In other words, the rational  $\omega$ -homology of a map may be zero at one type of term, yet non-zero at the other types of term.

**Theorem 4.5.** *Let  $X$  be a finite complex for which the map  $\varphi_X$  defined above is trivial and let  $Y$  be an  $H_0$ -space. Then  $H_*^{\omega}(X, Y; f) \otimes \mathbb{Q} = 0$  for any map  $f: X \rightarrow Y$ .*

*Proof.* Since  $Y$  is an  $H_0$ -space, its minimal model  $\mathcal{M}_Y \cong H^*(Y; \mathbb{Q})$  has trivial differential. Let  $\phi: H^*(Y; \mathbb{Q}) \rightarrow \mathcal{M}_X$  denote the minimal model of  $f$ . For a derivation  $\theta \in \text{Der}_*(H^*(Y; \mathbb{Q}), \mathcal{M}_X; \phi)$ , we have  $\delta(\theta) = \pm d_X \theta$ . Using this observation, we obtain a map of chain complexes

$$\mu: \text{Der}_*(H^*(Y; \mathbb{Q}), \mathcal{M}_X; \phi) \rightarrow \text{Der}_*(H^*(Y; \mathbb{Q}), H^*(X; \mathbb{Q}); H(\phi)),$$

defined by  $\mu(\theta)(\chi) = [\theta(\chi)]$ . Using the preceding observation, together with the free-ness of  $H^*(Y; \mathbb{Q})$ , it is straightforward to check that  $\mu$  induces an isomorphism on passing to homology (note that the right-hand term has trivial differential, and so is its own homology). Furthermore, the following diagram commutes:

$$\begin{array}{ccc} H_*(\text{Der}(\mathcal{M}_X, \mathcal{M}_X; 1)) & \xrightarrow{H(\phi^*)} & H_*(\text{Der}(H^*(Y; \mathbb{Q}), \mathcal{M}_X; \phi)) \\ \varphi_X \downarrow & & \cong \downarrow \mu \\ \text{Der}_*(H^*(X; \mathbb{Q}), H^*(X; \mathbb{Q}); 1) & \xrightarrow{(H(\phi))^*} & \text{Der}_*(H^*(Y; \mathbb{Q}), H^*(X; \mathbb{Q}); H(\phi)) \end{array}$$

Therefore, the assumption that  $\varphi_X = 0$  implies that the top map  $H(\phi^*)$  in the above diagram is zero. A straightforward diagram chase using the homology ladder that defines the rationalized  $G$ -sequence now gives the result.  $\square$

**Example 4.6.** Following Theorem 4.3 we remarked that the cellular inclusion  $S^4 \rightarrow \mathbb{H}P^\infty$  does not have an exact rationalized  $G$ -sequence. However, it does satisfy the hypotheses of Theorem 4.5, since here  $X = S^4$  has the property that all derivations of the cohomology algebra are trivial.

*Remark 4.7.* The hypothesis on  $X$  in Theorem 4.5, that  $\varphi_X = 0$ , deserves some comment. First, we observe that the nature of the hypothesis distinguishes structure at the minimal model level from structure at the cohomology level. This is a distinction that is made in rational homotopy for a wide variety of structures. Next, we observe that this condition is satisfied for many, if not all,  $F_0$ -spaces  $X$ . Indeed, Conjecture 2.10—the long-standing conjecture of Halperin concerning  $F_0$ -spaces—is equivalent to the assertion that all negative-degree derivations on the cohomology algebra of an  $F_0$ -space are trivial (see [Mei82] for details). Whenever this conjecture is true—and it has been verified in many cases—obviously we have  $\varphi_X = 0$ . Therefore, Theorem 4.5 can be compared with Theorem 4.3, as a result with weaker hypotheses, and correspondingly weaker conclusion. Finally, we note that the map  $\varphi_X$  makes an appearance in a completely different context, in the work of Belegradek and Kapovitch [BK03].

Our last set of results relate directly to Conjecture 2.10. First, we observe that for an inclusion of a summand of a product, the  $G$ -sequence behaves in a particularly nice way. Since it is no harder to do so, we state and prove this result in the integral setting.

**Proposition 4.8.** *Suppose that  $i_1: X \rightarrow X \times B$  is the inclusion into the first summand. Then the  $G$ -sequence of  $i_1$  is exact, and furthermore reduces to split short exact sequences*

$$0 \longrightarrow G_n(X) \xrightarrow{(i_1)_\#} G_n(X \times B, X; i_2) \begin{array}{c} \xleftarrow{(i_2)_\#} \\ \xrightarrow{(p_2)_\#} \end{array} \pi_n(B) \longrightarrow 0,$$

where  $p_2: X \times B \rightarrow B$  is projection onto the second summand and the splitting is induced by inclusion into the second summand  $i_2: B \rightarrow X \times B$ .

*Proof.* This follows from results in [LW88b] (see also [Woo97]), but we give a brief argument here. First, let  $X \xrightarrow{j} E \xrightarrow{p} B$  be any fibre sequence. Hilton's excision homomorphism for relative homotopy groups gives an isomorphism  $\pi_*(j) \cong \pi_*(B)$  [Hil65, Chap.3]. Thus we may view  $G_*^{rel}(E, X; j)$  as a subgroup of  $\pi_*(B)$  and the  $G$ -sequence of the fibre inclusion  $j$  as a subsequence the long exact homotopy sequence of the fibration.

Now apply this remark to the trivial fibration  $X \xrightarrow{i_1} X \times B \xrightarrow{p_2} B$ . The inclusion  $i_2: B \rightarrow X \times B$  induces a splitting of the long exact homotopy sequence of this trivial fibration in the usual way. The result now follows from the observation that  $(i_2)_\#(\pi_n(B)) \subseteq G_n(X \times B, X)$ , as is easily established from the definitions.  $\square$

Of course, this result and its proof can be rationalized, and it is in the rational setting that we will use it. Conjecture 2.10 concerns fibrations  $X \rightarrow E \rightarrow B$  with fibre an  $F_0$ -space and arbitrary base. However, it is well-known how to reduce the conjecture to consideration of such fibrations with base an odd-dimensional sphere [Mei82]. Furthermore, in [Lup98] it is pointed out that, for such fibrations with base an odd-dimensional sphere, Halperin's conjecture actually asserts that the fibration should be trivial.

From these remarks, we see that a necessary condition for Conjecture 2.10 to be true is that any fibration  $X \rightarrow E \rightarrow S^{2r+1}$  with  $X$  an  $F_0$ -space must have a fibre inclusion whose  $G$ -sequence reduces to the split short exact sequences corresponding via Proposition 4.8 to the inclusion  $i_1: X \rightarrow X \times S^{2r+1}$ . Perhaps surprisingly, the converse is true.

**Theorem 4.9.** *Let  $X \xrightarrow{j} E \xrightarrow{p} S^{2r+1}$  be any fibration with  $X$  an  $F_0$ -space. The following are equivalent:*

- (1) *The fibration is rationally TNCZ, that is,  $j^*: H^*(E; \mathbb{Q}) \rightarrow H^*(X; \mathbb{Q})$  is surjective.*
- (2) *The rationalized  $G$ -sequence of the fibre inclusion reduces to split short exact sequences*

$$0 \longrightarrow G_n(X) \otimes \mathbb{Q} \xrightarrow{(j)_\#} G_n(E, X; j) \otimes \mathbb{Q} \xrightarrow[(p)_\#]{(i_2)_\#} \pi_n(S^{2r+1}) \otimes \mathbb{Q} \longrightarrow 0$$

for each  $n \geq 2$ .

*Proof.* The implication (1)  $\Rightarrow$  (2) follows by the remarks preceding the enunciation. We prove (2)  $\Rightarrow$  (1).

Suppose the fibration  $X \rightarrow E \rightarrow S^{2r+1}$  has minimal model

$$\Lambda(u) \xrightarrow{i} \Lambda(u) \otimes \Lambda V, D \xrightarrow{\pi} (\Lambda V, d),$$

where  $i$  denotes the inclusion  $i(u) = u \otimes 1$  and  $\pi$  is the projection. The hypothesis that  $p_\#: G_{2r+1}(E, X; j) \otimes \mathbb{Q} \rightarrow \pi_{2r+1}(S^{2r+1}) \otimes \mathbb{Q}$  is onto—included in (2), when translated into our derivation setting, gives the existence of a  $\pi$ -derivation  $\psi \in \text{Der}_{2r+1}(\Lambda(u) \otimes \Lambda V, \Lambda V; \pi)$  that is a cocycle, and that satisfies  $\psi(u) = 1$ . Using this  $\psi$ , define a linear map  $\Phi: \Lambda(u) \otimes \Lambda V \rightarrow \Lambda(u) \otimes \Lambda V$  by setting  $\Phi(a + ub) = a + ub + u\psi(a)$  for a typical element  $a + ub \in \Lambda(u) \otimes \Lambda V$ . We will check that  $\Phi$  is

actually a DG algebra isomorphism  $(\Lambda(u) \otimes \Lambda V, D) \rightarrow (\Lambda(u) \otimes \Lambda V, d)$ . First,  $\Phi$  is an algebra map. This follows from the fact that  $\psi$  is a derivation. Suppose given two elements  $a + ub, a' + ub' \in \Lambda(u) \otimes \Lambda V$ . Then we have

$$\begin{aligned} \Phi((a + ub)(a' + ub')) &= \Phi(aa' + u((-1)^{|a|}ab' + ba')) \\ &= aa' + u((-1)^{|a|}ab' + ba') + u\psi(aa'). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \Phi(a + ub)\Phi(a' + ub') &= (a + ub + u\psi(a))(a' + ub' + u\psi(a')) \\ &= aa' + u((-1)^{|a|}ab' + ba' + (-1)^{|a|}a\psi(a') + \psi(a)a'). \end{aligned}$$

These agree, since  $\psi(aa') = \psi(a)a' + (-1)^{|a|}a\psi(a')$ . Next, we check that  $\Phi$  commutes with differentials, that is, that  $\Phi D = d\Phi$ . This will follow from the fact that  $\psi$  is a cocycle. First observe that  $d\Phi(u) = 0 = \Phi D(u)$ . It is thus sufficient to check that  $\Phi D(\chi) = d\Phi(\chi)$  for a typical element  $\chi \in \Lambda V$ . To this end, write  $D(\chi) = d(\chi) + u\theta(\chi)$ . Now we calculate

$$\Phi D\chi = \Phi(d(\chi) + u\theta(\chi)) = d(\chi) + u\theta(\chi) + u\psi(d\chi)$$

and

$$d\Phi\chi = d(\chi + u\psi(\chi)) = d(\chi) - u d\psi(\chi).$$

These agree since  $\psi$  is a cocycle, whence  $-d\psi(\chi) = \psi D(\chi) = \psi(d(\chi) + u\theta(\chi))$ . It is evident that  $\Phi$  is an isomorphism, since we have  $\Phi(u) = u$  and  $\pi \circ \Phi = \pi$ . Therefore,  $\Phi: (\Lambda(u) \otimes \Lambda V, D) \rightarrow (\Lambda(u) \otimes \Lambda V, d)$  is a DG isomorphism and the fibration is rationally trivial.  $\square$

This leads to the following equivalent phrasing of Conjecture 2.10:

**Corollary 4.10.** *Let  $X$  be an  $F_0$ -space. Then  $X$  satisfies Conjecture 2.10 if and only if the  $G$ -sequence of the fibre inclusion in every fibration of the form  $X \rightarrow E \rightarrow S^{2n+1}$  decomposes into split short exact sequences as in (2) of Theorem 4.9.*

*Proof.* In [Mei82], Meier showed that Halperin's conjecture for  $X$  is equivalent to the collapsing of the rational Serre spectral sequence for all fibrations with fibre  $X$  and base an odd sphere. By [Lup98, Th.2.3] this latter condition is equivalent to the rational homotopy triviality of the fibration. The result now follows from Theorem 4.9.  $\square$

*Remark 4.11.* It is possible to develop the ideas leading to Theorem 4.9 substantially beyond the application given above. Namely, one can use the splitting of the  $G$ -sequence of the fibre inclusion as a *measure of how close the fibration is to being trivial*. Now in the above results, we see that for fibrations of the form  $X \rightarrow E \rightarrow S^{2n+1}$ , with  $X$  an  $F_0$ -space, this notion coincides with the notion of the fibration being TNCZ. Furthermore, in this closely restricted setting, both coincide with the fibration actually being trivial. In general, however, this  $G$ -sequence point of view gives a new way of measuring how close a fibration is to being trivial. We intend to develop these ideas in a subsequent paper.

## APPENDIX A

In this appendix, we give careful proofs of the results from DG algebra homotopy theory that are used to establish Theorem 2.1. We rely heavily on the notion of pullback in the DG algebra setting. By this, we mean the following. Suppose given DG algebra maps  $f: A \rightarrow C$  and  $g: B \rightarrow C$ . Then we form the (*DG algebra pullback* or *fibre product*, as it is called in [FHT01]) as  $A \times_C B = \{(x, y) \in A \oplus B \mid f(x) = g(y)\}$ . Here  $A \oplus B$  denotes the direct sum of DG algebras. Together with the projections, the pullback forms the following (strictly) commutative square of DG algebra maps:

$$(10) \quad \begin{array}{ccc} A \times_C B & \xrightarrow{p_1} & A \\ p_2 \downarrow & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

This square possesses the usual universal property of pullbacks. Namely, suppose given DG algebra maps  $\alpha: Z \rightarrow A$  and  $\beta: Z \rightarrow B$  that satisfy  $f \circ \alpha = g \circ \beta$ . Then there exists a DG algebra map  $\phi = (\alpha, \beta): Z \rightarrow A \times_C B$ , which is the unique DG algebra map for which  $p_1 \circ \phi = \alpha$  and  $p_2 \circ \phi = \beta$ . We emphasize that throughout this appendix we distinguish carefully between diagrams that are strictly commutative and ones that are commutative only up to DG homotopy. Indeed, it is precisely this distinction that calls for the proofs of this appendix.

The following basic property of the pullback is readily gleaned from the discussion in [FHT01, Sec.13(a)]:

**Lemma A.1.** *Suppose that either  $f$  or  $g$  is surjective in the pullback diagram (10). If  $f$  is a quasi-isomorphism, then  $p_2$  is a quasi-isomorphism.*

We also make use of the so-called *surjective trick*, described in [FHT01, Sec.12(b)]. Given a DG algebra map  $\eta: B \rightarrow A$ , this manoeuvre results in a diagram

$$\begin{array}{ccc} B & \xrightleftharpoons[1 \cdot \varepsilon]{\lambda} & B \otimes E(A) \\ \eta \downarrow & \swarrow \gamma & \\ A & & \end{array}$$

in which  $\gamma$  is a surjection, and both  $1 \cdot \varepsilon$  and  $\lambda$  are quasi-isomorphisms. Some parts of the diagram commute, thus  $(1 \cdot \varepsilon) \circ \lambda = 1$  and  $\gamma \circ \lambda = \eta$ . Other compositions result in commutativity only up to DG homotopy. Recall that the notion of DG homotopy is only defined for DG algebra maps from a minimal model. Given any map  $\phi: \mathcal{M} \rightarrow B \otimes E(A)$  from a minimal model into  $B \otimes E(A)$ , we have  $\phi \sim \lambda \circ (1 \cdot \varepsilon) \circ \phi$ , where  $\sim$  denotes DG homotopy of maps from a minimal model. In particular, we thus have  $\eta \circ (1 \cdot \varepsilon) \circ \phi \sim \gamma \circ \phi$ .

Now suppose given a map  $f: X \rightarrow Y$ . We choose and fix a minimal model  $\mathcal{M}_f: \mathcal{M}_Y \rightarrow \mathcal{M}_X$  for  $f$  as follows ([FHT01, Sec.12(c)]): Let  $A^*(f): A^*(Y) \rightarrow A^*(X)$  denote the map induced by  $f$  on polynomial differential forms. Let  $\eta_X: \mathcal{M}_X \rightarrow A^*(X)$  and  $\eta_Y: \mathcal{M}_Y \rightarrow A^*(Y)$  denote minimal models for  $X$  and  $Y$ . As in [FHT01, Sec.12(b)], we convert  $\eta_X$  into a surjection  $\gamma_X: \mathcal{M}_X \otimes E(A^*(X)) \rightarrow A^*(X)$  and lift  $A^*(f) \circ \eta_Y$  through the surjective quasi-isomorphism  $\gamma_X$ , using [FHT01, Lem.12.4],

to obtain  $\phi_f: \mathcal{M}_Y \rightarrow \mathcal{M}_X \otimes E(A^*(X))$ . Now set  $\mathcal{M}_f = \gamma_X \circ \phi_f$ . All this is summarized in the following diagram.

$$\begin{array}{ccc}
 & & \mathcal{M}_X \otimes E(A^*(X)) \\
 & \nearrow \phi_f & \\
 \mathcal{M}_Y & \xrightarrow{\mathcal{M}_f} & \mathcal{M}_X \\
 \eta_Y \downarrow \simeq & & \eta_X \downarrow \simeq \\
 A^*(Y) & \xrightarrow{A^*(f)} & A^*(X)
 \end{array}
 \begin{array}{c}
 \\
 \nearrow \alpha \\
 \searrow \beta \\
 \nearrow \gamma_X
 \end{array}$$

By construction, we have  $\gamma_X \circ \phi_f = A^*(f) \circ \eta_Y$ ,  $\gamma_X \circ \alpha = \eta_X$ , and  $\beta \circ \alpha = 1$ . Remaining parts of the diagram only commute up to DG homotopy, however, thus we have  $\eta_X \circ \beta \sim \gamma_X$ ,  $\eta_X \circ \mathcal{M}_f \sim A^*(f) \circ \eta_Y$ , and so-on.

Now let  $\alpha: S^n \rightarrow \text{map}(X, Y; f)$  be a representative of a homotopy class in  $\pi_n(\text{map}(X, Y; f))$ . Let  $F: S^n \times X \rightarrow Y$  be an affiliated map for  $\alpha$ , that is,  $F(s, x) = \alpha(s)(x)$ . Since  $\alpha$  is a based map, we have  $F \circ i = f: X \rightarrow Y$ , where  $i: X \rightarrow S^n \times X$  denotes (based) inclusion into the second summand  $i(x) = (*, x)$ .

In the following result, we justify that the Sullivan minimal model of any affiliated map, and a DG homotopy between two such, have the restricted form that we require of them for the definition and well defined-ness of  $\Phi_f$ .

**Proposition A.2.** *Suppose given maps  $F, G: S^n \times X \rightarrow Y$  and  $H: S^n \times X \times I \rightarrow Y$  a homotopy from  $F$  to  $G$  that is stationary on  $X \times I$ . Suppose that  $H(*, x, t) = f(x)$  for  $f: X \rightarrow Y$  and let  $\mathcal{M}_f: \mathcal{M}_Y \rightarrow \mathcal{M}_X$  be a fixed choice of minimal model for  $f$ . There is a DG homotopy  $\mathcal{M}_H: \mathcal{M}_Y \rightarrow \mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt)$  from  $\mathcal{M}_F$  to  $\mathcal{M}_G$ , minimal models for  $F$  and  $G$ , of the form*

$$\mathcal{M}_H(\chi) = 1 \otimes \mathcal{M}_f(\chi) \otimes 1 + \text{terms in } (\mathcal{M}_{S^n})^+ \otimes \mathcal{M}_X \otimes \Lambda(t, dt).$$

In particular, any map  $F: S^n \times X \rightarrow Y$  that satisfies  $F \circ i = f$  has a minimal model  $\mathcal{M}_F: \mathcal{M}_Y \rightarrow \mathcal{M}_{S^n} \otimes \mathcal{M}_X$  of the form

$$\mathcal{M}_F(\chi) = 1 \otimes \mathcal{M}_f(\chi) + \text{terms in } (\mathcal{M}_{S^n})^+ \otimes \mathcal{M}_X.$$

*Proof.* Construct the following pullback:

$$\begin{array}{ccc}
 P & \xrightarrow{p_1} & \mathcal{M}_X \otimes \Lambda(t, dt) \otimes E(A^*(X \times I)) \\
 p_2 \downarrow & & \simeq \downarrow \gamma \\
 A^*(S^n \times X \times I) & \xrightarrow{A^*(i)} & A^*(X \times I)
 \end{array}$$

Here,  $i: X \times I \rightarrow S^n \times X \times I$  denotes the inclusion  $i(x, t) = (*, x, t)$  and  $\gamma$  denotes the surjective quasi-isomorphism obtained by converting the quasi-isomorphism  $\mathcal{M}_X \otimes \Lambda(t, dt) \rightarrow A^*(X \times I)$  to a surjection. Since  $i$  is an inclusion, the induced map  $A^*(i)$  is a surjection. From Lemma A.1, we have that  $p_2$  is a quasi-isomorphism. Now

consider the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt) \otimes E(A^*(S^n \times X)) & & \\
\begin{array}{c} \nearrow \phi_H \\ \downarrow \phi \\ \dashrightarrow \end{array} & \begin{array}{c} \downarrow \phi \\ \downarrow \gamma' \\ \downarrow \phi_f \end{array} & \begin{array}{c} \searrow \varepsilon \cdot (1 \otimes 1 \otimes E(A^*(i))) \\ \downarrow \gamma \end{array} \\
\mathcal{M}_Y & \xrightarrow{p_1} & \mathcal{M}_X \otimes \Lambda(t, dt) \otimes E(A^*(X \times I)) \\
\begin{array}{c} \dashrightarrow \psi \\ \searrow A^*(H) \circ \eta_Y \end{array} & \begin{array}{c} \dashrightarrow P \\ \downarrow p_2 \end{array} & \begin{array}{c} \dashrightarrow \\ \downarrow \gamma \end{array} \\
A^*(S^n \times X \times I) & \xrightarrow{A^*(i)} & A^*(X \times I),
\end{array}$$

Here,  $\gamma'$  is the surjective quasi-isomorphism obtained by converting the quasi-isomorphism  $\mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt) \rightarrow A^*(S^n \times X \times I)$  to a surjection,  $\phi_f: \mathcal{M}_Y \rightarrow \mathcal{M}_X \otimes E(A^*(X))$  denotes the lift used above to obtain our fixed choice of minimal model for  $f$ , and  $J: \mathcal{M}_X \rightarrow \mathcal{M}_X \otimes \Lambda(t, dt)$  denotes the map  $J(x) = x \otimes 1$ . From the pullback, we obtain the maps  $\phi$  and  $\psi$  indicated. Since both  $p_2$  and  $\gamma'$  are quasi-isomorphisms, it follows that  $\phi$  is a quasi-isomorphism. We claim that  $\phi$  is also surjective. For suppose  $(a, b) \in P$ , so that  $\gamma(a) = A^*(i)(b)$ . Since  $\varepsilon \cdot (1 \otimes 1 \otimes E(A^*(i)))$  is surjective, we can pick  $x \in \mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt) \otimes E(A^*(S^n \times X \times I))$  with  $\phi(x) = (a, b')$ , and  $A^*(i)(b') = \gamma(a) = A^*(i)(b)$ . Thus  $b - b' \in \ker A^*(i)$ . So now pick  $y \in E(A^*(S^n \times X \times I))$  with  $\gamma_{S^n}(1 \otimes 1 \otimes 1 \otimes y) = b - b'$ —recall that  $E(A^*(S^n \times X \times I))$  is freely generated by the vector space  $A^*(S^n \times X \times I)$  and its suspension. Then  $\phi(x + 1 \otimes 1 \otimes 1 \otimes y) = (a, b)$ ; so  $\phi$  is indeed surjective. Now lift  $\psi$  through the surjective quasi-isomorphism  $\phi$ , to obtain a map  $\phi_H: \mathcal{M}_Y \rightarrow \mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt) \otimes E(A^*(S^n \times X \times I))$  with  $\phi \circ \phi_H = \psi$ . As usual for the minimal model of a map, we now set  $\mathcal{M}_H = \beta \circ \phi_H: \mathcal{M}_Y \rightarrow \mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt)$ .

We now check that this DG homotopy starts and ends at minimal models for  $F$  and  $G$  respectively. Let  $j_0: S^n \times X \rightarrow S^n \times X \times I$  denote inclusion at  $t = 0$ , and  $\varepsilon_0: \Lambda(t, dt) \rightarrow \mathbb{Q}$  the augmentation given by  $\varepsilon_0(t) = 0$  and  $\varepsilon_0(dt) = 0$ . We must check that  $(1 \otimes 1) \cdot \varepsilon_0: \mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt) \rightarrow \mathcal{M}_{S^n} \otimes \mathcal{M}_X$  is a minimal model for  $F$ . For this, consider the following diagram:

$$\begin{array}{ccccc}
\mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt) \otimes E(A^*(S^n \times X \times I)) & & & & \\
\begin{array}{c} \nearrow \phi_H \\ \downarrow \alpha' \\ \dashrightarrow \end{array} & \begin{array}{c} \downarrow \alpha' \\ \downarrow \beta' \end{array} & \begin{array}{c} \searrow ((1 \otimes 1) \cdot \varepsilon_0) \otimes E(A^*(j_0)) \\ \downarrow \gamma' \end{array} & & \mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes E(A^*(S^n \times X)) \\
\mathcal{M}_Y & \xrightarrow{\mathcal{M}_H} & \mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt) & \xrightarrow{(1 \otimes 1) \cdot \varepsilon_0} & \mathcal{M}_{S^n} \otimes \mathcal{M}_X \\
\begin{array}{c} \dashrightarrow \eta_Y \\ \downarrow \eta' \end{array} & & \begin{array}{c} \dashrightarrow \\ \downarrow \eta' \end{array} & & \begin{array}{c} \dashrightarrow \\ \downarrow \eta'' \end{array} \\
A^*(Y) & \xrightarrow{A^*(H)} & A^*(S^n \times X \times I) & \xrightarrow{A^*(j_0)} & A^*(S^n \times X)
\end{array}$$

We have

$$\begin{aligned} \eta'' \circ (1 \otimes 1) \cdot \varepsilon_0 \circ \mathcal{M}_H &= A^*(j_0) \circ \eta' \circ \beta' \circ \phi_H \\ &\sim A^*(j_0) \circ \gamma' \circ \phi_H = A^*(j_0) \circ A^*(H) \circ \eta_Y \\ &= A^*(F) \circ \eta_Y. \end{aligned}$$

It follows that  $(1 \otimes 1) \cdot \varepsilon_0 \circ \mathcal{M}_H$  is a minimal model for  $F$ . A similar argument shows that  $(1 \otimes 1) \cdot \varepsilon_1 \circ \mathcal{M}_H$  is a minimal model for  $G$ , where  $\varepsilon_1: \Lambda(t, dt) \rightarrow \mathbb{Q}$  is the augmentation given by  $\varepsilon_1(t) = 1$  and  $\varepsilon_1(dt) = 0$ .  $\square$

Finally, we show that the map  $\Phi'_f: \pi_n(\text{map}(X, Y; f)) \rightarrow H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f))$  is a homomorphism. The key here is to translate addition in  $\pi_n(\text{map}(X, Y; f))$  into addition of homotopy classes of affiliated maps. Denote by  $[S^n \times X, Y]_f$  the set of homotopy classes of affiliated maps  $F$  that restrict to  $f: X \rightarrow Y$ . Now suppose  $\alpha, \beta \in \pi_n(\text{map}(X, Y; f))$  correspond to  $F, G \in [S^n \times X, Y]_f$ . Then the sum  $\alpha + \beta \in \pi_n(\text{map}(X, Y; f))$  corresponds to  $F + G \in [S^n \times X, Y]_f$ , described as follows. First consider the commutative diagram

$$\begin{array}{ccccc} & & X & \xrightarrow{i_2} & S^n \times X \\ & \swarrow i_2 & \downarrow f & & \downarrow G \\ S^n \times X & \xrightarrow{j_1} & P & \xrightarrow{j_2} & S^n \times X \\ \downarrow F & & \downarrow (F|G) & & \downarrow G \\ & & Y & \xrightarrow{1_Y} & Y \\ & \swarrow 1_Y & \downarrow 1_Y & & \downarrow 1_Y \\ & & Y & \xrightarrow{1_Y} & Y \end{array}$$

in which the top and bottom squares are pushouts, and the front vertical map is induced as a map of pushouts from the back three. The pushout  $P$  can readily be identified with  $(S^n \vee S^n) \times X$ . Let  $\sigma: S^n \rightarrow S^n \vee S^n$  denote the usual pinching comultiplication. Then we obtain the composition  $(F | G) \circ (\sigma \times 1): S^n \times X \rightarrow Y$ , which gives  $F + G$ .

Now we translate this addition in  $[S^n \times X, Y]_f$  into minimal model terms. Recall the notation established in the proof of Theorem 2.1.

**Proposition A.3.** *Suppose  $F$  and  $G$  have minimal models such that  $(\psi \otimes 1) \circ \mathcal{M}_F(\chi) = 1 \otimes \mathcal{M}_f(\chi) + s_n \otimes \theta_F(\chi)$  and  $(\psi \otimes 1) \circ \mathcal{M}_G(\chi) = 1 \otimes \mathcal{M}_g(\chi) + s_n \otimes \theta_G(\chi)$ . Then the map  $F + G = (F | G) \circ (\sigma \times 1)$  has minimal model such that  $(\psi \otimes 1) \circ \mathcal{M}_{F+G}(\chi) = 1 \otimes \mathcal{M}_f(\chi) + s_n \otimes (\theta_F + \theta_G)(\chi)$ .*

*Proof.* First consider the map  $(F | G): (S^n \vee S^n) \times X \rightarrow Y$ . As discussed in [FHT01, Sec.13(b)], a DG algebra model for the pushout  $P = (S^n \vee S^n) \times X$  can be obtained as a pullback of minimal models. Specifically, by forming the pullback of minimal models

$$\begin{array}{ccc} Q & \xrightarrow{p_1} & H^*(S^n; \mathbb{Q}) \otimes \mathcal{M}_X \\ p_2 \downarrow & & \downarrow \varepsilon \cdot 1 \\ H^*(S^n; \mathbb{Q}) \otimes \mathcal{M}_X & \xrightarrow{\varepsilon \cdot 1} & \mathcal{M}_X, \end{array}$$

we obtain a DG algebra  $Q = H^*(S^n; \mathbb{Q}) \otimes \mathcal{M}_X \oplus_{\mathcal{M}_X} H^*(S^n; \mathbb{Q}) \otimes \mathcal{M}_X$  quasi-isomorphic to the minimal model of  $P$ . Further, the same discussion can readily be extended to show that in the above situation, a minimal model for the induced map  $(F | G)$  can be obtained using the map induced on the pullback of minimal models. Now this induced map is the map

$$\xi: \mathcal{M}_Y \rightarrow H^*(S^n; \mathbb{Q}) \otimes \mathcal{M}_X \oplus_{\mathcal{M}_X} H^*(S^n; \mathbb{Q}) \otimes \mathcal{M}_X$$

given by  $\xi(\chi) = (1 \otimes \mathcal{M}_f(\chi) + s_n \otimes \theta_F(\chi), 1 \otimes \mathcal{M}_f(\chi) + s_n \otimes \theta_G(\chi))$ . Under the identification of the pullback  $Q$  with  $(H^*(S^n; \mathbb{Q}) \oplus H^*(S^n; \mathbb{Q})) \otimes \mathcal{M}_X$ , the composition  $(F | G) \circ (\sigma \times 1)$  evidently has minimal model of the described form.  $\square$

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