# COMPUTATION OF THE EQUIVARIANT 1-STEM BY A DECOMPOSITION OF EQUIVARIANT STABLE HOMOTOPY CLASSES

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ABSTRACT. For any compact Lie group $G$, we give a decomposition of the group $\{X, Y\}_G^k$ of (unpointed) stable $G$-homotopy classes as a direct sum of subgroups of fixed orbit types. This is done by interpreting the $G$-homotopy classes in terms of the generalized fixed point transfer and making use of conormal maps. Finally, we give a full computation of the first equivariant (stable) stem for $G$, $\pi_1^{G\,\mathrm{st}} = \{*, *\}_G^{-1}$.

## 0. INTRODUCTION

A description of the homotopy classes, or of the stable homotopy classes of maps between two topological spaces has been a classical question in topology. Particularly, the stable homotopy classes of (pointed) maps between spheres, namely the so-called stable stems, $\pi_*^{\mathrm{st}}$, have been important objects to study. Historically, via the Brouwer degree theory, the 0-stem was computed, namely $\pi_0^{\mathrm{st}} \cong \mathbb{Z}$. The Hopf map and the Pontryagin theorem provided $\pi_1^{\mathrm{st}} \cong \mathbb{Z}_2$. Nowadays, a lot on this subject is already known and the literature on it amounts to hundreds (maybe thousands) of papers.

A variant of the question arises when we assume that a compact Lie group $G$ acts on all spaces involved and that all the maps considered commute with the group action, that is, that the maps are $G$-equivariant –$G$-maps for short. Then the corresponding question is to provide a description of the stable $G$-homotopy classes between $G$-spaces. Especially, the stable homotopy classes of maps between unit spheres of orthogonal representations pose an important question.

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It is quite easy to show that the negative $G$-stems are zero, that is $\pi_k^{G\,\mathrm{st}} = 0$ if $k < 0$. In 1970, Segal [24], stated that for any finite group $G$, $\pi_0^{G\,\mathrm{st}} \cong A(G)$, where $A(G)$ is the Burnside ring of $G$. This result was proved by Kosniowski [15], and independently by Rubinsztein [22] with a gap that was filled later by Dancer [5]. T. tom Dieck [6] proved the same result for a general compact Lie group $G$, giving a convenient definition of the Burnside ring $A(G)$ for this case.

Lately, the groups $\pi_k^{G\,\mathrm{st}}$, $k > 0$, have been studied intensively by people working on nonlinear analysis since they provide very interesting applications to problems on bifurcations with symmetries (see [9]). Ize et al. have made many computations of $\pi_*^{G\,\mathrm{st}}$ when $G$ is abelian ([9, 10, 11, 12, 13, 14]).

Balanov and Krawcewicz [1] showed for a general compact Lie group $G$ that there is a direct sum decomposition

$$\pi_k^{G\,\mathrm{st}} \cong \bigoplus_{(H)} \Pi_k(H)\,,$$

where $\Pi_k(H)$ denotes the subgroup of $\pi_k^{G\,\mathrm{st}}$ corresponding to the isotropy type $(H)$, for a subgroup $H \subset G$; the sum ranks over all $(H)$ such that $\dim W(H) \leq k$. Here $W(H) = NH/H$ is the Weyl group of $H$. Moreover, this splitting is in the unstable range (see [17] for an alternative proof of this fact), unlike that given in in [16, V.9.1]. Following computations made in [8], where a construction of the equivariant degree is given, one obtains that if $\dim W(H) = k$, then $\Pi_1(H) \cong \mathbb{Z}$ or $\mathbb{Z}_2$, depending on whether $W(H)$ is biorientable or not. On the other hand, in the treatment of $\pi_1^{G\,\mathrm{st}}$ made in [1] it was shown that for $\Pi_1(H) \subset \pi_1^{G\,\mathrm{st}}$, with $\dim W(H) = 0$, there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \Pi_1(H) \longrightarrow W(H)_{\mathrm{ab}} \longrightarrow 0\,, \tag{0.1}$$

where $W(H)_{\mathrm{ab}}$ denotes the abelianization of the Weyl group. In [2], Balanov, Krawcewicz, and Steinlein, using results of Ize and purely algebraic arguments, proved that this sequence splits when $G$ is abelian.

J. Cruickshank [3] has also considered stable equivariant homotopy groups of spheres. One should beware, however, that his concept of equivariant 1-stem differs from that of our first equivariant stem.

In this paper we start giving a decomposition of the group of equivariant stable homotopy classes of maps between two $G$-spaces $X$ and $Y$, provided that $X$ has trivial $G$-action (Theorem 1.8). A similar result was proven by Lewis, Jr., May, and McClure in [16, V.10.1] under other assumptions (they consider more general symmetry but their space $X$ is a finite CW-complex) and using rather different methods. An advantage of our approach is that it gives a short proof showing the geometric interpretation of the maps that form a term of this decomposition, even in the unstable range as in [17]. In particular, we do not need the Adams and Wirthmüller isomorphisms to define the splitting homomorphism. To carry out the decomposition, we use the equivariant fixed point transfer given by the second author in [19], and the fixed point theoretical arguments used in [17].

In the second part of the paper we give another geometrical interpretation of the kernel in the short exact sequence (0.1) and then show that the sequence always splits (Theorem 2.18). This, together with well-known facts, leads to a complete description of $\pi_1^{G\,\mathrm{st}}$ for any compact Lie group $G$ in Theorem 2.6. It is worth to point out that this theorem works in the unstable range, provided that the representation fulfills some conditions (see Proposition 2.17).

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## 1. The general decomposition formula

In this section, we use the generalized fixed point transfer to give a direct sum decomposition of $\{X,Y\}_G^k$. All along the paper, $G$ will denote a compact Lie group. We shall assume that $X$ and $Y$ are metric spaces with a $G$-action.

Definition 1.1. Let $V$, $W$, $M$, and $N$ be finite dimensional real $G$-modules, namely, orthogonal representations of $G$, and let $\rho$ be the element $[M]-[N] \in \mathrm{RO}(G)$. Then the elements of $\{X,Y\}_G^\rho = G$-$\mathfrak{S}tab^\rho(X,Y)$ are *stable homotopy classes* represented by equivariant maps of pairs

$$\alpha : (N\times V, N\times V-0)\times X \longrightarrow (M\times V, M\times V-0)\times Y\,.$$

Such a map will be *stably homotopic* to another

$$\alpha' : (N\times V', N\times V'-0)\times X \longrightarrow (M\times V', M\times V'-0)\times Y\,,$$

if after taking the product of each map with the identity maps of some $(W, W-0)$ and $(W', W'-0)$, respectively, they become $G$-homotopic, where $V\times W \cong_G V'\times W'$. Denote the class of $\alpha$ by $\{\alpha\}$.

Remark 1.2. Taking the product of $X$ with a pair $(L, L-0)$ for some orthogonal representation $L$ of $G$ amounts to the same as smashing $X^+ = X \sqcup \{*\}$ with the sphere $\mathbb{S}^L$ that is obtained as the one-point compactification of $L$ (which is $G$-homeomorphic to the unit sphere $S(L\oplus\mathbb{R})$ in the representation $L\oplus\mathbb{R}$, with trivial action on the last coordinate). Thus $\{X,Y\}_G^\rho \cong \mathrm{colim}_V[\mathbb{S}^N\wedge\mathbb{S}^V\wedge X^+, \mathbb{S}^M\wedge\mathbb{S}^V\wedge Y^+]_G$, where the colimit of pointed $G$-homotopy classes is taken over a cofinal system of $G$-representations $V$.

In [19] (see also [20]) one proves that any $\{\alpha\} \in \{X, Y\}_G^\rho$ can be written as a composite

$$\{\alpha\} = \varphi \circ \tau(f)\,, \tag{1.3}$$

where $\tau(f)$ is the equivariant fixed point transfer of an equivariant fixed point situation

$$\begin{array}{ccc} N \times E \supset \mathcal{U} & \xrightarrow{\quad f \quad} & M \times E \\ & {\scriptstyle p\cdot \mathrm{proj}_E} \searrow \quad \swarrow {\scriptstyle p\cdot \mathrm{proj}_E} & \\ & X\,, & \end{array} \tag{1.4}$$

where $E \longrightarrow X$ is a $G$-ENR$_X$ and the *fixed point set* $\mathrm{Fix}(f) = \{(s,e) \in \mathcal{U} \mid f(s,e) = (0,e) \in M \times E\}$ lies properly over $X$, $\rho = [M] - [N] \in \mathrm{RO}(G)$. The transfer is a stable map

$$\tau(f) : (N \times V, N \times V - 0) \times X \longrightarrow (M \times V, M \times V - 0) \times \mathcal{U}\,,$$

for some orthogonal representation $V$, and $\varphi : \mathcal{U} \longrightarrow Y$ is a nonstable equivariant map (by the localization property of the fixed point transfer, $\mathcal{U}$ can always be assumed to be a very small open $G$-neighborhood of the fixed point set $\mathrm{Fix}(f)$; see [20, 4.4]), (the composite is made after suspending $\varphi$ by taking its product with the identity of $(M \times V, M \times V - 0))$.

We denote by $\mathrm{Or}_G$ the set of orbit types of $G$, that is the set of conjugacy classes $(H)$ of subgroups $H \subset G$. For any $G$-ENR$_X$ $E$, where $X$ has trivial $G$-action, the set of orbit types in $E$, denoted by $\mathrm{Or}_G(E)$, is always finite.

In what follows, we shall only be concerned with the special case $N = \mathbb{R}^n$, $M = \mathbb{R}^{n+k}$, $k \in \mathbb{Z}$, and we shall assume that $X$ is a space with trivial $G$-action.

For the statement of the main result of this section we need the following definitions. The first of them was originally given in [17, 5.4].

Definition 1.5. Consider the fixed point situation (1.4) above. We say that the map $f : \mathcal{U} \longrightarrow \mathbb{R}^{n+k} \times E$ is *conormal* if for every orbit type $(H) \in \mathrm{Or}_G(\mathbb{R}^n \times E) = \mathrm{Or}_G(E)$, there exist an open invariant neighborhood $\mathcal{V}$ of $\mathcal{U}^{(\underline{H})}$ in $\mathcal{U}^{(H)}$ and an equivariant retraction $r : \overline{\mathcal{V}} \longrightarrow \mathcal{U}^{(\underline{H})}$ such that for the restricted map $f^{(H)} = f|_{\mathcal{U}^{(H)}}$ we have

$$f^{(H)}|_{\overline{\mathcal{V}}} = f \circ r : \overline{\mathcal{V}} \longrightarrow \mathbb{R}^{n+k} \times E\,.$$

Here $\mathcal{U}^{(H)}$ consists of the points in $\mathcal{U}$ with isotropy larger than $(H)$ and $\mathcal{U}^{(\underline{H})}$ to those with isotropy **strictly** larger than $(H)$.

Definition 1.6. For any subgroup $H \subset G$, we define the subgroup $\{X,Y\}^k_{(H)}$ of $\{X,Y\}^k_G$ as the subgroup of those classes $\{\alpha\}$ such that $\{\alpha\} = \varphi \circ \tau(f)$, where

(a) $f$ is a conormal map, and
(b) $\mathrm{Fix}(f) \subset \mathcal{U}_{(H)}$, where $\mathcal{U}_{(H)}$ consists of the points in $\mathcal{U}$ with isotropy group conjugate to $H$.

Remark 1.7. The fact that $\{X,Y\}^k_{(H)}$ is a subgroup of $\{X,Y\}^k_G$ follows easily by observing that both properties (a) and (b) are preserved by the sum of two elements $\{\alpha\} = \varphi \circ \tau(f)$, $\{\beta\} = \psi \circ \tau(g)$, that, by the additivity property of the fixed point transfer, corresponds to the disjoint union $f+g$ of the fixed point situations (see [21, 1.17]).

The main result in this section is the following.

**Theorem 1.8.** *Let $X$ be a space with trivial $G$-action. Then there is an isomorphism*

$$\{X,Y\}^k_G \cong \bigoplus_{(H)} \{X,Y\}^k_{(H)} .$$

For the proof we need some preliminary results. Consider a fixed point situation as (1.4). First note that it is always possible to provide $\mathrm{Or}_G(E)$ with an order $(H_j)$, $j = 1,2,\dots,l$ such that $(H_i) \subset (H_j)$ implies $j \leq i$. Define $E_i \subset E$ as $\bigcup_{i \leq j} E^{(H_j)}$. These $G$-subspaces determine a filtration of $E$ such that $E_i - E_{i-1} = E_{(H_i)}$. Let $f_i = f|_{\mathcal{U}_i} : \mathcal{U}_i \longrightarrow \mathbb{R}^{n+k} \times E_i$, where $\mathcal{U}_i = \mathcal{U} \cap (\mathbb{R}^n \times E_i)$.

**Proposition 1.9.** *For every $i = 1,2,\dots,l$ there exists an invariant neighborhood $\mathcal{V}_i$ of $E_{i-1}$ in $E_i$ and an equivariant retraction $r_i : \overline{\mathcal{V}}_i \longrightarrow E_{i-1}$ that is admissibly homotopic to the identity. Thus $f_i$ is admissibly homotopic to $f'_{i-1} = f_{i-1} \circ (\mathrm{id}_{\mathbb{R}^n} \times r_i)$.*

The proof is similar to those of [17, 5.3 and 5.7]. □

**Proposition 1.10.** *The following hold:*

(a) *$f$ is equivariantly homotopic by an admissible homotopy $f_\tau$ to a conormal map $f' = f_1 : V \longrightarrow \mathbb{R}^m \times E$. Moreover, if $A \subset \mathcal{U}$ is a closed $G$-ENR subspace, then this homotopy can be taken relative to $A$.*
(b) *Furthermore, if $f_0$ and $f_1$ are equivariantly homotopic by an admissible homotopy, and each of them is equivariantly homotopic by an admissible homotopy to two conormal maps $f'_0$, $f'_1 : \mathcal{U} \longrightarrow \mathbb{R}^m \times E$, respectively, then these two conormal maps are equivariantly homotopic by an admissible conormal homotopy.*

The proof is the same as that of [17, 5.7] (see also [21, 2.10 and 2.11] or [25, II.6.8 and III.5.2]). □

We also need a lemma.

**Lemma 1.11.** *Let* $f : \mathcal{U} \longrightarrow \mathbb{R}^{n+k} \times E$ *be a fixed point situation over* $X$ *such that* $f$ *is a conormal map and take* $(H) \in \mathrm{Or}_G(E)$*. Then there is a neighborhood* $\mathcal{V}$ *of* $\mathrm{Fix}(f|_{\mathcal{U}_{(H)}})$ *such that* $g = f|_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathbb{R}^{n+k} \times E$ *is a conormal map with* $\mathrm{Fix}(g) = \mathrm{Fix}(f|_{\mathcal{U}_{(H)}})$*. Denote* $g$ *by* $f_{(H)}$*. Consequently,*

$$\tau(f) = \sum_{(H)\in \mathrm{Or}_G(E)} \tau(f_{(H)})\,. \tag{1.12}$$

*Proof.* Since $f$ is conormal, the set $F = \mathrm{Fix}(f|_{\mathcal{U}_{(H)}})$ is separated from all other fixed points. Then there is a neighborhood $\mathcal{V}$ of $F$ in $\mathcal{U}$ such that $\mathrm{Fix}(f) \cap \mathcal{V} = F$. Hence $g = f|_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathbb{R}^{n+k} \times E$ is a conormal map with the desired properties. By the additivity property of the transfer we obtain the decomposition (1.12). □

We now pass to the proof of Theorem 1.8.

*Proof.* Any $\{\alpha\} \in \{X, Y\}^k_G$ can be written as the composite (1.3) $\varphi \circ \tau(f)$, where $\tau(f)$ is the equivariant fixed point transfer of an equivariant fixed point situation (1.4). By Proposition 1.10 (a), $f$ can be assumed to be a conormal map, and by Lemma 1.11, $\tau(f) = \sum_{(H)\in \mathrm{Or}_G(E)} \tau(f_{(H)})$. Defining $\{\alpha_{(H)}\}$ by $\{\alpha_{(H)}\} = \varphi|_{\mathcal{U}_{(H)}} \circ \tau(f_{(H)})$, we have immediately

$$\{\alpha\} = \sum_{(H)\in \mathrm{Or}_G(E)} \{\alpha_{(H)}\}\,, \tag{1.13}$$

where $\{\alpha_{(H)}\} \in \{X, Y\}^k_{(H)}$. So, by Proposition 1.10 (b), we may define

$$\Phi : \{X, Y\}^k_G \longrightarrow \bigoplus_{(H)} \{X, Y\}^k_{(H)} \quad \text{by} \quad \Phi(\{\alpha\}) = \oplus_{(H)}\{\alpha_{(H)}\}\,.$$

If $(H) \neq (K)$, then $\{X, Y\}^k_{(H)} \cap \{X, Y\}^k_{(K)} = 0$ as easily follows with the same argument used in the proof of [17, 6.2]. Thus we may also define

$$\Psi : \bigoplus_{(H)} \{X, Y\}^k_{(H)} \longrightarrow \{X, Y\}^k_G \quad \text{by} \quad \Psi(\oplus_{(H)}\{\alpha_{(H)}\}) = \sum_{(H)} \{\alpha_{(H)}\}\,.$$

Then $\Phi$ and $\Psi$ are inverse isomorphisms. □

## 2. Computation of the first $G$-stem

In this section, we compute the first equivariant stem for any compact Lie group $G$.

Given any orthogonal representation $V$ of $G$, $\mathbb{S}^V$ will denote the one-point compactification of $V$ with the induced $G$-action (see Remark 1.2).

Definition 2.1. We define the *kth equivariant stem for a compact Lie group $G$* or briefly the *kth $G$-stem*, $k = 0, 1, 2, \ldots$, by

$$\pi_k^{G\,\mathrm{st}} = \operatorname*{colim}_V\,[\mathbb{S}^{V+k}, \mathbb{S}^V]_G\,.$$

where $V$ varies along a cofinal set of orthogonal $G$-representations and $[-,-]_G$ denotes the set of pointed $G$-homotopy classes of pointed $G$-maps. Of course, $V+k$ denotes the orthogonal representation $V \oplus \mathbb{R}^k$ with $G$ acting trivially in the second summand.

Remark 2.2. According to Definition 1.1 and to Remark 1.2, we have that the elements of $\pi_k^{G\,\mathrm{st}}$ are represented by maps of pairs

$$\alpha : (V \times \mathbb{R}^k, V \times \mathbb{R}^k - 0) \longrightarrow (V, V - 0)\,,$$

for some orthogonal representation $V$ of $G$. That is $\pi_k^{G\,\mathrm{st}} = \{*,*\}_G^{-k}$ in our notation of the previous section. In other words, $\pi_k^{G\,\mathrm{st}}$ is the $k$th homotopy group of the infinite loop space $Q_G = \Omega_G^\infty \mathbb{S}^\infty = \operatorname{colim}_V \Omega_V \mathbb{S}^V$, where $\Omega_V \mathbb{S}^V = \operatorname{Maps}_G(\mathbb{S}^V, \mathbb{S}^V)$.

Denote by $\Pi_k(H)$ the subgroup $\{*,*\}_{(H)}^{-k}$ of $\pi_k^{G\,\mathrm{st}}$ as in Definition 1.6. Let, moreover, $W(H)$ denote the *Weyl group* of $H \subset G$, defined by $W(H) = NH/H$, where $NH \subset G$ is the normalizer of $H$ in $G$. In the rest of the paper, we denote $W(H)$ by $\Gamma_H$ or simply by $\Gamma$ if there is no danger of confusion.

For the $k$th $G$-stem, one has a stronger form of Theorem 1.8 derived using an equivariant transversality argument in [1, 2.7]; namely

$$\pi_k^{G\,\mathrm{st}} \cong \bigoplus_{\substack{(H)\in \mathrm{Or}_G \\ \dim \Gamma \leq k}} \Pi_k(H)\,. \tag{2.3}$$

Recall that a compact Lie group $\Gamma$ is said to be *biorientable* if it has an orientation invariant under left and right translations (see [1], [8], or [18]). From considerations in [8] (see also [18]) the following can be proved.

**Proposition 2.4.** *Let* $\dim \Gamma = k$. *Then*

$$\Pi_k(H) \cong \begin{cases} \mathbb{Z} & \text{if } W(H) \text{ is biorientable,} \\ \mathbb{Z}_2 & \text{otherwise.} \end{cases}$$

□

NOTE 2.5. For instance, a compact Lie group $\Gamma$ is biorientable if it is either finite, abelian, or connected (cf. [8]). The simplest nonbiorientable group (of dimension 1) is O(2).

In what follows, we shall compute the subgroups $\Pi_1(H)$ of the first $G$-stem to obtain the main result of this section, as follows.

**Theorem 2.6.** *There is a sum decomposition of the first $G$-stem*

$$\pi_1^{G\,\mathrm{st}} = \bigoplus_{\substack{(H)\in \mathrm{Or}_G \\ \dim W(H)\leq 1}} \Pi_1(H)\,.$$

*Here, if* $\dim W(H) = 0$,

$$\Pi_1(H) \cong \mathbb{Z}_2 \oplus W(H)_{\mathrm{ab}}\,, \tag{2.7}$$

*where $W(H)_{\mathrm{ab}}$ is the abelianization of $W(H)$, and, if* $\dim W(H) = 1$,

$$\Pi_1(H) \cong \begin{cases} \mathbb{Z} & W(H) \text{ is biorientable,} \\ \mathbb{Z}_2 & \text{if } W(H) \text{ is not biorientable.} \end{cases} \tag{2.8}$$

In view of Proposition 2.4, we only need to prove Equation (2.7). For doing this, we shall make some general considerations.

ASSUMPTION 2.9. $V$ denotes a $G$-module and the elements in $\pi_k^{G\,\mathrm{st}}$ are represented by maps $(V\times\mathbb{R}^{l+k}, V\times\mathbb{R}^{l+k}-0) \longrightarrow (V\times\mathbb{R}^{l}, V\times\mathbb{R}^{l}-0)$, with $l \geq k+3$, where $G$ acts trivially on the second factor.

Note that the Weyl group $\Gamma$ of $H \subset G$ acts effectively on $V^H$. We denote by $U$ the representation $V^H\times\mathbb{R}^{l+k}$ of $\Gamma$, with the obvious action, and by $U_0$ the representation $V^H\times\mathbb{R}^l$. Let $(P)$ be the principal orbit type of the action of $\Gamma$ on $U$, and let $U_P = U - S$, where $S$ consists of all points in $U$ with isotropy group type different from $(P)$ (see [6]).

NOTE 2.10.

1. The set $U_P$ is in general disconnected; however, it is connected, provided that $\dim(U - U_P) \leq \dim U - 2$. This holds if $\dim U^{\Gamma'} \leq \dim U - 2$ for any $(\Gamma') > (P)$, and this can always be attained in the stable range. For this, it is enough to replace $V$ by $V \oplus V$.
2. Even being $U_P$ connected, it need not be simply connected. By Lefschetz duality, $U_P$ will be simply connected if $\dim(U - U_P) \leq \dim U - 3$. This holds if $\dim U^{\Gamma'} \leq \dim U - 3$ for any $(\Gamma') > (P)$. For this, it is enough to replace $V$ by $V \oplus V \oplus V$.
3. Increasing the size of $V$ further (summing again with itself) we may also assume that $U_P$ has an orientation-preserving $\Gamma$-action.

Denote by $\Omega_k^{\Gamma\,\mathrm{fr}}(U_P)$ the group of bordism classes of $\Gamma$-framed $k$-submanifolds of $U_P$. One has the following result of Balanov and Krawcewicz.

**Proposition 2.11.** ([1, 3.2]) *Let* $\dim\Gamma \leq k$. *Then* $\Pi_k(H) \cong \Omega_k^{\Gamma\,\mathrm{fr}}(U_P)$. □

To focus on the proof of Equation (2.7), assume in what follows that $\dim\Gamma = 0$; that is, $\Gamma$ is a finite group. Note that $\Gamma$ acts effectively on $U$, but since $\Gamma$ is finite, the principal orbit type corresponds to trivial isotropy, i.e., the action of $\Gamma$ on $U_P$ is in fact free.

Let $\overline{U}_P$ denote the quotient space $U_P/\Gamma$. If $[M,\eta] \in \Omega_k^{\mathrm{fr}\,\Gamma}(U_P)$, then $M$ is a framed $\Gamma$-submanifold of $U_P$ (and $\eta$ is a $\Gamma$-trivialization of the normal bundle), and thus $\Gamma$ acts freely on $M$ and $\overline{M} = M/\Gamma$ is an oriented submanifold of $\overline{U}_P$. Consequently, there is a homomorphism $\Phi_k : \Omega_k^{\Gamma\,\mathrm{fr}}(U_P) \longrightarrow \Omega_k(\overline{U}_P)$, where $\Omega_k$ denotes the usual oriented bordism of $k$-submanifolds. The image of the canonical homomorphism $\Omega_k^{\Gamma\,\mathrm{fr}}(U_P) \longrightarrow \Omega_k^{\mathrm{fr}}(U_P)$ lies inside $\Omega_k^{\mathrm{fr}}(U_P)^\Gamma$, where $\Omega_k^{\mathrm{fr}}(U_P)$ has the action of $\Gamma$ induced by that on $U_P$. By the Steenrod–Thom theorem, we know that there is a homomorphism $\Omega_k(\overline{U}_P) \longrightarrow H_k(\overline{U}_P;\mathbb{Z})$ that is an isomorphism for $k \leq 3$ and an epimorphism for $k = 4$ (see [23]). In the case $k = 1$, that we are concerned with, we thus have an isomoprphism.

An essential step in deriving $\Pi_1(H)$ when $\dim\Gamma = 0$ was done in [1, 4.3], where details on the previous comments can be seen; namely we have the following.

**Theorem 2.12.** $\ker\Phi \cong \mathbb{Z}_2$ *and thus one has a short exact sequence*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \Omega_1^{\Gamma\,\mathrm{fr}}(U_P) \xrightarrow{\Phi} H_1(\overline{U}_P;\mathbb{Z}) \longrightarrow 0\,. \tag{2.13}$$

*Moreover,* $\ker\Phi$ *consists of those bordism classes of* $G$*-framed invariant manifolds* $[M,\eta] \in \Omega_1^{\Gamma\,\mathrm{fr}}(U_P)$*, where* $M \approx_{\mathrm{diff}} \mathbb{S}^1$ *and* $\eta$ *is an equivariant trivialization of the normal bundle such that the quotient manifold* $\overline{M} = M/\Gamma \subset \overline{U}_P$*,* $\overline{M} \approx \mathbb{S}^1$*, is nullbordant.* □

In [2, 2.4], it is shown that if $G$ is abelian, then the sequence (2.13) splits. Their argument is purely algebraic and makes use of the computation in [12] of $\Pi_1(H)$ as a product of $p$-factors, $p$ prime. We show in what follows that (2.13) **always** splits.

Note 2.14. There is an isomorphism

$$\Omega_*^{\Gamma\,\mathrm{fr}}(\Gamma x) = \Omega_*^{\Gamma\,\mathrm{fr}}(\Gamma) \cong \Omega_*^{\mathrm{fr}}(*)\,, \tag{2.15}$$

that is a consequence of the following well-known fact (see [6]). Namely, there is a bijection $[\mathbb{S}^V \wedge X, \mathbb{S}^V \wedge Y \wedge \Gamma^+]_\Gamma \cong [\mathbb{S}^V \wedge X, \mathbb{S}^V \wedge Y]$, that provides the isomorphism (2.15), since the homology theory $\Omega_*^{\Gamma\,\mathrm{fr}}$ is equivalent to the theory $\pi_*^{\Gamma\ \mathrm{st}}$. In particular,

$$\Omega_1^{\Gamma\ \mathrm{fr}}(\Gamma x) \cong \Omega_1^{\mathrm{fr}}(*) \cong \mathbb{Z}_2 .$$

**Lemma 2.16.** *Take $x \in U_P$. If $i_x^\Gamma : \Gamma \cong \Gamma x \hookrightarrow U_p$ is the inclusion and $i_* = i_{x*}^\Gamma : \Omega_1^{\Gamma\ \mathrm{fr}}(\Gamma) \longrightarrow \Omega_1^{\Gamma\ \mathrm{fr}}(U_P)$ is the induced homomorphism, then*

$$\ker \Phi = \mathrm{im}(i_*) .$$

*Proof.* Recall first that $d = \dim U_P = \dim \overline{U}_P \geq 3$ (see Assumption 2.9), and assume that we have a metric on $U_P$ that is $\Gamma$-invariant and take $\varepsilon > 0$ sufficiently small, that $\pi^{-1}(D_\varepsilon(\overline{x})) = \bigsqcup_{\gamma\in\Gamma} \gamma D_\varepsilon(x) \approx \Gamma \times D_\varepsilon(x)$, where $D_\varepsilon$ denotes the corresponding $d$-balls of radius $\varepsilon$, and let $\overline{M}$ be the boundary $\partial D^2_{\varepsilon/2}(\overline{x}) \subset \overline{U}_P$ of a 2-disk of radius $\varepsilon/2$ contained in $D_\varepsilon(\overline{x})$. Hence $\overline{M}$ is diffeomorphic to $\mathbb{S}^1$.

Let $\overline{\eta}_0, \overline{\eta}_1 : \nu(\overline{M}) \longrightarrow \overline{M} \times \mathbb{R}^{d-1}$ be trivializations of the normal bundle of $\overline{M}$ such that $[\overline{M}, \overline{\eta}_0] = 0 \in \Omega_1^{\mathrm{fr}}(D_\varepsilon(\overline{x}))$ and $[\overline{M}, \overline{\eta}_1] \neq 0 \in \Omega_1^{\mathrm{fr}}(D_\varepsilon(\overline{x}))$. Let $j_x : D_\varepsilon(\overline{x})) \longrightarrow D_\varepsilon(x))$ be the inverse diffeomorphism to that induced by $\pi$, and call $M_x = j_x(\overline{M})$. Define $M = \bigsqcup_{\gamma\in\Gamma} \gamma M_x \subset U_P$. $M$ is homeomorphic to $\Gamma \times \overline{M}$.

Note that $\nu(M_x) \subset D_\varepsilon(x)$ is diffeomorphic to $\nu(\overline{M})$ via the map $(m, v) \mapsto (\pi(m), D\pi(m)v)$. On the other hand,

$$\gamma M_x \subset \gamma D_\varepsilon(x) = D_\varepsilon(\gamma x) \quad \text{and} \quad \nu(\gamma M_x) = \gamma(\nu(M_x)) ,$$

since $\gamma$ induces a diffeomorphism, because it is a linear orthogonal map. Consequently, the tubular neighborhood

$$\nu(M) = \bigsqcup_{\gamma\in\Gamma} \gamma(\nu(M_x)) ,$$

and thus we can define an equivariant trivialization $\eta_i : \nu(M) \longrightarrow M \times U_0$, $i = 0, 1$, by

$$\eta_i(\gamma m, \gamma v) = (\gamma m, \gamma\overline{\eta}_i(\pi(m), D\pi(m)v))$$

for $m \in M_x$ and $v \in \nu_m(M_x)$. Observe that $\eta_i$ is equivariant, since for $\mu \in \Gamma$ we have

$$\begin{aligned} \eta_i(\mu(\gamma m, \gamma v)) &= \eta_i((\mu\gamma)m, (\mu\gamma)v) \\ &= ((\mu\gamma)m, (\mu\gamma)(\overline{\eta}_i(\pi(m), D\pi(m)v))) \\ &= (\mu(\gamma m), \mu(\gamma\overline{\eta}_i(\pi(m), D\pi(m)v))) \\ &= \mu\eta_i(\gamma m, \gamma v) . \end{aligned}$$

Hence we get that $[M,\eta_0],[M,\eta_1] \in \Omega_1^{\Gamma\ \mathrm{fr}}(\Gamma\times D_\varepsilon(x)) \subset \Omega_1^{\Gamma\ \mathrm{fr}}(U_P)$. Consequently, $\overline{M} = M/\Gamma$ is nullbordant, thus implying that $[M,\eta_0],[M,\eta_1] \in \ker\Phi$. By construction, they lie in $\mathrm{im}(i_*) = \mathrm{im}(i^\Gamma_{D_\varepsilon\,*})$, where $i^\Gamma_{D_\varepsilon} : \Gamma\times D_\varepsilon \hookrightarrow U_P$, and obviously, $[M,\eta_1] \neq 0$ in $\Omega_1^{\Gamma\ \mathrm{fr}}(U_P)$. □

**Proposition 2.17.** *If $\Gamma$ is finite, $U_P$ is connected and the action of $\Gamma$ preserves the orientation, then*

$$\Omega_1^{\Gamma\ \mathrm{fr}}(U_P) \cong \mathbb{Z}_2 \oplus H_1(\overline{U}_P;\mathbb{Z})\,.$$

*If, moreover, $U_P$ is simply connected, then*

$$\Pi_1(H) \cong \mathbb{Z}_2 \oplus \Gamma_{\mathrm{ab}}\,.$$

*Proof.* First observe that for any manifold $U$ with a free action of a finite group $\Gamma$, and its projection $\pi : U \longrightarrow U/\Gamma$ onto its orbit space, there is a well-defined transfer homomorphism $\pi! : \Omega_*^{\mathrm{fr}}(U/\Gamma) \longrightarrow \Omega_*^{\Gamma\ \mathrm{fr}}(U)$ given as follows.

- Every manifold $M \subset U/\Gamma$ that represents an element in $\Omega_*^{\mathrm{fr}}(U/\Gamma)$ is mapped by $\pi!$ to the $\Gamma$-invariant submanifold $\widetilde{M} = \pi^{-1}M \subset U$.
- Observe that the tangent-bundle morphism $(\pi, D\pi) : \tau(U) \longrightarrow \tau(U/\Gamma)$ provides an isomorphism $\tau(U) \cong \pi!(\tau(U/\Gamma))$, where here $\pi!$ denotes the transfer on bundles (in KO-theory). In particular, we have $\tau(U)|_{\widetilde{M}} \cong \pi!(\tau(U/\Gamma))|_M = \pi!(\tau(M)) \oplus \pi!(\nu(M)) \cong \tau(\widetilde{M}) \oplus \nu(\widetilde{M})$, where $\nu$ stands for the normal bundle, as before.
- Note that for any bundle $\xi$, $\pi!(\xi)$ is always a $\Gamma$-bundle. Moreover, if $\xi$ is trivial, then $\pi!(\xi)$ is $\Gamma$-trivial. Besides, if $\eta = \{e_i(x)\}_{i=1}^k$, $e_i(x) : E_x(\xi) \longrightarrow V$, is a frame for the trivialization of $\xi$, then $\widetilde{\eta} = \{\widetilde{e}_i(x) = e_i(\pi(x))\}_{i=1}^k$ is a frame for the $\Gamma$-trivialization of $\pi!(\xi)$. In the special case of the tangent bundle $\tau(U)$ and its subbundles, the $\Gamma$-trivialization is obtained from the equality $\pi\circ\gamma = \pi$, for any $\gamma \in \Gamma$. Thus $D\pi \circ D\gamma = D\pi$.
- Consequently, an element $[M,\eta] \in \Omega_*^{\mathrm{fr}}(U/\Gamma)$ determines an element $[\widetilde{M},\widetilde{\eta}] \in \Omega_*^{\Gamma\ \mathrm{fr}}(U)$, since a bordism between two such manifolds lifts in the same way.
- Note that $\pi! : \Omega_*^{\mathrm{fr}}(\Gamma x/\Gamma) = \Omega_*^{\mathrm{fr}}(*) \longrightarrow \Omega_*^{\Gamma\ \mathrm{fr}}(\Gamma x) = \Omega_*^{\Gamma\ \mathrm{fr}}(\Gamma)$ is the inverse of the canonical isomorphism given above in Note 2.14.

Consider now the following diagram that commutes by the very definition of the homomorphisms.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_1^{\mathrm{fr}}(*) & \xrightarrow{j_*} & \Omega_1^{\mathrm{fr}}(\overline{U}_P) & \xrightarrow{\Psi} & H_1(\overline{U}_P;\mathbb{Z}) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \pi!} & & \Big\downarrow{\scriptstyle \pi!} & & \Big\downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & \Omega_1^{\Gamma\ \mathrm{fr}}(\Gamma) & \xrightarrow[i_*]{} & \Omega_1^{\Gamma\ \mathrm{fr}}(U_P) & \xrightarrow[\Phi]{} & H_1(\overline{U}_P;\mathbb{Z}) & \longrightarrow & 0\,,
\end{array}$$

where the horizontal homomorphisms on the left-hand side are induced by inclusions, $\Phi$ is the homomorphism given by Balanov and Krawcewicz described above, and $\Psi$ is the forgetful homomorphism mapping to oriented bordism $\Omega_1(\overline{U}_P) = H_1(\overline{U}_P;\mathbb{Z})$ also mentioned above. By the commutativity of the diagram and the five lemma, the homomorphism $\pi!$ in the middle is an isomorphism.

The homomorphism $j_*$ on the top row splits by $s^* : \Omega_1^{\mathrm{fr}}(\overline{U}_P;\mathbb{Z}) \longrightarrow \Omega_1^{\mathrm{fr}}(*)$ induced by $s : \overline{U}_P \longrightarrow *$. This gives us the splitting homomorphism $\sigma : \Omega_1^{\Gamma\ \mathrm{fr}}(U_P) \longrightarrow \Omega_1^{\Gamma\ \mathrm{fr}}(\Gamma)$, that is explicitly given by $\sigma = \pi! \circ s_* \circ \pi!^{-1}$. □

Therefore, we have the following.

**Theorem 2.18.** *The short exact sequence*

$$0 \longrightarrow \Omega_1^{\Gamma\ \mathrm{fr}}(\Gamma x) \underset{i_*}{\overset{\sigma}{\rightleftarrows}} \Omega_1^{\Gamma\ \mathrm{fr}}(U_P) \xrightarrow{\Phi} H_1(\overline{U}_P;\mathbb{Z}) \longrightarrow 0\,.$$

*splits.* □

Combining 2.11 and 2.12 with the previous theorem, we obtain our main theorem 2.6.

## 3. Some examples of the first $G$-stem

To finish the paper, we discuss briefly some examples of Theorem 2.6.

Examples 3.1.

1. Let $G = \mathbf{1}$ be the trivial group. Then there is only one $H \subset G$ and $W(H) = G/H = \mathbf{1}$ has dimension 0. Thus $\pi_1^{\mathrm{st}} = \mathbb{Z}_2$.
2. Historically, the first case of $\pi_1^{G\,\mathrm{st}}$ described was for $G = \mathbb{S}^1$, when
$$\pi_1^{\mathbb{S}^1\,\mathrm{st}} \cong \mathbb{Z}_2 \oplus \bigoplus_{H \subset \mathbb{S}^1} \mathbb{Z}\,,$$
and was given by Dylawerski [7].

3. Let $G$ be a finite group. Then for every $H \subset G$, $\dim W(H) = 0$. Thus

$$\pi_1^{G\,\mathrm{st}} \cong \bigoplus_{(H)\in \mathrm{Or}(G)} (\mathbb{Z}_2 \oplus W(H)_{\mathrm{ab}})\,. \tag{3.2}$$

If $G$ is abelian, then $W(H) = G/H$ and thus

$$\pi_1^{G\,\mathrm{st}} \cong \bigoplus_{H\subset G} (\mathbb{Z}_2 \oplus G/H)\,. \tag{3.3}$$

Particular special cases are $G = \mathbb{Z}_p$, where $p$ is prime. Then

$$\pi_1^{\mathbb{Z}_p\,\mathrm{st}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_p\,.$$

We leave to the reader the verification that the description (3.3) agrees with the decomposition in terms of prime factors of $G$ given by Ize and Vignoli [12, 13, 14].

4. Let $G$ be either $\mathrm{O}(2)$ or $\mathrm{SO}(3)$. Then $G$ has infinitely many conjugacy classes of closed subgroups $H$ such that $W(H)$ is finite (see [4, IV.(4.10) Ex.9]). Thus $\pi_1^{G\,\mathrm{st}}$ has infinitely many $\mathbb{Z}_2$-summands and for each of them also a $W(H)_{\mathrm{ab}}$-summand (see [2] for further details on the case $G = \mathrm{SO}(3)$).
5. Let $G = \mathrm{O}(k)$. Then $G$ has infinitely many finite conjugacy classes of subgroups $H$ generated by reflections such that $W(H)$ is finite (see [4, V.(2.18) Ex.6]). Thus, as in the previous example, $\pi_1^{\mathrm{O}(k)\,\mathrm{st}}$ has infinitely many $\mathbb{Z}_2$-summands and for each of them also a $W(H)_{\mathrm{ab}}$-summand.

Remark 3.4. Examples 3. and 4. above show that in general, for non-abelian compact Lie groups $G$, $\pi_1^{G\,\mathrm{st}}$ is an infinite –and quite complicated– group. We conjecture that the only case when the first stem is finite is for a finite group, that is

$$\pi_1^{G\,\mathrm{st}} \text{ is finite if and only if } G \text{ is finite.}$$

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