Contemporary Mathematics

Minami-Webb type decompositions for compact Lie groups

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1. Introduction

Let p be a fixed prime number. We extend to compact Lie groups some stable classifying space decompositions of Minami [**M**], following Webb [**W**]. One notable feature of [**W**] is the use of a combinatorial Möbius function to encode p-local information about the cohomology of a finite group. We wish to show similar phenomena hold for compact Lie groups. However, for a compact Lie group G one is faced with the problem of an infinite number of conjugacy classes of p-toral subgroups, that is, extensions of tori by finite p-groups. These groups are the analogs of p-groups for finite groups. We circumvent this problem by considering a certain finite G-complex which allows us to introduce combinatorial methods in the compact Lie group case. This complex is based on the notion of p-stubborn subgroups which arose earlier in modular representation theory of finite groups (where they were called p-radical groups) in connection with Alperin's conjecture [**A**], [**Bouc**], in group cohomology [**W**], and in the study of homotopy classes of maps between classifying spaces of compact Lie groups [**JMO**]. We also derive a decomposition based on the corresponding complex for elementary abelian p-subgroups. Several examples are given to illustrate the various decompositions.

2. Main Results

We begin by defining the G-sets used to construct our stable decompositions. Let $\Phi = \Phi(G)$ be the poset of non-trivial p-toral subgroups $P \leq G$ with finite Weyl groups $W_G(P) = N_G(P)/P$. Let

$\Delta(\Phi) = |Nerve(\Phi)|$ be the geometric realization of the nerve of Φ viewed as a category. Thus $\Delta(\Phi)$ is a simplicial complex associated with Φ. The n-simplices $\sigma = (P_0, P_1, ..., P_n)$ of $\Delta(\Phi)$ are sequences of inclusions $P_0 < P_1 < ... < P_n$ of elements of Φ. G acts on $\Delta(\Phi)$ via conjugation. Our goal is to replace $\Delta(\Phi)$ by a finite G-complex which can be used to study BG. We shall need a compact Lie group version of seminal results of Quillen [**Q**] (following Brown [**Bwn**]) regarding this complex and certain subcomplexes.

Throughout sums of classifying spaces will be taken in the Grothendieck group of spectra under wedge sum.

The complex $\Delta(\mathcal{S})$ of p-stubborn subgroups:

Let $\mathcal{S} = \mathcal{S}(G) \subset \Phi(G)$ be the poset of *p-stubborn* subgroups of G. Thus $P \in \mathcal{S}$ if and only if 1)P is p-toral with $W_G(P)$ finite, 2) $P = O_p(N_G(P))$, i.e., $W_G(P)$ has no non-trivial normal p-subgroups. For finite groups homological properties of these subgroups were studied by Bouc [**Bouc**]. In the compact Lie group case Jackowski-McClure-Oliver showed that up to conjugacy in G, $\mathcal{S}$ is finite [**JMO**]. Let $\Delta(\mathcal{S}) = |Nerve(\mathcal{S})|$. As we shall see $\Delta(\mathcal{S})$ has desirable homotopical properties.

LEMMA 2.1. *The complex $\Delta(\mathcal{S})/G$ is finite.*

The proof is given in Section 5.

If $\sigma = (P_0, P_1, ..., P_n)$ is a simplex of $\Delta(\mathcal{S})$ then the isotropy subgroup

$$G_\sigma = N_G(P_0) \cap ... \cap N_G(P_n).$$

THEOREM 2.2. *Stably*

$$BG_p^\wedge \simeq \sum_{\overline{\sigma} \in \Delta(\mathcal{S})/G} (-1)^{dim(\sigma)} (BG_\sigma)_p^\wedge.$$

Since $\Delta(\mathcal{S})/G$ is a finite complex by Lemma 2.1, the summation is finite. The proof of Theorem 2.2, given in Section 6, is a topological variation of the proof of Webb's Theorem A [**W2**] for Mackey functors from the category of finite G-sets to a category of modules.

The complex $\Delta(\mathcal{A})$ of elementary abelian p-subgroups:

By considering elementary abelian groups we obtain another decomposition. Let $\mathcal{A} = \mathcal{A}(G)$ be the poset of non-trivial elementary abelian p-subgroups of G and $\Delta(\mathcal{A}) = |Nerve(\mathcal{A})|$ the associated simplicial complex.

LEMMA 2.3. *The complex $\Delta(\mathcal{A})/G$ is finite.*

The proof is given in Section 5.

As in the case of finite groups we have

LEMMA 2.4. *If P is a non-trivial p-toral subgroup of G then the fixed point complex $\Delta(\mathcal{A})^P$ is contractible.*

PROOF. The proof given in [**Q**, 4.4] works equally well for compact Lie groups. □

With this lemma the proof of Theorem 2.2 applied to $\Delta(\mathcal{A})$ proves

THEOREM 2.5. *Stably*

$$BG_p^\wedge \simeq \sum_{\overline{\sigma}\in\Delta(\mathcal{A})/G} (-1)^{dim(\sigma)} (BG_\sigma)_p^\wedge.$$

A Möbius function and Minami-Webb type formula:

Let $\mathbf{Z}_p$ denote the p-adic integers. If G is a finite group, $H \leq G$ then u_H denotes the permutation module $\mathbf{Z}_p \otimes_{\mathbf{Z}_p H} \mathbf{Z}_p G$ viewed as an element of the Green ring which is the Grothendieck group (over $\mathbf{Q}$) of finitely generated indecomposable $\mathbf{Z}_p G$ modules [**W**]. We recall that a *cyclic mod-p group* is a extension of a finite p-group by a finite cyclic p'-group. Let $\mathcal{C}(G)$ be the collection of all cyclic mod-p subgroups of G. A Möbius function $f : \mathcal{C}(G) \to \mathbf{Z}$ is defined recursively by

$$\sum_{J\leq K\in\mathcal{C}(G)} f(K) = 1.$$

Computation of f is facilitated by P. Hall's observation [**Ha**] that f vanishes except on intersections of maximal subgroups.

By Webb's formula [**W**, Theorem D']

$$u_G = \sum_{H\in\mathcal{C}(G)^*} \frac{f(H)}{|W_G(H)|} \; u_H \tag{1}$$

where the sum is taken a set of representatives of conjugacy classes $\mathcal{C}(G)^*$ of cyclic mod-p subgroups of G and $W_G(H) =: N_G(H)/H$, the

Weyl group of H. From this Minami derives a corresponding formula for classifying spaces [**M**, Theorem 6.6]

$$BG_p^\wedge \simeq \sum_{H \in \mathcal{C}(G)^*} \frac{f(H)}{|W_G(H)|} \; BH_p^\wedge \tag{2}$$

In this formula we may omit any p'-groups of $\mathcal{C}(G)$ since completed at p their classifying space contributes nothing to the sum. However as we shall see in Examples 2 and 3 of Section 3 it is necessary to leave these groups in formula (1).

In the general compact Lie group case let $\mathcal{C}(\mathcal{S})$ be the set of all cyclic mod-p extensions of p-stubborn subgroups. Thus $H \in \mathcal{C}(\mathcal{S})$ if and only if H contains a normal subgroup $P \in \mathcal{S}$ such that H/P is a finite cyclic p'-group.

Since the conjugacy classes of $\mathcal{S}$ are finite and each element has finite Weyl group it follows that $\mathcal{C}(\mathcal{S})$ has a finite number of conjugacy classes. Let c_{JK} denote the number of conjugates of K which contain J. By Lemma 5.2, c_{JK} is finite for $J, K \in \mathcal{C}(\mathcal{S})$. Thus we can define a Möbius function $f : \mathcal{C}(\mathcal{S}) \longrightarrow \mathbf{Z}$ recursively by

$$\sum_{J \leq K \in \mathcal{C}(\mathcal{S})^*} c_{JK} f(K) = 1 \tag{3}$$

for fixed $J \in \mathcal{C}(\mathcal{S})$ with $O_p(J) \neq 1$. Here $\mathcal{C}(\mathcal{S})^*$ is a set of representatives for the conjugacy classes of $\mathcal{C}(\mathcal{S})$.

THEOREM 2.6. *Stably*

$$BG_p^\wedge \simeq \sum_{H \in \mathcal{C}(\mathcal{S})^*} \frac{f(H)}{|W_G(H)|} \; BH_p^\wedge .$$

Here we have tensored the Grothendieck group of spectra with the rationals $\mathbf{Q}$. The proof is given in Section 7.

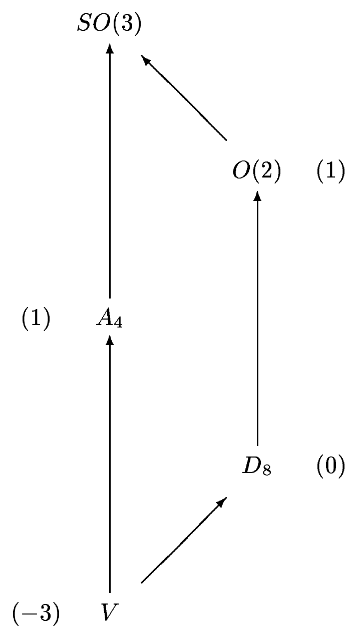
3. Applications

In these examples all spaces are stable and completed at $p = 2$.

EXAMPLE 3.1.

$G = SO(3)$. Then up to conjugacy $\mathcal{S}$ consists of $\{O(2), V\}$ where $V = O(1) \times O(1) \leq O(2)$ is an elementary abelian 2-group of rank 2. $N_G(O(2)) = O(2)$ and $N_G(V)$ is the octahedral group isomorphic to Σ_4, the symmetric group on four letters. This is easily checked from the information on the conjugacy classes of subgroups of G given in

[**TD2**]. Let D_8 be the dihedral group of order 8. Then up to conjugacy $\mathcal{C}(\mathcal{S})$ consists of $\{V, A_4, D_8, O(2)\}$:



where arrows represent inclusion and the values of the Möbius function f are given in parentheses. These are computed inductively from equation (3). Since $Q = O(2)$, A_4 are maximal, $f(Q) = 1$. By Lemma 5.2, $c_{D_8,O(2)} = 1$ hence

$$f(D_8) + c_{D_8,O(2)} f(O(2)) = 1$$

implies $f(D_8) = 0$. Similarly $c_{V,O(2)} = 3$, $c_{V,A_4} = 1$ hence B

$$f(V) + c_{V,A_4} f(A_4) + c_{V,O(2)} f(O(2)) = 1$$

implies $f(V) = -3$. By Theorem 2.6 we have

$$BSO(3) \simeq BO(2) + \frac{1}{2}(BA_4 - BV) \tag{4}$$

Similarly the elementary abelian 2-subgroups of G fall into two conjugacy classes $\{V, C\}$ where $C \leq V$ has order two.

LEMMA 3.2. *Let $C' \leq V$, $V' \leq O(2)$ be subgroups of $SO(3)$ isomorphic to C, V respectively. Then*

1) C' is $N(V)$-conjugate to C.

2) V' is $O(2)$-conjugate to V.

Proof. 1) $N(V) = V \rtimes GL_2(\mathbf{F}_2)$. Thus any two non-trivial involutions of V are conjugate by an element which normalizes V.

2) $V =< a, -I_2 \cdot a >$ where

$$a = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let

$$\tau = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Since $SO(2)$ does not contain an elementary abelian 2-group of rank 2 we may assume V' contains two generators $a', b' \in O(2)_{-1}$, the non-identity component of $O(2)$. Thus $V' = \langle a', b' \rangle$ where $a' = \tau x$, $b' = \tau y$ for some $x, y \in SO(2)$. However $a'b' = b'a'$, hence $xy^{-1} = yx^{-1}$. Thus $x = -I_2 \cdot y$ and $V' = \langle a', -I_2 \cdot a' \rangle$. Now since every non-trivial involution of $O(2)_{-1}$ is $O(2)$-conjugate to a and $-I_2$ is central, V' is $O(2)$-conjugate to V.

□

By Lemma 3.2(1), $\Delta(\mathcal{A})/G$ has a single 1-simplex corresponding to $C \leq V$ and two 0-simplices corresponding to C, V. We have $N_G(C) = O(2)$, $N_G(V) = \Sigma_4$ and $N_G(C) \cap N_G(V) = D_8$. Thus by Theorem 2.5

$$BSO(3) \simeq BO(2) + B\Sigma_4 - BD_8. \tag{5}$$

Using $\mathcal{S}$ and Lemma 3.2(2), Theorem 2.2 yields the same result. These formulas are consistent with those of [**Mitch-P**].

Example 3.3.

$G = SU(2) = S^3$. There are two conjugacy classes of 2-stubborn subgroups, $H = N_G(S^1) = \langle S^1, j \rangle$ and $K = Q_8$. It is easy to check that any pair of subgroups $H' \geq K'$ conjugate to H, K respectively is simultaneously conjugate. (This type of argument is illustrated in Lemma 3.2.) Thus $\Delta(\mathcal{S})/G$ has a single one simplex corresponding to $K \leq H$ and two zero simplices corresponding to H, K. Furthermore $N_G(H) = H$ and $N_G(K)$ is the binary octahedral group which is isomorphic to $\overline{\Sigma}_4$, the two-fold cover of Σ_4 and $N(H) \cap N(K) = Q_8$. Thus by the formula of Theorem 2.2

$$BS^3 \simeq BN_G(S^1) + B\overline{\Sigma}_4 - BQ_8$$

Applying Theorem 2.6 we have the refinement

$$BS^3 \simeq BN_G(S^1) + \frac{1}{2}B(Q_8 \rtimes \mathbf{Z}/3 - BQ_8)$$

The interested reader can verify that this decomposition relates well to that of BQ_8 given in [**Mitch-P**] i.e.,

$$\Sigma^{-1}BS^3/BN_G(S^1) \simeq \frac{1}{2}B(Q_8 \rtimes \mathbf{Z}/3 - BQ_8).$$

EXAMPLE 3.4.

$G = U(2)$ with standard maximal torus T. The center $C \leq T$ of G, i.e., matrices of the form zI_2, $z \in S^1$, is non-trivial so Theorem 2.5 does not yield a useful expression of BG. We could use Theorem 2.6. However, more simply $G/C = SU(2)/\langle \pm I_2 \rangle = SO(3)$ and C is in every maximal subgroup. Therefore the Möbius functions for G and G/C correspond. Hence we may pull back formula (4) to obtain

$$BU(2) \simeq B(T \rtimes \mathbf{Z}/2) + \frac{1}{2}[B(Q \rtimes \mathbf{Z}/3) - BQ] \tag{6}$$

where

$$Q = (\Delta S^1 \times \mathbf{Z}/2) \rtimes \mathbf{Z}/2,$$

i.e., Q is generated by the elements

$$\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}, \qquad z \in S^1$$

and the involutions a, τ defined in the proof of Lemma 3.2.

The action of $\mathbf{Z}/3$ is given by conjugation with the element

$$\beta = \frac{1}{2}\begin{bmatrix} -1-i & -1-i \\ 1-i & -1+i \end{bmatrix}.$$

Another decomposition can be obtained from $\mathcal{S}$ which (up to conjugacy) consists of $\{N(T), Q\}$ with $Q \leq N(T)$. This follows from Oliver's description of the p-stubborn subgroups of the classical groups [**O**] where Q is denoted by Γ_2^U.

LEMMA 3.5. *If $Q' \leq N(T)$ is conjugate to Q then Q' is T-conjugate to Q.*

PROOF. Since $\Delta S^1 = ZU(2)$, $Q' = \langle \Delta(S^1), a', \tau' \rangle$ where a', τ' correspond to a, τ under the given conjugation. Then a', τ' are non-commuting involutions. Since $\tau' \in N(T)$ it has the form $\tau' = \tau^\epsilon(z_1, z_2)$, $\epsilon = 0, 1$.

Case 1: $\epsilon = 1$.

$$\tau'^2 = [\tau(z_1, z_2)]^2 = (z_1 z_2, z_1 z_2) = 1$$

Hence $z_2 = {z_1}^{-1}$. Then $\tau' = \tau(z_1, {z_1}^{-1})$. However

$$(z_1, 1)\tau(z_1, {z_1}^{-1})({z_1}^{-1}, 1) = \tau(1, 1) = \tau$$

Thus we may assume $\tau' = \tau$. Now if $a' \in T$, then $a' = \pm a$. If not $a' = \tau(z_1, z_2)$, then arguing as above we have $z_2 = {z_1}^{-1}$. Hence

$$-1 = \tau a' \tau a' = ({z_1}^2, {z_1}^{-2})$$

Thus $z_1 = \pm i$ and $a' = \pm(i, i)a$. Thus either way Q' is T-conjugate to Q.

Case 2: $\epsilon = 0$. We have $-1 = det(\tau') = z_1 z_2$ and $1 = {\tau'}^2 = (z_1 z_2, z_1 z_2)$. Hence $z_2 = -{z_1}^{-1}$, $z_1 = \pm 1$. Thus $\tau' = \pm a$. In this case $a' \notin T$ since a' and τ' do not commute. Hence $a' = \tau(z_1, z_2)$ and so a' is T-conjugate to τ. This implies Q' is T-conjugate to Q.

□

By Lemma 3.5, $\Delta(\mathcal{S})$ has one 1-simplex and two 0-simplices. Computing normalizers we have $N(N(T)) = N(T) = T \rtimes \mathbf{Z}/2$, $N(Q) = Q \rtimes \Sigma_3$ where the Σ_3 action is generated by $\{\alpha, \beta\}$ with

$$\alpha = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then $N(T) \cap N(Q) = Q \rtimes \Sigma_2$. Thus by Theorem 2.2 we have

$$BU(2) \simeq B(T \rtimes \mathbf{Z}/2) + B(Q \rtimes \Sigma_3) - B(Q \rtimes \Sigma_2) \tag{7}$$

We note that the same formula is obtained by pulling back equation (5). This gives another description of $Q \rtimes \Sigma_3$. Finally we note that equation (7) transforms to equation (6) by simplifying $B(Q \rtimes \Sigma_3)$. This is done by pulling back Webb's formula (1) for Σ_3 to obtain

$$B(Q \rtimes \Sigma_3) \simeq B(Q \rtimes \Sigma_2) + \frac{1}{2}[B(Q \rtimes \mathbf{Z}/3) + BQ].$$

Example 3.6.

$G = SU(3)$. There are three conjugacy classes of 2-stubborn subgroups. By [**O**] they are represented by the subgroups $\{T, N_{U(2)}(T), Q\}$ of $U(2)$ defined in Example 3.4. We consider these as subgroups of $SU(3)$ by the usual monomorphism $U(2) \to SU(3)$. Computing normalizers we find

$$N_{SU(3)}(T) = T \rtimes \Sigma_3,$$
$$N_{SU(3)}(N_{U(2)}(T)) = N_{U(2)}(T)$$
$$N_{SU(3)}(Q) = Q \rtimes \Sigma_3.$$

From the inclusions $T \le N_{U(2)}(T) \ge Q$ we have

$$N_{SU(3)}(T) \cap N_{SU(3)}(N_{U(2)}(T)) = T \rtimes \Sigma_3$$

$$N_{SU(3)}(N_{U(2)}(T)) \cap N_{SU(3)}(Q) = Q \rtimes \Sigma_2$$

By Lemma 3.5, $Q \le N_{U(2)}(T)$ is unique up to conjugation (even in $U(2)$), while T is the unique maximal torus of $N_{U(2)}(T)$. Hence $\Delta(\mathcal{S})/G$ has exactly two 1-simplices corresponding to $T \le N_{U(2)}(T)$, $Q \le N_{U(2)}(T)$ and three 0-simplices. Thus by Theorem 2.2

$$BSU(3) \simeq B(T \rtimes \Sigma_3) + B(Q \rtimes \Sigma_3) + B(Q \rtimes \Sigma_2). \tag{8}$$

Theorem 2.6 yields

$$BSU(3) \simeq B(T \rtimes \mathbf{Z}/2) + \frac{1}{2}[B(T \rtimes \mathbf{Z}/3) - BT] + \frac{1}{2}[BQ \rtimes \mathbf{Z}/3 - BQ]. \tag{9}$$

Alternatively, the conjugacy classes of elementary abelian 2-subgroups of G are represented by W of rank 2 generated by the diagonal matrices with entries

$$(a, b, (ab)^{-1}), \qquad a, b = \pm 1$$

and by Z of rank one generated by $(-1, -1, 1)$. Then $N_G(Z) = U(2)$, $N_G(W) = T \rtimes \Sigma_3$, $N_G(Z) \cap N_G(W) = T \rtimes \mathbf{Z}/2$. Hence Theorem 2.5 gives

$$BSU(3) \simeq BU(2) + B(T \rtimes \Sigma_3) - B(T \rtimes \mathbf{Z}/2). \tag{10}$$

We can simplify $B(T \rtimes \Sigma_3)$ by pulling back Webb's formula (1) for Σ_3, as in Example 3.4, to obtain

$$B(T \rtimes \Sigma_3) \simeq B(T \rtimes \mathbf{Z}/2) + \frac{1}{2}[B(T \rtimes \mathbf{Z}/3) - BT]$$

Thus

$$BSU(3) \simeq BU(2) + \frac{1}{2}[B(T \rtimes \mathbf{Z}/3) - BT]$$

which combined with (6) yields another derivation of (9). Similarly (10) combined with (7) gives (8).

4. A G-homotopy equivalence of complexes

We shall need the following result in the proof of Theorem 2.2

PROPOSITION 4.1. *1) Inclusion $i : \Delta(\mathcal{S}) \to \Delta(\Phi)$ is a G-homotopy equivalence.*

2) Suppose $P \in \Phi$ is non-trivial p-toral subgroup then $\Delta(\mathcal{S})^P$ is contractible.

LEMMA 4.2. *If P is a non-trivial p-toral subgroup then $\Delta(\Phi)^P$ is contractible.*

PROOF. We adapt Quillen's method. Let $Q \in \Phi^P$ then $P \leq N_G(Q)$. Then $PQ \leq N_G(Q)$ is a compact Lie group which is a finite extension of Q since $W_G(Q)$ is finite. Let S, T be the maximal torus of P, Q respectively. Then $S \trianglelefteq P$, $T \trianglelefteq Q$ since P, Q are p-toral. Let T' be a maximal torus of $(PQ)_0$ which contains S. Since PQ is a finite extension of Q , $T' = T$. Thus $ST = T$ and so $S \leq T$. It follows easily that $\pi =: PQ/T$ is a finite p-group generated by Q/T and P/S. Hence PQ is p-toral.

We claim $PQ \in \Phi^P$. Since $P \leq N_G(PQ)$ it remains to show $W_G(PQ)$ is finite. If not there is a torus $T'' \leq N_G(PQ)$ such that $T'' \nleq PQ$ Then T'' normalizes PQ but acts trivially on π since $Aut(\pi)$ is finite. Thus T'' acts trivially on the quotient $PQ/Q = \pi/(Q/T)$. Thus T'' normalizes Q and hence $T'' \leq Q$ since $W_G(Q)$ is finite. This contradicts the existence of T'' proving the claim.

Thus we have $P \leq PQ \geq Q$ in Φ. This proves $\Delta(\Phi)^P$ is conically contractible by [**Q**, 1.5]. □

We recall a result of Thévenaz-Webb and some generalizations. From here through the proof of Proposition 4.5 we will identify a poset with its geometric realization.

If Y is a G-poset then

$$Y_{\leq y} = \{z \in Y \mid z \leq y\}$$

$$Y_{\geq y} = \{z \in Y \mid z \geq y\}.$$

THEOREM 4.3 (Thévanaz-Webb). [**TW**, Th. 1] *Let G be a group, let X, Y be G-posets and let $\phi : X \to Y$ be a map of G-posets. Suppose either*

1) $\phi^{-1}(Y_{\leq y})$ is G_y-contractible for all $y \in Y$

2) $\phi^{-1}(Y_{\geq y})$ is G_y-contractible for all $y \in Y$

Then ϕ is a G-homotopy equivalence.

LEMMA 4.4. *Let $P \in \Phi$. Then $\Phi_{>P}$ is $N_G(P)$-contractible if and only if $P \notin \mathcal{S}$.*

PROOF. This is [**TW**, Lemma 2.1]; the proof applies equally well to compact Lie groups. □

We shall also need the following generalization of another result of [**TW**] extended to infinite groups.

PROPOSITION 4.5. *Let $X \subset Y$ be G-posets. Asssume $X = \bigcap Y_i$ where $Y_0 = Y$ and Y_{i+1} is obtained by deleting from Y_i the minimal elements of $Y_i - X$. If $y \in Y - X$ implies $Y_{>y}$ is G_y-contractible then the inclusion $X \to Y$ is a G-homotopy equivalence.*

PROOF. Let $\phi_i : Y_{i+1} \to Y_i$ be the inclusion. If $y \in Y_{i+1}$ then ${\phi_i}^{-1}((Y_i)_{\geq y}) = (Y_{i+1})_{\geq y}$ which has y as a minimal element. Thus $(Y_{i+1})_{\geq y}$ is the cone on y which is G_y fixed. Hence $(Y_{i+1})_{\geq y}$ is G_y contractible. If $y \in Y_i - Y_{i+1}$ then ${\phi_i}^{-1}((Y_i)_{\geq y}) = Y_{>y}$ since y and no element above y is deleted when forming Y_{i+1}. $Y_{>y}$ is G_y contractible by hypothesis, thus ϕ_i is a G-homotopy equivalence by Theorem 4.3. Now for each closed subgroup H there is the usual Milnor exact sequence

$$0 \to lim^1\ \pi_{*+1}({Y_i}^H) \to \pi_*(X^H) \to lim\ \pi_*({Y_i}^H) \to 0$$

Since $\phi_{i*} : \pi_*({Y_{i+1}}^H) \to \pi_*({Y_i}^H)$ is an isomorphism, the lim^1 term is zero and $\pi_*(X^H) \to \pi_*(Y^H)$ is an isomorphism and so $X \to Y$ is a G-homotopy equivalence. □

Proof of Proposition 4.1. 1) Let $\mathcal{S}' = \mathcal{S} \cup \Phi_{pos}$ where Φ_{pos} denote the elements of Φ of positive dimension. We will show the inclusions of posets $\mathcal{S}' \leq \Phi$ and $\mathcal{S} \leq \mathcal{S}'$ induce G homotopy equivalences.

$$\Delta(\mathcal{S}') \to \Delta(\Phi) \tag{11}$$

$$\Delta(\mathcal{S}) \to \Delta(\mathcal{S}') \tag{12}$$

We wish to apply Prop. 4.5. Let $X = \mathcal{S}'$, $Y = \Phi$. Then the elements $\pi \in Y - X$ are finite p-groups By considering their order $|\pi|$, one sees that $\pi \in Y_i$ then $|\pi| \geq p^i$. Thus $X = \bigcap Y_i$. Hence Lemma 4.4 and Prop. 4.5 imply (11) is a G-homotopy equivalence. For (12) we induct on dimension. Since this induction is finite the full strength of Prop. 4.5 is not needed.

Part 2) follows from Part 1) and Lemma 4.2 □

5. Proof of Lemmas 2.1 and 2.3

The following result of Bredon [**Brd**, Cor. II 5.7] will be useful.

LEMMA 5.1. *Let $K \leq H \leq G$ be compact Lie groups. Then the orbit space*

$$(G/H)^K/W_G(K)$$

of the right translation action of $W_G(K)$ is finite.

LEMMA 5.2. *For subgroups $J, K \leq G$ the number, c_{JK}, of conjugates of K which contain J is finite if $W_G(J)$ is finite. Moreover if $W_G(K)$ is also finite then*

$$c_{JK} = |(G/K)^J|/|W_G(K)|$$

PROOF. Suppose $W_G(J)$ is finite. By definition $c_{JK} = |\{g \in G : gKg^{-1} \geq J\}/N(K)|$. On the other hand $(G/K)^J = \{gK : gKg^{-1} \geq J\} = \{g : gKg^{-1} \geq J\}/K$ is finite since it has a finite number of $W_G(J)$ orbits by Lemma 5.1. Thus c_{JK} is finite. Since $W_G(K)$ acts on $(G/K)^J$ with orbit space $\{g \in G : gKg^{-1} \geq J\}/N(K)$ it follows that

$$c_{JK} = |(G/K)^J|/|W_G(K)|$$

if $W_G(K)$ is also finite. □

Proof of Lemma 2.1: Consider a simplex $\sigma = (Q_1, Q_2, ..., Q_n)$, $Q_1 < Q_2 < ... < Q_n$. Since there are only finitely many conjugacy classes of subgroups in $\mathcal{S}$ there are only a finite number of choices of Q_1 up to conjugacy. Since the number of G-conjugacy classes of $\mathcal{S}$ is finite, Lemma 5.2 implies there are only a finite number of choices of Q_2 which contain Q_1. This process is finite and terminates after the conjugacy classes of $\mathcal{S}$ have been used. □

Proof of Lemma 2.3: Since G is compact there is a bound d for the rank of all maximal elementary abelian subgroups of G. Then the dimension of $\Delta(\mathcal{A}) < d$. Let $(E_0, E_1, ..., E_{(d-1)})$ be a simplex of highest dimension. Since G has only finitely many conjugacy classes of elementary abelian subgroups [**Q2**, Lemma 6.3], it follows that in $\Delta(\mathcal{A})/G$ the subgroup $E_{(d-1)}$ ranges over a finite set. Hence $\Delta(\mathcal{A})/G$ is a finite complex. □

6. Proof of Theorem 2.2

Let $\{Spectra\}$ be the Grothendieck group of spectra completed at p. Let $\{G\text{-}space\}$ be the category of G-spaces with a continuous left G-action and define a functor $F : \{G\text{-}space\} \to \{Spectra\}$ by $F(X) = \Sigma^\infty EG^+ \wedge_G X^+$. In what follows, as elsewhere in the paper, we work stably and omit the symbol Σ^∞ for suspension spectrum. Then F is a Mackey functor [**W2**] with restriction and induction given by

$$res{\downarrow}_K^H = F(i) : F(K) \to F(H)$$

$$ind{\uparrow}_K^H = transfer : F(H) \to F(K)$$

where $i : K \to H$ is an inclusion of closed subgroups of G. At this point we formulate a special case of Webb's theorems the proof of which is applicable to the case of compact Lie groups.

THEOREM 6.1. *[Webb] Let G be a finite group, M a Mackey functor, $\mathcal{X}$ and $\mathcal{Y}$ collections of subgroups of G closed under conjugation and taking subgroups, and Δ a finite G-complex. Suppose*

1) For every simplex σ of Δ the vertices of Δ lie in distinct G-orbits.

2) For every subgroup $H \in \mathcal{X} - \mathcal{Y}$, Δ^H is contractible.

3) A Sylow p-subgroup $G_p \in \mathcal{X}$ and $M_*(pr) : M(G_p \times T) \to M(T)$ is a split surjection natural in T.

4) For every $Y \in \mathcal{Y}$, $M(Y) = 0$. Then

$$M(G) \approx \bigoplus_{\overline{\sigma} \in \Delta/G} (-1)^{dim(\sigma)} M(G_\sigma)$$

PROOF. [**W2**] Theorem A. □

Proof of Theorem 2.2: The homotopy category $\mathcal{H}o\{Spectra\}$ is an additive category and Theorem 6.1 applies even though it is stated for Mackey functors with R-modules as target. Addition of stable maps gives the *hom* sets the structure of abelian groups and direct sum is given by the wedge product.

In order to define terms let $\Delta = \Delta(\mathcal{S})$, $\mathcal{X}$ equal the set of all p-toral subgroups of G, and $\mathcal{Y} = \{1\}$. Hypothesis (1) was observed in [**W2**]. For (2) we note $H \in \mathcal{X}$ implies Δ^H is contractible by Lemma 4.2. For (3) we recall $\chi(G/G_p)$ is prime to p. Thus the transfer for the (space level) fibration

$$G/G_p \to EG \times_G (G_p \times T) \xrightarrow{pr} EG \times_G T$$

implies

$$F(G_p \times T) = EG^+ \wedge_G (G_p \times T)^+ \xrightarrow{pr_*} EG^+ \wedge_G T^+ = F(T)$$

is a natural split surjection as required. We note $\mathcal{X}$ is closed under conjugation and under taking subgroups. Thus the proof of Theorem 6.1 applies. □

7. Proof of Theorem 2.6

Let G be a compact Lie group which we initially assume has a normal maximal torus T. Let $\mathcal{C}(\overline{G})$ be the set of cyclic mod-p subgroups of the finite group $\overline{G} =: G/T$. Let $W_G(H) =: N_G(H)/H$, the Weyl group of H.

THEOREM 7.1. *Suppose G has a normal maximal torus T. Then stably*

$$BG_p^\wedge \simeq \sum_{\overline{H}\in\mathcal{C}(\overline{G})^*} \frac{f(H)}{|W_G(H)|} \ BH_p^\wedge$$

where $\overline{H} = H/T$ runs over a set of representatives of the finite set of conjugacy classes $\mathcal{C}(\overline{G})^*$ of cyclic mod-p subgroups of $\overline{G}$ and $f(H) =: f(\overline{H})$ is the Möbius function $f : \mathcal{C}(\overline{G}) \to \mathbf{Z}$ satisfying

$$\sum_{\overline{J}\leq\overline{H}\in\mathcal{C}(\overline{G})} f(H) = 1.$$

for fixed $\overline{J} \in \mathcal{C}(\overline{G})$.

PROOF. By a slight variation on Feshbach's construction [**F**] one can construct a nested sequence of finite subgroups $G_k \leq G_{k+1} \leq G$ with normal subgroups $T_k \leq T$ such that $G_k/T_k = \overline{G}$ and

$$colim\ H_*(BG_k;\mathbf{Z}/p) \approx H_*(BG;\mathbf{Z}/p)$$

Let $\pi_k : G_k \to \overline{G}$ be projection and set $H_k = {\pi_k}^{-1}(\overline{H})$. Let u_H denote the permutation module $\mathbf{Z}_p \otimes_{\mathbf{Z}_pH} \mathbf{Z}_pG$. Then from [**W**, Theorem D' and Lemma 7.1]

$$u_{G_k} = \sum_{\overline{H}\in\mathcal{C}(\overline{G})^*} \frac{f(H)}{|W_G(H)|} \ u_{H_k}$$

From this and (2) it follows directly that

$$(BG_k)_p^\wedge \simeq \sum_{\overline{H}\in\mathcal{C}(\overline{G})^*} \frac{f(H)}{|W_G(H)|} \ (BH_k)_p^\wedge$$

Passing to the colimit over k gives the desired result since $BH_p^\wedge \simeq colim(BH_k)_p^\wedge$. □

Proof of Theorem 2.6: Let $\sigma = (P_0, P_1, ..., P_n) \in \Delta(\mathcal{S})$. Let T_0 denote the normal and hence unique maximal torus of P_0. Since $W_G(P_0)$ is finite T_0 is a maximal torus of $N_G(P_0)$ which is also normal in $N_G(P_0)$. Since $P_0 \leq \cap N_G(P_i) \leq N_G(P_0)$, T_0 is a normal maximal torus of $G_\sigma =$

$\cap N_G(P_i)$. Let $\overline{G}_\sigma = G_\sigma/T_0$. As indicated above we can apply Theorem 7.1 to decompose BG_σ. Thus Theorem 2.2 yields

$$BG_p^\wedge \simeq \sum_{\overline{\sigma}\in\Delta(\mathcal{S})/G} (-1)^{dim(\sigma)} (BG_\sigma)_p^\wedge \simeq \sum_{\overline{\sigma}\in\Delta(\mathcal{S})/G} (-1)^{dim(\sigma)} \sum_{\overline{K}\in\mathcal{C}(\overline{G_\sigma})^*} \frac{f_\sigma(K)}{|W_{G_\sigma}(K)|} BK_p^\wedge$$

Collecting terms in this last expression we have

$$\sum_{\overline{\sigma}\in\Delta(\mathcal{S})/G} (-1)^{dim(\sigma)} \sum_{\overline{K}\in\mathcal{C}(\overline{G_\sigma})^*} \frac{f_\sigma(K)}{|W_{G_\sigma}(K)|}\ BK_p^\wedge = \sum_{H\in\mathcal{C}(\mathcal{R})^*} \frac{f(H)}{|W(H)|}\ BH_p^\wedge$$

Thus we obtain a formula for $f(H)$

$$f(H) = \sum_{\substack{\overline{\sigma}\in\Delta(\mathcal{S})/G \\ H\leq G_\sigma}} (-1)^{dim(\sigma)} \sum_{\substack{\overline{K}\in\mathcal{C}(\overline{G_\sigma})^* \\ K\sim_G H}} f_\sigma(K)\frac{|W_G(K)|}{|W_{G_\sigma}(K)|}$$

Lemma 7.2. *Let $J \in \mathcal{C}$. If $K \leq H \leq G$ then*

$$(G/K)^J = \coprod_{g\in(G/H)^J} (H/K)^{J^g}$$

.

Proof. This follows from the bundle $H/K \to G/K \to G/H$ using the usual homeomorphism $\psi : G\times_H H/K \to G/K$ of G-spaces given by $\phi(g, hK) = ghK$. □

It remains to show $f(H)$ satisfies the requisite formula of the theorem. Computing we have

$$\sum_{J\leq H\in\mathcal{C}(\mathcal{R})^*} c_{JH} f(H) =$$

$$\sum_{J\leq H\in\mathcal{C}(\mathcal{R})^*} |(G/H)^J|/|W_G(H)| \sum_{\substack{\overline{\sigma}\in\Delta(\mathcal{S})/G \\ H\leq G_\sigma}} (-1)^{dim(\sigma)} \sum_{\substack{\overline{K}\in\mathcal{C}(\overline{G_\sigma})^* \\ K\sim_G H}} f_\sigma(K)\frac{|W_G(K)|}{|W_{G_\sigma}(K)|}$$

$$= \sum_{\substack{\overline{\sigma}\in\Delta(\mathcal{S})/G \\ J\leq G_\sigma}} (-1)^{dim(\sigma)} \sum_{g\in(G/G_\sigma)^J} \sum_{J^g\leq K\in\mathcal{C}(\overline{G_\sigma})^*} f_\sigma(K)\frac{|(G_\sigma/K)^{J^g}|}{|W_{G_\sigma}(K)|}$$

by Lemma 7.2. Summing over

$$\overline{\sigma}\in\Delta(\mathcal{S})/G,\quad J\leq G_\sigma$$

is equivalent to summing over $\sigma \in \Delta(\mathcal{S})^J$ and dividing by $|(G/G_\sigma)^J|$ which is finite. We note for use below that $\Delta(\mathcal{S})^J$ is thus also finite by Lemma 5.1. Therefore the last expression becomes

$$= \sum_{\sigma \in \Delta(\mathcal{S})^J} \frac{(-1)^{dim(\sigma)}}{|(G/G_\sigma)^J|} \sum_{g \in (G/G_\sigma)^J} \left(\sum_{J^g \leq K \in \mathcal{C}(\overline{G_\sigma})^*} f_\sigma(K) \frac{|(G_\sigma/K)^{J^g}|}{|W_{G_\sigma}(K)|} \right)$$

For each g the expression in parentheses is 1 by the defining property of f_σ. Thus the entire expression simplifies to

$$= \sum_{\sigma \in \Delta(\mathcal{S})^J} (-1)^{dim(\sigma)} = \chi(\Delta(\mathcal{S})^J) = 1$$

by Proposition 4.1(1).

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