Stiefel-Whitney classes, united K-theory and real embeddings of number rings

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1 Introduction

Let F be a number field with ring of integers $\mathcal{O}_F$, and let $A = S^{-1}\mathcal{O}_F$ for some set S of primes in $\mathcal{O}_F$. Each complex embedding $\beta : F \longrightarrow \mathbb{C}$ induces a map $BGLA \longrightarrow BGL\mathbb{C} \cong BU$, and hence by pulling back the universal Chern classes we obtain Chern classes $c_n(\beta) \in H^{2n}BGLA$. It turns out that for any such A, the mod 2 Chern classes are independent of the complex embedding β, and for fixed β these classes are algebraically independent.

Now suppose that F admits a real embedding $\alpha : F \longrightarrow \mathbb{R}$. Then Stiefel-Whitney classes $w_n(\alpha) \in H^n(BGLA; \mathbb{F}_2)$ are defined in the analogous way. These classes, however, definitely will depend on the choice of real embedding. For example, the first Stiefel-Whitney class can be identified with the induced homomorphism $A^\times \longrightarrow \mathbb{R}^\times/(\mathbb{R}^\times)^2$. Furthermore, although the $w_n(\alpha)$ are again algebraically independent for fixed α, various relations hold as α varies. This raises the question: What are the relations among the $w_n(\alpha)$?

In addition to its intrinsic interest, this question is relevant to the problem of computing $H^*(BGLA; \mathbb{F}_2)$. For example, if A is a ring of S-integers with $\frac{1}{2} \in A$, there is a fibre sequence of 2-completed spaces

$$X \longrightarrow BGLA^+ \xrightarrow{\underline{\alpha}} (BO)^{r_1},$$

where $\underline{\alpha} = (\alpha_1, ..., \alpha_{r_1})$ corresponds to the r_1 distinct real embeddings and X is a space whose homotopy-type is accessible—thanks to the validated Lichtenbaum-Quillen conjectures [8] [6] and the results of [4]. To make use of the Serre spectral sequence of this fibre sequence, one needs to at least know the kernel of $H^*(\underline{\alpha}; \mathbb{F}_2)$, and this is exactly the problem posed by our question.

The relations among the Stiefel-Whitney classes depend on classical number-theoretic invariants of A, such as the narrow ideal class group $NPic\, A$, and unfortunately we are able to obtain a complete answer only under a certain technical hypothesis on the narrow class group. The proofs, however, involve a result of independent interest: We compute the 2-adic complex, real and self-conjugate K-theory of the algebraic K-theory spectrum KA. These

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three groups, together with the various maps between them, constitute the *united K-theory* defined by Bousfield [1].

A more detailed outline of our results follows. *Unless otherwise indicated, all homology and cohomology groups in this paper have $\mathbb{Z}/2$-coefficients, and all class groups, K-groups, etc. are localized (usually completed) at the prime 2. All spectra and spaces are completed at the prime 2.*

1.1 Relations among Stiefel-Whitney classes

Since the mod 2 Chern classes of a complexified real bundle are just the squares of the Stiefel-Whitney classes, it is immediate that the $w_n^2(\alpha)$ are independent of the choice of real embedding. In Theorem 2.7 we show:

Proposition 1.1 *If $A = F$, the relations $w_n^2(\alpha_i) = w_n^2(\alpha_j)$ generate all relations among the Stiefel-Whitney classes of the real embeddings.*

Note that we can define Stiefel-Whitney classes for any map $BGLA \longrightarrow BO$, and that the above relations can also be written as $w_n^2(\alpha_i - \alpha_j) = 0$. For a general A, however, there are more relations. Before going further, it will be convenient to reformulate the problem in terms of homology. The existence of the relations stated in the above proposition is equivalent to having a factorization

$$\begin{array}{ccc} & & \square^{r_1}_{H_*BU} H_*BO \\ & \nearrow & \downarrow \\ H_*BGLF & \xrightarrow[\underline{\alpha}]{} & \square^{r_1}_{\mathbb{F}_2} H_*BO \end{array}$$

where $\square$ denotes the cotensor product or pullback in the category $\mathcal{H}$ of bicommutative Hopf algebras over $\mathbb{F}_2$; note that the cotensor product over $\mathbb{F}_2$ is just the tensor product. The assertion that these are the only relations is equivalent to the assertion that the lifted map is surjective.

Note that for general A, the proposition gives an upper bound for the image of $H_*\underline{\alpha}$. We obtain a lower bound as follows: Let BSC denote the classifying space for self-conjugate K-theory. Then the complexification map factors canonically as

$$BO \xrightarrow{\epsilon} BSC \xrightarrow{\zeta} BU,$$

leading to inclusions

$$\square^{r_1}_{H_*BSC} H_*BO \subset \square^{r_1}_{H_*BU} H_*BO \subset \square^{r_1}_{\mathbb{F}_2} H_*BO.$$

There are short exact sequences in $\mathcal{H}$

$$H_*(U/O) \longrightarrow H_*BO \longrightarrow H_*BU$$

and

$$H_*(Sp/U)\longrightarrow H_*BO\longrightarrow H_*BSC.$$

It follows that $\square^{r_1}_{H_*BU}H_*BO \cong H_*BO \otimes (H_*(U/O))^{r_1-1}$ and $\square^{r_1}_{H_*BSC}H_*BO \cong H_*BO \otimes (H_*(Sp/U))^{r_1-1}$.

Proposition 1.2 *Assume $\frac{1}{2} \in A$. Then $\square^{r_1}_{H_*BSC}H_*BO \subset Im\, H_*\underline{\alpha} \subset \square^{r_1}_{H_*BU}H_*BO$.*

The problem is to determine where $Im\, H_*\underline{\alpha}$ fits between these two extremes; we will see that already for real quadratic fields and $A = \mathcal{O}_F[\frac{1}{2}]$, both extremes can occur.

Let $\mathbb{R}_F$ denote the product of r_1 copies of $\mathbb{R}$, indexed by the real embeddings. Thus $\mathbb{R}_F^\times/(\mathbb{R}_F^\times)^2 \cong (\mathbb{Z}/2)^{r_1}$. Then Proposition 1.1 is valid for any A such that $A^\times\longrightarrow\mathbb{R}_F^\times/(\mathbb{R}_F^\times)^2$ is surjective. But in general there is an exact sequence

$$0\longrightarrow A^{pos}\longrightarrow A^\times\longrightarrow\mathbb{R}_F^\times/(\mathbb{R}_F^\times)^2\longrightarrow NPic\, A\longrightarrow Pic\, A\longrightarrow 0,$$

where $NPic\, A$ is the narrow ideal class group of divisors modulo totally positive principal divisors. We write κA for the kernel of $NPic\, A\longrightarrow Pic\, A$, the principal divisors modulo totally positive principal divisors. Note that κA is a $\mathbb{Z}/2$-vector space, and let $b = dim_{\mathbb{Z}/2}\kappa A$.

The precise statement of our main theorem can be found in Theorem 2.11 below; here we will state it somewhat imprecisely:

Theorem 1.3 *Suppose that the short exact sequence*

$$0\longrightarrow\kappa Pic\, R\longrightarrow NPic\, R\longrightarrow Pic\, R\longrightarrow 0$$

splits. Then

$$Im\, H_*\underline{\alpha} = H_*BO \otimes (H_*(U/O))^{r_1-b-1} \otimes (H_*(Sp/U))^b.$$

Here we have left it ambiguous exactly how the various factors $H_*(U/O)$, $H_*(Sp/U)$ are situated in the image.

Note that Stiefel-Whitney classes $w_n\beta$ are defined for any formal $\mathbb{Z}$-linear combination β of real embeddings. More precisely, there is a natural homomorphism

$$\mathbb{Z}Hom\,(F,\mathbb{R})\longrightarrow Hom_{\mathcal{H}}(H^*BO, H^*BGLA)$$

that factors through the quotient $\overline{\mathbb{Z}Hom\,(F,\mathbb{R})}$ by the relations $2(\alpha_i - \alpha_j) = 0$. Thus $\overline{\mathbb{Z}Hom\,(F,\mathbb{R})} = \mathbb{Z} \oplus (\mathbb{Z}/2)^{r_1-1}$, where the generators of the indicated summands can be taken as α_1 and $\beta_i = \alpha_1 - \alpha_i$, $2 \le i \le r_1$. Let $V = (\mathbb{Z}/2)^{r_1-1}$ denote the torsion subgroup, and let ϕ denote the composite

$$V \hookrightarrow \mathbb{Z}/2Hom\,(F,\mathbb{R})\longrightarrow Hom\,(A^\times, \mathbb{R}^\times/(\mathbb{R}^\times)^2).$$

Corollary 1.4 *The relations among the Stiefel-Whitney classes of the real embeddings are generated by the relations*

(i) $w_n^2(\alpha_i - \alpha_j) = 0$ for all $n \ge 1$;

(ii) $w_n(\beta) = 0$ for n odd and $\beta \in Ker\,\phi$.

The proof of Theorem 1.3 proceeds roughly as follows: In the diagram following Proposition 1.1, the factorization comes about because the composites

$$H_*BGLF \xrightarrow{\alpha_i - \alpha_j} H_*BO \longrightarrow H_*BU$$

are null in $\mathcal{H}$, and since the fibre sequence $U/O \longrightarrow BO \longrightarrow BU$ induces a short exact sequence in $\mathcal{H}$, we get a lift

$$\begin{array}{ccc} & & H_*(U/O) \\ & \nearrow & \downarrow \\ H_*BGLF & \xrightarrow[\alpha_i - \alpha_j]{} & H_*BO \end{array}$$

In fact the maps $\alpha_i - \alpha_j$ can be modified slightly, without changing the induced map on homology, so that this lifting exists on the level of infinite loop spaces. In Theorem 1.3, the idea is to produce enough further liftings of the form

$$\begin{array}{ccc} & & H_*(Sp/U) \\ & \nearrow & \downarrow \\ H_*BGLA & \xrightarrow[f]{} & H_*BO \end{array}$$

for suitable maps f constructed from the real embeddings. We will see that the map from real to self-conjugate K-theory is injective, so there is no hope of getting such lifts on the geometric level. Nevertheless, we find enough maps f such that the composite

$$H_*BGLA \xrightarrow{f} H_*BO \longrightarrow H_*BSC$$

is null in $\mathcal{H}$. This yields a lowering of the upper bound on $Im\, H_*\underline{\alpha}$.

Up to this point we have not used the splitting hypothesis, which now enters as follows: The number-theoretic part of our analysis has focused on the image of the natural map $\rho : A^\times/(A^\times)^2 \longrightarrow \mathbb{R}_F^\times/(\mathbb{R}_F^\times)^2$. But $A^\times/(A^\times)^2$ fits into a short exact sequence

$$0 \longrightarrow A^\times/(A^\times)^2 \longrightarrow H^1_{\acute{e}t}(A; \mathbb{F}_2) \longrightarrow Pic\, A[2] \longrightarrow 0,$$

where $Pic\, A[2]$ denotes the elements annihilated by 2 in the Picard group, and ρ is just the restriction to $A^\times/(A^\times)^2$ of the natural map $H^1_{\acute{e}t}(A; \mathbb{F}_2) \longrightarrow H^1_{\acute{e}t}(\mathbb{R}_F; \mathbb{F}_2)$. This latter map induces a map $\partial : Pic\, A[2] \longrightarrow \kappa A$, and if $\partial = 0$ then our upper and lower bounds can be seen to coincide. Since the vanishing of ∂ is equivalent to the splitting hypothesis, this completes the proof of Theorem 1.3.

1.2 United K-theory of KA

We compute the complex, real and self-conjugate K-theory of the algebraic K-theory spectrum KA, at least up to certain extensions. These three cohomology theories can be packaged into a single object called *united K-theory* [1], incorporating the Adams operations, the coefficient rings and the maps between the three theories. Our complete results are too technical to state here; we give two sample calculations that play a key role in the Stiefel-Whitney class results.

The ring KO^0KO of degree zero operations in 2-adic real K-theory can be identified with the *Iwasawa algebra* $\Lambda = \mathbb{Z}_2[[T]]$. Here T corresponds to $\psi^5 - 1$. Let $T_F = (1+T)^{2^{a_F}} - 1$, where 2^{a_F} is the order of the group of 2-power roots of unity in $F\sqrt{-1}$.

Theorem 1.5 *KO^0KA is generated as a Λ-module by the real embeddings α_i, subject only to the relations $2T_F\alpha_i = 0$ and $2(\alpha_i - \gamma_{ij}\alpha_j) = 0$ for certain elements $\gamma_{ij} \in \Lambda$. In particular, $KO^0KA \cong \Lambda/2T_F \oplus (\Lambda/2)^{r_1-1}$ as Λ-module.*

Note that KO^0KA depends only on F, not on A. For general n, KO^nKA will depend on the basic Iwasawa module M associated to A (see §3.1 for the definition).

The ring KSC^0KSC of degree zero operations in 2-adic self-conjugate K-theory can be identified with $\Lambda' = \Lambda[\sigma]$, where σ is the group of order two generated by the self-conjugate version of ψ^{-1}.

Theorem 1.6 *There are short exact sequences of Λ'-modules*

$$0 \longrightarrow eM_+(-1) \longrightarrow KSC^0KA \longrightarrow eM^+ \longrightarrow 0$$

and

$$0 \longrightarrow KO^0KA \longrightarrow KSC^0KA \longrightarrow eM^+(-1) \longrightarrow 0.$$

Here M is the basic Iwasawa module mentioned above, e is the appropriate extension of scalars functor, and $(-)^+, (-)_+$ refer to σ-invariants and σ-coinvariants, respectively.

Organization of the paper: §2 states the results on Stiefel-Whitney classes, and takes the proofs as far as possible without invoking the results on united K-theory. §3 computes the real K-theory and §4 the self-conjugate K-theory of KA. In §5 the K-theory results are used to complete the proof of the main Theorem 2.11 on Stiefel-Whitney classes.

We assume throughout that F has a real embedding, and that a basepoint embedding $\alpha_1 : F \subset \mathbb{R}$ has been fixed.

2 Stiefel-Whitney classes of real embeddings

We begin by considering the mod 2 Chern classes associated to a complex embedding of F, and deduce the relations $w_n^2\alpha_i = w_n^2\alpha_j$ for Stiefel-Whitney classes. In §2.2 we reformulate the problem in terms of homology. In §2.3 we digress to review the homology of BSC, the classifying space for self-conjugate K-theory. In §2.4 we give a general estimate on the size of the image of $H_*\underline{\alpha}$, where $\underline{\alpha}$ is the natural map $BGLA \longrightarrow (BO)^{r_1}$ induced by the real embeddings. In §2.5 we digress to review the narrow Picard group $NPic\,A$ and related invariants. Our main theorem is stated in §2.6; we completely determine the image of $H_*\underline{\alpha}$—and hence also the relations among the Stiefel-Whitney classes of the real embeddings—under a certain technical hypothesis on $NPic\,A$. We give an outline of the proof, postponing the details to §5.

2.1 Characteristic classes for real and complex embeddings

Let $\alpha : F \longrightarrow \mathbb{C}$ be a complex embedding. Then α induces a map $BGL(\alpha) : BGLF \longrightarrow BGL\mathbb{C}^\delta$, where $\mathbb{C}^\delta$ means $\mathbb{C}$ with the discrete topology. Let A be a subring of F. Pulling back the universal mod 2 Chern classes along the composite

$$BGLA \longrightarrow BGLF \longrightarrow BGL\mathbb{C}^\delta \longrightarrow BGL\mathbb{C}^{top} \cong BU,$$

where $\mathbb{C}^{top}$ means $\mathbb{C}$ with the classical topology, we obtain mod 2 Chern classes $c_n(\alpha) \in H^{2n}BGLA$. The following result is well-known.

Proposition 2.1 *The mod 2 Chern classes $c_n(\alpha)$ are independent of the choice of complex embedding α. Furthermore, for fixed α the $c_n(\alpha)$ are algebraically independent.*

Proof: For the first statement, it suffices to take $A = F$. Suppose β is another complex embedding. Then there is a field automorphism ϕ of $\mathbb{C}$ with $\beta = \phi\alpha$. Hence it is enough to show that ϕ induces the identity map on $H_*BGL\mathbb{C}^\delta$. By Suslin's theorem [9] $BGL\mathbb{C}^\delta \longrightarrow BGL\mathbb{C}^{top}$ induces an isomorphism on mod 2 homology. Since the map $\mathbb{R}P^\infty \longrightarrow BU$ classifying the complexification of the canonical line bundle is a generating complex for BU, it follows that the natural map $i : \mathbb{R}P^\infty \longrightarrow BGL\mathbb{C}^\delta$ induced by the inclusion $\pm 1 \subset \mathbb{C}^\times$ is a generating complex for $BGL\mathbb{C}^\delta$. But $\phi i = i$, so $H_*BGL(\phi)$ is the identity, as desired.

For the second statement, it suffices to take $A = \mathbb{Z}$. Note that i factors canonically through $BGL\mathbb{Z}$, and induces a canonical map of Hopf algebras $S(\tilde{H}_*\mathbb{R}P^\infty) \longrightarrow H_*BGL\mathbb{Z}$. Then the composite

$$S(\tilde{H}_*\mathbb{R}P^\infty) \longrightarrow H_*BGL\mathbb{Z} \longrightarrow H_*BU$$

is surjective, and dualizing yields the result.

Similarly, each real embedding $\alpha : F \longrightarrow \mathbb{R}$ defines Stiefel-Whitney classes $w_n(\alpha) \in H^nBGLA$. In this case, however, $w_n(\alpha)$ may depend on α. In fact $w_1(\alpha)$ can be identified with the homomorphism $\alpha^\times/2 : F^\times \longrightarrow \mathbb{R}^\times/(\mathbb{R}^\times)^2$, and it is clear that this homomorphism depends on α. For fixed α the $w_n(\alpha)$ will be algebraically independent, by the argument used for Chern classes, but there will be relations as α varies. For example, Proposition 2.1 yields the following corollary.

Corollary 2.2 *Let $\alpha, \beta : F \longrightarrow \mathbb{R}$ be real embeddings. Then $w_n^2(\alpha) = w_n^2(\beta)$ for all n.*

2.2 A reformulation in terms of homology

Corollary 2.2 can be reformulated in terms of the Hopf algebras H_*BO, H_*BU and their duals. These are connected bicommutative Hopf algebras of finite type over the field $\mathbb{F}_2$; recall that the category $\mathcal{H}$ of all such Hopf algebras is an abelian category. Given objects $A_1, ..., A_n, B \in \mathcal{H}$ and morphisms $B \longrightarrow A_i$, we can form the multiple pushout or tensor product

$$\otimes_B A_i = A_1 \otimes_B A_2 ... \otimes_B A_n.$$

Dually, we can form the multiple pullback or cotensor product

$$\square_{B^*} A_i^* = A_1^* \square_{B^*} A_2^* \ldots \square_{B^*} A_n^*.$$

Note that when $B = \mathbb{F}_2$, the cotensor product and tensor product coincide.

Corollary 2.3 *Let $\alpha_1, ..., \alpha_{r_1}$ denote the distinct real embeddings of F. Then the natural map $(\alpha_1, ..., \alpha_{r_1})_* : H_*BGLA \longrightarrow \otimes^{r_1} H_*BO$ factors:*

$$\begin{array}{ccc} & & \square^{r_1}_{H_*BU} H_*BO \\ & \nearrow^{\cdots} & \downarrow \\ H_*BGLA & \longrightarrow & \square^{r_1}_{\mathbb{F}_2} H_*BO \end{array}$$

Dually, $(\alpha_1, ..., \alpha_{r_1})^* : \otimes^{r_1} H^*BO \longrightarrow H^*BGLF$ factors through $\otimes^{r_1}_{H^*BU} H^*BO$.

The structure of the tensor and cotensor products in this last corollary can be made more explicit. Let $\mathcal{P}$ denote the Hopf kernel of $H_*BO \longrightarrow H_*BU$. It is a polynomial algebra on odd-dimensional primitive generators $s_1, s_3, ...$. In fact, the fibre sequence of spectra $\Sigma KO \xrightarrow{\eta} KO \longrightarrow K$ induces a fibre sequence of spaces $U/O \longrightarrow BO \longrightarrow BU$ that in turn induces a short exact sequence in $\mathcal{H}$

$$\mathcal{P} = H_*(U/O) \longrightarrow H_*BO \longrightarrow H_*BU.$$

Dually, $\mathcal{P}^*$ is an exterior algebra on the universal Stiefel-Whitney classes. Now it is a trivial fact, valid in any abelian category, that if $A \longrightarrow B$ is any morphism, with kernel K, then the r-fold multiple pullback $A \times_B \times ... \times_B A$ is naturally isomorphic to $A \times K^{r-1}$. So here we conclude:

Proposition 2.4 *There are natural isomorphisms*

$$\square^{r_1}_{H_*BU} H_*BO \cong H_*BO \otimes \mathcal{P}^{r_1 - 1}$$

and

$$\otimes^{r_1}_{H^*BU} H^*BO \cong H^*BO \otimes \mathcal{P}^{*r_1 - 1}.$$

2.3 A digression on self-conjugate K-theory

The spectrum KSC of self-conjugate K-theory is the homotopy fibre of the map $\psi^{-1} - 1 : K \longrightarrow K$. Thus there is a fibre sequence

$$\Sigma^{-1} K \xrightarrow{\Sigma^{-1}\gamma} KSC \xrightarrow{\zeta} K.$$

There is also a well-known fibre sequence

$$\Sigma^2 \wedge KO \xrightarrow{\eta^2} KO \xrightarrow{\epsilon} KSC.$$

The notation for the maps follows [1].

Passing to basepoint-components of zero-th spaces, we get fibre sequences of spaces

$$U \longrightarrow BSC \longrightarrow BU$$

and

$$Sp/U \longrightarrow BO \longrightarrow BSC.$$

Since $\zeta\epsilon = c : KO \longrightarrow K$, the map $H_*BSC \longrightarrow H_*BU$ is onto and hence the Serre spectral sequence of the first fibre sequence collapses. On the other hand, $H_*(Sp/U)$ is a polynomial algebra on primitive generators in degrees congruent to 2 mod 4; dimension counting then shows that the Serre spectral sequence of the second fibre sequence also collapses. We conclude that there is a short exact sequence in $\mathcal{H}$

$$H_*Sp/U \longrightarrow H_*BO \longrightarrow H_*BSC$$

in which the first map is an isomorphism onto $\mathcal{FP}$. Here $\mathcal{F}$ denotes the Frobenius endomorphism $x \mapsto x^2$ on objects of $\mathcal{H}$. Thus:

Proposition 2.5 *$H_*BSC = H_*BO/\mathcal{FP}$.*

Corollary 2.6 *Let*

$$q_k = w_{2k-1} + w_1 w_{2k-2} + ... w_{k-1} w_k.$$

Then H^*BSC is a polynomial algebra on the even Chern classes c_{2k} and the q_k. (Note that $q_k^2 = c_{2k-1} + c_1 c_{2k-2} + ... c_{k-1} c_k$.)

The corollary follows by a standard calculation; see [7], where this same Hopf algebra arises as $H_*BGL\mathbb{F}_3$.

2.4 Stiefel-Whitney classes for rings of S-integers, I

Our next step is an estimate on the size of $Im\, H_*\underline{\alpha}$.

Theorem 2.7 *Suppose $\frac{1}{2} \in A$. Then*

$$\Box^{r_1}_{H_*BSC} H_*BO \subset Im\, H_*\underline{\alpha} \subset \Box^{r_1}_{H_*BU} H_*BO.$$

Moreover, equality holds for the second inclusion if and only if $A^\times \longrightarrow \mathbb{R}_F^\times/(\mathbb{R}_F^\times)^2$ is surjective. In particular, equality always holds for the second inclusion when $A = F$.

Proof: The inclusion $H_*\underline{\alpha} \subset \Box^{r_1}_{H_*BU} H_*BO$ was shown above. If equality holds, then $H_1\underline{\alpha}$ is surjective and hence $A^\times \longrightarrow \mathbb{R}^\times_F/(\mathbb{R}^\times_F)^2$ is surjective.

Conversely, suppose $A^\times \longrightarrow \mathbb{R}^\times_F/(\mathbb{R}^\times_F)^2$ is surjective. The canonical map $j : \mathbb{R}P^\infty \longrightarrow BGLF$ induces a homomorphism of Hopf algebras $\mathcal{S}(\tilde{H}_*\mathbb{R}P^\infty) \cong H_*BO \longrightarrow H_*BGLA$. It follows that the diagonal $\Delta H_*BO \subset \Box^{r_1}_{H_*BU} H_*BO$ is in the image of $(\alpha_1, ..., \alpha_{r_1})_*$. The proof of Proposition 2.4 shows that $\Box^{r_1}_{H_*BU} H_*BO$ is generated by ΔH_*BO together with the subalgebra $\otimes^{r_1}\mathcal{P}$ generated by all primitives. Hence it suffices to show that $Prim\,(H_*BO)^{r_1}$ is in the image.

In order to do this, we need some additional structure. All of the homology Hopf algebras considered in this paper are Hopf modules over the Hopf ring H_*BO, and induced maps are Hopf module maps. For a detailed exposition of this point, see [2]; for present purposes the following *ad hoc* construction will suffice.

Tensor product of vector bundles leads to a ring space structure on $BO \times \mathbb{Z}$, so that $BO = BO \times \{0\}$ becomes a ring space without identity. This yields a second product on H_*BO, denoted $a \circ b$, Similarly, the tensor product of projective modules leads to a natural ring space structure on $BGLA^+$ for any commutative ring A. Now the canonical Hopf algebra homomorphism $S(\tilde{H}_*\mathbb{R}P^\infty) \longrightarrow H_*BGL\mathbb{Z}$ discussed earlier is in fact a homomorphism of Hopf rings. This can be seen by algebraic calculation, or by observing that the H-space structure on $\mathbb{R}P^\infty$ makes $Q\mathbb{R}P^\infty_+$ a ring space, and $Q\mathbb{R}P^\infty_+ \longrightarrow BGL\mathbb{Z}^+ \times \mathbb{Z}$ is a ring map. In any case, the conclusion is that for any module space X over $BGL\mathbb{Z}^+$, H_*X is naturally a Hopf module over the Hopf ring H_*BO. Furthermore, the primitives in H_*X always form a submodule for this structure.

Returning to the proof, it is easy to show that $Prim\,H_*BO$ is generated by H_1BO under the Hopf module structure (see for example [2]). Since $H_*BGLA \longrightarrow (H_*BO)^{r_1}$ is a map of Hopf modules, we conclude that the image contains $\mathcal{P}^{r_1}$, as desired.

The Approximation Theorem (see for example [5], II, 3.4) shows that $F^\times \longrightarrow \mathbb{R}^\times_F/(\mathbb{R}^\times_F)^2$ is surjective, proving the last assertion of the theorem.

It remains to show that $\Box^{r_1}_{H_*BSC} H_*BO \subset Im\,H_*\underline{\alpha}$. This is where the assumption $\frac{1}{2} \in A$ will be used. As in the argument above, it suffices to show that $\mathcal{F}Prim\,H_*BO^{r_1}$—the squares of the primitives—lie in $Im\,H_*(\underline{\alpha})$. Since $\mathcal{F}Prim\,H_*BO$ is generated by s_1^2 as Hopf module over H_*BO, it suffices to show that the primitives of dimension 2 lie in the image. Since π_2BO maps isomorphically onto the primitives in H_2BO, we have reduced to the following standard lemma.

Lemma 2.8 *If 2 is a unit in A, then* $K_2A/2 \longrightarrow (K_2\mathbb{R}/2)^{r_1}$ *is surjective.*

Briefly, the Mercurjev-Suslin theorem implies that there is a commutative diagram

$$\begin{array}{ccc} K_2A/2 & \longrightarrow & (K_2\mathbb{R}/2)^{r_1} \\ \downarrow & & \downarrow \\ Br\,A[2] & \longrightarrow & (Br\,\mathbb{R})^{r_1} \end{array}$$

in which the left vertical map is onto and the right vertical map is an isomorphism. Class field theory tells us that the bottom map is surjective. This proves the lemma, and completes the proof of the proposition.

Theorem 2.7 has the following corollary for Stiefel-Whitney classes.

Corollary 2.9 *The ideal of all relations among the Stiefel-Whitney classes $w_n\alpha_i$ contains the relations $w_n^2\alpha_i = w_n^2\alpha_j$, and if $A^\times \longrightarrow \mathbb{R}_F^\times/(\mathbb{R}_F^\times)^2$ is surjective then these are the only relations. In particular this holds for $A = F$.*

In the general case there are at most the further relations $q_k(\alpha_i) = q_k(\alpha_j)$, or equivalently $w_n(\alpha_i) = w_n(\alpha_j)$ for n odd.

2.5 The narrow Picard group and related invariants

Let F^{pos} denote the group of totally positive elements of F; that is, the elements that map to a positive number under every real embedding of F. Note that there is a short exact sequence

$$0 \longrightarrow F^{pos} \longrightarrow F^\times \longrightarrow \mathbb{R}_F^\times/(\mathbb{R}_F^\times)^2 \longrightarrow 0.$$

Similarly, let A^{pos} denote $A^\times \cap F^{pos}$, the group of totally positive units of A. Again there is a short exact sequence

$$0 \longrightarrow A^{pos} \longrightarrow A^\times \longrightarrow \mathbb{R}_F^\times/(\mathbb{R}_F^\times)^2,$$

but now the last map need not be onto. For example, when $F = \mathbb{Q}\sqrt{m}$ is real quadratic and $A = \mathcal{O}_F$, the map $\mathcal{O}_F^\times \longrightarrow \mathbb{R}_F^\times/(\mathbb{R}_F^\times)^2 = (\mathbb{Z}/2)^2$ will have rank one precisely when the fundamental unit ϵ is totally positive. In the range $2 \leq m \leq 10$, for instance, one can easily check by hand that this happens precisely when $m = 3, 6, 7$.

Now let $Div\,A$, $Prin\,A$ denote the divisor group and principal divisor group respectively, so that $Pic\,A = Div\,A/Prin\,A$. A principal divisor is said to be totally positive if it admits a totally positive generator. The group of all such totally positive principal divisors will be denoted $Prin^{pos}A$. Let $\kappa A = Prin\,A/Prin^{pos}A$. Then κA has exponent 2, and in fact $\kappa A \cong coker\,(A^\times \longrightarrow \mathbb{R}_F^\times/(\mathbb{R}_F^\times)^2)$. Thus there is an exact sequence

$$0 \longrightarrow A^{pos} \longrightarrow A^\times \longrightarrow \mathbb{R}_F^\times/(\mathbb{R}_F^\times)^2 \longrightarrow \kappa A \longrightarrow 0,$$

where the last map is the obvious one arising from the identification $\mathbb{R}_F^\times/(\mathbb{R}_F^\times)^2 = F^\times/F^{pos}$.

Finally, the *narrow Picard group* $NPic\,A$ is defined by

$$NPic\,A = Div\,A/Prin^{pos}A.$$

Hence there is a short exact sequence

$$0 \longrightarrow \kappa A \longrightarrow NPic\,A \longrightarrow Pic\,A \longrightarrow 0.$$

Under the isomorphisms of class field theory, $NPic\,A$ corresponds to the maximal abelian extension $\tilde{H}_F$ of F that is unramified at all finite primes and in which the primes in S split completely.

We can splice the last two exact sequences together to get

$$0 \longrightarrow A^{pos} \longrightarrow A^{\times} \longrightarrow \mathbb{R}_F^{\times}/(\mathbb{R}_F^{\times})^2 \longrightarrow NPic\, A \longrightarrow Pic\, A \longrightarrow 0.$$

Here the map $\mathbb{R}_F^{\times}/(\mathbb{R}_F^{\times})^2 \longrightarrow NPic\, A$ has the following Galois-theoretic interpretation: There is a canonical identification $\mathbb{R}_F^{\times}/(\mathbb{R}_F^{\times})^2 = H_1(\mathbb{R}_F;\mathbb{Z}_2) = \oplus^{r_1} H_1(\mathbb{R};\mathbb{Z}_2)$. Each real embedding α_i determines a well-defined "complex conjugation" σ_i in $G(F_{ab}/F)$, and α_i maps to σ_i in $G(\tilde{H}_F/F)$.

For our purposes, there is a further complication that must be taken into account. Assume $1/2 \in A$, and recall the short exact sequence

$$0 \longrightarrow A^{\times}/(A^{\times})^2 \longrightarrow H^1_{\acute{e}t}(A;\mathbb{Z}/2) \longrightarrow Pic\, A[2] \longrightarrow 0.$$

Then the map $A^{\times}/(A^{\times})^2 \longrightarrow \mathbb{R}_F^{\times}/(\mathbb{R}_F^{\times})^2$ considered above is the restriction of the natural map $H^1_{\acute{e}t}(A;\mathbb{Z}/2) \longrightarrow H^1_{\acute{e}t}(\mathbb{R}_F;\mathbb{Z}/2)$. In particular, there is an induced map $Pic\, A[2] \longrightarrow \kappa A$. The rank of this map is $b-a$, where

$$a = dim_{\mathbb{Z}/2} coker\, (H^1_{\acute{e}t}(A;\mathbb{Z}/2) \longrightarrow \mathbb{R}_F^{\times}/(\mathbb{R}_F^{\times})^2)$$

and

$$b = dim_{\mathbb{Z}/2} coker\, (A^{\times} \longrightarrow \mathbb{R}_F^{\times}/(\mathbb{R}_F^{\times})^2) = dim_{\mathbb{Z}/2} \kappa A.$$

Proposition 2.10 *$a = b$ if and only if the short exact sequence*

$$0 \longrightarrow \kappa A \longrightarrow NPic\, A \longrightarrow Pic\, A \longrightarrow 0$$

splits.

Proof: The nonzero elements of $H^1_{\acute{e}t}(A;\mathbb{Z}/2)$ correspond to quadratic extensions E of F that are unramified away from S and the infinite places. If $[I] \in Pic\, A$ with $I^2 = (x)$, we obtain such an extension by taking $E = F\sqrt{x}$. Of course x is only defined up to units in A—that is, as an element of $Prin\, A$—but in any case we see that the map $j : Pic\, A[2] \longrightarrow \kappa A$ has the following description: Given $[I] \in Pic\, A$ such that I^2 is principal, choose a generator x for I^2 as above. Regarding x as a principal divisor, we then get a well-defined element $[x] \in \kappa A$. In other words, j coincides with the boundary map ∂ in the six-term exact sequence obtained from the short exact sequence by reducing mod 2. Since the short exact sequence splits if and only if $\partial = 0$, this completes the proof.

We conclude this section with a few special cases and examples.

Example 1: Take $A = \mathcal{O}_F[\frac{1}{2}]$, and suppose that (i) $Pic\, A = 0 = Pic\, A_0$, and (ii) there is a unique prime $\mathcal{P}_0$ over 2 in A_0. (These conditions hold, for example, when F is the maximal real subfield of $\mathbb{Q}(\mu_{2^n})$.) Then $NPic\, A = 0$.

To see this, suppose $Pic\, A = 0$ but $NPic\, A \neq 0$. Then there is a quadratic extension L of F that is ramified only at ∞ and with the unique prime $\mathcal{P}$ over 2 in $\mathcal{O}_F$ splitting in L. In particular, $L \cap F_0 = 0$. Hence $L_0 = LF_0$ is a quadratic extension of F_0, unramified at all finite primes and with $\mathcal{P}_0$ split. Hence $Pic\, A_0 \neq 0$.

Example 2: Let $F = \mathbb{Q}\sqrt{7}$. Then the fundamental unit $\epsilon = 8 + 3\sqrt{7}$ is totally positive. Since $Pic\,\mathcal{O}_F = 0$, we have $\kappa\mathcal{O}_F = NPic\,\mathcal{O}_F = \mathbb{Z}/2$. The ideal over 2 is generated by $3 + \sqrt{7}$, which again is totally positive. Hence for $A = \mathcal{O}_F[\frac{1}{2}]$, we again find $\kappa A = NPic\,A = \mathbb{Z}/2$.

Example 3: (The author learned this example from Ralph Greenberg.) Consider a real quadratic field $\mathbb{Q}\sqrt{m}$, and suppose (i) the fundamental unit is totally positive; and (ii) the prime divisors of m are all congruent to 1 mod 4. Then in the case $A = \mathcal{O}_F$, it can be shown that the short exact sequence of Proposition 2.10 does not split. If in addition $m = 5\,mod\,8$, the splitting still fails for $A = \mathcal{O}_F[\frac{1}{2}]$. For an explicit example, one can take $m = 205$.

2.6 Stiefel-Whitney classes for rings of S-integers, II

Objects in $\mathcal{H}$ with a compatible H_*BO-module structure form a category $\mathcal{H}H_*BO$; if there is also a compatible action of the Steenrod algebra $\mathcal{A}$ we get a category $\mathcal{H}H_*BO\mathcal{A}$ (see [2]). In the category $\mathcal{H}H_*BO$,

$$Hom\,(\mathcal{P}, H_*BO) = \mathbb{Z}/2 = Hom\,(\mathcal{FP}, H_*BO),$$

where the isomorphisms are given by restricting to $\mathcal{P}_1$, $\mathcal{F}(\mathcal{P}_1)$, respectively. Hence subobjects of $(H_*BO)^{r_1}$ of the form $\mathcal{P}^r \otimes (\mathcal{FP})^s$ will be uniquely determined by specifying a pair of independent subspaces of $(H_1BO)^{r_1}$, of dimensions r, s respectively. The H_*BO summand occurring below always comes for the diagonal embedding, and contains a distinguished copy of $\mathcal{P}$ corresponding to the diagonal subspace of $(\mathbb{Z}/2)^{r_1}$. We remark also that in the category $\mathcal{H}H_*BO\mathcal{A}$, $End\,H_*BO = \mathbb{Z}_2$, and hence $Aut\,(H_*BO)^{r_1} = GL_n\mathbb{Z}_2$.

Theorem 2.11 *Suppose that the short exact sequence*

$$0 \longrightarrow \kappa Pic\,A \longrightarrow NPic\,A \longrightarrow Pic\,A \longrightarrow 0$$

splits. Then the image of $H_*\underline{\alpha} : H_*BGLA \longrightarrow (H_*BO)^{r_1}$ has the form

$$H_*BO \otimes \mathcal{P}^{r_1-b-1} \otimes (\mathcal{FP})^b,$$

where the tensor products are to be interpreted as internal direct sums in the category $\mathcal{H}H_*BO\mathcal{A}$. In particular, $Im\,H_*\underline{\alpha}$ is completely determined by $H_1\underline{\alpha}$.

The proof of this theorem will yield partial results even when the splitting hypothesis on $NPic\,R$ does not hold.

To restate Theorem 2.11 in terms of Stiefel-Whitney classes, consider the natural homomorphism

$$\mathbb{Z}/2[Hom\,(F, \mathbb{R})] \longrightarrow Hom\,(A^\times, \mathbb{R}^\times/(\mathbb{R}^\times)^2)$$

that maps $\alpha : F \longrightarrow \mathbb{R}$ to the induced map $A^\times \longrightarrow \mathbb{R}^\times/(\mathbb{R}^\times)^2$, and note that the kernel has dimension b. As explained in the introduction, there is a composite map ϕ:

$$V \hookrightarrow \overline{\mathbb{Z}[Hom\,(F, \mathbb{R})]} \longrightarrow \mathbb{Z}/2[Hom\,(F, \mathbb{R})] \longrightarrow Hom\,(A^\times, \mathbb{R}^\times/(\mathbb{R}^\times)^2),$$

where $\overline{\mathbb{Z}[Hom\,(F,\mathbb{R})]}$ is $\mathbb{Z}[Hom\,(F,\mathbb{R})]$ mod $2(\alpha_i - \alpha_j)$, and $V \cong (\mathbb{Z}/2)^{r_1-1}$ is generated by the $\alpha_i - \alpha_j$.

Corollary 2.12 *The relations among the Stiefel-Whitney classes are generated by the relations*
(i) $w_n^2(\alpha_i - \alpha_j) = 0$
and
(ii) $w_n(\beta) = 0$ *for all odd* n *and* $\beta \in Ker\,\phi$.
Alternatively, (ii) may be replaced by
(iii) $q_n(\beta) = 0$ *for all* n*, where* β *ranges over a basis for* $Ker\,\phi$.

The rest of this section is devoted to an outline of the proof of Theorem 2.11. The details are postponed to §5. We assume the splitting hypothesis of Theorem 2.11 only where indicated.

Lemma 2.13

$$H_*BO \otimes \mathcal{P}^{r_1-b-1} \otimes (\mathcal{FP})^b \subset Im\,H_*\underline{\alpha}.$$

Proof: By Theorem 2.7 we have $H_*BO \otimes (\mathcal{FP})^{r_1-1} \subset Im\,H_*\underline{\alpha}$. On the other hand, since $r_1 - b = dim_{\mathbb{Z}/2} Im\,(R^\times \longrightarrow \mathbb{R}_F^\times/(\mathbb{R}_F^\times)^2)$, the argument of Theorem 2.7 shows that $\mathcal{P}^{r_1-b} \subset Im\,H_*\underline{\alpha}$. Recall that one of the copies of $\mathcal{P}$ lies in the diagonal H_*BO.

Now by Corollary 2.3 we know that

$$Im\,H_*\underline{\alpha} \subset H_*BO \otimes \mathcal{P}^{r_1}.$$

The problem is to sharpen this upper bound until it agrees with the lower bound of Lemma 2.13.

Let us temporarily write $f_1 = \alpha_1$ and for $i \geq 2$ $f_i = \alpha_1 - \alpha_i$, regarded as maps $KR \longrightarrow KO$. Then for $i \geq 2$ the composite

$$H_*BGLR \xrightarrow{f_i} H_*BO \xrightarrow{c} H_*BU$$

is null in $\mathcal{H}$, and hence there is a lift

$$\begin{array}{ccc} & & H_*(U/O) \\ & \nearrow & \downarrow \\ H_*BGLR & \xrightarrow[f_i]{} & H_*BO \end{array}$$

In other words, if $f = (f_1, ..., f_{r_1})$, then we have a lift of the form

$$\begin{array}{ccc} & & H_*BO \otimes (H_*(U/O))^{r_1-1} \\ & \nearrow & \downarrow \\ H_*BGLR & \xrightarrow[f]{} & (H_*BO)^{r_1} \end{array}$$

Remark: Let $i \geq 2$. The f_i's themselves do not satisfy $cf_i = 0$, but they can be modified slightly to arrange this; see Proposition 3.7.

We will show that up to a further automorphism of $(KO)^{r_1}$, we may write $\underline{\alpha} : KR \longrightarrow (KO)^{r_1}$ as $g = (g_1, ..., g_{r_1})$ so that there is a further lifting

$$\begin{array}{ccc} & & H_*BO \otimes (H_*(U/O))^{r_1-a-1} \otimes (H_*(Sp/U))^a \\ & \nearrow & \downarrow \\ H_*BGLR & \xrightarrow{g} & H_*BO \otimes (H_*BO)^{r_1-a-1} \otimes (H_*BO)^a \end{array}$$

Now if the splitting hypothesis of Theorem 2.11 holds, then $a = b$ by Proposition 2.10 and hence the lower and upper bounds agree, proving Theorem 2.11.

To obtain the refined lifting displayed above, the main point is to produce enough maps $g_i \in KO^0KR$ such that the composite

$$H_*BGLR \xrightarrow{g_i} H_*BO \xrightarrow{\epsilon} H_*BSC$$

is null. Here it will definitely not be possible to arrange that $\epsilon g_i = 0$ in KSC^0KR, and in fact we will see that $\epsilon : KO^0KR \longrightarrow KSC^0KR$ is injective. Instead we will proceed as follows: The ring of operations KSC^0KSC is a commutative local ring, abstractly isomorphic to $K^0K \cong \Lambda'$. We will choose the g_i's so that for $r_1 - a + 1 \leq i \leq r_1$ there is a lift

$$\begin{array}{ccc} & & KSC \\ & \nearrow & \downarrow \theta \\ KR & \xrightarrow[\epsilon g_i]{} & KSC \end{array}$$

where θ lies in the maximal ideal of KSC^0KSC. On the other hand, it is easy to see that any such θ has the property $Im\, H_*\Omega_0^\infty\theta \subset \mathcal{F}H_*BSC$. But for all $i \geq 2$, the g_i's will be chosen so that $Im\, H_*\Omega_0^\infty\theta \subset \mathcal{P}H_*BSC$. Since $\mathcal{F}H_*BSC \cap \mathcal{P}H_*BSC = \mathbb{F}_2$, this will show that $H_*\Omega_0^\infty \epsilon g_i$ is null, as desired.

3 Real K-theory of KA

In this section we compute KO^*KA as a module over KO^*KO, up to certain extensions. In fact, we compute (K^*KA, KO^*KA) as an object in Bousfield's category ACR [1]; this entails keeping track of the realification map r and the complexification map c.

We assume throughout that $A = S^{-1}\mathcal{O}_F$ with S finite and $\frac{1}{2} \in A$. The results can be extended to the case S infinite by passing to inverse limits. As usual, all spectra are completed at the prime 2.

3.1 Notation

The ring of operations K^0K is isomorphic to the pro-group ring $\Lambda' = \mathbb{Z}_2[[\Gamma']]$, where ${\mathbb{Z}_2}^\times \cong \Gamma'$. Here $k \in {\mathbb{Z}_2}^\times$ corresponds to $\psi^k \in K^0K$. To avoid superfluous notation, we will not distinguish between a group of order two and its unique nontrivial element, usually denoted σ. Thus $\Gamma' = \Gamma \times \sigma$, where σ corresponds to the Adams operation ψ^{-1} and $\Gamma \cong \mathbb{Z}_2$ has topological generator γ corresponding to ψ^5. In fact γ is topologically generated by ψ^{-1} and ψ^k where k is any integer congruent to ± 3 mod 8; topologists usually take $k = 3$, but for Galois-theoretic reasons we take ψ^5 as our standard generator unless otherwise mentioned. Then $\Lambda' = \Lambda[\sigma]$ with $\Lambda \cong \mathbb{Z}_2[[T]]$, $T = \psi^5 - 1$.

The full ring of operations K^*K is a twisted tensor product $\Lambda' \tilde{\otimes} K^*S^0$, where $K^*S^0 = \pi_{-*}K = \mathbb{Z}_2[\beta, \beta^{-1}]$, $\beta \in \pi_2 K$. The twisting is given by $\psi^k\beta = k\beta\psi^k$; in the notation of Tate twisting this is written $\pi_{2n}K = \mathbb{Z}_2(n)$.

Similarly, $KO^*KO \cong \Lambda' \tilde{\otimes} KO^*S^0$, where Λ corresponds to the real Adams operations and

$$KO^*S^0 = \pi_{-*}KO = \mathbb{Z}_2[\eta, \xi, \beta_{\mathbb{R}}, \beta_{\mathbb{R}}^{-1}]/(2\eta, \eta^3, \xi^2 - 2\beta_{\mathbb{R}}, \eta\xi).$$

Here $\eta \in \pi_1 KO$, $\xi \in \pi_4 KO$, $\beta_{\mathbb{R}} \in \pi_8 KO$.

Remark: In the computations below, we will usually not bother to record the ξ-multiplications, because of the formula $\xi = r\beta^2 c$ in $KO^{-4}KO$.

The group Γ' is also canonically isomorphic to $Aut\,\mu_\infty(\mathbb{C})$, where $\mu_\infty(-)$ denotes the group of all 2-power roots of unity. Hence if we fix a real embedding of the number field F, and let $F_\infty = F(\mu_\infty\mathbb{C})$ denote the 2-adic cyclotomic extension, we have a monomorphism of finite index $\Gamma'_F \equiv G(F_\infty/F) \longrightarrow \Gamma'$. Moreover, $\Gamma'_F = \Gamma_F \times \sigma$, where $\Gamma_F = G(F_\infty/F_0)$. Here $F_0 = F\sqrt{-1}$ is the first stage in the usual filtration $F \subset F_0 \subset F_1 \subset ... \subset F_\infty$.

Let $a_F = \nu_2 \mid \mu_\infty(F_0) \mid$. Then Γ_F has index $2^{a_F - 2}$ in Γ, and $\Gamma'_F \subset \Gamma'$ is generated by the elements σ, γ_F corresponding to ψ^{-1}, ψ^q; here q is any integer such that $q = \pm 1\, mod\, 2^{a_F}$ but not $mod\, 2^{a_F+1}$. We have $\Lambda' \otimes_{\Lambda'_F} \mathbb{Z}_2 = \Lambda \otimes_{\Lambda_F} \mathbb{Z}_2 = \Lambda/\omega_F$, where $\omega_F = (1+T)^{2^{a_F - 2}} - 1$.

Let E_S denote the maximal abelian 2-extension of F_∞ that is unramified away from S. Then for $A = S^{-1}\mathcal{O}_F$, the *basic Iwasawa module* M associated to A is $M = G(E_S/F_\infty)$.

If L is a Λ'_F-module, we will write eL for $\Lambda' \otimes_{\Lambda'_F} L$ (e is for "extension of scalars"). We are thinking of e as a functor from compact Λ'_F-modules to compact Λ'-modules. It is easy to see that e commutes with Tate twisting, so that the expression $eL(n)$ is unambiguous.

Now write L^+, L_+ respectively for the kernel and cokernel of $1 - \sigma$. Similarly, write L^-, L_- respectively for the kernel and cokernel of $1 + \sigma$. Then e commutes with all four of these functors, so that expressions such as eL^+ are unambiguous. Furthermore, e is evidently an exact functor, and so commutes with both ordinary and Tate homology/cohomology: $H^p(\sigma; eL) = eH^p(\sigma; L)$, and so on.

Finally, if n is even then $(L^+)(n) = (L(n))^+$, while if n is odd $(L^+)(n) = (L(n))^-$. The notation $L^+(n)$ will always mean $(L^+)(n)$. Similar remarks apply to $L^-, L_\pm$.

3.2 Complex K-theory

In this notation we have [4]:

Theorem 3.1

$$K^{-2n}KA \cong e\mathbb{Z}_2(n)$$

$$K^{-2n-1}KA \cong eM(n)$$

Of course these isomorphisms are determined by the two cases with $n = 0$.

We need to take a closer look at the isomorphism $K^0KA \cong e\mathbb{Z}_2$. First of all, the proof of the theorem above shows:

Proposition 3.2 *Let $\beta : F \longrightarrow \mathbb{C}$ be any field embedding. Then the induced map $K^\wedge\beta$ generates K^0KA as Λ'-module (after pulling back to KA).*

Second, the relation between generators coming from different embeddings β can be explained as follows: Let $Hom\,(F, \mathbb{C})$ denote the set of all such field embeddings. Note that a fixed basepoint embedding β_1 determines a bijection $Hom\,(F, \mathbb{C}) \cong G_{\mathbb{Q}}/G_F$.

Proposition 3.3 *There is a natural isomorphism of Λ'-modules*

$$\mathbb{Z}_2[G_{\mathbb{Q}_\infty}\backslash Hom\,(F, \mathbb{C})] \cong K^0KA,$$

or equivalently, fixing β_1 as above,

$$\mathbb{Z}_2[G_{\mathbb{Q}_\infty}\backslash G_{\mathbb{Q}}/G_F] \cong K^0KA.$$

Proof: Up to homotopy, the natural action of $Aut\,\mathbb{C}$ on $K\mathbb{C}$ factors through the epimorphism $Aut\,\mathbb{C} \longrightarrow G(\mathbb{Q}_\infty/\mathbb{Q}) = \Gamma_F$. To see this, recall Suslin's equivalence $K\mathbb{C} \cong bu$ (remember that all spectra are completed at 2), which implies that the homotopy action is determined by its effect on π_*. In fact, since the action is by automorphisms of ring spectra, it is even determined by its action on $\pi_2 K\mathbb{C} \cong Hom\,(\mathbb{Z}/2^\infty, \mu_{2^\infty})$. This proves the claim.

It follows that the natural map

$$\mathbb{Z}_2[G_{\mathbb{Q}}/G_F] \longrightarrow K^0KA$$

determined by β_1 factors through a map from the double-coset module

$$\phi : \mathbb{Z}_2[G_{\mathbb{Q}_\infty}\backslash G_{\mathbb{Q}}/G_F] \longrightarrow K^0KA.$$

Now $G_{\mathbb{Q}_\infty}\backslash G_{\mathbb{Q}}/G_F = \Gamma'/H$, where H is the image of G_F in Γ'; i.e., $H = \Gamma'_F$. Hence the source and target of ϕ are finitely-generated $\mathbb{Z}_2$-modules of the same rank; since ϕ is surjective by Proposition 3.2, this completes the proof.

At the risk of belaboring the point, we can partially rephrase the last proposition as follows.

Corollary 3.4 *Define an equivalence relation on* $Hom\,(F,\mathbb{C})$ *by* $\beta \sim \beta'$ *if* $K\beta = K\beta'$. *Then* Γ' *acts transitively on the set of equivalence classes, with isotropy* Γ'_F.

Now recall that $K^{rel}A$ is the fibre of the natural map $KA \longrightarrow \bigvee^{r_1} bo$.

Proposition 3.5 $KO^{2n}K^{rel}A = 0$ *for all* n, *and*

$$KO^{-2n-1}K^{rel}A \cong \begin{cases} eN^+(n) & \text{if } n \text{ even} \\ eM^-(n) & \text{if } n \text{ odd} \end{cases}$$

Proof: By [4] Theorem 4.9, which applies here thanks to [6], we have

$$K^mK^{rel}A \cong \begin{cases} eN(n) & \text{if } m = -2n-1 \\ 0 & \text{if } m \text{ even} \end{cases}$$

Furthermore, $K^*K^{rel}A$ is σ-acyclic. Hence

$$c : KO^mK^{rel}A \xrightarrow{\cong} (K^mK^{rel}A)^+.$$

This proves the proposition for m even or $m = -2n-1$ with n even. If $m = -2n-1$ with n odd, we get $KO^{-2n-1}K^{rel}A \cong eN^-(n)$. Then the short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow \Lambda_F^{r_1} \longrightarrow 0$$

shows that $M^- = N^-$, completing the proof.

Let $\delta : \bigvee^{r_1} bo \longrightarrow \Sigma K^{rel}A$ denote the connecting map in the cofibre sequence defining $K^{rel}A$. Since $K^*K^{rel}A$ has projective dimension one as Λ'-module ([4], Theorem 4.9) and is concentrated in odd degrees, while K^*bo is concentrated in even degrees, δ is uniquely determined by $K^0\delta$. Furthermore $K^0\delta$ factors as

$$eN \longrightarrow eN/eM \longrightarrow K^0(\bigvee^{r_1} bo) \cong \Lambda^{r_1},$$

where $eN/eM \cong \Lambda^r$ although we do not have a canonical basis. On the other hand, the computation of K^0KR above shows:

Proposition 3.6 *There is a* Λ*-basis for* eN/eM *such that* $eN/eM \longrightarrow K^0 \bigvee^{r_1} bo$ *has diagonal matrix* $(T_F, 1, ..., 1)$.

3.3 Computation of KO^mKA for m even

By Corollary 3.4, for $2 \le i \le r_1$ there are elements $\gamma_i \in \Gamma$, well defined mod Γ_F, such that $cK(\alpha_i) = c(\gamma_i K(\alpha_1)) \in K^0KA$. Define $f_i \in KO^0KA$ by $f_1 = K(\alpha_1)$, $f_i = K(\alpha_i) - \gamma_i K(\alpha_1)$.

Proposition 3.7 $KO^0KA \cong \Lambda/2T_F \oplus (\Lambda/2)^{r_1-1}$. *In fact* KO^0KA *has a presentation as* Λ*-module with generators* f_i *and relations* $2T_F f_1 = 0$, $2f_i = 0$ $(2 \le i \le r_1)$.

Proof: Note that by construction, $c(T_F f_1) = 0 = c(f_i)$, $i \geq 2$. Hence the formula $rc = 2$ shows that the f_i's satisfy the indicated relations. Furthermore, since $K^0 K^{rel} A = 0$, the resulting homomorphism $\Lambda/2T_F \oplus (\Lambda/2)^{r_1 - 1} \longrightarrow KO^0 KA$ is surjective. Hence it is enough to show that $KO^0 KA$ is abstractly isomorphic to $\Lambda/2T_F \oplus (\Lambda/2)^{r_1 - 1}$. To see this consider the coboundary map $KO^0\delta : eN^+ \longrightarrow \Lambda^{r_1}$, and recall that there is an exact sequence

$$0 \longrightarrow M^+ \longrightarrow N^+ \longrightarrow \Lambda^{r_1} \longrightarrow (\Lambda/2)^{r_1} \longrightarrow 0.$$

It follows from Proposition 3.6 that $eN^+/eM^+ \cong (\Lambda)^{r_1}$ has a Λ-basis such that the map $eN^+/eM^+ \longrightarrow \Lambda^r$ induced by $KO^0\delta$ is a diagonal matrix $(2T_F, 2, ..., 2)$. This completes the proof.

Note that $c : KO^0 KA \longrightarrow K^0 KA$ is the obvious epimorphism $\Lambda/2T_F \oplus (\Lambda/2)^{r_1-1} \longrightarrow \Lambda/T_F$, while $r : K^0 KA \longrightarrow KO^0 KA$ is the obvious monomorphism

$$\Lambda/T_F \subset \Lambda/2T_F \subset \Lambda/2T_F \oplus (\Lambda/2)^{r_1 - 1}.$$

Proposition 3.8 *$KO^{-2}KA \cong e\mathbb{F}_2(1)$, generated by $\eta^2 f_1$, and $\eta^2 f_i = 0$ for $2 \leq i \leq r_1$.*

Proof: Consider the commutative square

$$\begin{array}{ccc} K^{-2}\bigvee^{r_1} bo & \longrightarrow & K^{-2}KA \\ \downarrow r & & \downarrow r \\ KO^{-2}\bigvee^{r_1} bo & \longrightarrow & KO^{-2}KA \end{array}$$

in which the horizontal maps and the left vertical map c are onto; hence the right vertical map is also onto. This shows that $KO^{-2}KA$ is generated by $r\beta c f_1 = \eta^2 f_1$, and that $r\beta c f_i = 0 = \eta^2 f_i$, $0 \leq i \leq 2$. On the other hand Proposition 3.6 implies that the image of the lower map is isomorphic to $e\mathbb{F}_2(1)$.

Note that $c : KO^{-2}KA \longrightarrow K^{-2}KA$ is the zero map, while $r : K^{-2}KA \longrightarrow KO^{-2}KA$ is the obvious epimorphism $e\mathbb{Z}_2(1) \longrightarrow e\mathbb{F}_2(1)$.

Proposition 3.9 *$KO^{-4}KA \cong e\mathbb{Z}_2(2)$, generated by $r\beta^2 c(f_1) = \xi f_1$.*

Proof: Consider the diagram analogous to the one used in the previous proposition. Algebraically it has the form

$$\begin{array}{ccc} (\Lambda)^{r_1} & \longrightarrow & e\mathbb{Z}_2(2) \\ \downarrow r & & \downarrow r \\ (\Lambda)^{r_1} & \longrightarrow & KO^{-4}KA \end{array}$$

where now the left vertical map is an isomorphism and both horizontal maps are surjective. Hence the right vertical map is surjective and $KO^{-4}KA$ is generated by ξf_1. As in the previous proposition, Proposition 3.6 shows that $KO^{-4}KA$ is abstractly isomorphic to $e\mathbb{Z}_2(2)$, and the result follows.

Note that $c : KO^{-4}KA \longrightarrow K^{-4}KA$ is multiplication by two, while $r : K^{-4}KA \longrightarrow KO^{-4}KA$ is an isomorphism.

The final case is trivial:

Proposition 3.10 $KO^{-6}KA = 0$.

Note that the results of this section imply:

Corollary 3.11 *For $m = 2, 3, 4, 5, 6 \, mod \, 8$, $\eta : KO^{-m}KA \longrightarrow KO^{-m-1}KA$ is zero.*

3.4 Computation of $KO^m KA$ for m odd

Proposition 3.12

$$KO^{-3}KA \cong ((1-\sigma)eM)(1)$$

and

$$KO^{-5}KA \cong ((1+\sigma)eM)(2) = eM^+(2).$$

In each case the maps c, r are given respectively by the evident inclusion and norm maps.

Proof: Consider the commutative squares

$$\begin{array}{ccc} K^{-m}KA & \xrightarrow{r} & KO^{-m}KA \\ \downarrow{\scriptstyle 1+\sigma} & & \downarrow{\scriptstyle c^+} \\ (1+\sigma)K^{-m}KA & \xrightarrow[i]{} & (K^{-m}KA)^+ \end{array}$$

where c^+ is the natural factorization of c and i is the inclusion. For $m = 3, 5$, Corollary 3.11 implies that r is onto and c^+ is injective. Hence c_+ is an isomorphism onto $(1+\sigma)K^{-m}KA$. The remaining claims of the proposition follow easily from this, where in the case $m = 5$ we use the fact that $\hat{H}^0(\sigma; M) = 0$ (see [4], 4.8).

Proposition 3.13 *There are short exact sequences*

$$0 \longrightarrow (\Lambda/2)^{r_1} \longrightarrow KO^{-1}KA \longrightarrow eM^+ \longrightarrow 0$$

and

$$0 \longrightarrow eM_+ \longrightarrow KO^{-1}KA \longrightarrow e\mathbb{F}_2 \longrightarrow 0$$

In fact the first short exact sequence corresponds to

$$0 \longrightarrow \eta KO^0KA \longrightarrow KO^{-1}KA \longrightarrow c(KO^{-1}KA) \longrightarrow 0$$

and the second to

$$0 \longrightarrow r(K^{-1}KA) \longrightarrow KO^{-1}KA \xrightarrow{\eta} KO^{-2}KA \longrightarrow 0.$$

Proof: We have $\eta KO^0KA = (KO^0KA)/(Im\, c) \cong (\Lambda/2)^{r_1}$, using Proposition 3.7. Now consider the commutative square in the proof of Proposition 3.12, with $m = -1$. Although r is no longer surjective and c^+ is no longer injective, the bottom arrow is an equality and c^+ is surjective. Hence $c(KO^{-1}KA) = eM^+$, yielding the first exact sequence.

By Corollary 3.11 the sequence

$$0 \longrightarrow KO^{-3}KA \xrightarrow{\beta^{-1}c} K^{-1}KA \xrightarrow{r} KO^{-1}KA \xrightarrow{\eta} KO^{-2}KA \longrightarrow 0$$

is exact. Since $r : K^{-3}KA \longrightarrow KO^{-3}KA$ is onto, we can compute the image of $\beta^{-1}c$ by computing the image of the composite $\beta^{-1}cr = \beta^{-1}(1+\sigma) = (1-\sigma)\beta^{-1}$. Hence $Im\,(\beta^{-1}c) = Im\,(1-\sigma)$, yielding the second exact sequence.

Proposition 3.14 $KO^1KA \cong eM^-(-1)$. *Furthermore,* $\eta : KO^1KA \longrightarrow KO^0KA$ *maps onto the 2-torsion submodule* $(\Lambda/2)^{r_1}$, *generated by* $T_F f_1, f_2, ..., f_{r_1}$, *and in fact the exact sequence*

$$K^1KA \xrightarrow{r} KO^1KA \xrightarrow{\eta} \eta KO^1KA \longrightarrow 0$$

can be identified with

$$eM(-1) \xrightarrow{1+\sigma} e(M(-1))^+ \longrightarrow \hat{H}^0(\sigma; eM(-1)) = e\hat{H}^1(\sigma; M)(-1) \cong (\Lambda/2)^{r_1} \longrightarrow 0.$$

Proof: There is a short exact sequence

$$0 \longrightarrow KO^1KA \xrightarrow{c} K^1KA \xrightarrow{r\beta^{-1}} KO^3KA \longrightarrow 0.$$

Since KO^3KA injects into K^3KA, the kernel of $r\beta^{-1}$ is the same as the kernel of $cr\beta^{-1} = (1+\sigma)\beta^{-1} = \beta^{-1}(1-\sigma)$. Hence KO^1KA maps isomorphically onto the kernel of $1-\sigma$, proving the first statement.

The remaining assertions follow easily using Proposition 3.7.

Note we have shown that c and r are given by inclusion of the fixed points and the norm $1+\sigma$, respectively.

4 Self-conjugate and united K-theory of KA

Note: For details on united K-theory, see [1].

4.1 (K^*KA, KO^*KA) as CR-object

United K-theory consists of real, complex and self-conjugate K-theory, together with the various operations and maps relating the three theories. It has the homological advantage that its Adams spectral sequence vanishes above filtration 2. Furthermore, if the real K-theory of a spectrum satisfies a certain condition called *CR-acyclicity*, the self-conjugate theory is redundant for homological purposes and one can use the "CR" theory consisting of real and complex K-theory only. Although we do not make use of the united K-theory Adams spectral sequence in this paper, it is interesting to note that KA is CR-acyclic.

Theorem 4.1 *KA is CR-acyclic in the sense of Bousfield.*

Proof: We must show that the chain complex

$$...\longrightarrow \eta KO^{-m}KA \xrightarrow{\eta} \eta KO^{-m-1}KA \longrightarrow ...$$

is exact. In view of Corollary 3.11, this reduces to showing that

$$0\longrightarrow \eta KO^1KA \xrightarrow{\eta} \eta KO^0KA \xrightarrow{\eta} \eta KO^{-1}KA \longrightarrow 0$$

is short exact. The second η is surjective by Proposition 3.8. By Proposition 3.7, ηKO^1KA is a free $\Lambda/2$-module on $T_F f_1, f_2, ..., f_{r_1}$. By Proposition 3.13, ηKO^0KA is a free $\Lambda/2$-module on ηf_i, $1 \leq i \leq r_1$. Hence the first η above is injective with cokernel isomorphic to $\Lambda/(2, T_F) = e\mathbb{F}_2$, proving the proposition.

4.2 Self-conjugate K-theory

4.2.1 Operations in self-conjugate K-theory

The Adams operations in self-conjugate K-theory (see [1]) lead to a continuous group homomorphism from Γ' to the group of units in $[KSC, KSC]$. This in turn yields a continuous ring homomorphism $\Lambda' \longrightarrow KSC^0KSC$.

Proposition 4.2 *$\Lambda' \longrightarrow KSC^0KSC$ is an isomorphism of topological rings.*

Hence the functor $KSC^*(-)$ takes values in graded compact Λ'-modules, as it does for $K^*(-)$ and $KO^*(-)$. Now consider the natural transformations induced by the standard maps

$$KO \xrightarrow{\epsilon} KSC \xrightarrow{\zeta} K \xrightarrow{\gamma} \Sigma KSC \xrightarrow{\tau} KO.$$

Proposition 4.3 *For any spectrum X, there are short exact sequences of Λ'-modules*

$$0\longrightarrow (K^{n-1}X)_+(-1)\longrightarrow KSC^nX\longrightarrow (K^nX)^+\longrightarrow 0$$

and

$$0\longrightarrow KO^nX/\eta^2\longrightarrow KSC^nX\longrightarrow KO^{n+3}X(1)\longrightarrow 0.$$

The resulting maps $(K^{n-1}X)_+(-1)\longrightarrow KO^{n+3}X(1)$ and $KO^nX/\eta^2\longrightarrow (K^nX)^+$ are induced by $r\beta^{-2}$ and c, respectively.

4.3 Self-conjugate K-theory of KA

In this section we compute KSC^*KA up to extensions, using Proposition 4.3. One could also make use of $K^{rel}KA$, as in §3.2, but this does not seem to help much in resolving the extension problems. For the application to Stiefel-Whitney classes, we only need KSC^0KA. All exact sequences and isomorphisms below are as Λ'-modules. The proofs amount to substituting the results of our real and complex K-theory calculations into Proposition 4.3.

Proposition 4.4 *There are short exact sequences*

$$0 \longrightarrow eM_+(-1) \longrightarrow KSC^0KA \longrightarrow eM^+ \longrightarrow 0$$

and

$$0 \longrightarrow KO^0KR \longrightarrow KSC^0KA \longrightarrow eM^+(-1) \longrightarrow 0.$$

Proposition 4.5 *There is a short exact sequence*

$$0 \longrightarrow e\mathbb{F}_2 \longrightarrow KSC^{-1}KA \longrightarrow eM^+ \longrightarrow 0$$

and an isomorphism

$$KO^{-1}KA/\eta^2 \xrightarrow{\cong} KSC^{-1}KA.$$

Proposition 4.6 *There are isomorphisms*

$$eM_- \xrightarrow{\cong} KSC^{-2}KA$$

and

$$KSC^{-2}KA \xrightarrow{\cong} eM^-.$$

Proposition 4.7 *There are short exact sequences*

$$0 \longrightarrow e\mathbb{Z}_2(-1) \longrightarrow KSC^{-3}KA \longrightarrow eM^-(-1) \longrightarrow 0$$

and

$$0 \longrightarrow (e(1-\sigma)M)(1) \longrightarrow KSC^{-3}KA \longrightarrow KO^0X(1) \longrightarrow 0.$$

To compute the η multiplications in KSC^*KA, we use the formula $\eta = \Sigma^{-1}\gamma\beta\zeta$. In other words, $\eta : KSC^nX \longrightarrow KSC^{n-1}X$ is the composite

$$KSC^nX \longrightarrow K^nX \longrightarrow K^{n-2}X(-1) \longrightarrow KSC^{n-1}X.$$

Proposition 4.8 *For $n = -1, -2$, $\eta : KSC^nKA \longrightarrow KSC^{n-1}KA$ is zero.*

For $n = 0$, η is the evident composite

$$KSC^0KA \longrightarrow e\mathbb{Z}_2 \longrightarrow e\mathbb{Z}/2 \longrightarrow KSC^{-1}KA,$$

where the first two maps are surjective and the last is injective.

For $n = 1$, η is given by the composite

$$KSC^1KA \longrightarrow eM^-(-1) \longrightarrow (eM^-/(1-\sigma))(-1) = (\Lambda/2)^{r_1} \longrightarrow KSC^0KA,$$

where again the first two maps are surjective and the last is injective.

The proof is by direct inspection.

Multiplication by the generator $\omega \in \pi_3 KSC$ can also be computed, at least up to extensions, by using the formula $\omega = \gamma\beta^2\zeta$. For example, $\omega : KSC^0KA \longrightarrow KSC^{-3}KA$ factors as

$$KSC^0KA \longrightarrow e\mathbb{Z}_2 \longrightarrow e\mathbb{Z}_2(-1) \longrightarrow KSC^{-3}KA,$$

where a twist has been introduced because $\pi_3 KSC \cong \mathbb{Z}_2(1)$ as Λ'-module. Details will be left to the reader.

5 Proof of Theorem 2.11

We follow the outline given in §2.6. Since $a = b$ under our splitting hypothesis (see Proposition 2.10), what remains to be shown is:

Lemma 5.1

$$Im\, H_*\underline{\alpha} \subset H_*BO \otimes \mathcal{P}^{r_1-a-1} \otimes (\mathcal{FP})^a.$$

We recall that $a = dim_{\mathbb{Z}/2} Ker\,(H_1(\mathbb{R}_F; \mathbb{Z}/2) \longrightarrow H_1(A; \mathbb{Z}/2))$.

Now $\epsilon : KO^0KA \longrightarrow KSC^0KA$ is a homomorphism of Λ'-modules, and therefore induces a map $\epsilon/\mathcal{M}' : KO^0KA/\mathcal{M} \longrightarrow KSC^0KR/\mathcal{M}'$ of $\mathbb{Z}/2$-vector spaces.

Lemma 5.2 $dim_{\mathbb{Z}/2} Ker\,(\epsilon/\mathcal{M}') \geq a$.

Assuming this, we may choose a minimal generating set $g_1, ..., g_{r_1}$ for KO^0KR such that (i) $g_1 = f_1$; (ii) $g_2, ..., g_{r_1}$ generate $Ker\, c$; and (iii) $\overline{g}_{r_1-a-1}, ..., \overline{g}_{r_1}$ are in the kernel of $\epsilon/\mathcal{M}'$. Here $\overline{g}_i$ means the reduction of g_i mod $\mathcal{M}'$.

Lemma 5.3 *a) Γ' acts trivially on H_*BSC.*

b) If $\theta \in \mathcal{M}'$, then $Im\, H_*\Omega_0^\infty\theta \subset \mathcal{F}H_*BSC$.

Proof: a) It is clear that Γ' fixes the canonical map $j : \mathbb{R}P^\infty \longrightarrow BO$, and hence also fixes the composite $\epsilon j : \mathbb{R}P^\infty \longrightarrow BSC$. But ϵj is a generating complex for H_*BSC, so (a) follows.

b) By part (a), T and $1 - \sigma$ induce the null map on H_*BSC. So we need only check that the image of the H-space squaring map—denoted [2]—lies in the squares $\mathcal{F}H_*BSC$. But the analogous statement for H_*BO is obviously true, and the assertion for H_*BSC follows immediately.

It follows that there is a lift

$$
\begin{array}{ccc}
 & & H_*BO \otimes (H_*U/O)^{r_1-a-1} \otimes (H_*Sp/U)^a \\
 & \nearrow & \big\downarrow \\
H_*BGLA & \xrightarrow[g]{} & H_*BO \otimes (H_*BO)^{r_1-a-1} \otimes (H_*BO)^a
\end{array}
$$

completing the proof of Lemma 5.1 and hence also the proof of Theorem 2.11.

It remains to prove Lemma 5.2.

Let $\mathcal{A}_0$ denote $Ker\,(KO^0KR/\mathcal{M} \longrightarrow KSC^0KR/\mathcal{M}'_{SC})$, and let a_0 denote its dimension over $\mathbb{Z}/2$. In the proof that follows we will successively replace A_0 by certain $\mathbb{Z}/2$-vector spaces $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ with dimensions a_1, a_2, a_3, with $a_0 \geq a_1 - 1$ and $a_1 = a_2 = a_3 = a + 1$.

Step 1: There is a commutative diagram of Λ'-modules

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Ker\,c & \longrightarrow & KO^0KA & \xrightarrow{c} & K^0KA & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & (K^{-1}KA)_+(-1) & \longrightarrow & KSC^0KA & \longrightarrow & K^0KA & \longrightarrow & 0
\end{array}
$$

in which $Ker\,c \cong (\Lambda/2)^{r_1}$ and the vertical maps are injective. Reducing modulo $\mathcal{M}'$, we get a diagram of the form

$$
\begin{array}{ccccccc}
(\mathbb{Z}/2)^{r_1} & \longrightarrow & (\mathbb{Z}/2)^{r_1} & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\
\big\downarrow h_1 & & \big\downarrow h_0 & & \big\downarrow = & & \\
K^{-1}KA/\mathcal{M}' & \longrightarrow & KSC^0KA/\mathcal{M}' & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0
\end{array}
$$

Let $\mathcal{A}_1 = Ker\,h_1$, $a_1 = dim_{\mathbb{Z}/2}\mathcal{A}_1$. Then $\mathcal{A}_1 \longrightarrow \mathcal{A}_0$ is onto, and hence $a_0 \geq a_1 - 1$. (Note that this last inequality holds whether or not the kernel $\mathbb{Z}/2$ of the upper left horizontal map in the diagram lies in $\mathcal{A}_1$.)

Step 2. We next equate a_1 with a purely number-theoretic invariant. Let $\mathcal{A}_2$ denote the kernel of the natural map

$$[M^-/(1-\sigma)M]/\mathcal{M} \longrightarrow M/\mathcal{M}'$$

and let $a_2 = dim_{\mathbb{Z}/2}\mathcal{A}_2$. We claim that $a_1 = a_2$.

To see this, first note that multiplication by $\eta : KO^1KA \longrightarrow KO^0KR$ induces an isomorphism $KO^1KR/rK^1KA \cong Ker\, c$. By Proposition 3.14, the source of this isomorphism can be identified with $e[M^-/(1-\sigma)M](-1)$, where the twist is irrelevant since the module being twisted is isomorphic to $(\Lambda/2)^{r_1}$. On the other hand, $\epsilon\eta = \eta\epsilon = \Sigma^{-1}\gamma\beta c$ and hence $\epsilon\eta : KO^1KA \longrightarrow KSC^0KA$ can be identified with the composite

$$eM^-(-1) \longrightarrow eM(-1) \longrightarrow eM_+/(-1) = (K^{-1}KA)_+/(-1) \subset KSC^0KA$$

.

In other words, $(Ker\, c)/\mathcal{M} \longrightarrow K^{-1}KA/\mathcal{M}'$ can be identified with

$$e([M^-/(1-\sigma)M]/\mathcal{M} \longrightarrow M/\mathcal{M}'),$$

proving that $a_1 = a_2$ as desired.

Step 3. Recall that we have fixed an embedding $\alpha_1 : F \subset \mathbb{R}$, so that the 2-adic cyclotomic extension F_∞ is a subfield of $\mathbb{C}$. Let F_n^+ denote the real subfield of F_n for $0 \leq n \leq \infty$, so in particular $G(F_\infty^+/F) = \Gamma_F$. Similarly, let A_n^+ denote the corresponding ring of S-integers in F_n^+. Let

$$\mathbb{R}_n = \mathbb{R}_{F_n^+} = \coprod_{Hom\,(F_n^+,\mathbb{R})} \mathbb{R}.$$

Note that $H_1(\mathbb{R}_\infty; \mathbb{Z}_2) \cong (\Lambda_F/2)^{r_1}$, and there is a natural augmentation $H_1(\mathbb{R}_\infty; \mathbb{Z}_2) \longrightarrow \mathbb{Z}/2$ given by applying $H_1(-; \mathbb{Z}_2)$ to the composite

$$Spec\, \mathbb{R}_\infty \longrightarrow Spec\, \mathbb{R}_F = \coprod^{r_1} Spec\, \mathbb{R} \longrightarrow Spec\, \mathbb{R},$$

where the last map is the folding map. Let $\tilde{H}_1(\mathbb{R}_\infty; \mathbb{Z}_2)$ denote the kernel of this augmentation. The Serre spectral sequence for A_∞^+/A yields a short exact sequence

$$0 \longrightarrow H_1(A_\infty^+; \mathbb{Z}_2)/T \longrightarrow H_1(A; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

which, when reduced mod 2, becomes

$$0 \longrightarrow H_1(A_\infty^+; \mathbb{Z}_2)/\mathcal{M} \longrightarrow H_1(A; \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Thus the natural map $H_1(\mathbb{R}_\infty; \mathbb{Z}_2) \longrightarrow H_1(A_\infty^+; \mathbb{Z}_2)$ induces a map

$$h_3 : \tilde{H}_1(\mathbb{R}_\infty; \mathbb{Z}_2)/\mathcal{M} \longrightarrow H_1(A_\infty^+; \mathbb{Z}_2)/\mathcal{M}.$$

Let $\mathcal{A}_3 = Ker\, h_3$, $a_3 = dim_{\mathbb{Z}/2}\mathcal{A}_3$.

Claim: $a_2 = a_3$.

Note first that the Serre spectral sequence of A_∞/A_∞^+ yields a short exact sequence

$$0 \longrightarrow M_+ \longrightarrow H_1(A_\infty^+;\mathbb{Z}_2) \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Hence we get an injective map $M^-/(1-\sigma)M \longrightarrow H_1(A_\infty^+;\mathbb{Z}_2)$.

We next show that the image of this map coincides with the image of $\tilde{H}_1(\mathbb{R}_\infty;\mathbb{Z}_2) \longrightarrow H_1(A_\infty^+;\mathbb{Z}_2)$. Recall that the homology Serre spectral sequence of A_∞/A_∞^+ is a spectral sequence of modules over $H^*(\sigma;\mathbb{Z}_2)$ via the cap product. If $x \in H^2(\sigma;\mathbb{Z}_2)$ is a generator, we have in particular a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(\sigma;M) & \longrightarrow & H_3(A_\infty^+;\mathbb{Z}_2) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow = & & \\
0 & \longrightarrow & H_0(\sigma;M) & \longrightarrow & H_1(A_\infty^+;\mathbb{Z}_2) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0
\end{array}$$

where the vertical maps are given by cap product with x. This shows that the image of M_+ in $H_1(A_\infty^+;\mathbb{Z}_2)$ coincides with the image of $Ker\,(H_3(A_\infty^+;\mathbb{Z}_2) \longrightarrow \mathbb{Z}/2)$ under the cap product.

On the other hand, the commutative diagram

$$\begin{array}{ccc}
H_3(\mathbb{R}_\infty;\mathbb{Z}_2) & \xrightarrow{\cong} & H_3(A_\infty^+;\mathbb{Z}_2) \\
\cong \downarrow & & \downarrow \\
H_1(\mathbb{R}_\infty;\mathbb{Z}_2) & \longrightarrow & H_1(A_\infty^+;\mathbb{Z}_2)
\end{array}$$

where again the vertical maps are given by cap product with x, and the top horizontal arrow is an isomorphism by Tate's theorem, shows that this last image can be identified with the image of $\tilde{H}_1(\mathbb{R}_\infty;\mathbb{Z}_2)$. It follows that $a_2 = a_3$, as claimed.

Step 4. In this final step, we show that $a_3 = a+1$. This will complete the proof of the key lemma, and hence also the proof of Theorem 2.11.

Consider the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
\mathbb{Z}/2 & \longrightarrow & \tilde{H}_1(\mathbb{R}_\infty;\mathbb{Z}_2)/\mathcal{M} & \longrightarrow & H_1(\mathbb{R}_\infty;\mathbb{Z}_2)/\mathcal{M} & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\
\downarrow & & h_3 \downarrow & & h_4 \downarrow & & \downarrow = & & \\
0 & \longrightarrow & H_1(A_\infty^+;\mathbb{Z}_2)/\mathcal{M} & \longrightarrow & H_1(A;\mathbb{Z}/2) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0
\end{array}$$

Here we note that $H_1(\mathbb{R}_\infty;\mathbb{Z}_2) = H_1(\mathbb{R}_\infty;\mathbb{Z}/2)$, and that $H_1(\mathbb{R}_\infty;\mathbb{Z}_2)/\mathcal{M} = H_1(\mathbb{R}_\infty;\mathbb{Z}_2)/T_F$. Now let $a_4 = dim_{\mathbb{Z}/2}Ker\,h_4$. Then since the arrow $\mathbb{Z}/2 \longrightarrow \tilde{H}_1(\mathbb{R}_\infty;\mathbb{Z}_2)/\mathcal{M}$ in the upper left is injective, diagram chase shows $a_3 = a_4 + 1$. On the other hand, h_4 can be identified with the natural map $H_1(\mathbb{R}_F;\mathbb{Z}/2) \longrightarrow H_1(A;\mathbb{Z}/2)$. Hence $a_4 = a$, completing the proof of Lemma 5.2.

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