On the chain-level intersection pairing for PL manifolds.

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1 Introduction.

Let M be a compact oriented PL manifold. The chain-level intersection pairing was introduced by Lefschetz in [9] as a tool for constructing the intersection pairing on the homology of M. The chain-level pairing is a basic ingredient in Chas and Sullivan's construction [2] of a Batalin-Vilkovisky structure on the homology of the free loop space of M.

For a complete understanding of the construction in [2], it seems helpful to have the following theorem. Let C_*M be the PL chain complex of M (see Section 3 for the definition). Let us say that a subcomplex of a chain complex is *full* if the inclusion map is a quasi-isomorphism.

Theorem 1.1. *The domain of the chain-level intersection pairing is a full subcomplex of* $C_*M \otimes C_*M$.

It might seem at first that something like Theorem 1.1 would have been needed already by Lefschetz to define the intersection pairing on homology, but for that purpose two weaker facts suffice:

(i) For any cycles C and D in C_*M, $C \otimes D$ is homologous to an element in the domain of the intersection pairing.

(ii) If C', D' are two other cycles with $C' \otimes D'$ homologous to $C \otimes D$, then $C' \otimes D' - C \otimes D$ is the boundary of an element in the domain of the intersection pairing. (This is needed to show that the intersection pairing on homology is well-defined.)

Theorem 1.1 is harder to prove than (i) and (ii) because (among other reasons) a cycle in $C_*M \otimes C_*M$ cannot in general be written in the form $\sum C_i \otimes D_i$ with C_i and D_i cycles.

One goal of this paper is to prove Theorem 1.1 and, more generally, the analogous statement for the k-fold iterate of the intersection pairing; see Proposition 12.2 and Remark 12.3.

It seems useful to go farther and to show that the intersection pairing gives C_*M a structure of "partially defined commutative DGA;" this is the second (and main) goal of

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this paper (see Theorem 12.1). A consequence of this (to be proved in [13]) is that C_*M is canonically quasi-isomorphic to an E_∞ chain algebra.

The third goal of this paper is to give a new treatment of the chain-level intersection pairing, based on the account in [5] but with some improvements (in particular, the version given here avoids the need for the rather complicated arguments in [5, Section 7] by making fuller use of the technical felicities of [3]).

The results of this paper will be applied in [13] to prove two theorems about the Chas-Sullivan operations. Let S_* denote the singular chain functor and let $\mathfrak{F}$ be the framed little 2-disks operad [4].

Theorem A. *The Batalin-Vilkovisky structure on the homology of LM is induced by a natural action of an operad quasi-isomorphic to $S_*\mathfrak{F}$ on a chain complex quasi-isomorphic to $S_*(LM)$.*

(Theorem A is the analog for $H_*(LM)$ of Deligne's Hochschild cohomology conjecture [12, Section I.1.19].)

Theorem B. *The Eilenberg-Moore spectral sequence converging to the homology of LM is a spectral sequence of Batalin-Vilkovisky algebras.*

The paper is organized as follows.

Section 2 gives a brief discussion of the definitions of the chain-level intersection pairing given in [9], [10] and [5], and explains why these versions of the definition are not convenient as a starting-point for proving Theorem 1.1.

Section 3 recalls the definition of the PL chain complex of a PL space.

Section 4 recalls from [5] a method for making chain-level constructions by using relative homology.

Section 5 constructs the backwards (Umkehr) map in relative homology induced by a PL map between compact oriented PL manifolds. In Section 6 a chain-level backwards map is deduced from this, using the method of Section 4.

Section 7 recalls the definition of exterior product for PL chains.

In Section 8, the chain-level intersection pairing is defined as the composite of the exterior product and the chain-level backwards map induced by the diagonal map; the motivation for this definition is that the intersection of two subsets of a set S is the intersection of their Cartesian product with the diagonal in $S \times S$.

Section 9 gives the formal definition of "partially defined commutative algebra." I use Leinster's concept of homotopy algebra [11] for this purpose rather than the Kriz-May definition of partial algebra [8] (but I will use the term "Leinster partial algebra" instead of "homotopy algebra," since the latter term seems excessively generic). Leinster's definition seems to have all of the advantages and none of the disadvantages of the Kriz-May definition (cf. Remark 9.4(b)).

Sections 10–13 give the proof that the intersection pairing and its iterates determine a Leinster partial commutative DGA structure on C_*M. The proof uses a general-position result which is proved in Sections 14 and 15.

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2 The Lefschetz and Goresky-MacPherson definitions of the chain-level intersection pairing.

This section is not needed logically for the rest of the paper; it is offered as motivation for Sections 3–8. The reader may also find it helpful to consult Steenrod's account of Lefschetz's work on the intersection pairing ([16, pages 28–30]).

This section uses some technical terms which will be defined in Sections 3–7.

Lefschetz's first account of the chain-level intersection pairing $C \cdot D$ was in [9]. In this paper he uses the obvious definition: if $C = \sum m_i \sigma_i$ and $D = \sum n_i \tau_i$ then

$$C \cdot D = \sum \pm\, m_i n_j\, \sigma_i \cap \tau_j, \tag{2.1}$$

where the signs are determined by the orientations of σ_i, τ_j and M. This formula does not in fact give a chain unless all of the intersections $\sigma_i \cap \tau_j$ have the same dimension, so some restriction on the pair (C, D) is necessary. Generically, the intersection of a p-dimensional PL subspace and a q-dimensional PL subspace has dimension $\leq p + q - \dim M$; pairs of PL subspaces with this property are said to be in *general position*. Lefschetz restricts the domain of the intersection pairing to pairs (C, D) for which all of the pairs (σ_i, τ_j) are in general position, and he interprets terms $\sigma_i \cap \tau_j$ which are in dimension less than $\dim C + \dim D - \dim M$ as 0.

In order to prove the crucial formula

$$\partial(C \cdot D) = (\partial C) \cdot D \,\pm\, C \cdot \partial D, \tag{2.2}$$

Lefschetz imposes a further restriction on the domain of the intersection pairing: he requires that all of the pairs $(\partial\sigma_i, \tau_j)$ and $(\sigma_i, \partial\tau_j)$ should also be in general position.[1] This assumption allows him to prove equation (2.2) by working with one pair of simplices at a time and extending additively.

This definition has the disadvantage that the domain of the intersection pairing is not invariant under subdivision. For example, if σ and τ are 1-simplices in a 2-manifold which intersect at a point in the interior of both, then the pair (σ, τ) is in the domain, but if we subdivide σ and τ at the intersection point we obtain a pair of chains $(\sigma' + \sigma'', \tau' + \tau'')$ which is not in the domain (because for example the pair $(\partial\sigma', \tau')$ is not in general position).[2] This phenomenon is general: if (C, D) is in the domain of Lefschetz's intersection pairing with $C \cdot D \neq 0$ then (unless C and D are both in dimension $\dim M$) there will always be a subdivision in which the pair of chains determined by C and D is not in the domain.

Since the PL chain complex C_*M is defined to be a direct limit over subdivisions (see Section 3) it would be difficult to create a partially defined operation on $C_*M \otimes C_*M$ using the definition in [9]. This would clearly be a significant obstacle to proving (or even formulating) Theorem 1.1.

[1] If C and D are chains on the *same triangulation* and are not both in dimension $\dim M$, this condition forces $C \cdot D$ to be 0, because σ_i and τ_j will intersect along a common face and therefore $\sigma_i \cap \tau_j$ will be contained in $\partial\sigma_i \cap \tau_j$.

[2] Note also that, if the intersection point is P, then formula (2.1) gives $\sigma \cdot \tau = \pm P$ but $(\sigma' + \sigma'') \cdot (\tau' + \tau'') = \pm 4P$.

Lefschetz returned to the chain-level intersection pairing in [10, Section IV.6]. He gave a formula more general than (2.1) (equation (46) on page 212) in which the coefficients are "looping coefficients" [10, Section IV.5]. This allowed him to enlarge the domain of the intersection pairing as follows: if we write $\mathrm{supp}(C)$ for the union of the simplices that occur in C, then $C \cdot D$ is defined when the three pairs $(\mathrm{supp}(C), \mathrm{supp}(D))$, $(\mathrm{supp}(\partial C), \mathrm{supp}(D))$, $(\mathrm{supp}(C), \mathrm{supp}(\partial D))$ are in general position; note that this condition is invariant under subdivision.

The "looping coefficients" used in Lefschetz's second definition are tricky to define explicitly (see [10, Subsection 58 on page 216]). The theory has been worked out carefully in [7] (which I have not had an opportunity to consult) and seems to be rather complicated (see the Math Review: 41 # 2663).

The chain-level intersection pairing became temporarily obsolete when the cup product was discovered and it was noticed that the intersection pairing in homology could be defined using only Poincaré duality and the cup product, without any recourse to the chain level.

Goresky and MacPherson returned to the chain-level intersection pairing as a tool for constructing an intersection pairing in intersection homology ([5, Section 2]). They gave an elegant construction in which the procedure of the previous paragraph is reversed: the chain-level intersection pairing is derived from the relative versions of Poincaré duality and the cup product. Their version of the chain-level intersection pairing has the same domain as Lefschetz's second definition and is probably equivalent to it.

Note that, with either the Lefschetz or Goresky-MacPherson definitions, the intersection pairing is defined on a subset of $C_*M \times C_*M$ rather than $C_*M \otimes C_*M$. It would be natural to define it on a subset of $C_*M \otimes C_*M$ by letting the domain be all elements that can be written in the form

$$\sum C_i \otimes D_i$$

with every pair $(\mathrm{supp}(C_i), \mathrm{supp}(D_i))$, $(\mathrm{supp}(\partial C_i), \mathrm{supp}(D_i))$, $(\mathrm{supp}(C_i), \mathrm{supp}(\partial D_i))$ in general position. For such an element, one could define the intersection pairing to be $\sum C_i \cdot D_i$. But this raises two issues: it is hard to tell when an element can be written in the required form, and it would be hard to prove that two different ways of writing an element in this form lead to the same value for the intersection pairing.

The definition to be given in Section 8 resolves both of these issues by defining the intersection pairing (up to a dimension shift) as the composite of the exterior product

$$\varepsilon : C_*M \otimes C_*M \to C_*(M \times M)$$

(see Section 7) and the chain-level backwards map

$$\Delta_! : C_*^{\Delta}(M \times M) \to C_*M$$

induced by the diagonal (see Section 6); here $C_*^{\Delta}(M \times M)$ denotes the set of chains E in $C_*(M \times M)$ for which both E and ∂E are in general position with respect to the diagonal. With this definition, the domain of the intersection pairing (up to a dimension shift) is

$$\varepsilon^{-1}(C_*^{\Delta}(M \times M)).$$

The analog of equation (2.2) is immediate from the fact that ε and $\Delta_!$ are chain maps.

3 PL chains.

We begin by reviewing some basic definitions.

A *simplicial complex* K is a set of simplices in $\mathbb{R}^n$ (for some n) with two properties: every face of a simplex in K is in K and the intersection of two simplices in K is a common face. (A face of a simplex σ is the simplex spanned by some subset of the vertices of σ.)

The *simplicial chain complex of* K, denoted c_*K, is defined by letting c_pK be generated by pairs (σ, o), where σ is a p-simplex of K and o is an orientation of σ, subject to the relation $(\sigma, o) = -(\sigma, -o)$ where $-o$ denotes the opposite orientation. We leave it as an exercise to formulate the definition of the boundary map ∂ (or see [15, page 159]). If we choose orientations for the simplices of K (with no requirement of consistency among the orientations) then every nonzero element c of c_*K can be written uniquely in the form $\sum n_i\sigma_i$ with all $n_i \neq 0$.

The *realization* of K, denoted $|K|$, is the union of the simplices of K.

A *subdivision* of K is a simplicial complex L with two properties: $|L| = |K|$ and every simplex of L is contained in a simplex of K.

The *subdivision category* of K has an object for each subdivision L of K and a morphism $L \to L'$ whenever L' is a subdivision of L.

If L' is a subdivision of L there is an induced monomorphism $c_*L \to c_*L'$ which takes (σ, o) to $\sum(\tau, o_\tau)$, where the sum runs over all $\tau \in L'$ which are contained in σ and have the same dimension as σ, and o_τ is the orientation induced by o. This makes c_* a covariant functor on the subdivision category of K.

A subspace X of $\mathbb{R}^n$ will be called a *PL space* if there is a simplicial complex K with $X = |K|$. K will be called a *triangulation* of X; note that X determines K up to subdivision by [1, page 222].

The *PL chain complex* of a PL space $|K|$, denoted $C_*|K|$, is the direct limit

$$\operatorname*{colim}_L c_*L$$

taken over the subdivision category of K.

Remark 3.1. This definition is taken from [5, Subsection 1.2], which seems to be the first place where the PL chain complex was defined.

Note that the direct system defining $C_*|K|$ is a rather simple one: the subdivision category is a directed set (because any two subdivisions have a common refinement [1, page 222]), and all of the maps $c_*L \to c_*L'$ are monomorphisms. It follows that each of the maps $c_*L \to C_*|K|$ is a monomorphism.

Remark 3.2. The homology of c_*L is canonically isomorphic to the singular homology of $|K|$ by [15, Theorems 4.3.8 and 4.4.2]; since homology commutes with colimits over directed sets, the homology of $C_*|K|$ is also canonically isomorphic to the singular homology of $|K|$.

Now let C be a nonzero element of $C_*|K|$. There is a subdivision L of K with $C \in c_*L$, so (after choosing orientations for the simplices in L) we can write $C = \sum n_i\sigma_i$ where the σ_i are simplices in L and the n_i are nonzero. We define the *support of* C, denoted $\mathrm{supp}(C)$, to be $\bigcup \sigma_i$; this is independent of the choice of L. The support of 0 is defined to be the empty set.

4 A useful lemma.

Let K be a simplicial complex. A *subcomplex* of K is a subset K' of K with the property that every face of every simplex in K' is also in K'.

A *PL subspace* of $|K|$ is a space of the form $|L|$ where L is a subcomplex of a subdivision of K.

The next lemma is taken from Section 1.2 of [5]; it gives a way of using relative homology to make chain-level constructions.

Lemma 4.1. *Let K be a simplicial complex and let A and B be PL subspaces of $|K|$ such that $B \subset A$ and $\dim(B) = \dim(A) - 1$. Let $p = \dim(A)$.*

(a) *There is a natural isomorphism $\alpha_{A,B}$ from $H_p(A, B)$ to the abelian group*

$$\{\, C \in C_p(K) \,:\, \operatorname{supp}(C) \subset A \text{ and } \operatorname{supp}(\partial C) \subset B \,\}.$$

(b) *The diagram*

$$\begin{array}{ccc} H_p(A,B) & \xrightarrow{\alpha_{A,B}} & \{\, C \in C_p(K) \,:\, \operatorname{supp}(C) \subset A \text{ and } \operatorname{supp}(\partial C) \subset B \,\} \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} \\ H_{p-1}(B,\emptyset) & \xrightarrow{\alpha_{B,\emptyset}} & \{\, D \in C_{p-1}(K) \,:\, \operatorname{supp}(D) \subset B \text{ and } \partial D = 0 \,\} \end{array}$$

commutes.

Proof. For part (a), note that $H_p(A, B)$ is isomorphic to the p-th homology of the complex C_*A/C_*B, and this in turn is isomorphic to the quotient of the relative cycles by the relative boundaries. The module of relative cycles is the set S of the lemma and the module of relative boundaries is $\partial(C_{p+1}A) + C_pB$, which is zero because of the hypotheses. Part (b) is immediate from the definitions. □

5 A backwards map in relative homology.

A *PL map* from $|K|$ to $|K'|$ is a continuous function f with the property that, for some subdivision L of K, the restriction of f to each simplex of L is an affine map with image in a simplex of K'.

A *PL homeomorphism* is a PL map which is a homeomorphism.

An *m-dimensional PL manifold* is a PL space M with the property that each point of M is contained in the interior of a PL subspace which is PL homeomorphic to the m simplex.

Let M be a compact oriented m-dimensional PL and let A and B be PL subspaces of M with $B \subset A$. Let N be a compact oriented PL manifold of dimension n and let $f : N \to M$ be a PL map. Let $A' = f^{-1}(A)$ and $B' = f^{-1}(B)$.

We want to construct a map

$$f_! : H_*(A, B) \to H_{*+n-m}(A', B') \tag{5.1}$$

(one should think of this as taking a homology class to its inverse image with respect to f).

First we need a lemma, which will be proved at the end of this section.

Lemma 5.1. *Let X be a PL subspace of M, let $X' = f^{-1}(X)$, and let W' be an open neighborhood of X'. Then there is an open neighborhood W of X with $f^{-1}(W) \subset W'$.*

Now let (U', V') be an open pair in N with $A' \subset U'$ and $B' \subset V'$. Using the lemma, we choose an open pair (U, V) in M with $A \subset U$, $B \subset V$, $f^{-1}(U) \subset U'$, and $f^{-1}(V) \subset V'$.

Consider the composite

$$H_*(A, B) \to H_*(U, V) \cong \check{H}^{m-*}(M - U, M - V) \xrightarrow{f^*} \check{H}^{m-*}(N - U', N - V') \cong H_{*+n-m}(U', V'),$$

where the second and fourth arrows are Poincaré-Lefschetz duality isomorphisms ([3, Proposition VIII.7.2]). By the naturality of the cap product ([3, VIII.7.6]) this composite is independent of the choice of (U, V) and is natural with respect to (U', V'). We therefore get a map

$$H_*(A, B) \to \lim H_{*+n-m}(U', V')$$

where the inverse limit is taken over all open pairs $(U', V') \supset (A', B')$. This inverse limit is isomorphic to $H_{*+n-m}(A', B')$ by [3, Exercise 4 at the end of Section VIII.13]; here we use the fact that the realization of a simplicial complex is an ENR (see for example [3, Proposition IV.8.12]). This completes the construction of the map (5.1).

For use in the next section, we need:

Lemma 5.2. *The diagram*

$$\begin{array}{ccc} H_*(A, B) & \xrightarrow{f_!} & H_{*+n-m}(A', B') \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\ H_{*-1}(B) & \xrightarrow{f_!} & H_{*+n-m-1}(B') \end{array}$$

commutes.

Proof. This follows easily from [3, VII.12.22]. □

Proof of Lemma 5.1. Fix triangulations of M and N. By subdividing the triangulation of N, we can ensure that every simplex that intersects X' is contained in W'. By a further subdivision in both M and N, we can ensure that the map f is simplicial (see [6, top of page 15 and Lemma 1.10]). Since f is simplicial it maps simplices onto simplices, and any simplex in N that maps onto a simplex which intersects X must intersect X' and must therefore be contained in W'. Now let $\tilde{X}$ be the union of all simplices that intersect X and let W be the interior of $\tilde{X}$. Then W is an open neighborhood of X and $f^{-1}(W)$ is contained in W'. □

6 A backwards map at the chain level.

Let M, N and $f : N \to M$ be as in the previous section.

We say that a PL subspace A of M is in *general position* with respect to f if

$$\dim(f^{-1}(A)) \le \dim(A) + n - m.$$

(The dimension of the empty set is defined to be $-\infty$, so if $f^{-1}(A)$ is empty then A is in general position.)

Remark 6.1. For later use we make two observations.

(a) Suppose that f is a composite gh, that A is in general position with respect to g, and that $g^{-1}(A)$ is in general position with respect to h. Then A is in general position with respect to f.

(b) Suppose that N is a Cartesian product $M \times M_1$ and $f : N \to M$ is the projection. Then every A is in general position with respect to f.

A p-chain C in C_*M is said to be in general position with respect to f if

$$\dim(f^{-1}(\mathrm{supp}(C))) \le p + n - m.$$

Let $C_*^f M$ be the set of all chains $C \in C_*M$ for which both C and ∂C are in general position with respect to f. Note that $C_*^f M$ is a subcomplex of C_*M.

We want to construct a chain map

$$f_! : C_*^f M \to C_{*+n-m}N.$$

So let $C \in C_q^f M$. Let $[C]$ be the homology class of C in $H_q(\mathrm{supp}(C), \mathrm{supp}(\partial C))$. Let T be the abelian group

$$\{\, D \in C_{q+n-m}N \mid \mathrm{supp}(D) \subset f^{-1}(\mathrm{supp}(C)) \text{ and } \mathrm{supp}(\partial D) \subset f^{-1}(\mathrm{supp}(\partial C)) \,\}.$$

We define $f_!(C)$ to be the image of $[C]$ under the following composite:

$$H_q(\mathrm{supp}(C), \mathrm{supp}(\partial C)) \xrightarrow{f_!} H_{q+n-m}(f^{-1}(\mathrm{supp}(C)), f^{-1}(\mathrm{supp}(\partial C))) \cong T \hookrightarrow C_{q+n-m}N$$

Here the first map was constructed in Section 5 and the isomorphism is from Lemma 4.1 (which applies because of the hypothesis that both C and ∂C are in general position with respect to f). $f_!$ is a chain map by Lemmas 4.1(b) and 5.2.

Remark 6.2. Note that, by the definition of T, we have $\mathrm{supp}(f_!(C)) \subset f^{-1}(\mathrm{supp}(C))$.

7 The exterior product for PL chains.

Let σ_1 and σ_2 be simplices. It is easy to see that $\sigma_1 \times \sigma_2$ is a PL space; that is, there is a simplicial complex J with $|J| = \sigma_1 \times \sigma_2$. Note that there is no canonical way to choose J, but that any two choices of J have a common subdivision.

It follows that the product of any two PL spaces is a PL space.

Let $|K_1|$ and $|K_2|$ be PL spaces. We want to construct a map

(7.1) $$\varepsilon : C_*|K_1| \otimes C_*|K_2| \to C_*(|K_1| \times |K_2|),$$

called the *exterior product.*

As a first step, let L_1 and L_2 be subdivisions of K_1 and K_2 respectively. We define a map

(7.2) $$\varepsilon' : c_*L_1 \otimes c_*L_2 \to C_*(|K_1| \times |K_2|)$$

(see Section 3 for the definition of c_*).

It suffices to define ε' on generators, so for $i = 1, 2$ let σ_i be a simplex of L_i with orientation o_i. Let J be a simplicial complex with

$$|J| = \sigma_1 \times \sigma_2.$$

Then $\varepsilon'((\sigma_1, o_1) \otimes (\sigma_2, o_2))$ is defined to be

$$\sum (\tau, o_\tau)$$

where τ runs through the simplices of J with dimension $\dim(\sigma_1) + \dim(\sigma_2)$, and o_τ is the orientation of τ induced by $o_1 \times o_2$.

The maps ε' are consistent as L_1 and L_2 vary; passage to colimits gives the map ε.

Remark 7.1. (a) It is easy to check that ε is a monomorphism.

(b) The quasi-isomorphism relating c_* to singular chains ([15, Theorems 4.3.8 and 4.4.2]) takes ε to the Eilenberg-MacLane shuffle product ([3, VI.12.26.2]). Since the latter is a quasi-isomorphism, so is ε.

(c) For singular chains, the shuffle product has an explicit natural homotopy inverse, namely the Alexander-Whitney map ([3, VI.12.26.2]). Unfortunately the Alexander-Whitney map is not compatible with subdivision, so it seems to have no analog for PL chains.

8 The chain-level intersection pairing.

We now have the ingredients needed to define the chain-level intersection pairing.

Let M be a compact oriented PL manifold of dimension m and let $\Delta : M \to M \times M$ be the diagonal map. As in Section 6, let $C_*^\Delta(M \times M)$ be the subcomplex of $C_*(M \times M)$ consisting of chains C for which both C and ∂C are in general position with respect to Δ.

It is convenient to shift degrees so that the intersection pairing preserves degree. For a chain complex C_* and an integer n, we will write $\Sigma^n C_*$ for the n-fold suspension of C_*, that is, the chain complex with C_i in degree $i + n$.

Let us define

$$G_2 \subset \Sigma^{-2m}(C_*M \otimes C_*M)$$

to be $\Sigma^{-2m}(\varepsilon^{-1}(C_*^\Delta(M \times M))$, where ε is the exterior product (the G stands for "general position" and the subscript 2 will be explained in Section 10).

The chain-level intersection pairing μ is the composite

$$G_2 \xrightarrow{\varepsilon} \Sigma^{-2m} C_*^\Delta(M \times M) \xrightarrow{\Delta_!} \Sigma^{-m} C_* M.$$

Remark 8.1. It is not difficult to check that, if C and D are chains for which the Goresky-MacPherson intersection pairing $C \cap D$ is defined (see [5, pages 141–142]), then (up to the dimension shifts in the definitions of G_2 and μ) $C \otimes D$ is in G_2 and $\mu(C \otimes D) = C \cap D$.

9 Leinster partial commutative DGA's.

Our main goal in the rest of the paper is to show that the chain-level intersection pairing and its iterates determine a partially defined commutative DGA structure on $\Sigma^{-m}C_*M$.

First we need a precise definition of "partially defined commutative algebra." We will use the definition given by Leinster in [11, Section 2.2] (but note that Leinster uses the term "homotopy algebra" instead of "partial algebra").

Let Φ be the category of finite sets (including the empty set) and let Ch be the category of chain complexes.

Given a functor A defined on Φ, we will write A_S (instead of $A(S)$) for the value of A at S.

Definition 9.1. A Leinster partial commutative DGA is a functor A from Φ to Ch together with chain maps

$$\xi_{S,T} : A_{S \coprod T} \to A_S \otimes A_T$$

for each S, T and

$$\xi_\emptyset : A_\emptyset \to \mathbb{Z}$$

(where $\mathbb{Z}$ is considered as a chain complex concentrated in degree 0) such that

(i) the collection $\xi_{S,T}$ is a natural transformation of functors from $\Phi \times \Phi$ to Ch,

(ii) the diagram

$$\begin{array}{ccc} A_{S\coprod T\coprod U} & \xrightarrow{\xi_{S\coprod T,U}} & A_{S\coprod T}\otimes A_U \\ {\scriptstyle \xi_{S,T\coprod U}}\downarrow & & \downarrow{\scriptstyle \xi_{S,T}\otimes 1} \\ A_S\otimes A_{T\coprod U} & \xrightarrow{1\otimes\xi_{T,U}} & A_S\otimes A_T\otimes A_U \end{array}$$

commutes for all S, T, U,

(iii) the diagram

$$\begin{array}{ccc} A_{S\coprod T} & \xrightarrow{\xi_{S,T}} & A_S\otimes A_T \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\ A_{T\coprod S} & \xrightarrow{\xi_{T,S}} & A_T\otimes A_S \end{array}$$

commutes for all S,

(iv) the diagram

$$\begin{array}{ccc} A_S & \xrightarrow{\xi_{\emptyset,S}} & A_\emptyset\otimes A_S \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \xi_\emptyset\otimes 1} \\ & & \mathbb{Z}\otimes A_S \end{array}$$

commutes for all S, and

(v) $\xi_\emptyset$ and each $\xi_{S,T}$ are quasi-isomorphisms.

Remark 9.2. Definition 9.1 is the precise analog, for the category *Ch*, of Segal's Γ-spaces [14]. This is not immediately obvious, since a Γ-space is a functor on the category $\mathcal{F}$ of *based* finite sets; the point is that the maps $\xi_{S,T}$ in Definition 9.1 encode the same information as the projection maps in Segal's definition.

Notation 9.3. For $k \geq 1$ let $\overline{k}$ denote the set $\{1, \ldots, k\}$. Let $\overline{0}$ be the empty set. For $k \geq 0$ let A_k denote $A_{\overline{k}}$.

Note that a functor A on Φ is entirely determined by its restriction to the full subcategory with objects $\overline{k}$, $k \geq 0$.

Remark 9.4. (a) An ordinary commutative DGA B determines a Leinster partial commutative DGA with $A_k = B^{\otimes k}$.

(b) Conversely, it will be shown in [13] that Leinster partial commutative DGA's can be functorially replaced by quasi-isomorphic E_∞ DGA's. This is one reason for using Definition 9.1 instead of the Kriz-May definition of partial algebra [8, Definition II.2.4]; Kriz and May prove an analogous result for partial commutative *simplicial* algebras (in their sense), but they explain [8, bottom of page 35 and top of page 51] that their method of proof does not work for partial commutative *differential graded* algebras.

10 The functor G.

As a first step in showing that the intersection pairing on $\Sigma^{-m}C_*M$ extends to a Leinster partial commutative DGA structure, we define a functor G from Φ to *Ch* with $G_1 = \Sigma^{-m}C_*M$ (see Notation 9.3). The G stands for "general position."

G_2 has already been defined in Section 8. In order to define G_k for $k \geq 3$ we need a definition.

Let $R : \overline{k} \to \overline{k'}$ be any map. Define

$$R^* : M^{k'} \to M^k$$

to be the composite

$$M^{k'} = \mathrm{Map}(\overline{k'}, M) \to \mathrm{Map}(\overline{k}, M) = M^k$$

where the second arrow is induced by R. Thus the projection of $R^*(x_1, \ldots, x_{k'})$ on the i-th factor is $x_{R(i)}$.

If $R : \overline{k} \twoheadrightarrow \overline{k'}$ is a surjection then we think of R^* as a generalized diagonal map. For example, if k' is 1 and R is the constant map then $R^* : M \to M^k$ is the usual diagonal map.

Let ε_k denote the k-fold exterior product

$$(C_*M)^{\otimes k} \hookrightarrow C_*(M^k).$$

Definition 10.1. Define G_0 to be $\mathbb{Z}$ and G_1 to be $\Sigma^{-m}C_*M$. For $k \geq 2$ define G_k to be the subcomplex of $\Sigma^{-mk}((C_*M)^{\otimes k})$ consisting of the elements $\Sigma^{-mk}C$ for which both $\varepsilon_k(C)$ and $\varepsilon_k(\partial C)$ are in general position with respect to all generalized diagonal maps, that is,

$$G_k = \bigcap_{k' < k} \ \bigcap_{R:\overline{k} \twoheadrightarrow \overline{k'}} \Sigma^{-mk}(\varepsilon_k^{-1}(C_*^{R^*} M^k)).$$

Lemma 10.2. *If $\Sigma^{-mk}C$ is in G_k then both $\varepsilon_k(C)$ and $\varepsilon_k(\partial C)$ are in general position with respect to all maps R^*.*

Proof. Any R factors as R_1R_2, where R_1 is an inclusion and R_2 is a surjection. Then $R^* = R_2^*R_1^*$, and R_1^* is a composite of projection maps. The lemma now follows from Remarks 6.1(a) and 6.1(b). $\square$

It remains to define the effect of G on morphisms in Φ. First we need three lemmas.

Lemma 10.3. *Let*

$$N_1 \xrightarrow{g} N_2 \xrightarrow{f} N_3$$

be a diagram of compact oriented PL manifolds and PL maps. Let $C \in C_*^f N_3 \cap C_*^{fg} N_3$. Then
(a) *$f_! C \in C_*^g N_2$ and*
(b) *$g_! f_! C = (fg)_! C$.*

Proof. Part (a) is immediate from Remark 6.2 and part (b) is immediate from the definitions. $\square$

Lemma 10.4. *Let $f : N_1 \to M_1$ and $g : N_2 \to M_2$ be PL maps between compact oriented PL manifolds. Then*
(a) *The exterior product*

$$\varepsilon : C_*M_1 \otimes C_*M_2 \to C_*(M_1 \times M_2)$$

takes $C_*^f M_1 \otimes C_*^g M_2$ to $C_*^{f\times g}(M_1 \times M_2)$, and
(b) *the diagram*

$$\begin{array}{ccc} C_p^f M_1 \otimes C_q^g M_2 & \xrightarrow{\quad\varepsilon\quad} & C_{p+q}^{f\times g}(M_1 \times M_2) \\ \big\downarrow{\scriptstyle f_!\otimes g_!} & & \big\downarrow{\scriptstyle (f\times g)_!} \\ C_{p+n_1-m_1} N_1 \otimes C_{q+n_2-m_2} N_2 & \xrightarrow{\varepsilon} & C_{p+q+n_1+n_2-m_1-m_2}(N_1 \times N_2) \end{array}$$

commutes for all p and q, where m_i (resp., n_i) is the dimension of M_i (resp., N_i).

Proof. Part (a) is obvious from the definitions, and part (b) follows from [3, VII.12.17]. $\square$

Lemma 10.5. *Let $R : \overline{k} \to \overline{k'}$ be any map. Then $(R^*)_! \circ \varepsilon_k$ takes G_k to $\varepsilon_{k'}(G_{k'})$.*

Proof. We need to prove two things: that

$$(R^*)_!(\varepsilon_k(G_k)) \subset \Sigma^{-mk'}\varepsilon_{k'}(C_*(M)^{\otimes k'})$$

and that

$$(R^*)_!(\varepsilon_k(G_k)) \subset \Sigma^{-mk'} C_*^{S^*}(M^{k'})$$

for all surjections $S : \overline{k'} \to \overline{k''}$. The first follows from Lemma 10.4 and the second from Lemmas 10.2 and 10.3(a). □

We can now define the effect of G on morphisms by letting

$$G_R : G_k \to G_{k'}$$

be $\varepsilon_{k'}^{-1} \circ (R^*)_! \circ \varepsilon_k$ (recall that $\varepsilon_{k'}$ is a monomorphism). Lemma 10.3(b) implies that $G_{R\circ S} = G_R \circ G_S$ for all R and S.

11 The maps $\xi_{k,l}$.

In order to construct the maps

$$\xi_{k,l} : G_{k+l} \to G_k \otimes G_l$$

it suffices to show

Lemma 11.1. *The inclusion*

$$G_{k+l} \hookrightarrow \Sigma^{-m(k+l)}(C_*M)^{\otimes(k+l)} \cong \Sigma^{-mk}(C_*M)^{\otimes k} \otimes \Sigma^{-ml}(C_*M)^{\otimes l}$$

has its image in $G_k \otimes G_l$.

We can then define $\xi_{k,l}$ to be the inclusion $G_{k+l} \hookrightarrow G_k \otimes G_l$.

To prove Lemma 11.1 we need a criterion for deciding when an element of $\Sigma^{-mk}(C_*M)^{\otimes k} \otimes \Sigma^{-ml}(C_*M)^{\otimes l}$ is in $G_k \otimes G_l$; we will build up to this in stages, culminating in Corollary 11.4.

Let K be a triangulation of M^k. Choose orientations for the simplices of K (with no consistency required among the choices). Recall that (since orientations have been chosen) c_pK is the free abelian group generated by the p-simplices of K. Let $R : \overline{k} \to \overline{k'}$ be any map and define $c_p(K, R)$ to be the free abelian group generated by the p-simplices of K that are *not* in general position with respect to R^*. Let

$$\Upsilon_p^{K,R} : c_pK \to c_p(K, R)$$

be the homomorphism which is the identity on $c_p(K, R)$ and which takes the p-simplices that are in general position with respect to R^* to 0. Let

$$\Psi_p^{K,R} : c_pK \to c_p(K, R) \oplus c_{p-1}(K, R)$$

be $\Upsilon_p^{K,R} + \Upsilon_{p-1}^{K,R} \circ \partial$.

As an immediate consequence of the definitions, we have:

Lemma 11.2. (a) *An element of c_pK is in general position with respect to R^* if and only if it is in the kernel of $\Upsilon_p^{K,R}$.*

(b) *An element of c_pK is in $C_*^{R^*}(M^k)$ (that is, the element and its boundary are both in general position with respect to R^*) if and only if it is in the kernel of $\Psi_p^{K,R}$.*

(c) *An element of c_pK is in $\Sigma^{mk}G_k$ if and only if it is in the kernel of*

$$\sum_R \Psi_p^{K,R} : c_pK \to \bigoplus_R (c_p(K,R) \oplus c_{p-1}(K,R)).$$

□

Next let L be a triangulation of M^l. We would like to characterize the elements of $\Sigma^{-mk}c_pK \otimes \Sigma^{-ml}c_qL$ that are in $G_k \otimes G_l$. First we need some algebra.

Lemma 11.3. *Let*

$$0 \to A \to B \xrightarrow{g} C$$

and

$$0 \to D \to E \xrightarrow{h} F$$

be exact sequences of abelian groups, with C and F torsion free. Then $A \otimes D$ is the kernel of

$$g \otimes 1 + 1 \otimes h : B \otimes E \to (C \otimes E) \oplus (B \otimes F)$$

Proof. We may assume without loss of generality that g and h are surjective. Then the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A \otimes D & \to & B \otimes D & \to & C \otimes D & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A \otimes E & \to & B \otimes E & \to & C \otimes E & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A \otimes F & \to & B \otimes F & \to & C \otimes F & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

has exact rows and columns. The lemma follows by an easy diagram chase. □

Corollary 11.4. *Let $C \in (C_*M)^{\otimes k} \otimes (C_*M)^{\otimes l}$ be such that $(\varepsilon_k \otimes \varepsilon_l)(C)$ is in $c_*K \otimes c_*L$. Then $\Sigma^{-m(k+l)}C$ is in $G_k \otimes G_l$ if and only if $(\varepsilon_k \otimes \varepsilon_l)(C)$ is in the kernel of*

$$\sum_{R:\overline{k}\to\overline{k'}} \Psi_p^{K,R} \otimes 1 + \sum_{S:\overline{l}\to\overline{l'}} 1 \otimes \Psi_q^{L,S}$$

□

Proof of Lemma 11.1. Let $\Sigma^{-m(k+l)}C \in G_{k+l}$. Then there are triangulations K of M^k and L of M^l such that $(\varepsilon_k \otimes \varepsilon_l)(C) \in c_*K \otimes c_*L$. Let $R : \overline{k} \to \overline{k'}$. Then both $\varepsilon_{k+l}(C)$ and $\varepsilon_{k+l}(\partial C)$ are in general position with respect to $(R \times 1)^*$ (by definition of G_{k+l}), and it is easy to see that this implies

$$(\Psi_p^{K,R} \otimes 1)(\varepsilon_k \otimes \varepsilon_l)(C) = 0$$

Similarly

$$(1 \otimes \Psi_q^{L,S})(\varepsilon_k \otimes \varepsilon_l)(C) = 0$$

for all S. Thus $\Sigma^{-m(k+l)}C$ is in $G_k \otimes G_l$ by Corollary 11.4. □

12 The main theorem.

Theorem 12.1. *The functor G defined in Section 10, with the maps $\xi_{k,l}$ defined in Section 11, is a Leinster partial commutative DGA.*

To prove Theorem 12.1, we need to verify the five parts of Definition 9.1. Part (i) follows easily from the definitions and Lemma 10.4. Parts (ii)–(iv) are immediate from the definition of $\xi_{k,l}$. Part (v) is an easy consequence of the following result, which will be proved in Sections 13–15.

Proposition 12.2. *The inclusion*

$$G_k \hookrightarrow \Sigma^{-mk}(C_*M)^{\otimes k}$$

is a quasi-isomorphism for all k.

Remark 12.3. When $k = 2$ this is Theorem 1.1 of the introduction, up to the dimension shift introduced in Section 8.

13 Proof of Proposition 12.2.

Throughout this section and the next we fix an integer $k \geq 2$.

A *PL homotopy* is a PL map $h : X \times I \to Y$, where X and Y are PL spaces and I is the interval $[0, 1]$ with its usual PL structure.

It will be convenient to have notation for the standard inclusion maps $X \to X \times I$. We write i_0 (resp., i_1) for the map which takes x to $(x, 0)$ (resp., $(x, 1)$).

We need a supply of PL homotopies that preserve the image of

$$\varepsilon_k : (C_*M)^{\otimes k} \hookrightarrow C_*(M^k).$$

Definition 13.1. Suppose that we are given a number l with $1 \leq l \leq k$ and a PL homotopy

$$\phi : M \times I \to M.$$

The *l-th factor PL homotopy* determined by this data is the composite

$$M^k \times I \cong M^{l-1} \times (M \times I) \times M^{k-l} \xrightarrow{1\times\phi\times 1} M^k.$$

Let ι be the canonical element of $C_1(I)$.

Lemma 13.2. *Let*

$$h : M^k \times I \to M^k$$

be an l-th factor PL homotopy for some l and let C be in the image of

$$\varepsilon_k : (C_*M)^{\otimes k} \hookrightarrow C_*(M^k).$$

Then
(a) $(h \circ i_1)_*(C)$ *is in the image of* ε_k, *and*
(b) $h_*(\varepsilon(C \otimes \iota))$ *is in the image of* ε_k.

Proof. This is an easy consequence of the definitions. □

For the proof of Proposition 12.2 we will use a filtration of $\Sigma^{-mk}(C_*M)^{\otimes k}$.

Definition 13.3. (i) For $0 \leq j \leq k$ define Λ_j to be the set of all surjections $R : \overline{k} \twoheadrightarrow \overline{k'}$ such that for each $i > j$ the set $R^{-1}(R(i))$ has only one element.

(ii) For $0 \leq j \leq k$ define G_k^j to be the subcomplex of $\Sigma^{-mk}(C_*M)^{\otimes k}$ consisting of the chains C for which both $\varepsilon_k(C)$ and $\varepsilon_k(\partial C)$ are in general position with respect to R^* for all $R \in \Lambda_j$.

Thus we have a filtration

$$G_k = G_k^k \subset G_k^{k-1} \subset \cdots \subset G_k^0 = \Sigma^{-mk}(C_*M)^{\otimes k}$$

Proposition 12.2 follows immediately from:

Proposition 13.4. *For each* $1 \leq j \leq k$ *the inclusion* $G_k^j \subset G_k^{j-1}$ *is a quasi-isomorphism.*

For this we need a lemma which will be proved in Sections 14 and 15.

Lemma 13.5. *Suppose that* $D \in \Sigma^{mk} G_k^{j-1}$ *and* $\partial D \in \Sigma^{mk} G_k^j$. *Then there is a j-th factor homotopy*

$$h : M^k \times I \to M^k$$

such that $h \circ i_0$ *is the identity and the chains* $(h \circ i_1)_*(\varepsilon_k D)$, $(h \circ i_1)_*(\varepsilon_k(\partial D))$ *and* $h_*(\varepsilon(\partial D \otimes \iota))$ *are in general position with respect to* R^* *for all* $R \in \Lambda_j$.

Proof of Proposition 13.4. We have to show two things:

(i) If D is a cycle in $\Sigma^{mk} G_k^{j-1}$ then there is a cycle C in $\Sigma^{mk} G_k^j$ homologous to D.

(ii) If C is a cycle in $\Sigma^{mk} G_k^j$ which is the boundary of an element of $\Sigma^{mk} G_k^{j-1}$ then C is the boundary of an element of $\Sigma^{mk} G_k^j$.

To show (i), choose a homotopy h as in Lemma 13.5. Then $(h \circ i_1)_*(\varepsilon_k D)$ is in the image of ε_k by Lemma 13.2, so we may define

$$C = \varepsilon_k^{-1}((h \circ i_1)_*(\varepsilon_k D)).$$

C is a cycle, and the general-position property given in Lemma 13.5 implies that C is in $\Sigma^{mk}G_k^j$. Let $\kappa, \lambda \in C_0 I$ be the 0-chains associated to $0, 1 \in I$; then

$$\partial\iota = \lambda - \kappa.$$

Now

$$\begin{aligned}\partial(h_*(\varepsilon(D \otimes \iota))) &= h_*(\varepsilon(\partial D \otimes \iota)) + (-1)^{|D|} h_*(\varepsilon(D \otimes \lambda)) - (-1)^{|D|} h_*(\varepsilon(D \otimes \kappa)) \\ &= 0 + (-1)^{|D|}(h \circ i_1)_*(\varepsilon_k D) - (-1)^{|D|}(h \circ i_0)_*(\varepsilon_k D) \\ &= (-1)^{|D|}\varepsilon_k(C - D).\end{aligned}$$

Since ε_k is a quasi-isomorphism, this implies that C is homologous to D.

To show (ii), let $D \in \Sigma^{mk}G_k^{j-1}$ with $\partial D = C$. Choose a homotopy h as in Lemma 13.5. Then $(h \circ i_1)_*(\varepsilon_k D)$ and $h_*(\varepsilon(\partial D \otimes \iota))$ are in the image of ε_k by Lemma 13.2, so we may define

$$E_1 = \varepsilon_k^{-1}((h \circ i_1)_*(\varepsilon_k D))$$

and

$$E_2 = \varepsilon_k^{-1}(h_*(\varepsilon(\partial D \otimes \iota))).$$

The general-position property given in Lemma 13.5 imply that E_1 and E_2 are in $\Sigma^{mk}G_k^j$. Now

$$\begin{aligned}\varepsilon_k(\partial E_2) &= (-1)^{|D|+1}(h_*(\varepsilon(\partial D \otimes \lambda)) - h_*(\varepsilon(\partial D \otimes \kappa))) \\ &= (-1)^{|D|+1}((h \circ i_1)_*(\varepsilon_k \partial D) - (h \circ i_0)_*(\varepsilon_k \partial D)) \\ &= (-1)^{|D|+1}\varepsilon_k(\partial E_1 - C).\end{aligned}$$

Since ε_k is a monomorphism, this implies

$$\partial((-1)^{|D|}E_2 + E_1) = C.$$

□

14 Background for the proof of Lemma 13.5.

In this section we collect the tools used in the proof of Lemma 13.5. First we have two simple facts about affine geometry which are the heart of the proof. Recall that the *affine span* of a subset of $\mathbb{R}^n$ is the smallest affine subspace containing it.

Lemma 14.1. *Let σ and τ be simplices in $\mathbb{R}^n$ such that the affine span of $\sigma \cup \tau$ is all of $\mathbb{R}^n$. Then*

$$\dim(\sigma \cap \tau) \le \dim(\sigma) + \dim(\tau) - n.$$

Proof. Let U (resp., V) be the affine span of σ (resp., τ). If $U \cap V$ is empty the statement is obvious. Otherwise we can choose a point in $U \cap V$ and move it to the origin by a translation; then U and V become ordinary subspaces which span $\mathbb{R}^n$ and we have

$$\dim(U \cap V) = \dim(U) + \dim(V) - n,$$

which proves the lemma. □

Notation 14.2. If σ is a simplex in $\mathbb{R}^n$ and u is an element of $\mathbb{R}^n$ which is not in σ, the convex hull of σ and u will be denoted by $\langle \sigma, u \rangle$.

Lemma 14.3. *Let σ and τ be simplices in $\mathbb{R}^n$. Let u be a point which is not in the affine span of $\sigma \cup \tau$. Then*

$$\langle \sigma, u \rangle \cap \tau = \sigma \cap \tau.$$

Proof. Let $v \in \langle \sigma, u \rangle \cap \tau$. Since $v \in \langle \sigma, u \rangle$, we can write v in the form $\alpha u + (1-\alpha)s$, with $s \in \sigma$. If α were nonzero we would have

$$u = \frac{1}{\alpha} v - \frac{1-\alpha}{\alpha} s.$$

Since $v \in \tau$, this would imply that u is in the affine span of $\sigma \cup \tau$. Therefore α must be 0, so v is in σ, and hence in $\sigma \cap \tau$, which proves the lemma. □

Next we recall a well-known way of triangulating $\sigma \times I$. By an *ordered* simplex we will mean a simplex with a total ordering of its vertices.

Lemma 14.4. *Let σ be an ordered simplex and let $v_0 < \cdots < v_l$ be the ordering of its vertices. For $0 \leq i \leq l$ let $\sigma[i] \subset \sigma \times I$ be the convex hull of*

$$\{(v_j, 0) \mid j \leq i\} \cup \{(v_j, 1) \mid j \geq i\}.$$

Then

(a) Each $\sigma[i]$ is an $(l+1)$-simplex.

(b) The set L whose elements are the $\sigma[i]$ and their faces is a triangulation of $\sigma \times I$. □

Remark 14.5. With the notation of Lemma 14.4, let τ be the simplex spanned by $v_1, \ldots, v_{l-1}$. Then

$$\sigma[i] = \langle \tau[i], (v_l, 1) \rangle$$

for each $i < l$, and

$$\sigma[l] = \langle \sigma \times \{0\}, (v_l, 1) \rangle.$$

Finally, we need a tool for extending PL maps and homotopies.

Construction 14.6. Let ρ be a simplex in $\mathbb{R}^n$ and let u be an element of $\mathbb{R}^n$ which is not in ρ.

Let Ω be a PL space with a PL homeomorphism $\omega : \Omega \to \Delta^m$.

(i) Let $f : \rho \to \Omega$ be a PL map and let w be an element of Ω. We can extend f to a PL map

$$\bar{f} : \langle \rho, u \rangle \to \Omega$$

by the formula

$$\bar{f}(\alpha x + (1-\alpha)u) = \omega^{-1}(\alpha\omega(f(x)) + (1-\alpha)\omega(w)).$$

(ii) Next suppose we are given an ordering of the vertices of ρ; extend this to $\langle \rho, u \rangle$ by letting u be the maximal element. Let $\phi : \rho \times I \to \Omega$ be a PL homotopy and let z and z' be elements of Ω. We can extend ϕ to a PL homotopy

$$\bar{\phi} : \langle \rho, u \rangle \times I \to \Omega$$

as follows. Let $l-1$ be the dimension of ρ. For $i < l$ we have

$$\langle \rho, u \rangle[i] = \langle \rho[i], u \rangle$$

by Remark 14.5. ϕ is already defined on $\rho[i]$, and we can extend it to $\langle \rho[i], u \rangle$ by using the construction in part (i). For $i = l$, Remark 14.5 gives

$$\begin{aligned} \langle \rho, u \rangle[l] &= \langle \langle \rho, u \rangle \times \{0\}, (u,1) \rangle \\ &= \langle \langle \rho \times \{0\}, (u,0) \rangle, (u,1) \rangle \end{aligned}$$

ϕ is already defined on $\rho \times \{0\}$, and we can extend it to $\langle \langle \rho \times \{0\}, (u,0) \rangle, (u,1) \rangle$ by applying the construction in part (i) twice, taking $(u,0)$ to z and $(u,1)$ to z'.

15 Proof of Lemma 13.5.

We will assume that $j = k$, since the other cases are essentially the same and the notation is simpler in this case. So suppose we are given a D satisfying:

Assumption 15.1. (i) D is in $\Sigma^{mk} G_k^{k-1}$.
(ii) ∂D is in $\Sigma^{mk} G_k^k$.

With the assumption that $j = k$, Lemma 13.5 specializes to:

Lemma 15.2. *There is a k-th variable homotopy $h : M^k \times I \to M^k$ such that $h \circ i_0$ is the identity and the three chains $(h \circ i_1)_*(\varepsilon_k D)$, $(h \circ i_1)_*(\varepsilon_k(\partial D))$ and $h_*(\varepsilon(\partial D \otimes \iota))$ are in general position with respect to R^* for all surjections $R : \overline{k} \twoheadrightarrow \overline{k'}$.*

By the definition of PL manifold, M has a covering by (the interiors of) PL subspaces Ω_i which are PL homeomorphic to Δ^m. Choose PL homeomorphisms

$$\omega_i : \Omega_i \to \Delta^m.$$

Recall that, by the definition in Section 3, a PL space is given as a subspace of some $\mathbb{R}^n$, and therefore inherits a metric. In particular, this is true for the PL manifold M.

Notation 15.3. The Lebesgue number of the covering $\{\Omega_i\}$ (with respect to the metric just mentioned) will be denoted by η.

Choose a triangulation K of M such that

(i) each Ω_i is a union of simplices of K,

(ii) each simplex of K is contained in some Ω_i,

(iii) the restriction of each ω_i to each simplex of K in Ω_i is affine,

(iv) the diameter of each simplex of K is less than $\frac{\eta}{2}$, and

(v) $D \in (c_* K)^{\otimes k}$.

Notation 15.4. Let $\tau_1, \dots, \tau_r$ be the simplices of K.

We fix orientations for $\tau_1, \dots, \tau_r$ (with no requirement of consistency among the choices); this allows us to think of the τ_j as generators of $c_* K$.

Property (v) of K implies that D can be written as a sum

$$D = \sum_{\mathbf{a}} n_{\mathbf{a}} \, \tau_{a_1} \otimes \cdots \otimes \tau_{a_k}, \tag{15.1}$$

where $\mathbf{a}$ runs through multi-indices

$$(a_1, \dots, a_k) \in \{1, \dots, r\}^k$$

and $n_{\mathbf{a}} \in \mathbb{Z}$. For $1 \le j \le r$ we define E_j by

$$\sum_{\mathbf{a} \text{ such that } a_k = j} n_{\mathbf{a}} \, \tau_{a_1} \otimes \cdots \otimes \tau_{a_{k-1}}; \tag{15.2}$$

with this notation we can rewrite equation (15.1) as

$$D = \sum_j E_j \otimes \tau_j. \tag{15.3}$$

Similarly, ∂D can be written as

$$\partial D = \sum_{\mathbf{b}} p_{\mathbf{b}} \, \tau_{b_1} \otimes \cdots \otimes \tau_{b_k}$$

and we define

$$F_j = \sum_{\mathbf{b} \text{ such that } b_k = j} p_{\mathbf{b}}\, \tau_{b_1} \otimes \cdots \otimes \tau_{b_{k-1}}$$

which gives

$$\partial D = \sum_j F_j \otimes \tau_j. \tag{15.4}$$

Now choose an ordering

$$v_1, \ldots, v_s$$

for the vertices of K.

Definition 15.5. For $1 \leq p \leq s$ let A_p be the union of the simplices of K whose vertices are in the set $\{v_1, \ldots, v_p\}$. Let A_0 be the empty set.

Note that A_s is M.

Let us denote the metric on M by d and the standard norm on $\mathbb{R}^m$ by $|| \; ||$.

Definition 15.6. (i) For each Ω_i, choose numbers γ_i and δ_i with

$$||\omega_i(x) - \omega_i(y)|| \leq \gamma_i\, d(x, y)$$

and

$$d(x, y) \leq \delta_i\, ||\omega_i(x) - \omega_i(y)||$$

for all $x, y \in \Omega_i$ (such numbers exist because ω_i and its inverse are PL maps).

(ii) Let λ be the greater of $\max_i \gamma_i \delta_i$ and 1.

We will construct, by induction over p with $0 \leq p \leq s$, a PL homotopy

$$\phi_p : A_p \times I \to M$$

with the following properties:

(1) The restriction of ϕ_p to $A_{p-1} \times I$ is ϕ_{p-1}.

(2) $\phi_p \circ i_0$ is the identity.

(3) For each $x \in A_p, t \in I$ we have

$$d(\phi_p(x, t), x) \leq \frac{\eta}{2\lambda^{s-p}}$$

(see Notation 15.3 and Definitions 15.5 and 15.6).

(4) Let

$$h_p = 1 \times \phi_p : M^{k-1} \times A_p \times I \to M^k$$

be the k-th factor homotopy induced by ϕ_p. If σ is a simplex of K and j is a number such that $\sigma \subset \tau_j \subset A_p$, then each of the following chains is in general position with respect to R^* for all surjections $R : \overline{k} \twoheadrightarrow \overline{k'}$:

(a) $(h_p \circ i_1)_*(\varepsilon_k(E_j \otimes \sigma))$,
(b) $(h_p \circ i_1)_*(\varepsilon_k(F_j \otimes \sigma))$,
(c) $(h_p)_*(\varepsilon(F_j \otimes \sigma \otimes \iota))$.

This will complete the proof of Lemma 15.2, because the homotopy h_s will have the properties required by the lemma.

The first step of the induction (the case $p = 0$) is trivial. Suppose that ϕ_{p-1} has been constructed.

Notation 15.7. Let

$$\pi_1, \ldots, \pi_t$$

be the simplices of K which have a vertex at v_p and are in A_p but not A_{p-1}. For each π_j, let ρ_j be the face opposite v_p; thus

$$\pi_j = \langle \rho_j, v_p \rangle.$$

Combining property (iv) of the triangulation K with property (3) of ϕ_{p-1} and the fact that $\lambda \geq 1$, we see that for each j the diameter of the set

$$\pi_j \cup \phi_{p-1}(\rho_j \times I)$$

is less than η. It follows that for each j we can choose a number $i(j)$ with

$$\pi_j \cup \phi_{p-1}(\rho_j \times I) \subset \Omega_{i(j)}. \tag{15.5}$$

If z' is any point in the intersection of the $\Omega_{i(j)}$ we can apply Construction 14.6(ii) (with $z = v_p$ and $\Omega = \Omega_{i(j)}$) to extend ϕ_{p-1} over each $\pi_j \times I$. The resulting homotopy ϕ_p will automatically satisfy properties (1) and (2) above. It only remains to enumerate the conditions that z' must satisfy in order for properties (3) and (4) to hold, and to show that there is a z' satisfying these conditions.

First we consider property (3). Let $x \in \pi_j$ and $t \in I$. Let l be the dimension of π_j. With the notation of Lemma 14.4, we have $(x,t) \in \pi_j[e]$ for some e with $1 \leq e \leq l$. There are two cases to consider: $e < l$ and $e = l$.

In the first case, Remark 14.5 allows us to write (x,t) as

$$\alpha\,(y, t') + (1-\alpha)\,(v_p, 1)$$

with $0 \le \alpha \le 1$, $y \in \rho_j$ and $t' \in I$. By Construction 14.6(ii) we have

$$\omega_{i(j)}(\phi_p(x,t)) = \alpha\, \omega_{i(j)}(\phi_{p-1}(y,t')) + (1-\alpha)\, \omega_{i(j)}(z') \tag{15.6}$$

and by property (iii) of the triangulation K we have

$$\omega_{i(j)}(x) = \alpha\, \omega_{i(j)}(y) + (1-\alpha)\, \omega_{i(j)}(v_p). \tag{15.7}$$

Now we have

$$\begin{aligned} d(\phi_p(x,t),x) &\le \delta_{i(j)}\, ||\omega_{i(j)}(\phi_p(x,t)) - \omega_{i(j)}(x)|| \quad \text{by Definition 15.6(i)} \\ &\le \delta_{i(j)}(\alpha\, ||\omega_{i(j)}(\phi_{p-1}(y,t')) - \omega_{i(j)}(y)|| + (1-\alpha)\, ||\omega_{i(j)}(z') - \omega_{i(j)}(v_p)||) \\ &\qquad \text{by equations (15.6) and (15.7)} \\ &\le \delta_{i(j)}\gamma_{i(j)}(\alpha\, d(\phi_{p-1}(y,t'),y) + (1-\alpha)\, d(z',v_p)) \quad \text{by Definition 15.6(i)} \\ &\le \lambda \left(\alpha \frac{\eta}{2\lambda^{s-p+1}} + (1-\alpha)\, d(z',v_p)\right) \\ &\qquad \text{by Definition 15.6(ii) and property (3) of } \phi_{p-1} \end{aligned}$$

and this will be $\le \frac{\eta}{2\lambda^{s-p}}$ if $d(z',v_p) \le \frac{\eta}{2\lambda^{s-p+1}}$.

For the second case of property (3) we have $e = l$, and Remark 14.5 gives

$$(x,t) = \alpha\,(y,0) + (1-\alpha)\,(v_p,1)$$

for some $y \in \pi_j$. Equation (15.7) is still valid, and equation (15.6) is replaced by

$$\omega_{i(j)}(\phi_p(x,t)) = \alpha\, \omega_{i(j)}(y) + (1-\alpha)\, \omega_{i(j)}(z') \tag{15.8}$$

Now we have

$$\begin{aligned} d(\phi_p(x,t),x) &\le \delta_{i(j)}\, ||\omega_{i(j)}(\phi_p(x,t)) - \omega_{i(j)}(x)|| \quad \text{by Definition 15.6(i)} \\ &\le \delta_{i(j)}(1-\alpha)\, ||\omega_{i(j)}(z') - \omega_{i(j)}(v_p)|| \quad \text{by equations (15.7) and (15.8)} \\ &\le \delta_{i(j)}\gamma_{i(j)} d(z',v_p) \quad \text{by Definition 15.6(i)} \\ &\le \lambda d(z',v_p) \quad \text{by Definition 15.6(ii)} \end{aligned}$$

and for this to be $\le \frac{\eta}{2\lambda^{s-p}}$ it again suffices to have $d(z',v_p) \le \frac{\eta}{2\lambda^{s-p+1}}$.

It remains to determine the conditions that z' must satisfy in order for ϕ_p to have property (4).

We begin with some general observations. Let $S : \overline{k} \twoheadrightarrow \overline{l}$ be a surjection and let $\mathrm{im}(S^*)$ denote the image of S^*; note that the dimension of $\mathrm{im}(S^*)$ is lm (where m is the dimension of M). Since S^* is 1-1, the definition of general position simplifies somewhat: a chain C is in general position with respect to S^* if and only if

$$\dim(\mathrm{supp}(C) \cap \mathrm{im}(S^*)) \le \dim(\mathrm{supp}(C)) + (l-k)m.$$

Let us observe that, if $\chi_1, \ldots, \chi_k$ are simplices of M and C is $\varepsilon_k(\chi_1 \otimes \cdots \otimes \chi_k)$, then $\mathrm{supp}(C) \cap \mathrm{im}(S^*)$ is homeomorphic to the subspace

$$\prod_{j \in \overline{l}} \bigcap_{S(i)=j} \chi_i$$

of M^l. Thus

$$\dim(\operatorname{supp}(C)\cap \operatorname{im}(S^*)) = \sum_{j\in \bar{l}} \dim\Big(\bigcap_{S(i)=j} \chi_i\Big). \tag{15.9}$$

It follows that C is in general position with respect to S^* if and only if

$$\sum_{j\in \bar{l}} \dim\Big(\bigcap_{S(i)=j} \chi_i\Big) \le \sum_i \dim(\chi_i) + (l-k)m.$$

Combining this with Assumption 15.1(i), Definition 13.3, and equation (15.1), we have

$$\sum_{j\in \bar{l}} \dim\Big(\bigcap_{S(i)=j} \tau_{a_i}\Big) \le \sum_i \dim(\tau_{a_i}) + (l-k)m \tag{15.10}$$

for all $\mathbf{a}$ with $n_{\mathbf{a}} \neq 0$ and all S such that $S^{-1}(k)$ is a single point. In particular, let P be a subset of $\bar{k}$ which doesn't contain k and let S be any surjection which takes P to a point and is 1-1 on the rest of $\bar{k}$. In this situation (15.10) simplifies to

$$\dim\Big(\bigcap_{i\in P} \tau_{a_i}\Big) \le \sum_{i\in P} \dim(\tau_{a_i}) - (|P|-1)m, \tag{15.11}$$

where $|P|$ is the cardinality of P.

Now let us consider what conditions z' must satisfy in order for part (a) of property (4) to be valid. Let $R : \bar{k} \twoheadrightarrow \bar{k'}$ be a surjection. By the inductive hypothesis, property (4a) is already valid when σ is a simplex of A_{p-1}, so we may assume that σ is a simplex of A_p which is not in A_{p-1} (see Definition 15.5).

Denote the set $R^{-1}(R(k))$ by Q.

With the notation of equation (15.2), $(h_p \circ i_1)_*(\varepsilon_k(E_j\otimes\sigma))\cap \operatorname{im}(R^*)$ is homeomorphic to the union over all $\mathbf{a}$ such that $n_{\mathbf{a}} \neq 0$ of

$$\Big(\prod_{j\neq R(k)} \bigcap_{R(i)=j} \tau_{a_i}\Big) \times \Big(\phi_p(\sigma\times\{1\}) \cap \bigcap_{i\in Q-\{k\}} \tau_{a_i}\Big).$$

Thus we want to choose z' so that the inequality

$$\begin{aligned}\sum_{j\neq R(k)} \dim\Big(\bigcap_{R(i)=j} \tau_{a_i}\Big) + \dim\Big(\phi_p(\sigma\times\{1\}) \cap \bigcap_{i\in Q-\{k\}} \tau_{a_i}\Big) \\ \le \sum_{i<k} \dim(\tau_{a_i}) + \dim(\phi_p(\sigma\times\{1\})) + (k'-k)m\end{aligned} \tag{15.12}$$

is valid for all $\mathbf{a}$ with $n_{\mathbf{a}} \neq 0$. Applying (15.11) to each of the sets $R^{-1}(j)$ with $j \neq R(k)$ reduces (15.12) to

$$\begin{aligned}\dim\Big(\phi_p(\sigma\times\{1\}) \cap \bigcap_{i\in Q-\{k\}} \tau_{a_i}\Big) \\ \le \dim(\phi_p(\sigma\times\{1\})) + \sum_{i\in Q-\{k\}} \dim(\tau_{a_i}) - (|Q|-1)m.\end{aligned} \tag{15.13}$$

Applying (15.11) once more to the set $Q - \{k\}$ reduces (15.13) to

$$\text{(15.14)} \qquad \dim\Big(\phi_p(\sigma \times \{1\}) \cap \bigcap_{i \in Q-\{k\}} \tau_{a_i}\Big) \leq \dim(\phi_p(\sigma \times \{1\})) + \dim\Big(\bigcap_{i \in Q-\{k\}} \tau_{a_i}\Big) - m,$$

that is, it suffices to choose z' so that the following property holds for each $\mathbf{a}$ with $n_{\mathbf{a}} \neq 0$:

(*) The subspace

$$\phi_p(\sigma \times \{1\})$$

of M is in general position with respect to the simplex

$$\bigcap_{i \in Q-\{k\}} \tau_{a_i}.$$

To show that it is possible to choose z' satisfying (*), we need to recall how the subspace $\phi_p(\sigma \times \{1\})$ was constructed. σ is one of the simplices $\pi_1, \ldots, \pi_t$ (see Notation 15.7), say π_q, and ρ_q is the face opposite to the vertex v_p. By (15.5), the set $\phi_{p-1}(\rho_q \times 1)$ is contained in $\Omega_{i(q)}$. The image of $\phi_{p-1}(\rho_q \times 1)$ under the PL homeomorphism

$$\omega_{i(q)} : \Omega_{i(q)} \to \Delta^m$$

is a union of simplices which we will denote by $\chi_1, \ldots, \chi_u$. The subspace $\phi_p(\sigma \times \{1\})$ is the inverse image under $\omega_{i(q)}$ of the union of the simplices

$$\langle \chi_1, \omega_{i(q)}(z') \rangle, \ldots, \langle \chi_u, \omega_{i(q)}(z') \rangle.$$

Let us write $\psi_{\mathbf{a}}$ for the simplex $\bigcap_{i \in Q-\{k\}} \tau_{a_i}$ that occurs in property (*). In order for z' to satisfy property (*), each of the pairs consisting of a simplex $\langle \chi_l, \omega_{i(q)}(z') \rangle$ and a simplex $\omega_{i(q)}(\psi_{\mathbf{a}})$ must be in general position. By Lemma 14.1, this condition is automatically satisfied (with no restriction on z') by those pairs for which the affine span of $\chi_l \cup \omega_{i(q)}(\psi_{\mathbf{a}})$ is all of $\mathbb{R}^m$. For the remaining pairs, it suffices by Lemma 14.3 that $\omega_{i(q)}(z')$ should not be in the affine span of $\chi_l \cup \omega_{i(q)}(\psi_{\mathbf{a}})$ (note that χ_l and $\omega_{i(q)}(\psi_{\mathbf{a}})$ are in general position by the inductive hypothesis). Since this affine span is nowhere dense, the set of allowable z' for each such pair contains an open dense subset of a neighborhood of v_p. If we now let l, $\mathbf{a}$ and σ vary through all relevant choices, we still have an open dense subset U of a neighborhood of v_p for which property (*), and hence property (4a), is valid.

A similar argument shows that there is an open dense subset V of a neighborhood of v_p for which property (4b) is valid and an open dense subset W of a neighborhood of v_p for which property (4c) is valid. If z' is chosen in the intersection of U, V and W, all parts of property (4) are valid. This concludes the proof.

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