# THE 2-COMPACT GROUPS IN THE $A$-FAMILY ARE $N$-DETERMINED

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ABSTRACT. The 2-compact groups associated to central quotients of $\mathrm{SU}(n+1)$, $n \geq 1$, are shown to be determined up to isomorphism by their maximal torus normalizers.

## 1. INTRODUCTION

A 2-compact group is a 2-complete connected based space $BX$ such that $H^*(\Omega BX;\mathbf{F}_2)$ is finite where $\Omega BX$ is the loop space [6]. It is customary, though sometimes confusing, to refer to $BX$ by the symbol $X$.

Any 2-compact group $BX$ comes equipped with a maximal torus normalizer $BN(X) \to BX$ where $BN(X)$ is the Borel construction

$$BT(X) \to BN(X) \to BW(X)$$

for the action of the Weyl group $W(X)$ on the maximal torus $T(X)$ [6, 9.8]. Does $BN(X)$ determine $BX$?

The answer to this question is "no" for the following reason. Let $G$ be a Lie group and $N(G) \to G$ its Lie group maximal torus normalizer. Assuming that the component group $\pi_0(G)$ is a finite 2-group, $B\widehat{G}$ is a 2-compact group and $B\widehat{N}(G) \to B\widehat{G}$ its 2-compact group maximal torus normalizer. (For any Lie group $H$, $B\widehat{H}$ stands for the partial 2-completion of the classifying space $BH$ for $H$.) Since there are distinct Lie groups, such as $\mathrm{O}(2n)$ and $\mathrm{SO}(2n+1)$, with isomorphic maximal torus normalizers, there are also distinct 2-compact groups, such as $\widehat{\mathrm{O}}(2n)$ and $\widehat{\mathrm{SO}}(2n+1)$, with isomorphic maximal torus normalizers. Thus we need to replace the maximal torus normalizer by a more delicate invariant which retains information about component groups. The maximal torus normalizer pair is a candidate for such a more delicate invariant.

For a 2-compact group $BX$ let $BX_0$, the identity component of $X$, denote the universal covering space of $BX$. Since $BX_0$ is again a 2-compact group, it has a maximal torus normalizer $BN(X_0) \to BX_0$. The maximal torus normalizers of $X$ and $X_0$ are related by a commutative diagram

$$\begin{tikzcd}
BN(X_0) \arrow[r] \arrow[d] & BX_0 \arrow[d] \\
BN(X) \arrow[r] \arrow[d] & BX \arrow[d] \\
B\pi_0(X) \arrow[r, equal] & B\pi_0(X)
\end{tikzcd}$$

where the columns are fibration sequences. The fibration $BN(X_0) \to BN(X) \to B\pi_0(X)$, called the *maximal torus normalizer pair* associated to $BX$, has the built-in property that it fully informs about the component group of $X$. Does the maximal torus normalizer pair determine the 2-compact group up to isomorphism?

Focusing on the following properties for a 2-compact group $X$,

(1) $X$ is determined by $(N(X), N(X_0))$
(2) Automorphisms of $X$ are determined by their restrictions to $N(X)$
(3) Automorphisms of $X$ are determined by their restrictions to $T(X)$

we shall say that

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- $X$ is totally $N$-determined if it satisfies (1) and (2)
- $X$ is uniquely $N$-determined if it satisfies (1) and (3)

In this terminology, one might formulate the conjecture that all 2-compact groups are totally $N$-determined, all *connected* 2-compact groups even uniquely $N$-determined. Here is an infinite family of simple 2-compact groups corroborating the conjecture.

1.1. **Theorem.** *The simple* 2*-compact group* $\mathrm{PGL}(n+1,\mathbf{C})$*,* $n \geq 1$*, is uniquely* $N$*-determined and its automorphism group* $\mathrm{Aut}(\mathrm{PGL}(n+1,\mathbf{C}))$ *equals* $\mathbf{Z}_2^\times$ *for* $n>1$ *and* $\mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$ *for* $n=1$*.*

It immediately follows (3.2, 4.3) that the Lie group $\mathrm{PGL}(n+1,\mathbf{C}) = \mathrm{PSL}(n+1,\mathbf{C})$ occuring in Theorem 1.1 can be replaced by any central quotient of $\mathrm{SL}(n+1,\mathbf{C})$. Indeed, the methods used here are not confined to simple, or semi-simple, 2-compact groups.

1.2. **Corollary.** [22, 1.9] *The* 2*-compact group* $\mathrm{GL}(n,\mathbf{C})$ *is uniquely* $N$*-determined and its automorphism group* $\mathrm{Aut}(\mathrm{GL}(n,\mathbf{C}))$ *equals* $\mathrm{Aut}_{\mathbf{Z}_2\Sigma_n}(\mathbf{Z}_2^n)$ *for* $n>2$ *and* $\mathbf{Z}^\times\backslash\mathrm{Aut}_{\mathbf{Z}_2\Sigma_2}(\mathbf{Z}_2^2)$ *for* $n=2$*.*

The methods are not even confined to the connected cases. For instance, it follows from Lemma 4.1 that the 2-compact group $\mathrm{GL}(n,\mathbf{C}) \rtimes C_2$, where $C_2$ acts on $\mathrm{GL}(n,\mathbf{C})$ by complex conjugation, is totally $N$-determined.

See [31, 33, 34] for classification results for other 2-compact groups (with polynomial $\mathbf{F}_2$-cohomology). The results for the automorphism groups are not new [18] but reproved here.

## 2. GENERALITIES

This sections contains the fundamental definitions and the first general results. Whereas $p$-compact groups are determined by their maximal torus normalizers [29, 1] when $p>2$, a finer invariant is needed for 2-compact groups as there are examples (2.2) of distinct 2-compact groups with identical maximal torus normalizers.

2.1. **Maximal torus normalizer pairs.** Let $N_0 \to N$ be a maximal rank normal monomorphism between two extended 2-compact tori, meaning simply that there exists a short exact sequence of loop spaces $N_0 \to N \to \pi$ for some finite group $\pi$. We say that $(N,N_0)$ is a maximal torus normalizer pair for the 2-compact group $X$, and we write $N(X,X_0) = (N,N_0)$, if there exists a morphism of loop space short exact sequences

$$\begin{tikzcd} N_0 \arrow[r] \arrow[d, "j_0"] & N \arrow[r] \arrow[d, "j"] & \pi \arrow[d, "\cong"] \\ X_0 \arrow[r] & X \arrow[r] & \pi_0(X) \end{tikzcd}$$

where $j$ and $j_0$ are maximal torus normalizers for $X$ and its identity component $X_0$. A maximal torus normalizer pair for $X$ determines the maximal torus $T(X)$, the Weyl groups, $W(X)$ and $W(X_0)$, of $X$ and $X_0$, the component group $\pi_0(X) = N(X)/N(X_0) = W(X)/W(X_0)$, and [7, 7.5] the center $Z(X_0) \to X_0$ of $X_0$.

2.2. **Example.** 1. Since $N(\mathrm{SO}(2n+1)) \subseteq \mathrm{O}(2n) \subsetneq \mathrm{SO}(2n+1)$, $\mathrm{O}(2n)$ and $\mathrm{SO}(2n+1)$ have the same maximal torus normalizer. Their maximal torus normalizer pairs are distinct, however, for $\mathrm{SO}(2n+1)$ is connected and $\mathrm{O}(2n)$ disconnected.
2. More generally [14], let $G$ be any compact connected Lie group and $N(G)$ its maximal torus normalizer. If $N(G)$ is not maximal, there exists a compact Lie group $H$ such that $N(G) \subseteq H \subsetneq G$. The two compact Lie groups, $G$ and $H$, have isomorphic maximal torus normalizers but distinct maximal torus normalizer pairs as $H$ is non-connected.
3. The Weyl groups for $\mathrm{SO}(2n+1)$ and $\mathrm{Sp}(n)$, $n \geq 3$, are isomorphic as reflection groups but $N(\mathrm{SO}(2n+1))$ is a split and $N(\mathrm{Sp}(n))$ a non-split extension [3]. Thus connected 2-compact groups can not be classified by their Weyl group alone.

2.3. **The Adams–Mahmud homomorphism.** For a 2-compact group (or extended 2-compact torus) $X$, we let $\mathrm{End}(X) = [BX,*;BX]$ denote the monoid of homotopy classes of endomorphism of $X$. The *automorphism group* $\mathrm{Aut}(X) \subseteq [BX,*;BX]$ of $X$ is the group of invertible elements in $\mathrm{End}(X)$ and the *outer automorphism group* $\mathrm{Out}(X) = \mathrm{Aut}(X)/\pi_0(X) \subseteq [BX;BX]$ is the group of conjugacy classes of automorphisms of $X$.

Let $X$ be a 2-compact group with maximal torus normalizer pair $(N, N_0)$. Turn the maximal torus normalizer $Bj\colon BN \to BX$ into a fibration. Any automorphism $f\colon X \to X$ of the 2-compact group $X$ restricts to an automorphism $\mathrm{AM}(f)\colon N \to N$ of the maximal torus normalizer, unique up to the action of the Weyl group $W(X_0) = \pi_1(X/N)$ of the identity component $X_0$ of $X$, such that the diagram

$$\begin{tikzcd} BN \arrow[r, "B(\mathrm{AM}(f))"] \arrow[d, "Bj"'] & BN \arrow[d, "Bj"] \\ BX \arrow[r, "Bf"'] & BX \end{tikzcd}$$

commutes up to based homotopy [26, §3]. The Adams–Mahmud homomorphism is the resulting homomorphism

$$\mathrm{AM}\colon \mathrm{Aut}(X) \to W(X_0)\backslash \mathrm{Aut}(N) \tag{2.4}$$

of automorphism groups.

The automorphism group of $N$ sits [24, 5.2] in a short exact sequence

$$0 \to H^1(W(X); \check{T}(X)) \to \mathrm{Aut}(N) \xrightarrow{\pi_*} \mathrm{Aut}(W(X), \check{T}(X), e(X)) \to 1 \tag{2.5}$$

where the normal subgroup to the left consists of all automorphisms of $N$ that induce the identity on homotopy groups and the group to the right consists of all pairs $(\alpha, \theta) \in \mathrm{Aut}(W(X)) \times \mathrm{Aut}(\check{T}(X))$ such that $\theta$ is $\alpha$-linear and the induced automorphism $H^2(\alpha^{-1}, \theta)$ [35, 6.7.6] preserves the extension class $e(X) \in H^2(W(X); \check{T}(X))$. The image of $W(X_0)$ in $\mathrm{Aut}(N)$ does not intersect the subgroup $H^1(W(X); \check{T}(X))$ (as $W(X_0)$ is represented faithfully in $\mathrm{Aut}(\check{T}(X))$ [6, 9.7]) so there is an induced short exact sequence

$$0 \to H^1(W(X); \check{T}(X)) \to W(X_0)\backslash \mathrm{Aut}(N) \xrightarrow{\pi_*} W(X_0)\backslash \mathrm{Aut}(W(X), \check{T}(X), e(X)) \to 1 \tag{2.6}$$

whose middle term is the target of the Adams–Mahmud homomorphism. In particular, if $X$ is *connected*, this short exact sequence

$$0 \to H^1(W(X); \check{T}(X)) \to \mathrm{Out}(N) \to W(X)\backslash \mathrm{Aut}(W(X), \check{T}(X), e(X)) \to 1 \tag{2.7}$$

has the group $\mathrm{Out}(N) = W(X)\backslash \mathrm{Aut}(N)$ of outer automorphisms of $N$ as its middle term. The group $\mathrm{Aut}(W(X), \check{T}(X), 0)$, which is the normalizer $N_{\mathrm{GL}(L(X))}(W(X))$ of $W(X)$ in $\mathrm{GL}(L(X))$, $L(X) = \pi_2(BT(X))$, fits into an exact sequence

$$Z(W(X))\backslash \mathrm{Aut}_{\mathbf{Z}_2 W(X)}(L(X)) \to W(X)\backslash N_{\mathrm{GL}(L(X))}(W(X)) \to \mathrm{Out}(W(X))$$

where, by Schur's lemma, $\mathrm{Aut}_{\mathbf{Z}_2 W(X)}(L(X)) = \mathbf{Z}_2^\times$ if $X$ is simple.

### 2.8. Totally $N$-determined 2-compact groups.

We are now ready to formulate the concept of $N$-determinism that will be used in this paper.

**2.9. Definition.** *Let $X$ be a 2-compact group with maximal torus normalizer pair $(N, N_0)$.*

1. $X$ has $N$-determined automorphisms *if the Adams–Mahmud homomorphism* (2.4) *for $X$ is injective and* $\pi_*(N)$-determined automorphisms *if* $\mathrm{AM}^{-1}(H^1(W(X); \check{T}(X))$ *is trivial.*
2. *$X$ is* $N$-determined *if for any other 2-compact group $X'$ with maximal torus normalizer pair $(N, N_0)$ there exist an isomorphism $f\colon X \to X'$ and an automorphism $\alpha \in \pi_0(N)\backslash \mathrm{Aut}(N)$ with $\pi_*(B\alpha) = 1$ such that the diagram*

$$\begin{tikzcd} BN \arrow[r, "B\alpha", "\cong"'] \arrow[d, "Bj"'] & BN \arrow[d, "Bj'"] \\ BX \arrow[r, "\cong", "Bf"'] & BX' \end{tikzcd} \tag{2.10}$$

   *commutes up to based homotopy.*
3. *$X$ is* totally $N$-determined *if it has $N$-determined automorphisms and is $N$-determined.*

A totally $N$-determined 2-compact group is

- *uniquely* $N$-determined if it has $\pi_*(N)$-determined automorphisms (i.e. $H^1(W(X); \check{T}(X)) \cap \mathrm{AM}(\mathrm{Aut}(X)) = \{1\}$)

- *strongly* $N$-determined if $H^1(W(X);\check{T}(X)) \subset \mathrm{AM}(\mathrm{Aut}(X))$

Thus a totally $N$-determined $p$-compact group is both uniquely and strongly $N$-determined if $H^1(W(X);\check{T}(X)) = 0$.

For a compact connected Lie group $G$, the cohomology group $H^1(W(G);\check{T}(G))$ is always an elementary abelian 2-group [20, 1.1]. For instance, this first cohomology group has order 2 for $G = \mathrm{PSU}(4)$ [19, Appendix B] (7.2), generated by an involution $\alpha$, say, of $N(\mathrm{PSU}(4))$. The unique solution to diagram (2.10) is

$$\begin{tikzcd}
N(\mathrm{PSU}(4)) \arrow[r, "\alpha"] \arrow[d, "j"'] & N(\mathrm{PSU}(4)) \arrow[d, "j'"] \\
\mathrm{PSU}(4) \arrow[r, equal] & \mathrm{PSU}(4)
\end{tikzcd}$$

when we use the morphisms $j$, induced by an inclusion of Lie groups, and $j' = j\alpha$ for maximal torus normalizers. PSU(4) is a uniquely but not strongly $N$-determined 2-compact group.

2.11. **Proposition.** *Suppose that the* 2*-compact group* $X$ *is totally* $N$*-determined.*

1. *For fixed* $\alpha \in \mathrm{Aut}(N)$ *with* $\pi_*(B\alpha) = 1$ *there is at most one isomorphism* $f\colon X \to X'$ *such that diagram in* 2.9.*(*2*) based homotopy commutes.*
2. *The pair* $(f,\alpha)$ *in in* 2.9.*(*2*) is unique* $\Leftrightarrow$ $X$ *is uniquely* $N$*-determined.*
3. *It is always possible to use* $\alpha = 1$ *in* 2.9.*(*2*)* $\Leftrightarrow$ $H^1(W(X);\check{T}(X)) \subset \mathrm{AM}(\mathrm{Aut}(X))$.
4. $W(X_0)\backslash \mathrm{Aut}(N) = H^1(W(X);\check{T}(X)) \cdot \mathrm{AM}(\mathrm{Aut}(X))$

*Proof.* 1. If $(f_1,\alpha)$ and $(f_2,\alpha)$ are two solutions to (2.10), then $\mathrm{AM}(f_2^{-1}f_1)$ is the identity and $f_1 = f_2$ as AM is assumed injective.
2. Suppose that the condition is satisfied and let $(f_1,\alpha_1)$ and $(f_2,\alpha_2)$ be two solutions to 2.9.(2). Then $\mathrm{AM}(f_2^{-1}f_1) = \alpha_2^{-1}\alpha_1 \in W(X_0)\backslash \mathrm{Aut}(N)$ belongs to both $\mathrm{AM}(\mathrm{Aut}(X))$ and $H^1(W(X);\check{T}(X))$ and is therefore trivial. Thus $\mathrm{AM}(f_2^{-1}f_1) = 1$ and $f_2 = f_1$ as AM is injective. Conversely, if $\mathrm{AM}(f) \neq 0$ lies in $H^1(W(X);\check{T}(X))$ for some $f \in \mathrm{Aut}(X)$ then $(f, \mathrm{AM}(f))$ and $(1,0)$ are two solutions to 2.9.(2) with $X' = X$ and $j' = j$.
3. Let $\alpha \in H^1(W(X);\check{T}(X))$. If we can always find an isomorphism under $N$, then there exists an isomorphism $f \in \mathrm{Aut}(X)$ such that $fj = j\alpha$. This means that $\mathrm{AM}(f) = \alpha$. Conversely, let $(f,\alpha)$ be a solution to 2.9.(2). If $H^1(W(X);\check{T}(X)) \subset \mathrm{AM}(\mathrm{Aut}(X))$ then $\mathrm{AM}(g) = \alpha$ for an automorphism $g \in \mathrm{Aut}(X)$. According to the commutative diagram

$$\begin{tikzcd}
BN \arrow[d, "Bj"'] & BN \arrow[l, "B\alpha"'] \arrow[r, "B\alpha"] \arrow[d, "Bj"'] & BN \arrow[d, "Bj'"] \\
BX & BX \arrow[l, "Bg"] \arrow[r, "Bf"'] & BX'
\end{tikzcd}$$

$fg^{-1}\colon X \to X'$ is an isomorphism under $N$.
4. For any automorphism $g$ of $N$ it is possible to find an automorphism $f$ of $X$ and and automorphism $\alpha$ of $N$ with $\pi_*(B\alpha) = 1$ such that the diagram

$$\begin{tikzcd}
BN \arrow[r, "B\alpha"] \arrow[d, "Bj"'] & BN \arrow[r, "Bg"] & BN \arrow[d, "Bj"] \\
BX \arrow[rr, "Bf"'] & & BX
\end{tikzcd}$$

commutes up to based homotopy. Thus $g = \mathrm{AM}(f)\alpha$. □

The subgroup $H^1(W(X);\check{T}(X))$ is clearly normal so that

$$W(X_0)\backslash \mathrm{Aut}(N) \cong H^1(W(X);\check{T}(X)) \rtimes \mathrm{Aut}(X), \quad \mathrm{Aut}(X) \cong W(X_0)\backslash \mathrm{Aut}(W(X),\check{T}(X),e(X))$$

for a uniquely $N$-determined 2-compact group $X$. (The corresponding statement for compact connected Lie groups is true [14, 3.10]. It is already known that compact connected Lie groups perceived as 2-compact groups have $\pi_*(N)$-determined automorphisms [18, 2.5].)

2.12. **Lemma.** *Let $X$ be a 2-compact group. Assume that the identity component $X_0$ is completely reducible* [23, 3.4, 3.10] *and that $\check{Z}(X_0) = \check{T}(X_0)^{W(X_0)}$.*

1. $H^1(W(X);\check{T}(X)) \cap \mathrm{AM}(\mathrm{Aut}(X)) = H^1(W/W_0;\check{T}^{W_0})$.
2. *If $H^1(W/W_0;\check{T}^{W_0}) \neq 0$ then $X$ does not have $\pi_*(N)$-determined automorphisms.*
3. *If $H^1(W/W_0;\check{T}^{W_0}) = 0$ and $X_0$ has $\pi_*(N)$-determined automorphisms, so does $X$.*
4. *If the monomorphism* $\mathrm{inf}\colon H^1(W/W_0;\check{T}^{W_0}) \to H^1(W;\check{T})$ *is an isomorphism,* $H^1(W;\check{T}) \subseteq \mathrm{AM}(\mathrm{Aut}(X))$.

*Proof.* (1) and (2). This follows from the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(W/W_0;\check{T}^{W_0}) \arrow[r] \arrow[d, hook] & \mathrm{Aut}(X) \arrow[r] \arrow[d, "\mathrm{AM}"] & \mathrm{Aut}(\pi_0, X_0)_X \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & H^1(W;\check{T}) \arrow[r] & W_0\backslash\mathrm{Aut}(N) \arrow[r] & W_0\backslash\mathrm{Aut}(W,\check{T};e) \arrow[r] & 1
\end{tikzcd}$$

with exact rows. The upper row is [24, 5.2].
(3). We must show that

$$\begin{tikzcd}
\mathrm{Aut}(X) \arrow[r, "\mathrm{AM}"] & W_0\backslash\mathrm{Aut}(N) \arrow[r] & W_0\backslash\mathrm{Aut}(W,\check{T};e)
\end{tikzcd}$$

is injective. The image of this homomorphism is contained in the subgroup $W_0\backslash\mathrm{Aut}(W,W_0,\check{T};e)$ where $\mathrm{Aut}(W,W_0,\check{T};e)$ consists of those pairs $(\alpha,\theta) \in \mathrm{Aut}(W;\check{T};e)$ for which $\alpha(W_0) = W_0$. In the commutative diagram

$$\begin{tikzcd}
\mathrm{Aut}(X) \arrow[r, "\cong"] \arrow[dr] & \mathrm{Aut}(\pi_0, X_0)_X \arrow[r, hook] \arrow[d] & \mathrm{Aut}(\pi_0) \times \mathrm{Aut}(X_0) \arrow[d, hook, "1\times\mathrm{AM}"] \\
& W_0\backslash\mathrm{Aut}(W,W_0,\check{T};e) \arrow[r] & \mathrm{Aut}(\pi_0) \times W_0\backslash\mathrm{Aut}(W_0,\check{T})
\end{tikzcd}$$

the slanted arrow must be injective. We know from [24, 5.2] that $\mathrm{Aut}(X) \cong \mathrm{Aut}(\pi_0,X_0)_X$.
(4). This is clear from (2) and (3). □

2.13. **Example.** The 2-compact group $\mathrm{GL}(2n,\mathbf{R}) = \mathrm{SL}(2n,\mathbf{R}) \rtimes \mathbf{Z}/2$, $n > 1$, does not have $\pi_*(N)$-determined automorphisms for $H^1(W/W_0;\check{T}^{W_0}) = H^1(\mathbf{Z}/2;\mathbf{Z}/2) = \mathbf{Z}/2$ is non-trivial. The maximal torus normalizer for $\mathrm{GL}(2n,\mathbf{R})$ is the same as the one for $\mathrm{SL}(2n+1,\mathbf{R})$ so $H^1(W;\check{T})$ equals $\mathbf{Z}/2$ for $n = 2$ and $(\mathbf{Z}/2)^2$ for $n \geq 3$ (2.2, 7.3). The 2-compact group $\mathrm{GL}(2n+1,\mathbf{R}) = \mathrm{SL}(2n+1,\mathbf{R}) \times \mathbf{Z}/2$, $n > 0$, has $\pi_*(N)$-determined automorphisms for [24, 5.4.(1)] $\mathrm{Aut}(\mathrm{GL}(2n+1,\mathbf{R})) = \mathrm{Aut}(\mathrm{SL}(2n+1,\mathbf{R}))$ and $\mathrm{SL}(2n+1,\mathbf{R})$ has $\pi_*(N)$-determined automorphisms (as does any compact connected Lie group [18, 2.5]). The 2-compact group $\mathrm{PGL}(2n,\mathbf{R}) = \mathrm{PSL}(2n,\mathbf{R}) \rtimes \mathbf{Z}/2$ has $\pi_*(N)$-determined automorphisms since the identity component has trivial center. In fact, $H^1(W(\mathrm{PSL}(2n,\mathbf{R})) \rtimes \mathbf{Z}/2;\check{T}) \subseteq H^1(\mathbf{Z}/2;H^0(W;\check{T})) + H^0(\mathbf{Z}/2;H^1(W;\check{T})) = 0$ (for $n \geq 5$) since $H^0(W;\check{T}) = 0 = H^1(W;\check{T})$ for $\mathrm{PSL}(2n,\mathbf{R})$ by [13].

2.14. **Lemma.** *Let $X$ be a connected 2-compact group with maximal torus normalizer $j\colon N \to X$. Then $X$ is (uniquely) $N$-determined if and only if for any other connected 2-compact group $X'$ with maximal torus normalizer $j'\colon N \to X'$ there exists a (unique) morphism $f\colon X \to X'$ such that*

$$\begin{tikzcd}
& T \arrow[dl, "j|T"'] \arrow[dr, "j'|T"] & \\
X \arrow[rr, "f"'] & & X'
\end{tikzcd}$$

*commutes up to conjugacy.*

*Proof.* The morphism $f\colon X \to X'$ in the above commutative diagram is in fact an isomorphism [8, 5.6] [27, 3.11]. The assumption of the lemma that $f$ be a morphism under $T$ means (use

$W\backslash[BT,BX] = [BT,BX]$ [25, 3.4] [8, 3.4]) that $f$ admits a restriction $N(f)$ to $N$ which is the identity on $T$, i.e. such that

$$\begin{tikzcd}
BT \arrow[r] \arrow[d, equal] & BN \arrow[r, "Bj"] \arrow[d, "BN(f)"'] & BX \arrow[d, "Bf"] \\
BT \arrow[r] & BN \arrow[r, "Bj'"'] & BX'
\end{tikzcd}$$

is homotopy commutative. But then also $\pi_0 N(f)\colon W \to W$ is the identity map for $W$ is faithfully represented as a group of operators on $T$ [6, 9.7]. Thus $\pi_*(BN(f))$ is the identity automorphism of $\pi_*(BN)$.

Assume that the isomorphism $f$ exists and is uniquely determined. In particular, the identity of $X$ is the only automorphism under $T$. That $f \in \mathrm{Aut}(X)$ is a map under $T$ means precisely that $\mathrm{AM}(f) \in H^1(W;\check{T})$. Thus $X$ is uniquely $N$-determined by (2.11.2). Suppose, conversely, that $X$ has this property and let $f_0, f_1\colon X \to X'$ be two isomorphisms under $T$. Then $f_1^{-1}f_0 \in \mathrm{Aut}(X)$ is an isomorphism under $T$ so equals the identity. □

2.15**. Remark.** When the 2-compact group $X$ has $N$-determined automorphisms, also the unbased Adams–Mahmud homomorphism $\mathrm{Out}(X) = W(X)\backslash \mathrm{Aut}(X) \to \mathrm{Out}(N) = \pi_0(N)\backslash \mathrm{Aut}(N)$ is injective [26, 3.7–3.9].

2.16. **LHS 2-compact groups.** Let $N_0 \to N$ be maximal rank normal monomorphism between two extended 2-compact tori, i.e. a commutative diagram with rows and columns that are short exact sequences of loop spaces

$$\begin{tikzcd}
T \arrow[r, equal] \arrow[d] & T \arrow[r] \arrow[d] & \{1\} \arrow[d] \\
N_0 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & W/W_0 \arrow[d, equal] \\
W_0 \arrow[r] & W \arrow[r] & W/W_0
\end{tikzcd}$$

where $T$ is a 2-compact torus and $W_0$ a normal subgroup of the finite group $W$. The 5-term exact sequence

$$0 \to H^1(W/W_0;\check{T}^{W_0}) \xrightarrow{\mathrm{inf}} H^1(W;\check{T}) \xrightarrow{\mathrm{res}} H^1(W_0;\check{T})^{W/W_0} \xrightarrow{d_2} H^2(W/W_0;\check{T}^{W_0}) \xrightarrow{\mathrm{inf}} H^2(W;\check{T})$$

is part of the Lyndon–Hochschild–Serre spectral sequence [15] converging to $H^*(W;\check{T})$.

2.17**. Definition.** *The pair $(N,N_0)$ of extended 2-compact tori is LHS if the initial segment*

$$0 \to H^1(W/W_0;\check{T}^{W_0}) \xrightarrow{\mathrm{inf}} H^1(W;\check{T}) \xrightarrow{\mathrm{res}} H^1(W_0;\check{T})^{W/W_0} \to 0$$

*is a short exact sequence. A 2-compact group is LHS if its maximal torus normalizer pair is LHS.*

Here are two ways to check if a given $p$-compact group $X$ is LHS (besides the evident situations where $\check{T}^{W_0} = 0$ or $W = W_0 \times W/W_0$ is a direct product).

The inflation homomorphism is the composition

$$H^2(W/W_0;\check{T}^{W_0}) \to H^2(W/W_0;\check{T}) \xrightarrow{H^2(W\to W/W_0)} H^2(W;\check{T})$$

of a coefficient group homomorphism followed by the restriction homomorphism induced by the projection of $W$ onto the group of components $W/W_0$. If the Weyl group $W = W_0 \rtimes W/W_0$ is a semi-direct product, $H^2(W \to W/W_0)$ is injective and therefore

$$(2.18)\quad H^1(W;\check{T}) \to H^1(W_0;\check{T})^{W/W_0} \text{ is surjective } \Leftrightarrow$$
$$H^2(W/W_0;\check{T}^{W_0}) \to H^2(W/W_0;\check{T}) \text{ is injective}$$

by exactness of the Lyndon–Hochschild–Serre spectral sequence.

Another possibility is to use the description of $H^1(W_0;\check{T})$ from [13]. The short exact sequence $1 \to W_0 \to W \to W/W_0 \to 1$ of groups yields an exact sequence

$$H_2(W) \to H_2(W/W_0) \to ((W_0)_{\mathrm{ab}})_{W/W_0} \to W_{\mathrm{ab}} \to (W/W_0)_{\mathrm{ab}} \to 0$$

of abelian groups (where $H_2(W/W_0)=0$ if $W/W_0$ has order two). The middle arrow in this exact sequence can be used to define a homomorphism

$$\begin{aligned}\mathrm{Hom}(W,\check{T}^W) = \mathrm{Hom}(W_{\mathrm{ab}},(\check{T}^{W_0})^{W/W_0}) &\to \mathrm{Hom}(((W_0)_{\mathrm{ab}})_{W/W_0},(\check{T}^{W_0})^{W/W_0}) \\ &\to \mathrm{Hom}(W_0,\check{T}^{W_0})^{W/W_0}\end{aligned}$$

which fits into the commutative diagram

(2.19)

$$\begin{tikzcd}
H^1(W;\check{T}) \arrow[r] & H^1(W_0;\check{T})^{W/W_0} \\
\mathrm{Hom}(W,\check{T}^W) \arrow[u] \arrow[r] & \mathrm{Hom}(W_0,\check{T}^{W_0})^{W/W_0} \arrow[u]
\end{tikzcd}$$

Here, the left vertical arrow, say, takes a homomorphism $W \to \check{T}^W$ to the cohomology class represented by the crossed homomorphism $W \to \check{T}^W \hookrightarrow \check{T}$. Since the right vertical arrow is an epimorphism in many cases [13, 1.2, 1.3], this can sometimes be used to show that $H^1(W;\check{T}) \to H^1(W_0;\check{T})^{W/W_0}$ is surjective.

2.20**. Example.** 1. The 2-compact group $\frac{\mathrm{GL}(m,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})} \rtimes C_2$, $m \geq 1$, where the $C_2$-action switches the two $\mathrm{GL}(m,\mathbf{C})$-factors, is LHS because (2.18) the map

$$H^2\Big(C_2;\frac{\check{S}\times\check{S}}{\check{S}}\Big) \to H^2\Big(C_2;\frac{\check{S}^m\times\check{S}^m}{\check{S}}\Big), \quad \check{S}=\mathbf{Z}/2^\infty,$$

can be identified to the identity on $H^3(C_2;\check{S})=\mathbf{Z}/2$ since $H^{>0}(C_2;\check{S}\times\check{S})=0=H^{>0}(C_2;\check{S}^m\times\check{S}^m)$ by Shapiro's lemma. Moreover, $H^1(W/W_0;\check{T}^{W_0}) = H^2\Big(C_2;\frac{\check{S}\times\check{S}}{\check{S}}\Big) = H^2(C_2;\check{S})=0$.

2. The 2-compact group $\frac{\mathrm{GL}(i_0,\mathbf{C})^2\times\mathrm{GL}(i_1,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})} \rtimes C_2$, $i_0,i_1 \geq 1$, where $C_2$ acts diagonally by switching the two $\mathrm{GL}(i_0,\mathbf{C})$-factors and the two $\mathrm{GL}(i_1,\mathbf{C})$-factors, is LHS, again, because

$$H^2\Big(C_2;\frac{\check{S}^2\times\check{S}^2}{\check{S}}\Big) \to H^2\Big(C_2;\frac{(\check{S}^{i_0})^2\times(\check{S}^{i_1})^2}{\check{S}}\Big), \quad \check{S}=\mathbf{Z}/2^\infty,$$

can be identified to the identity on $H^3(C_2;\check{S})$. Moreover, $H^1(W/W_0;\check{T}^{W_0}) = H^2\Big(C_2;\frac{\check{S}^2\times\check{S}^2}{\check{S}}\Big) = H^2(C_2;\check{S})=0$.

3. The 2-compact group $\frac{\mathrm{GL}(m,\mathbf{C})^4}{\mathrm{GL}(1,\mathbf{C})} \rtimes (C_2\times C_2)$, $m\geq 1$, where $C_2\times C_2 = \langle (12)(34),(13)(24)\rangle$ permutes the four $\mathrm{GL}(m,\mathbf{C})$-factors, is LHS. Again,

$$H^2\Big(C_2\times C_2;\frac{\check{S}^2\times\check{S}^2}{\check{S}}\Big) \to H^2\Big(C_2\times C_2;\frac{(\check{S}^m)^2\times(\check{S}^m)^2}{\check{S}}\Big), \quad \check{S}=\mathbf{Z}/2^\infty,$$

identifies to the identity on $H^3(C_2\times C_2;\check{S})$ by means of Shapiro's lemma and the Künneth isomorphism. Moreover, $H^1(W/W_0;\check{T}^{W_0}) = H^2\Big(C_2\times C_2;\frac{\check{S}^2\times\check{S}^2}{\check{S}}\Big) = H^2(C_2\times C_2;\check{S}) = H^2(C_2;\check{S}) + H^1(C_2;\mathbf{Z}/2) = H^1(C_2;\check{S}) = \mathbf{Z}/2$.

4. The 2-compact group $\mathrm{GL}(2n,\mathbf{R}) = \mathrm{SL}(2n,\mathbf{R}) \rtimes C_2$, $n\geq 2$, is LHS by (2.18). The homomorphism $\mathbf{Z}/2 = H^2(C_2;\mathbf{Z}/2) \to H^2(C_2,\check{T}) = \mathbf{Z}/2$ is injective because the action of $C_2$ on $\check{T}=(\mathbf{Z}/2^\infty)^n$ has $(-1,1,\dots,1)$ as its matrix. The 2-compact group $\mathrm{GL}(4,\mathbf{R}) = \mathrm{SL}(4,\mathbf{R})\rtimes C_2 = (\mathrm{SL}(2,\mathbf{C})\circ\mathrm{SL}(2,\mathbf{C}))\rtimes C_2$, in particular, is strongly, but not uniquely $N$-determined because $0 \neq H^1(W/W_0;\check{T}^{W_0}) = H^1(W;\check{T})$ (2.12, 7.3). For $n>2$, $\mathrm{GL}(2n,\mathbf{R})$ can be neither uniquely nor strongly $N$-determined.

2.21. **The center of the maximal torus normalizer.** We need criteria to ensure that the center of the 2-compact group $X$ agrees with the center of its maximal torus normalizer.

2.22. **Proposition.** [29, 4.12] *Let $X$ be a 2-compact group. If $Z(X_0) = Z(N(X_0))$ and $X_0$ has $N$-determined automorphisms, then $Z(X) = Z(N(X))$.*

Assume from now on that $X$ is a *connected* 2-compact group. Let $N(X) \to X$ be the maximal torus normalizer and $Z \to N$ a central monomorphism such that also the composition $Z \to N(X) \to X$ is central. The action map $BZ \times BN(X) \to BN(X)$ induces an action $[BN(X), BZ] \times \mathrm{Out}(N(X)) \to \mathrm{Out}(N(X))$ of the group $[BN(X), BZ] \cong H^1(\check{N}(X); \check{Z})$ on the set $\mathrm{Out}(N(X))$. Let $[BN(X), BZ]_{(1)}$ denote the isotropy subgroup at $(1) \in \mathrm{Out}(N(X))$.

2.23. **Lemma.** *If $Z(X) = Z(N(X))$ and $[BN(X), BZ]_{(1)} = 0$, then $Z(X/Z) = ZN(X/Z)$.*

*Proof.* Using [21, 4.6.4], the assumption of the lemma, and [29, 5.11], we get $Z(X/Z) = Z(X)/Z = Z(N(X))/Z = Z(N(X)/Z) = ZN(X/Z)$. □

2.24. **Remark.** Inspection shows that $Z(G) = ZN(G)$ for any *simply connected* compact Lie group $G$; see [5, 1.4] for a conceptual proof of this fact. In fact, $Z(G) = ZN(G)$ for any compact connected Lie group $G$ containing no direct factors isomorphic to $\mathrm{SO}(2n + 1)$ [20, 1.6].

2.25. **Example.** Let $X = \prod \mathrm{GL}(n_i, \mathbf{C})$ be a product of general linear groups and $Z = \mathbf{C}^\times \cdot (1, \dots, 1)$. Then $Z(X/Z) = ZN(X/Z)$ (2.24), unless $X = \mathrm{GL}(2, \mathbf{C})$, and, assuming that $X/Z$ has $N$-determined automorphisms, $Z(X_{h\pi}) = ZN(X_{h\pi})$ for any 2-compact group $X_{h\pi}$ with $X$ as its identity component (2.22). Indeed, the discrete approximation to $N(X)$ has the form $\check{N}(X) = \prod(\check{T}_i \rtimes \Sigma_{n_i}) = \check{T} \rtimes W$. Suppose that $(t, w) \in \check{N}(X)$ is such that $[(t, w), (s, 1)] \in \check{Z} = \mathbf{Z}/2^\infty$ for all $s \in \check{T}$. Then $(w - 1)\check{T} \subseteq \check{Z}$, which means that $w$ acts trivially on $\check{T}/\check{Z}$. But $W$ is faithfully represented as a group of automorphisms of this maximal torus, so $w = 1$. Suppose therefore that $t \in \check{T}$ is such that $[(t, 1), (s, v)] \in \check{Z}$ for all $(s, v) \in \check{N}(X)$. Then $(v - 1)t \in \check{Z}$ for all $v \in W$ and $v \to (v - 1)t$ is an element of $H^1(W; \check{Z})$ which becomes trivial in $H^1(W; \check{T})$ where it is a principal crossed homomorphism. Actually, $H^1(W; \check{Z}) = \bigoplus H^1(\Sigma_{n_i}; \check{Z})$ is isomorphic to the subgroup $\bigoplus H^1(\Sigma_{n_i}; \check{T})$ of $H^1(W; \check{T})$.

## 3. 2-COMPACT GROUPS WITH $N$-DETERMINED AUTOMORPHISMS

Let $X$ be a 2-compact group with maximal torus normalizer pair $N(X, X_0) = (N, N_0)$.

3.1. **Lemma.** [26, 4.2] *Suppose that*

1. *$X_0$ has $N$-determined automorphisms*
2. *$H^1\big(W/W_0; \check{Z}(X_0)\big) \to H^1\big(W/W_0; Z(\check{N}_0)\big)$ is injective*

*Then $X$ has $N$-determined automorphisms.*

3.2. **Lemma.** [26, 4.8] *Suppose that $X$ is connected. If the adjoint form $PX = X/Z(X)$ has $\pi_*(N)$-determined automorphisms, so does $X$.*

*Proof.* If $f \in \mathrm{Aut}(X)$ is an automorphism under $T(X)$, the induced automorphism $Pf \in \mathrm{Aut}(PX)$ is an automorphism under $T(PX)$, hence equals the identity, and the induced automorphism $Z(f) \in \mathrm{Aut}(ZX)$ is also the identity since the center $ZX \to X$ factors through the maximal torus $T(X) \to X$ [7, 7.5] [21, 4.3]. But then $f$ itself is the identity for $\mathrm{Aut}(X)$ embeds into $\mathrm{Aut}(PX) \times \mathrm{Aut}(ZX)$ [25, 4.3]. □

The functor $BC_X \colon \mathbf{A}(X) \to \mathbf{Top}$ takes an object $(V, \nu)$ of the Quillen category $\mathbf{A}(X)$ to its centralizer $BC_X(V, \nu) = \mathrm{map}(BV, BX)_{B\nu}$. The functor $\pi_j(BZC_X) \colon \mathbf{A}(X) \to \mathbf{Ab}$ takes $(V, \nu)$ into the abelian group $\pi_j(\mathrm{map}(BC_X(V, \nu), BX)_{e(\nu)}$ where $e(\nu) \colon BC_X(V, \nu) \to BX$ is the evaluation map.

3.3. **Lemma.** [26, 4.9] *Suppose that $X$ is connected and centerless. If*

1. *$C_X(L, \lambda)$ has $N$-determined ($\pi_*(N)$-determined) automorphisms for each rank 1 object $(L, \lambda)$ of $\mathbf{A}(X)$*
2. $\lim^1(\mathbf{A}(X); \pi_1(BZC_X)) = 0 = \lim^2(\mathbf{A}(X); \pi_2(BZC_X))$

*Then $X$ has $N$-determined ($\pi_*(N)$-determined) automorphisms.*

*Proof.* Suppose first that each line centralizer has $\pi_*(N)$-determined automorphisms. Let $f\colon X \to X$ be an automorphism under the maximal torus $T \to X$. Since any monomorphism $\lambda\colon L \to X$, $L = \mathbf{Z}/2$, factors through the maximal torus, the commutative diagram

$$\begin{tikzcd}
 & & N \arrow[r] \arrow[dd, "\mathrm{AM}(f)"] & X \arrow[dd, "f"] \\
L \arrow[r, "\lambda^T"] & T \arrow[ur] \arrow[dr] & & \\
 & & N \arrow[r] & X
\end{tikzcd}$$

shows that $f\lambda = \lambda$ and gives a commutative diagram

$$\begin{tikzcd}
 & C_N(L) \arrow[r] \arrow[dd, "C_{\mathrm{AM}(f)}(L)"] & C_X(L) \arrow[dd, "C_f(L)"] \\
T \arrow[ur] \arrow[dr] & & \\
 & C_N(L) \arrow[r] & C_X(L)
\end{tikzcd}$$

of automorphisms under $T$. Thus $\mathrm{AM}(C_f(L)) = C_{\mathrm{AM}(f)}(L)\colon C_N(L) \to C_N(L)$. Now, $\pi_*(C_N(L))$ is a subgroup of $\pi_*(N)$ (for $\pi_1(C_N(L)) = \pi_1(N)$ and $\pi_0(C_N(L)) = W(X)(L)$ is [7, 7.6] [25, 3.2.(1)] the stabilizer subgroup at $L < \check{T}$ for the action of $W(X)$ on $\check{T}$) so $\pi_*(C_{\mathrm{AM}(f)}(L)) = 1$ and $C_f(L) \simeq 1_{C_X(L)}$ since $C_X(L)$ has $\pi_*(N)$-determined automorphisms. For any other object $(V, \nu)$ of $\mathbf{A}(X)$ of rank $> 1$, choose a line $L$ in $V$. Since the monomorphism $\nu\colon V \to X$ canonically factors through $C_X(L)$ [6, 8.2] [29, 3.18], the commutative diagram

$$\begin{tikzcd}
 & & X \arrow[dd, "f"] \\
V \arrow[r] \arrow[urr, bend left, "\nu"] \arrow[drr, bend right, "\nu"'] & C_X(L) \arrow[ur] \arrow[dr] & \\
 & & X
\end{tikzcd}$$

shows that $f\nu = \nu$ and the induced diagram

$$\begin{tikzcd}
 & C_X(V) \arrow[dd, "C_f(V)"] \\
C_{C_X(L)}(V) \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & \\
 & C_X(V)
\end{tikzcd}$$

that $C_f(V)\colon C_X(V) \to C_X(V)$ is conjugate to the identity. The second assumption of the lemma assures that there are no obstructions to conjugating $f$ to the the identity now that we know that the restriction of $f$ to each of the centralizers is conjugate to the identity, see [26, 4.9].

Suppose next that each line centralizer has $N$-determined automorphisms. Let $f\colon X \to X$ be an automorphism such that the diagram

$$\begin{tikzcd}
 & X \arrow[dd, "f"] \\
N \arrow[ur] \arrow[dr] & \\
 & X
\end{tikzcd}$$

commutes up to conjugacy. For each line $L$ in $T$, the induced diagram

$$\begin{tikzcd}
 & C_X(L) \arrow[dd, "C_f(L)"] \\
C_N(L) \arrow[ur] \arrow[dr] & \\
 & C_X(L)
\end{tikzcd}$$

also commutes up to conjugacy. By assumption, this means (2.15) that the induced automorphisms $C_f(L)$ of line centralizers are conjugate to the identity. As above, this implies that the induced map $C_f(V)\colon C_X(V)\to C_X(V)$ is conjugate to the identity for any object $(V,\nu)$ of the Quillen category for $X$ and that $f$ is conjugate to the identity. $\square$

3.4. **Lemma.** [29, 9.4] *If the two connected 2-compact groups $X_1$ and $X_2$ have $N$-determined ($\pi_*(N)$-determined) automorphisms, so does the product $X_1\times X_2$.*

*Proof.* Since the statement conecerning $N$-determined automorphisms is proved in [29, 9.4] we deal here only with the case of $\pi_*(N)$-determined automorphisms. Let $f$ be an automorphism under $T_1\times T_2$ of the product 2-compact group $X_1\times X_2$. Then

$$f_1\colon X_1\to X_1\times X_2\xrightarrow{f}X_1\times X_2\to X_1$$

$$f_2\colon X_2\to X_1\times X_2\xrightarrow{f}X_1\times X_2\to X_2$$

are endomorphisms under the maximal tori and therefore conjugate to the respective identity maps. But $f$ is [29, 9.3] in fact conjugate to the product morphism $(f_1,f_2)$ which is the identity. $\square$

## 4. $N$-DETERMINED 2-COMPACT GROUPS

Let $X$ be a 2-compact group with maximal torus normalizer pair $N(X,X_0)=(N,N_0)$.

4.1. **Lemma.** *Suppose that*

(1) *$X_0$ is uniquely $N$-determined.*
(2) *$X$ is LHS.*
(3) *$H^2\bigl(W/W_0,\check{Z}(X_0)\bigr)\to H^2\bigl(W/W_0,Z(\check{N}_0)\bigr)$ is injective.*

*Then $X$ is $N$-determined.*

*Proof.* Let $X'$ be another 2-compact group with maximal torus normalizer pair $(N,N_0)$. The assumption on the identity component $X_0$ means (2.14) that there exists an isomorphism $f_0\colon X_0\to X_0'$ under $T$. For any $\xi\in W/W_0=N/N_0=X/X_0=X'/X_0'$, the isomorphism $\xi f_0\xi^{-1}$ is also an isomorphism under $T$ and thus $\xi f_0=f_0\xi$ as $X_0$ is uniquely $N$-determined. By the second assumption, the automorphism $\alpha_0=\mathrm{AM}(f_0)\colon N_0\to N_0$ with $\pi_*(B\alpha_0)=1$ extends to an isomorphism $\alpha\colon N\to N$ with $\pi_*(B\alpha)=1$.

Our aim is to find an isomorphism $f\colon X\to X'$ to fill in the based homotopy commutative diagram

$$\begin{tikzcd}
BX_0 \arrow[r, "Bf_0", "\cong"'] \arrow[d] & BX_0' \arrow[d] \\
BX \arrow[r, dotted] \arrow[d] & BX' \arrow[d] \\
B\pi_0(X) \arrow[r, "\cong"'] & B\pi_0(X')
\end{tikzcd}$$

where the isomorphism between the base 2-compact groups is given by the isomorphisms $\pi_0(X)\leftarrow N/N_0\to\pi_0(X')$. Since $f_0$ is $W/W_0$-equivariant up to homotopy, $\mathrm{map}(BX_0,BX_0')_{Bf_0}$ is a $W/W_0$-space. Composition with $BX\xleftarrow{Bj}BN\xrightarrow{Bj'}BX'$ gives maps

$$\mathrm{map}(BX_0,BX_0';Bf_0)^{hW/W_0}\xrightarrow{Bj^*}\mathrm{map}(BN_0,BX_0';B(j_0'\alpha))^{hW/W_0}$$
$$\xleftarrow[\simeq]{Bj'_*}\mathrm{map}(BN_0,BN_0;B\alpha_0)^{hW/W_0}$$

of homotopy fixed point spaces. The space to the right is non-empty for it contains the isomorphism $B\alpha\colon BN\to BN$. Using obstruction theory and the second assumption of the lemma, we see that also the homotopy fixed point space to the left is non-empty; it contains a morphism $Bf\colon BX\to BX'$ under $Bf_0\colon BX_0\to BX_0'$ and over $B\pi_0(X)\xrightarrow{\cong}B\pi_0(X')$ such that $Bf\circ Bj$ and $Bj\circ B\alpha$ are homotopic over $B(N/N_0)\to B\pi_0(X')$. But since the fibre $BX_0'$ of $BX'\to B\pi_0(X')$ is simply connected this means that $Bf\circ Bj$ and $Bj\circ B\alpha$ are based homotopic maps $BN\to BX'$. $\square$

4.2. **Example.** 1. Any 2-compact torus $T$ is strongly $N$-determined for if $j\colon T \to X$ is the maximal torus normalizer for the connected 2-compact group $X$, then $j$ is an isomorphism. Indeed, $H^*(BT;\mathbf{Q}_2) \cong H^*(BX;\mathbf{Q}_2)$ [6, 9.7.(3)] and the connected space $X/T$ has cohomological dimension $\mathrm{cd}_{\mathbf{F}_2}(X/T) = 0$ [7, 4.5, 5.6] so is a point.
2. Any 2-compact toral group $G$ is strongly $N$-determined: $G$ clearly has $N$-determined automorphisms as $G$ is its own maximal torus normalizer. If the 2-compact group $X$ has the same maximal torus normalizer pair $(G,T)$ as $G$, then $X$ is a 2-compact toral group and $j'\colon G \to X$ is an isomorphism. $G$ is uniquely $N$-determined if and only if $H^1(\pi_0(G);\check{T}) = 0$. In particular, $\mathrm{GL}(2,\mathbf{R})$ is uniquely and strongly $N$-determined.

4.3. **Lemma.** *Let $X$ be a connected 2-compact group and $Z \to X$ its center. If $X/Z$ is $N$-determined, so is $X$.*

*Proof.* Let $j\colon N \to X$ be the maximal torus normalizer for $X$ and $j'\colon N \to X'$ the maximal torus normalizer for some other connected 2-compact group $X'$. It suffices (2.14) to find a morphism $f\colon X \to X'$ under the maximal tori $X \xleftarrow{i} T \xrightarrow{i'} X'$. The 2-discrete center $\check{Z}$ of $X$ and $X'$ is contained in the the 2-discrete maximal torus $\check{T}$ [7, 7.5]. Factoring out [6, 8.3] these central monomorphisms we obtain the commutative diagram

$$\begin{tikzcd}
B\check{X} \arrow[d] & B\check{T} \arrow[l, "Bi"'] \arrow[r, "Bi'"] \arrow[d] & B\check{X}' \arrow[d] \\
B(X/Z) \arrow[rr, bend right, "B(f/Z)"'] & B(T/Z) \arrow[l, "B(i/Z)"'] \arrow[r, "B(i'/Z)"] & B(X'/Z)
\end{tikzcd}$$

where the vertical maps are fibrations with fibre $B\check{Z}$, the total spaces, such as $B\check{X}$, are the fibrewise discrete approximations, and $f/Z\colon X/Z \to X'/Z$ is the isomorphism under $T/Z$ that exists because $X?Z$ is $N$-determined. Construct the fibration

$$\mathrm{map}(B\check{Z}, B\check{Z}; B1) \to B\check{Z}_{h(X/Z)} \to B(X/Z)$$

whose sections are maps $BX \to BX'$ over $B(f/Z)$ and under $B\check{Z}$. There are two other such fibrations related to this one as shown in the commutative diagram

$$\begin{tikzcd}
\mathrm{map}(B\check{Z}, B\check{Z}; B1) \arrow[r, equal] \arrow[d] & \mathrm{map}(B\check{Z}, B\check{Z}; B1) \arrow[r, equal] \arrow[d] & \mathrm{map}(B\check{Z}, B\check{Z}; B1) \arrow[d] \\
B\check{Z}_{h(X/Z)} \arrow[d] & B\check{Z}_{h(T/Z)} \arrow[l] \arrow[r, "Bi^*"] \arrow[d] & B\check{Z}_{h(T/Z)} \arrow[d] \\
B(X/Z) & B(T/Z) \arrow[l, "B(i/Z)"] \arrow[r, equal] & B(T/Z)
\end{tikzcd}$$

where the middle fibration is the pull-back along $B(i/Z)$ of the left fibration and the fibre over $b \in B(T/Z)$ of the right fibration consists of one component (remark about equivariance?) of the space of maps of the fibre $B\check{T}_b$ over $b$ into the fibre $B\check{X}'_{B(i'/Z)(b)}$ over $B(i'/Z)(b)$. The fibre equivalence $Bi^*$ is induced by $Bi\colon B\check{T} \to B\check{X}$. The middle fibration has a section $u'$ such that $Bi^* \circ u'$ is the section $Bi'\colon B\check{T} \to B\check{X}'$ of the right fibration. We now have fibre maps

$$\begin{tikzcd}
X/T \arrow[rr, "u|X/T"] \arrow[d] & & B\check{Z} \arrow[d] \\
B\check{T} \arrow[rr] \arrow[dr, "B(i/Z)"'] & & B\check{Z}_{h(X/Z)} \arrow[dl] \\
& B(X/Z) &
\end{tikzcd}$$

where $u$ is the composition of $u'$ and $B\check{Z}_{h(T/Z)} \to B\check{Z}_{h(X/Z)}$. The canonical map, given by constants, $B\check{Z} \to \mathrm{map}(X/T, B\check{Z})$ is a homotopy equivalence since $X/T$ is simply connected [21, 5.6] and hence a version [26, 6.6] of the Zabrodsky lemma implies that $u = v \circ B(i/Z)$ for some section $v\colon B(X/Z) \to B\check{Z}_{h(X/Z)}$ of the left fibration. The section $v$ is, after fibre-wise completion, a fibre map $BX \to BX'$ under $BT$. □

Let $X_1$ and $X_2$ be two connected 2-compact groups with trivial centers and $j_1\colon N_1 \to X_1$, $j_2\colon N_2 \to X_2$ their maximal torus normalizers. The Splitting Theorem [8, 1.4] says that if the monomorphism $j\colon N_1 \times N_2 \to X$ is the maximal torus normalizer for some connected 2-compact group $X$ then there exist an isomorphism $s\colon X \to X_1 \times X_2$ and an automorphism $\alpha$ of $N_1 \times N_2$ such that the diagram

$$\begin{tikzcd}
N_1 \times N_2 \arrow[r, "\alpha", "\cong"'] \arrow[d, "j"'] & N_1 \times N_2 \arrow[d, "j_1 \times j_2"] \\
X \arrow[r, "\cong", "s"'] & X_1 \times X_2
\end{tikzcd}$$

commutes up to conjugacy. We record this in

4.4. **Lemma.** *The product of two $N$-determined connected 2-compact groups is $N$-determined.*

The problem is now reduced to the connected and center-less case. Consider therefore an extended 2-compact torus $N$ and two connected, center-less 2-compact groups $X$ and $X'$ both having $N$ as their maximal torus normalizer

$$\begin{tikzcd}
X & N \arrow[l, "j"'] \arrow[r, "j'"] & X'
\end{tikzcd} \tag{4.5}$$

For each toral object $(V,\nu)$ of $\mathbf{A}(X)$, let $\nu^N\colon V \to N$ be the unique preferred lift [27, 4.10] of $\nu$ (which factors through the identity component of $N$) and let $(V,\nu')$ be the object defined by $\nu' = j \circ \nu^N\colon V \to X'$ as in the commutative diagram

$$\begin{tikzcd}
 & V \arrow[dl, "\nu"'] \arrow[d, "\nu^N"] \arrow[dr, "\nu'"] & \\
X & N \arrow[l, "j"] \arrow[r, "j'"'] & X'
\end{tikzcd}$$

The functor $\mathbf{A}(X)^{\leq t} \to \mathbf{A}(X')^{\leq t}$ that takes the object $(V,\nu)$ to the object $(V,\nu')$ and is the identity on morphisms is an equivalence of toral Quillen categories [29, 2.8].

4.6. **Theorem.** *In the situation of (4.5), assume the following:*

(1) *Centralizers of all toral rank $\leq 2$ objects of $\mathbf{A}(X)$ have $N$-determined automorphisms.*

(2) *There exists a self-homotopy equivalence $\alpha \in H^1(W;\check{T}) \subseteq \mathrm{Out}(N)$ such that for every object $(L,\lambda) \in \mathrm{Ob}(\mathbf{A}(X))$ of rank 1 the diagram*

$$\begin{tikzcd}
C_N(\lambda^N) \arrow[r, "\alpha|C_N(\lambda^N)"] \arrow[d, "j|C_N(\lambda^N)"'] & C_N(\lambda^N) \arrow[d, "j'|C_N(\lambda^N)"] \\
C_X(\lambda) \arrow[r, "f_\lambda"'] & C_{X'}(\lambda')
\end{tikzcd}$$

*commutes for some isomorphism $f_\lambda$.*

(3) *For any non-toral rank 2 object $(V,\nu)$ of $\mathbf{A}(X)$ the composite monomorphism*

$$\nu'_L\colon\ V \xrightarrow{\overline{\nu}(L)} C_X(L,\nu|L) \xrightarrow[\cong]{f(L,\nu|L)} C_{X'}(L,(\nu|L)') \xrightarrow{\mathrm{res}} X'$$

*and the induced isomorphism* $f_{\nu,L}\colon C_X(V,\nu)\to C_{X'}(V,\nu'_L)$ *defined by the commutative diagram*

$$\begin{tikzcd}
C_{C_X(L,\nu|L)}(V,\overline{\nu}(L)) \arrow[r, "C_{f(L,\nu|L)}"] \arrow[d, "\cong"'] & C_{C_{X'}(L,(\nu|L)')}(V,f(L,\nu|L)\circ\overline{\nu}(L)) \arrow[d, "\cong"] \\
C_X(V,\nu) \arrow[r, "f_{\nu,L}"'] & C_{X'}(V,\nu'(L))
\end{tikzcd}$$

*do not depend on the choice of line* $L<V$*.*

(4) $\lim^2(\mathbf{A}(X);\pi_1(BZC_X))=0=\lim^3(\mathbf{A}(X);\pi_2(BZC_X))$.

*Then there exists an isomorphism* $f\colon X\to X'$ *under* $T$ *(*2.14*).*

*Proof.* The idea is that the isomorphisms $f_\lambda\colon C_X(\lambda)\to C_{X'}(\lambda')$ on the line centralizers restrict to isomorphisms $f_\nu\colon C_X(\nu)\to C_{X'}(\nu')$ for all centralizers in the $\mathbf{F}_2$-homology decomposition

$$\operatorname{hocolim}_{\mathbf{A}(X)} BC_X(\nu)\to BX$$

of $BX$. These locally defined isomorphisms combine to a globally defined isomorphism $BX\to BX'$.

First observe that the isomorphisms $f_\lambda$ on the line centralizers are uniquely determined by the cohomology class $\alpha\in H^1(W;\check{T})$ (2.11.(1)).

Let now $(V,\nu)$ be a rank 2 object of $\mathbf{A}(X)$ and $L$ a line in the plane $V$. If $(V,\nu)$ is *toral*, define $f_\nu\colon C_X(V,\nu)\to C_{X'}(V,\nu')$ to be the isomorphism induced by $f_{\nu|L}\colon C_X(L,\nu|L)\to C_{X'}(L,(\nu|L)'$. Since $f_\nu$ is an isomorphism under $\alpha|C_N(V,\nu^N)$ it does not depend on the choice of $L$ in $V$ (2.11.(1)). If $(V,\nu)$ is *non-toral*, define $\nu'$ to be $\nu'_L$ and define $f_\nu\colon C_X(V,\nu)\to C_{X'}(V,\nu')$ to be $f_{\nu,L}$. By assumption 4.6.(3), the monomorphism $\nu'$ and the isomorphism $f_{\nu,L}$ are independent of the choice of $L$.

This construction respects morphisms in $\mathbf{A}(X)$. Consider first, for instance, a morphism $\beta\colon(L_1,\lambda_1)\to(L_2,\lambda_2)$ between two lines in $X$. Then $\lambda_1=\lambda_2\beta$ and $\lambda_1^N=\lambda_2^N\beta$. The commutative diagram of isomorphisms

$$\begin{tikzcd}
C_X(\lambda_1) \arrow[ddd, "f_{\lambda_1}"'] & & & C_X(\lambda_2) \arrow[lll, "\beta^*"'] \arrow[ddd, "f_{\lambda_2}"] \\
& C_N(\lambda_1^N) \arrow[ul] \arrow[d, "\alpha|C_N(\lambda_1^N)"'] & C_N(\lambda_2^N) \arrow[l, "\beta^*"'] \arrow[ur] \arrow[d, "\alpha|C_N(\lambda_2^N)"] & \\
& C_N(\lambda_1^N) \arrow[dl] & C_N(\lambda_2^N) \arrow[l, "\beta^*"] \arrow[dr] & \\
C_{X'}(\lambda'_1) & & & C_{X'}(\lambda'_2) \arrow[lll, "\beta^*"]
\end{tikzcd}$$

shows that $(\beta^*)^{-1}\circ f_{\lambda_1}\circ\beta^*=f_{\lambda_2}$ for they are both isomorphism under $(\beta^*)^{-1}\circ\alpha|C_N(\lambda_1^N)\circ\beta^*=\alpha|C_N(\lambda_2^N)$. Second, by the very definition of $f_\nu$, the diagram

$$\begin{tikzcd}
C_X(V,\nu) \arrow[r, "f_\nu"] \arrow[d] & C_{X'}(V,\nu') \arrow[d] \\
C_X(L,\nu|L) \arrow[r, "f_{\nu|L}"'] & C_{X'}(L,(\nu|L)')
\end{tikzcd}$$

commutes whenever $L<V$ and $(V,\nu)$ is (toral or non-toral) rank 2 object of $\mathbf{A}(X)$.

We have now defined natural isomorphisms $f_\nu\colon C_X(V,\nu)\to C_{X'}(V,\nu')$ for all objects $(V,\nu)\in\operatorname{Ob}(\mathbf{A}(X))$ of rank $\leq 2$. For any other object $(E,\varepsilon)$ of $\mathbf{A}(X)$, choose a line $L<E$ and proceed as for toral rank 2 objects. That is, define $\varepsilon'\colon E\to X'$ to be the monomorphism

$$\begin{tikzcd}
E \arrow[r, "\overline{\varepsilon}(L)"] & C_X(E,\varepsilon|L) \arrow[r, "f_{\varepsilon|L}"] & C_{X'}(E,(\varepsilon|L)') \arrow[r, "\mathrm{res}"] & X'
\end{tikzcd}$$

and define $f_\varepsilon\colon C_X(E,\varepsilon) \to C_{X'}(E,\varepsilon')$ to be the isomorphism

$$\begin{tikzcd}
C_{C_X(E,\varepsilon|L)}(\overline{\varepsilon}(L)) \arrow[r, "(f_{\varepsilon|L})_*"] \arrow[d, "\cong"'] & C_{C_{X'}(E,(\varepsilon|L)')}(f_{\varepsilon|L} \circ \overline{\varepsilon}(L)) \arrow[d, "\cong"] \\
C_X(E,\varepsilon) \arrow[r, "f_\varepsilon"'] & C_{X'}(E,\varepsilon')
\end{tikzcd}$$

induced by $f_{\varepsilon|L}$. If $L_1$ and $L_2$ are two distinct lines in E, let $P = \langle L_1, L_2 \rangle$ be the plane generated by them. Then the commutative diagram

$$\begin{tikzcd}
& C_X(L_1,\varepsilon|L_1) \arrow[r, "f_{\varepsilon|L_1}", "\cong"'] & C_{X'}(L_1,(\varepsilon|L_1)') \arrow[dr, "\mathrm{res}"] & \\
P \arrow[ur, "\overline{\varepsilon}(L_1)"] \arrow[r, "\overline{\varepsilon}(P)"] \arrow[dr, "\overline{\varepsilon}(L_2)"'] & C_X(P,\varepsilon|P) \arrow[u] \arrow[d] \arrow[r, "\cong", "f_{\varepsilon|P}"'] & C_{X'}(P,(\varepsilon|P)') \arrow[u] \arrow[d] \arrow[r, "\mathrm{res}"] & X' \\
& C_X(L_2,\varepsilon|L_2) \arrow[r, "\cong", "f_{\varepsilon|L_2}"'] & C_{X'}(L_2,(\varepsilon|L_2)') \arrow[ur, "\mathrm{res}"'] &
\end{tikzcd}$$

shows that neither $(E,\varepsilon') \in \mathrm{Ob}(\mathbf{A}(X'))$ nor the isomorphism $f_\varepsilon$ depend on the choice of line in $E$. Thus we have constructed a collection of centric [4] maps

$$\mathrm{B}C_X(V,\nu) \to \mathrm{B}X', \quad (V,\nu) \in \mathrm{Ob}(\mathbf{A}(X)), \tag{4.7}$$

that are homotopy invariant under $\mathbf{A}(X)$-morphisms. The vanishing (4.6.(4)) of the obstruction groups means [36] that these homotopy $\mathbf{A}(X)$-invariant maps can be realized by a map

$$\mathrm{B}f\colon \mathrm{B}X \xleftarrow{\simeq} \operatorname{hocolim} \mathrm{B}C_X \to \mathrm{B}X'$$

such that $f \circ \mathrm{res} = \mathrm{res} \circ f_\nu$ for all $(V,\nu) \in \mathrm{Ob}(\mathbf{A}(X))$. In particular, $f$ is a map under $T$ and an isomorphism (2.14). $\square$

4.8. **Verification of condition 4.6.(2).** Define $\mathbf{A}_{\mathrm{LHS}}(X)^{\leq t}$ to be the full subcategory of the toral Quillen category $\mathbf{A}(X)^{\leq t} = \mathbf{A}(W,t)$ [29, 2.2] generated by all objects $\nu$ whose centralizers $C_X(\nu)$ are LHS and totally $N$-determined. For such an object, the solutions to the isomorphism problem

$$\begin{tikzcd}
C_N(\nu^N) \arrow[r, "\alpha_\nu"] \arrow[d] & C_N(\nu^N) \arrow[d] \\
C_X(\nu) \arrow[r, "f_\nu"'] & C_{X'}(\nu')
\end{tikzcd} \tag{4.9}$$

define a subset $\{\alpha_\nu\}$ of $H^1(W;\check{T})(\nu)$ and (2.12.(1)) an element $\overline{\alpha}_\nu$ of $H^1(W_0;\check{T})^{W/W_0}(\nu)$. These elements respect the morphisms in $\mathbf{A}_{\mathrm{LHS}}(X)^{\leq t}$ (because the restriction of a solution is a solution) so they represent an element $(\overline{\alpha}_\nu)$ of the limit group. If the two homomorphisms

$$H^1(W(X);\check{T}(X)) \to \lim{}^0(\mathbf{A}_{\mathrm{LHS}}(X)^{\leq t}; H^1(W;\check{T})) \to \lim{}^0(\mathbf{A}_{\mathrm{LHS}}(X)^{\leq t}; H^1(W_0;\check{T})^{W/W_0})$$

are surjective, this elements is the image og an element $\alpha \in H^1(W(X);\check{T}(X))$. This means that th isomorphism problems (4.9) have a coherent solution where $\alpha_\nu = \alpha|C_N(\nu^N)$ is the restriction of $\alpha$ for all objects $\nu$ of $\mathbf{A}_{\mathrm{LHS}}(X)^{\leq t}$.

We can therefore replace 4.6.(1) and 4.6.(2) by

- $C_X(\nu)$ is LHS and totally $N$-determined for each toral elementary abelian 2-subgroup $(V,\nu)$ of $X$ of rank $\leq 2$
- $\lim^1(\mathbf{A}_{\mathrm{LHS}}(X)^{\leq t}; H^1(W_0;\check{T})^{W/W_0}) = 0$

The first property ensures that

$$\begin{aligned} H^1(W(X);\check{T}(X)) \to \lim{}^0(\mathbf{A}_{\mathrm{LHS}}(X)^{\leq t}; H^1(W;\check{T})) &\cong \lim{}^0(\mathbf{A}(X)^{\leq t}_{\leq 2}; H^1(W;\check{T})) \\ &\cong \lim{}^0(\mathbf{A}(X)^{\leq t}; H^1(W;\check{T})) \end{aligned}$$

is an isomorphism [10, 8.1] [32] and the second property that there is an exact sequence

$$0 \to \lim^0 H^1(W/W_0; \check{T}^{W_0}) \to \lim^0 H^1(W; \check{T}) \to \lim^0 H^1(W_0; \check{T})^{W/W_0} \to 0,$$

where the limits are taken over over $\mathbf{A}_{\mathrm{LHS}}(X)^{\leq t}$ or $\mathbf{A}(X)^{\leq t}_{\leq 2}$. It is sometimes possible to compute the above $\lim^1$-term by means of Oliver's cochain complex [32].

4.10. **Verification of condition 4.6.(3).** The following observation can sometimes be useful in the verification of condition 4.6.(3).

4.11**. Lemma.** *Let $(V, \nu)$ be a non-toral rank 2 object of $\mathbf{A}(X)$ and $L < V$ a line in $V$. Write $C_3$ for the Sylow 3-subgroup of $\mathrm{GL}(V)$. Suppose that*

1. $C_3 \subseteq \mathbf{A}(X)(V, \nu) \cap \mathbf{A}(X')(V, \nu')$
2. $f_{\nu,L}\colon C_X(V, \nu) \to C_{X'}(V, \nu')$ *is $C_3$-equivariant*

*Then condition 4.6.(3) is satisfied.*

*Proof.* Let $L_1$ and $L_2 = L$ be lines in $V$. Choose an automorphism $\alpha$ of $(V, \nu)$ that takes $L_1$ to $L_2$. Then $\nu'_{L_2}\alpha = \nu' - L_1$ and $f_{\nu,L_1} = C_X(\alpha) \circ f_{\nu,L_2} \circ C_X(\alpha)^{-1}$ (4.15). □

The following lemma assures that condition 4.11.(1) holds.

4.12**. Lemma.** *Suppose that*

1. *There is (up to conjugacy) a unique monomorphism $\lambda\colon \mathbf{Z}/2 \to X$ with non-connected centralizer*
2. *There is (up to conjugacy) a unique non-toral monomorphism $\nu\colon (\mathbf{Z}/2)^2 \to X$*

*Then the same holds for $X'$ and $\mathbf{A}(X)(V, \nu) = \mathrm{GL}(V) = \mathbf{A}(X')(V, \nu')$ for the non-toral object $(V, \nu)$ of $\mathbf{A}(X)$.*

*Proof.* Let $\nu'\colon (\mathbf{Z}/2)^2 \to X'$ be a non-toral monomorphism from a rank elementary abelian 2-group into $X'$ and let $i_1\colon \mathbf{Z}/2 \to (\mathbf{Z}/2)^2$ be the inclusion into the first summand. Then $\nu' i_1$ corresponds to $\lambda$ under the bijection between rank 1 objects of $\mathbf{A}(X)$ and $\mathbf{A}(X')$, i.e. $\nu' i_1 = \lambda'$. Moreover, the diagram

(4.13)
$$\begin{tikzcd}
X & C_X(\mathbf{Z}/2, \lambda) \arrow[l, "\mathrm{res}"'] \arrow[rr, "f_\lambda", "\cong"'] & & C_{X'}(\mathbf{Z}/2, \nu' i_1) \arrow[r, "\mathrm{res}"] & X' \\
& & (\mathbf{Z}/2)^2 \arrow[ull, bend left, "\nu"] \arrow[ul, "\overline{\nu}(\mathbf{Z}/2)"'] \arrow[ur, "\overline{\nu}'(\mathbf{Z}/2)"] \arrow[urr, bend right, "\nu'"'] & &
\end{tikzcd}$$

is commutative. To see this, observe that $\nu = \mathrm{res} \circ f_\lambda^{-1} \circ \overline{\nu}'(\mathbf{Z}/2)$ by uniqueness of $\nu$, and $\overline{\nu}(\mathbf{Z}/2) = f_\lambda^{-1} \circ \overline{\nu}'(\mathbf{Z}/2)$ by uniqueness of canonical factorizations under $\mathbf{Z}/2$ [28, 3.9]. We conclude that $\nu' = \mathrm{res} \circ \overline{\nu}'(\mathbf{Z}/2) = \mathrm{res} \circ f_\lambda \circ \overline{\nu}(\mathbf{Z}/2)$ is uniquely determined up to conjugacy. □

Note in connection with condition 4.11.(2) that by mapping $(\mathbf{Z}/2)^2$ into middle part of diagram (4.13) we see that $f((\mathbf{Z}/2)^2, \nu)$ is a map under the canonical factorization in the sense that

(4.14)
$$\begin{tikzcd}
& (\mathbf{Z}/2)^2 \arrow[dl, "\overline{\nu}((\mathbf{Z}/2)^2)"'] \arrow[dr, "\overline{\nu}'((\mathbf{Z}/2)^2)"] & \\
C_X((\mathbf{Z}/2)^2, \nu) \arrow[rr, "\cong", "f((\mathbf{Z}/2)^2, \nu)"'] & & C_{X'}((\mathbf{Z}/2)^2, \nu')
\end{tikzcd}$$

commutes where the canonical monomorphisms, $\overline{\nu}((\mathbf{Z}/2)^2)$ and $\overline{\nu}'((\mathbf{Z}/2)^2)$, are $\mathrm{GL}(E)$-equivariant. Thus the restriction of $f(V, \nu)$ to $V$ is $C_3$-equivariant.

4.15. **Canonical factorizations.** Let $\nu\colon V \to X$ be a monomorphism from an elementary abelian $p$-group to the $p$-compact group $X$. The canonical factorization of $\nu$ through its centralizer is the central monomorphism $\overline{\nu}(V)\colon V \to C_X(V,\nu)$ whose adjoint is $V \times V \xrightarrow{+} V \xrightarrow{\nu} X$ [6, 8.2]. If $\alpha\colon (V_1,\nu_1) \to (V_2,\nu_2)$ is a morphism in $\mathbf{A}(X)$ then the canonical factorizations are related by a commutative diagram

(4.16)
$$\begin{tikzcd}
V_1 \arrow[r, "\overline{\nu}_1(V_1)"] \arrow[d, "\alpha"'] & C_X(V_1,\nu_1) \arrow[r, "\mathrm{res}"] & X \arrow[d, equal] \\
V_2 \arrow[r, "\overline{\nu}_2(V_2)"'] & C_X(V_2,\nu_2) \arrow[r, "\mathrm{res}"'] \arrow[u, "C_X(\alpha)"] & X
\end{tikzcd}$$

and we shall write $\overline{\nu}_2(V_1)\colon V_2 \to C_X(V_1,\nu_1)$ for $C_X(\alpha)\circ\overline{\nu}_2(V_2)$ and call it the canonical factorization of $\nu_2$ through the centralizer of $\nu_1$. The induced diagram

(4.17)
$$\begin{tikzcd}
C_{C_X(V_2,\nu_2)}(V_2,\overline{\nu}_2(V_2)) \arrow[r, "C_{C_X(\alpha)}", "\cong"'] \arrow[d, "\cong"'] & C_{C_X(V_1,\nu_1)}(V_2,\overline{\nu}_2(V_1)) \arrow[r, "C_{C_X(V_1,\nu_1)}(\alpha)"] & C_{C_X(V_1,\nu_1)}(V_1,\overline{\nu}_1(V_1)) \arrow[d, "\cong"] \\
C_X(V_2,\nu_2) \arrow[rr, "C_X(\alpha)"'] & & C_X(V_1,\nu_1)
\end{tikzcd}$$

is a factorization of $C_X(\alpha)$.

## 5. The Quillen category of PGL$(n+1,\mathbf{C})$

For $W$ is a finite group acting on a finite $\mathbf{F}_2$-vector space $V$, define $\mathbf{A}(W,V)$ to be the category whose objects are non-trivial subspaces of $V$ and whose morphisms are group homomorphisms induced by the $W$-action; the morphism set $\mathbf{A}(W,V)(V_1,V_2)$ is the set of orbits $\overline{W}(V_1,V_2)/W(V_1,V_1)$ for the action of the point-wise stabilizer group $W(V_1,V_1) = \{w \in W | wv = v \text{ for all } v \in V\}$ on the set $\overline{W}(V_1,V_2) = \{w \in W | wV_1 \subseteq V_2\}$.

5.1. **The toral subcategory of $\mathbf{A}(\mathrm{PGL}(n{+}1,\mathbf{C}))$.** An object $(V,\nu)$ of the Quillen category of the 2-compact group $X$ is *toral* if the monomorphism $\nu\colon V \to X$ is conjugate to a monomorphism that factors through the maximal torus $T(X)$ of $X$. Let $\mathbf{A}(X)^{\leq t}$ denote the full subcategory of $\mathbf{A}(X)$ generated by all toral objects. We shall determine this toral subcategory in case $X = \mathrm{PGL}(n{+}1,\mathbf{C})$.

5.2. **Lemma.** *The monomorphism $\nu\colon V \to \mathrm{PGL}(n+1,\mathbf{C})$ is toral if and only if it lifts to a morphism $V \to \mathrm{GL}(n+1,\mathbf{C})$. If $n$ is even, all objects of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$ are toral.*

*Proof.* All objects of $\mathbf{A}(\mathrm{GL}(n+1,\mathbf{C}))$ are toral by complex representation theory. Any monomorphism $V \to (\mathbf{C}^\times)^{n+1}/\mathbf{C}^\times$ lifts to $(\mathbf{C}^\times)^{n+1}$ since $\mathbf{C}^\times$ is a divisible abelian group. If $n$ is even, $\mathrm{PGL}(n+1,\mathbf{C}) = \mathrm{SL}(n+1,\mathbf{C})$ as 2-compact groups and all monomorphisms $V \to \mathrm{SL}(n+1,\mathbf{C})$ are toral by complex representation theory. □

5.3. **Proposition.** [29, 2.8] *The inclusion $t(\mathrm{PGL}(n+1,\mathbf{C})) \to T(\mathrm{PGL}(n+1,\mathbf{C}))$ induces an equivalence of categories $\mathbf{A}(\Sigma_{n+1}, t(\mathrm{PGL}(n+1,\mathbf{C}))) \to \mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}$.*

*Proof.* The functor is the identity on morphisms. Any morphism between two toral objects $V_1 \to V_2$ of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$ is induced from the action by a Weyl group element. □

5.4. **Corollary.** *When $n > 1$, the limits $\lim^i(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}; \pi_j(\mathrm{B}ZC_{\mathrm{PGL}(n+1,\mathbf{C})})) = 0$ and $\lim^i(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C})); \pi_j(\mathrm{B}ZC_{\mathrm{PGL}(n+1,\mathbf{C})}))$ is isomorphic to*

$$\lim{}^i(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))_{\not\leq t}; \pi_j(\mathrm{B}ZC_{\mathrm{PGL}(n+1,\mathbf{C})})_{\not\leq t}) \cong \lim{}^i(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C})); \pi_j(\mathrm{B}ZC_{\mathrm{PGL}(n+1,\mathbf{C})})_{\not\leq t})$$

*for all $i \geq 0$ and $j = 1, 2$.*

*Proof.* For any non-trivial toral subgroup $V \subseteq t(\mathrm{PGL}(n+1,\mathbf{C}))$ we have by (2.25) that

$$\pi_j(\mathrm{B}ZC_{\mathrm{PGL}(n+1,\mathbf{C})}) = H^{2-j}\big(\Sigma_{n+1}(V); L(\mathrm{PGL}(n+1,\mathbf{C}))\big), \quad j=1,2,$$

because the 2-discrete toral group

$$\begin{aligned}\check{Z}C_{\mathrm{PGL}(n+1,\mathbf{C})}(V) = ZC_{\check{N}(\mathrm{PGL}(n+1,\mathbf{C}))}(V) &= Z\big(\check{T}(\mathrm{PGL}(n+1,\mathbf{C})) \rtimes \Sigma_{n+1}(V)\big)\\ &= H^0(\Sigma_{n+1}(V); \check{T}(\mathrm{PGL}(n+1,\mathbf{C})))\end{aligned}$$

and consequently the higher limits of these functors $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t} \to \mathbf{Ab}$ are trivial while for $i=0$ we get $H^{2-j}(\Sigma_{n+1}; L(\mathrm{PGL}(n+1,\mathbf{C})))$ which is trivial for $n>1$. Apply [29, 2.11] to get the isomorphisms. $\square$

Let $E$ be a non-trivial elementary abelian 2-group and $\mathrm{Rep}(E, \mathrm{GL}(n+1,\mathbf{C}))$ the set of functions $i\colon E^\vee \to \mathbf{N}$ taking the dual $E^\vee = \mathrm{Hom}(E, \mathbf{C}^\times)$ of $E$ into the natural numbers such that $\sum_{f\in E^\vee} i(f) = n+1$. This set supports group actions

$$\begin{tikzcd} E^\vee \times \mathrm{Rep}(E,\mathrm{GL}(n+1,\mathbf{C})) \arrow[r] & \mathrm{Rep}(E,\mathrm{GL}(n+1,\mathbf{C})) & \mathrm{Rep}(E,\mathrm{GL}(n+1,\mathbf{C})) \times \mathrm{GL}(E) \arrow[l] \end{tikzcd}$$

given by $g \cdot i = i \circ \tau_g$, $g \in E^\vee$, and $i \cdot A = i \circ A^\vee$, $A \in \mathrm{GL}(E)$, where $\tau_g(f) = g+f$ and $A^\vee(f) = f \circ A^{-1}$ for all linear forms $f \in E^\vee$. The identity $\tau_{A^\vee(g)} A^\vee = A^\vee \tau_g$ gives $(g\cdot i)\cdot A = ((A^{-1})^\vee g)\cdot(i \cdot A)$.

We say that a subset $S$ of linear forms on $E$ has trivial equalizer, and write $\mathrm{Eq}(S) = 0$, if $S$ contains at least two elements and all the elements of $S$ agree only on the trivial element of $E$.

5.5. **Proposition.** *Let $E$ be a non-trivial elementary abelian 2-group.*

1. *The set of conjugacy classes of toral monomorphisms $\nu\colon E \to \mathrm{PGL}(n+1,\mathbf{C})$ corresponds bijectively to the set*
$$E^\vee \backslash \{i \in \mathrm{Rep}(E,\mathrm{GL}(n+1,\mathbf{C})) \mid \mathrm{Eq}(S(i)) = 0\}$$
*of $E^\vee$-orbits.*
2. $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(E^\vee i) = \{A \in \mathrm{GL}(E) \mid (E^\vee i)\cdot A^\vee = E^\vee i\}$.
3. $\pi_0(C_{\mathrm{PGL}(n+1,\mathbf{C})}(E^\vee i)) = \{\zeta \in E^\vee \mid \zeta \cdot i = i\}$.
4. *The set of isomorphism classes of $\dim_{\mathbf{F}_p} E$-dimensional toral objects of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$ corresponds bijectively to the set*
$$E^\vee \backslash \{i \in \mathrm{Rep}(E,\mathrm{GL}(n+1,\mathbf{C})) \mid \mathrm{Eq}(S(i)) = 0\}/\mathrm{GL}(E)$$
*of $E^\vee \times \mathrm{GL}(E)$-orbits.*

*Proof.* 1. Let $\nu\colon E \to \mathrm{PGL}(n+1,\mathbf{C})$ be a toral monomorphism and $\mu\colon E \to \mathrm{GL}(n+1,\mathbf{C})$ any lift of $\nu$ to $\mathrm{GL}(n+1,\mathbf{C})$. The representation $\mu$ is a sum of linear characters

$$\mu = \sum_{f\in E^\vee} i_\mu(f) f$$

for some function $i_\mu \in \mathrm{Rep}(E,\mathrm{GL}(n+1,\mathbf{C}))$. The condition that $\mu(E)$ intersects the center $\mathbf{C}^\times$ trivially, translates to $\mathrm{Eq}(S(i_\mu)) = 0$ (or, equivalently, $S(i_\mu)$ spans $V^\vee$ and $V = \bigcup_{f\in S(i_\mu)} \ker f$).

Any other lift of $\nu$ has the form $\zeta\mu$ for some $\zeta \in E^\vee$. We have $i_{\zeta\mu} = \zeta \cdot i_\mu$ for

$$(\zeta\mu)(v) = \sum i_\mu(f)\zeta(v)f(v) = \sum (i_\mu \circ \tau_\zeta)(f)f(v) = \sum (\zeta\cdot i_\mu)(f)f(v)$$

for all $v \in V$. (Also, $S(\tau_\zeta i_\mu) = \tau_\zeta S(i_\mu)$ so the equalizer subspace does not change.)
2. An automorphism $A \in \mathrm{GL}(E)$ preserves the conjugacy class of $\nu\colon E \to \mathrm{PGL}(n+1,\mathbf{C})$ if and only if $\mu A(v) = \zeta(v)\mu(v)$ for some $\zeta \in E^\vee$ (depending on $A$). Since

$$(\mu A)(v) = \sum i_\mu(f)(fA)(v) = \sum (i_\mu \circ A^\vee)(f)f(v) = \sum (i_\mu \cdot A)(f)f(v)$$

for all $v \in V$, this means that $i_\mu \cdot A = \zeta \cdot i_\mu$. Then $(g\cdot i_{mu})\cdot A = ((A^{-1})^\vee g)\cdot(i_{mu}\cdot A) = ((A^{-1})^\vee g)\cdot(\zeta\cdot i_{mu}) = ((A^{-1})^\vee g + \zeta)\cdot i_{mu} \in E^\vee i_{mu}$ for all $g \in E^\vee$.
3. The component group of $C_{PGL(n+1,\mathbf{C})}(E^\vee i_\mu)$ is [29, 5.11.(2)] isomorphic to the group of $\zeta \in E^\vee$ for which $\mu$ and $\zeta\mu$ are conjugate in $\mathrm{GL}(n+1,\mathbf{C})$. For the traces, this means that $i_\mu = \zeta \cdot i_\mu$. $\square$

5.6. **Remark.** 1. Since $i\tau_f\tau_\zeta = i\tau_f \Leftrightarrow i\tau_\zeta = i$, the right hand side for the equation in 5.5.(3) remains the same for all elements of the orbit $E^\vee i$.
2. Let $A \in \mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(E^\vee i)$ so that $iA^\vee = i\tau_\zeta$ for some $\zeta \in E^\vee$. Then

$$i\tau_{A^\vee(g)} = i \Leftrightarrow i\tau_{A^\vee(g)}A^\vee = iA^\vee \Leftrightarrow iA^\vee\tau_g = iA^\vee \Leftrightarrow i\tau_\zeta\tau_g = i\tau_\zeta \Leftrightarrow i\tau_g = i$$

for any $g \in E^\vee$, meaning that $A^\vee(g) \in \pi_0(C_{\mathrm{PGL}(n+1,\mathbf{C})}(E^\vee i)) \Leftrightarrow g \in \pi_0(C_{\mathrm{PGL}(n+1,\mathbf{C})}(E^\vee i))$. Thus $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(E^\vee i)^{\mathrm{op}}$ acts on $\pi_0(C_{\mathrm{PGL}(n+1,\mathbf{C})}(E^\vee i))$.

5.7. **Example.** (Toral lines and planes in $\mathrm{PGL}(m,\mathbf{C})$) Let $P(m,k)$ be the number of ways to write $m = i_0 + i_1 + \cdots + i_k$ as a sum of $k$ integers $i_0, i_1, \ldots, i_k$ such that $1 \leq i_0 \leq i_1 \leq \cdots \leq i_k$. There are $P(m,2) = [m/2]$ toral lines and $P(m,3) + P(m,4)$ toral planes in $\mathrm{PGL}(m,\mathbf{C})$. The $P(m,2)$ toral lines of type $i = (i_0, i_1)$ with $i_0, i_1 > 0$ and $i_0 + i_1 = m$ have these Quillen automorphism groups and centralizer component groups:

$(i_0, i_1)$**:** $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(L) = 1$, $\pi_0(C_{\mathrm{PGL}(m,\mathbf{C})}(L)) = 1$
$(i_0, i_0)$**:** $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(L) = 1$, $\pi_0(C_{\mathrm{PGL}(m,\mathbf{C})}(L)) = L^\vee$

The non-connected rank 1 centralizer is

$$C_{\mathrm{PGL}(m,\mathbf{C})}(L) = \frac{\mathrm{GL}(i_0,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})} \rtimes L^\vee, \quad ZC_{\mathrm{PGL}(m,\mathbf{C})}(L) \overset{7.7}{=} L$$

The $P(m,3) + P(m,4)$ toral planes of type $i = (i_0, i_1, i_2, i_3)$ with $i_0, i_1, i_2 > 0$, $i_3 \geq 0$, and $i_0 + i_1 + i_2 + i_3 = m$ have these Quillen automorphism groups and centralizer component groups:

$(i_0, i_1, i_2, i_3)$**:** $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(V) = 1$, $\pi_0(C_{\mathrm{PGL}(m,\mathbf{C})}(V)) = 1$
$(i_0, i_0, i_2, i_3)$**:** $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(V) = C_2$, $\pi_0(C_{\mathrm{PGL}(m,\mathbf{C})}(V)) = 1$
$(i_0, i_0, i_0, i_3)$**:** $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(V) = \mathrm{GL}(V)$, $\pi_0(C_{\mathrm{PGL}(m,\mathbf{C})}(V)) = 1$
$(i_0, i_0, i_2, i_2)$**:** $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(V) = C_2$, $\pi_0(C_{\mathrm{PGL}(m,\mathbf{C})}(V)) = L^\vee$ for some line $L < V$.
$(i_0, i_0, i_0, i_0)$**:** $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(V) = \mathrm{GL}(V)$, $\pi_0(C_{\mathrm{PGL}(m,\mathbf{C})}(V)) = V^\vee$

If $V = \mathbf{F}_2^2$ is a plane, then $V^\vee$ and $\mathrm{GL}(V) \cong \Sigma_3$ together generate all permutations of the four letters $(i_0, i_1, i_2, i_3)$. Thus there are $P(m,3)$ isomorphism classes of the form $(i_0, i_1, i_2, 0)$, $i_0, i_1, i_2 > 0$, $i_0 + i_1 + i_2 = m$ and $P(m,4)$ isomorphism classes of the form $(i_0, i_1, i_2, i_3)$, $i_0, i_1, i_2, i_3 > 0$, $i_0 + i_1 + i_2 + i_3 = m$. If, for instance $i = (i_0, i_0, i_0, i_3)$ with $i_0 \neq i_3$, then $V^\vee i = i\mathrm{GL}(V)$ contains four elements, so $\pi_0 \cong V^\vee$ and $\mathrm{Aut} = \mathrm{GL}(V)$. In all cases, $\pi_0 C_{\mathrm{PGL}(m,\mathbf{C})}(V,\nu) = \pi_0 ZC_{\mathrm{PGL}(m,\mathbf{C})}(V,\nu)$; this is clear in case $\pi_0(C_{\mathrm{PGL}(m,\mathbf{C})}(V)) = 1$ is trivial and in the remaining two cases it is a direct check.

The character table for $V = C_2 \times C_2 = \{e_0, e_2, e_2, e_3 = e_1 + e_2\}$

| | $e_0$ | $e_1$ | $e_2$ | $e_3$ |
|---|---|---|---|---|
| $\rho_0$ | 1 | 1 | 1 | 1 |
| $\rho_1$ | 1 | 1 | $-1$ | $-1$ |
| $\rho_2$ | 1 | $-1$ | 1 | $-1$ |
| $\rho_3$ | 1 | $-1$ | $-1$ | 1 |

contains four linear characters $V^\vee = \{\rho_0, \rho_1, \rho_2, \rho_3\}$. In the list above, $(i_0, i_1, i_2, i_3)$ means $i_0\rho_0 + i_1\rho_1 + i_2\rho_2 + i_3\rho_3$. Non-connected $\mathrm{PGL}(m,\mathbf{C})$-centralizers only occur for induced $\mathrm{GL}(m,\mathbf{C})$-representations:

$$(i_0, i_0, i_2, i_2) = \begin{cases} \mathrm{ind}_{\langle e_1 \rangle}^V (i_0\rho_0 + i_2\rho_1) & i_0 \neq i_2 \\ \mathrm{ind}_{\{0\}}^V (i_0\rho_0) = i_0\mathrm{reg}_V & i_0 = i_2 \end{cases}$$

In the first case, the centralizer

$$C_{\mathrm{PGL}(m,\mathbf{C})}(V,\rho) = \frac{\mathrm{GL}(i_0,\mathbf{C})^2 \times \mathrm{GL}(i_1,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})} \rtimes L^\vee, \quad ZC_{\mathrm{PGL}(m,\mathbf{C})}(V,\rho) \overset{7.7}{=} \frac{\mathrm{GL}(1,\mathbf{C}) \times \mathrm{GL}(1,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})} \times L,$$

is LHS and has $\pi_*(N)$-determined automorphisms (2.20). In the second case, we have a pure rank 2 object, the only rank 1 sub-object is $2i_0$ times the regular representation of $C_2$. Its centralizer

$$C_{\mathrm{PGL}(m,\mathbf{C})}(V,\rho) = \frac{\mathrm{GL}(i_0,\mathbf{C})^4}{\mathrm{GL}(1,\mathbf{C})} \rtimes V^\vee, \quad ZC_{\mathrm{PGL}(m,\mathbf{C})}(V,\rho) \overset{7.5}{=} V,$$

is LHS but does not does not have $\pi_*(N)$-determined automorphisms (2.20).

5.8. **The non-toral subcategory of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$.** For 2-compact group $X$, let $\mathbf{A}(X)_{\not\leq t}$ denote the full subcategory of $\mathbf{A}(X)$ on all non-toral objects and their sub-objects. We determine this non-toral subcategory $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))_{\not\leq t}$ in case $X = \mathrm{PGL}(n+1,\mathbf{C})$.

For any non-trivial elementary abelian 2-group $V$ in $\mathrm{PGL}(n+1,\mathbf{C})$, let $[\ ,\ ]\colon V \times V \to \mathbf{F}_2$ be the symplectic bilinear form [16, II.9.1] given by $[u\mathbf{C}^\times, v\mathbf{C}^\times] = [u,v]$ for all $u\mathbf{C}^\times$, $v\mathbf{C}^\times \in V$. (The elements $[u,v]$ and $u^2$ lie in the center $\mathbf{C}^\times$ of $\mathrm{GL}(n+1,\mathbf{C})$ so that $E = [u^2,v] = [u,v]^u[u,v] = [u,v]^2$ and thus $[u,v] \in \mathbf{C}^\times$ has order 2. Therefore $[u,v] = [u,v]^{-1} = [v,u]$.)

5.9. **Lemma.** *$V$ in $\mathrm{PGL}(n+1,\mathbf{C})$ is toral* $\Leftrightarrow [V,V] = 0$

*Proof.* Let $e_i\mathbf{C}^\times$, $1 \leq i \leq d$, be a basis for $V$. Since $\mathbf{C}^\times$ is divisible, we can assume that each $e_i \in \mathrm{GL}(n+1,\mathbf{C})$ has order 2. If $[V,V] = 0$, these $e_i$s commute and span a lift to $\mathrm{GL}(n+1,\mathbf{C})$ of $V \subseteq \mathrm{PGL}(n+1,\mathbf{C})$. □

An extra special 2-group is of *positive type* if it is isomorphic to a central product of dihedral groups $D_8$ of order 8.

5.10. **Lemma.** [12, 3.1] [29, 5.4] *Let $\nu\colon V \to \mathrm{PGL}(n,\mathbf{C})$ be a non-toral monomorphism of a non-trivial elementary abelian 2-group $V$ into $\mathrm{PGL}(n+1,\mathbf{C})$. Then there exists a morphism of short exact sequences of groups*

$$\begin{tikzcd}
1 \arrow[r] & Z(P) \arrow[r] \arrow[d, hook] & PE \arrow[r] \arrow[d, hook] & V \arrow[r] \arrow[d, hook, "\nu"] & 1 \\
1 \arrow[r] & \mathbf{C}^\times \arrow[r] & \mathrm{GL}(n+1,\mathbf{C}) \arrow[r] & \mathrm{PGL}(n+1,\mathbf{C}) \arrow[r] & 1
\end{tikzcd}$$

*where $PE$ is the direct product of an extra special 2-group $P \subseteq \mathrm{GL}(n+1,\mathbf{C})$ of positive type and an elementary abelian 2-group $E \subseteq \mathrm{GL}(n+1,\mathbf{C})$ with $P \cap E = \{1\} = [P,E]$.*

Write $\mathbf{C}^{n+1} = \mathbf{C}^{2^d} \otimes \mathbf{C}^m$ for some $d > 0$ and some $m \geq 0$. Let the extra-special 2-group $2_+^{1+2d}$ act faithfully on the first factor of the tensor product and let the (possibly trivial) elementary abelian 2-group $E$ act faithfully on the second factor such that no non-trivial element of $E$ acts as scalar multiplication. This makes $\mathbf{C}^{n+1}$ a $\mathbf{C}[2_+^{1+2d} \times E]$-module. The image of the group $2_+^{1+2d} \times E \subseteq \mathrm{GL}(n+1,\mathbf{C})$ in $\mathrm{PGL}(n+1,\mathbf{C})$ is a non-toral elementary abelian 2-group (5.9) and any non-toral elementary abelian 2-group in $\mathrm{PGL}(n+1,\mathbf{C})$ has this form (5.10).

Let $G = \langle P, E, i\rangle = P \circ C_4 \times E$ be the group generated by $E$ and the central product $P \circ C_4$ of $P$ and the cyclic group $C_4 = \langle i \rangle \subseteq \mathbf{C}^\times$ with $\mathbf{Z}/2$ amalgamated. The image of $G$ in $\mathrm{PGL}(n+1,\mathbf{C})$ is $V$ and $q(v\mathbf{C}^\times) = v^2$, $v \in G$, is a quadratic form on $V$ such that $q(u\mathbf{C}^\times + v\mathbf{C}^\times) = q(u\mathbf{C}^\times) + q(v\mathbf{C}^\times) + [u\mathbf{C}^\times, v\mathbf{C}^\times]$ for all $u\mathbf{C}^\times, v\mathbf{C}^\times \in V$.

5.11. **Lemma.** *$\mathbf{A}(\mathrm{GL}(n+1,\mathbf{C}))(G,G) \to \mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(V,V)$ is surjective.*

*Proof.* Suppose that $B \in \mathrm{GL}(n+1,\mathbf{C})$ is such that $V^{B\mathbf{C}^\times} = V$. Then $G^B \subseteq G \cdot \mathbf{C}^\times$: For any $g \in G$ there exist $h \in G$ and $z \in \mathbf{C}^\times$ such that $g^B = hz$. But since $G$ has exponent 4, $z^4 = 1$ so $z \in C_4$ and $g^B \in G$. □

A monomorphic conjugacy class $\nu\colon V \to \mathrm{PGL}(n+1,\mathbf{C})$ is said to be a $(2d+r,r)$ object of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$ if the underlying symplectic vector space of $(V,\nu)$ is isomorphic to $V = H^d \times V^\perp$ where $H$ denotes the symplectic plane over $\mathbf{F}_2$ and $\dim_{\mathbf{F}_p} V^\perp = r$ [16, II.9.6] (so that $\dim_{\mathbf{F}_p} V = r + 2d$). An $(r,r)$ object is the same thing as an $r$-dimensional toral object. We write $\mathrm{Sp}(V)$ or $\mathrm{Sp}(2d+r,r)$ (abbreviated to $\mathrm{Sp}(2d)$ if $r = 0$) for the group of linear automorphisms of $V$ that preserve the symplectic form.

5.12. **Corollary.** *Suppose that $n+1 = 2^d m$ for some natural numbers $d \geq 1$ and $m \geq 1$.*

(1) *There is up to isomorphism a unique $(2d,0)$ object $H^d$ of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$, and*

$$\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(H^d) = \mathrm{Sp}(2d), \quad C_{\mathrm{PGL}(n+1,\mathbf{C})}(H^d) = H^d \times \mathrm{PGL}(m,\mathbf{C})$$

*for this object.*

(2) *Isomorphism classes of* $(2d+r,r)$*,* $r>0$*, objects* $V$ *of* $\mathbf{A}(\mathrm{PGL}(2^dm,\mathbf{C}))$ *correspond bijectively to isomorphism classes of* $(r,r)$ *objects* $V^\perp$ *of* $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))$*, and*

$$\mathbf{A}(\mathrm{PGL}(2^dm,\mathbf{C}))(V)=\begin{pmatrix}\mathrm{Sp}(2d) & 0\\ * & \mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(V^\perp)\end{pmatrix}$$

$$C_{\mathrm{PGL}(2^dm,\mathbf{C})}(V)=V/V^\perp\times C_{\mathrm{PGL}(m,\mathbf{C})}(V^\perp)$$

*for these objects.*

*Proof.* 1. The group $2_+^{1+2d}\circ 4$ has [17, 7.5] $2^{1+2d}$ characters of degree 1 and 2 irreducible characters of degree $2^d$ (interchanged by the action of $\mathrm{Out}(2_+^{1+2d}\circ 4)\cong\mathrm{Sp}(2d)\times\mathrm{Aut}(C_4)$ [11, pp. 403–404]) given by

$$\chi_\lambda(g)=\begin{cases}2^d\lambda(g) & g\in C_4\\ 0 & g\notin C_4\end{cases}$$

where $\lambda\colon C_4\to\mathbf{C}^\times$ is an injective group homomorphism ($\lambda(i)=\pm i$). The linear characters vanish on the derived group $2=[2_+^{1+2d}\circ 4,2_+^{1+2d}\circ 4]$ but the irreducible characters of degree $2^d$ do not. Thus the only faithful representations of $2_+^{1+2d}\circ 4$ with central centers are multiples $m\chi_\lambda$ of $\chi_\lambda$ for a fixed $\lambda$. Phrased slightly differently, $\mathrm{GL}(m2^d,\mathbf{C})$ contains up to conjugacy a unique subgroup with central center isomorphic to $2_+^{1+2d}\circ 4$. For this group and its image $H^d$ in $\mathrm{PGL}(2^dm,\mathbf{C})$ we have

$$\mathbf{A}(\mathrm{GL}(m2^d,\mathbf{C}))(2_+^{1+2d}\circ 4,2_+^{1+2d}\circ 4)\cong\mathrm{Sp}(2d)\cong\mathbf{A}(\mathrm{PGL}(m2^d,\mathbf{C}))(H^d,H^d)$$

$$C_{\mathrm{GL}(m2^d,\mathbf{C})}(2_+^{1+2d}\circ 4)\cong\mathrm{GL}(m,\mathbf{C}),\quad C_{\mathrm{PGL}(m2^d,\mathbf{C})}(H^d)\cong H^d\times\mathrm{PGL}(m,\mathbf{C})$$

where the last isomorphism is a consequence of [29, 5.9].

2. The $(2d+r,r)$ object $(V,\nu)$ of $\mathbf{A}(\mathrm{PGL}(2^dm,\mathbf{C}))$ and the $(r,0)$ object $(V^\perp,\nu^\perp)$ of $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))$ correspond to each other iff there is an $m$-dimensional representation $\mu\colon V^\perp\to\mathrm{GL}(m,\mathbf{C})$ such that $\mathbf{C}^{2^d}\otimes\mu$ is a lift of $\nu|V^\perp$ and $\mu$ a lift of $\nu^\perp$. According to 5.10 any lift of $\nu|V^\perp$ has this form for some $\mu$ uniquely determined up to the action of $(V^\perp)^\vee$.

We use 5.11 to calculate the Quillen automorphism group of a $(2d+r,r)$ object $H^d\times V^\perp$ of $\mathbf{A}(\mathrm{PGL}(2^dm,\mathbf{C}))$. Let $H^d\times V^\perp$ be covered by the group $P\circ C_4\times V^\perp$ as in 5.10. Let $\alpha$ be an automorphism of $P\circ C_4$, let $\beta$ be any homomorphism of the form $P\circ C_4\to H^d\to V^\perp$, and let $\gamma$ be any Quillen automorphism of $(V^\perp,\nu^\perp)$. Choose a homomorphism $\zeta_1\colon P\circ C_4\to H^d\times C_4/C_2\to C_4$ such that $\lambda(\zeta_1(x)\alpha(x))=\lambda(x)$ for all $x\in C_4$ and a homomorphism $\zeta_2\colon V^\perp\to C_4$ such that $\lambda(\zeta_2(v))\mu(\gamma(v))=\mu(v)$ for all $v\in V^\perp$. Then the automorphism of $P\circ C_4$ that takes $(x,v)$ to $(\zeta_1(x)\zeta_2(v)\alpha(x),\beta(x)+\gamma(v))$ preserves the trace of $\chi_\lambda\#\mu$ and therefore the automorphism induced on the quotient is a Quillen automorphism of $H^d\times V^\perp$. Conversely, any automorphism of $P\circ C_4\times V^\perp$ takes the center $C_4\times V^\perp$ isomorphically to itself and hence it is of the form $(x,v)\to(\zeta(x,v)\alpha(x),\beta(x)+\gamma(v))$ for some automorphism $\alpha$ of $P\circ C_4$, some homomorphism $\beta\colon P\circ C_4\to V^\perp$ vanishing on $C_4$, and some homomorphism $\zeta\colon P\circ C_4\times V^\perp\to C_4$. Such an automorphism preserves the trace of $\chi_\lambda\#\mu$ iff $\lambda(\zeta(x,v)\alpha(x))=\mu(\gamma(v))$ for all $(x,v)\in Z(P\circ C_4\times V^\perp)=C_4\times V^\perp$. But this means that the induced automorphism of $H^d\times V^\perp$ is of the stated form. □

5.13. **Example.** (Oliver's cochain complex [32]) The non-toral objects of $\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))$ of rank $\leq 4$ are

- One $(2,0)$ object $H$, $\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))(H)=\mathrm{Sp}(2)=\mathrm{GL}(2,\mathbf{F}_2)$, $\pi_0=H$.
- $P(m,2)$ $(3,1)$ objects $V$, $\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))(V)=\mathrm{Sp}(3,1)$, $\pi_0=V/V^\perp$ or $V$.
- $P(m,3)+P(m,4)$ $(4,2)$ objects $E$, $\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))(E)=\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & \mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(E^\perp)\end{pmatrix}$, $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(E^\perp)=1$, $C_2$, $\mathrm{GL}(E)$, $\pi_0=E/E^\perp$, $E/E^\perp$ or $E/L$, $E/E^\perp$ or $E$.
- One $(4,0)$ object if $m$ is even.

The $(2,0)$ object $H$ contributes

$$\mathrm{Hom}_{\mathrm{Sp}(2)}(\mathrm{St}(H),H)\cong\mathbf{F}_2$$

The $(3,1)$ objects $V$ contribute

$$\mathrm{Hom}_{\mathrm{Sp}(3,1)}(\mathrm{St}(V),V)\cong\mathrm{Hom}_{\mathrm{Sp}(3,1)}(\mathrm{St}(V),V/V^\perp)\cong\mathbf{F}_2$$

The $(4,2)$ objects $E$ with $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(E^\perp)=1$ contribute

$$\mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & 1\end{pmatrix}}(\mathrm{St}(E),E/E^\perp)\cong \mathbf{F}_2^2$$

and the $(4,2)$ objects $E$ with $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(E^\perp)=C_2$ contribute

$$\mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & C_2\end{pmatrix}}(\mathrm{St}(E),E/L)\cong \mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & C_2\end{pmatrix}}(\mathrm{St}(E),E/E^\perp)\cong \mathbf{F}_2$$

The $(4,0)$ object (if it exists) and the $(4,2)$ objects with $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(E^\perp)=\mathrm{GL}(E)$ do not contribute to the cochain complex for the corresponding Hom-groups are trivial. Thus the cochain complex for computing higher limits of the functor $\pi_1(BZC_{\mathrm{PGL}(2m,\mathbf{C})})$ will have the form

$$\begin{aligned}(5.14)\quad 0\to \mathrm{Hom}_{\mathrm{Sp}(2)}(\mathrm{St}(H),H)\xrightarrow{\delta^1}\prod_{[m/2]}\mathrm{Hom}_{\mathrm{Sp}(3,1)}(\mathrm{St}(V),V/V^\perp)\xrightarrow{\delta^2}\\ \prod \mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & 1\end{pmatrix}}(\mathrm{St}(E),E/E^\perp)\times\prod \mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & C_2\end{pmatrix}}(\mathrm{St}(E),E/E^\perp)\to\cdots\end{aligned}$$

To show vanishing of the relevant higher limits it suffices to show that $\delta^1$ is injective and that the rank of $\delta^2$ is $P(m,2)-1$.

## 6. $N$-DETERMINISM OF THE $A$-FAMILY

By inductively applying 3.3 and 4.6 we show that the 2-compact groups $\mathrm{PGL}(n+1,\mathbf{C})$, $n\geq 1$, are uniquely $N$-determined.

**6.1. Lemma.** *Suppose that $n+1=2m\geq 2$ is even.*

1. *There is a unique monomorphism conjugacy class $\lambda\colon \mathbf{Z}/2\to\mathrm{PGL}(n+1,\mathbf{C})$ with disconnected centralizer. The centralizer of this monomorphism is $\mathrm{GL}(m,\mathbf{C})^2/\mathbf{C}^\times\rtimes\mathbf{Z}/2$*
2. *There is a unique monomorphism conjugacy class $\nu\colon H\to\mathrm{PGL}(n+1,\mathbf{C})$, $H=(\mathbf{Z}/2)^2$, such that $\nu$ is non-toral. The centralizer of this monomorphism is $H\times\mathrm{PGL}(m,\mathbf{C})$ and the Quillen automorphism group is $\mathrm{GL}(H)$.*

*Proof.* Use that any monomorphism of $\mathbf{Z}/2$ into $\mathrm{PGL}(n+1,\mathbf{C})$ lifts to $\mu\colon\mathbf{Z}/2\to\mathrm{GL}(n+1,\mathbf{C})$. The only possibility is that $\mu=m\cdot\mathrm{reg}$ is a direct sum of regular representations. The result for non-toral rank 2 objects in $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$ is a special case of 5.10. □

**6.2. Lemma.** *Suppose that* $\mathrm{PGL}(r+1,\mathbf{C})$ *is uniquely $N$-determined for all $0\leq r<n$. Then* $\mathrm{PGL}(n+1,\mathbf{C})$*, $n\geq 1$, satisfies conditions* 4.6.*(*1*),* 4.6.*(*2*), and* 4.6.*(*3*).*

*Proof.* We shall verify 4.6.(1) and 4.6.(2) by establishing the alternative two conditions from 4.8.

Let $(V,\nu)$ be a toral elementary abelian 2-subgroup of $\mathrm{PGL}(n+1,\mathbf{C})$ of rank $\leq 2$ and $C(\nu)=C_{\mathrm{PGL}(n+1,\mathbf{C})}(\nu)$ its centralizer. We have seen that $C(\nu)$ is LHS (2.20) and that $\check{Z}(C(\nu)_0)=\check{Z}(N_0(C(\nu)))$ as $C(\nu)_0$ does not contain a direct factor isomorphic to $\mathrm{GL}(2,\mathbf{C})/\mathrm{GL}(1,\mathbf{C})=\mathrm{SO}(3)$ (2.24, 5.7). The identity component $C(\nu)_0$ has $\pi_*(N)$-determined automorphisms according to 3.2 and 3.4, and $C(\nu)$ has $N$-determined automorphisms by 3.1. The identity component $C(\nu)_0$ is $N$-determined according to 4.3 and 4.4, and $C(\nu)$ is $N$-determined by 4.1. Thus $C(\nu)$ is LHS and totally $N$-determined.

The functor $H^1(W/W_0;\check{T}_0^W)$ is zero on $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}_{\leq 2}$ except on the object $(V,\nu)=(i_0,i_0,i_0,i_0)$, when $n+1=4i_0$, where it has value $\mathbf{Z}/2$. However, this object has Quillen automorphism group $\mathrm{GL}(V)$ and since the only $\mathrm{GL}(V)$-equivariant homomorphism $\mathrm{St}(V)=V\to\mathbf{Z}/2$ is the trivial homomorphism, $\lim^1(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}_{\leq 2};H^1(W/W_0;\check{T}_0^W))=0$ follows from Oliver's cochain complex [32].

When $n+1=2m$ is even, we verify condition 4.6.(3) by applying 4.11. Let $X'$ be a connected 2-compact group with maximal torus normalizer $j'\colon N(\mathrm{PGL}(n+1,\mathbf{C})\to X$. Since the first item in 4.11 is satisfied by 4.12 and 6.1, it suffices to show that the isomorphism (from 4.6.(3))

$$f_{\nu,L}\colon C_{\mathrm{PGL}(2m,\mathbf{C})}(H)=H\times\mathrm{PGL}(m,\mathbf{C})\to C_{X'}(H,\nu')$$

defined by choosing one of the three lines $L$ in $H$, is $C_3$-equivariant. Now [24]

$$\operatorname{Aut}(H \times \mathrm{PGL}(m, \mathbf{C})) = \mathrm{GL}(H) \times \operatorname{Aut}(\mathrm{PGL}(m, \mathbf{C}))$$

so that $f_{\nu,L}$ is $C_3$-equivariant if $\pi_0 f_{\nu,L}$ and the restriction of $f_{\nu,L}$ to the identity components are $C_3$-equivariant. Here, $\operatorname{Aut}(\mathrm{PGL}(m, \mathbf{C})) = \mathbf{Z}_2^\times$ (or $\mathbf{Z}_2^\times/\{\pm 1\}$ if $m = 2$) since $\mathrm{PGL}(m, \mathbf{C})$ has $\pi_*(N)$-determined automorphisms by induction hypothesis so $C_3$ must act trivially on the identity components for purely group theoretic reasons. The commutative triangle (4.14)

$$\begin{tikzcd}
& \pi_0(H) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
\pi_0(C_{\mathrm{PGL}(2m,\mathbf{C})}(H,\nu)) \arrow[rr, "\pi_0(f_{\nu,L})"'] & & \pi_0(C_{X'}(H,\nu'))
\end{tikzcd}$$

in which the slanted arrows, representing the canonical factorizations, are $C_3$-equivariant (even $\mathrm{GL}(H)$-equivariant) shows that $\pi_0(f_{\nu,L})$ is $C_3$-equivariant. □

We shall next compute the higher limits from 3.3.(2) and 4.6.(4) by means of 5.4 and the cochain complex 5.14 from [32]. As 5.4 is not valid for $\mathrm{PGL}(2, \mathbf{C})$ we first consider this case separately.

6.3. **Proposition.** *The* 2*-compact group* $\mathrm{PGL}(2, \mathbf{C})$ *is uniquely* $N$*-determined.*

*Proof.* The functor $C_{\mathrm{PGL}(2,\mathbf{C})}$ takes the Quillen category of $\mathrm{PGL}(2, \mathbf{C})$, consisting (5.7, 5.13, 6.1) of one toral line, $L$, and one non-toral plane, $H$,

(6.4)
$$\begin{tikzcd}
L \arrow[r] & H \arrow[loop right, "\mathrm{GL}(H)"]
\end{tikzcd}$$

to the diagram

(6.5)
$$\begin{tikzcd}
\mathrm{GL}(1,\mathbf{C})^2/\mathrm{GL}(1,\mathbf{C}) \rtimes C_2 & H \arrow[l] \arrow[loop right, "\mathrm{GL}(H)^{\mathrm{op}}"]
\end{tikzcd}$$

of uniquely $N$-determined 2-compact groups. The 2-compact toral group to the left is uniquely $N$-determined (4.2) because $H^1(C_2; \mathbf{Z}/2^\infty) = 0$ for the non-trivial action. The center functor takes this diagram back to the starting point (6.4) for which the higher limits vanish [29, 12.7.4]. $\mathrm{PGL}(2, \mathbf{C})$ is thus uniquely $N$-determined by 3.3 and 4.6. □

6.5. **Lemma.** *The low degree higher limits of the functors* $\pi_j(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})$, $j = 1, 2$, *are:*

(1) $\lim^i(\mathbf{A}(\mathrm{PGL}(n+1, \mathbf{C})), \pi_1(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})) = 0$ *for* $i = 1, 2$,

(2) $\lim^i(\mathbf{A}(\mathrm{PGL}(n+1, \mathbf{C})), \pi_2(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})) = 0$ *for* $i = 2, 3$,

*for all* $n \geq 1$.

Let $V = \mathbf{F}_2 e_1 + \mathbf{F}_2 e_2 + \mathbf{F}_2 e_3$ be a 3-dimensional vector space over $\mathbf{F}_2$ with basis $\{e_1, e_2, e_3\}$ and (degenerate) symplectic inner product matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let $\mathbf{F}_2[1]$ be the 21-dimensional $\mathbf{F}_2$-vector space on all length one flags $[P > L]$ and $\mathbf{F}_2[0]$ the 14-dimensional $\mathbf{F}_2$-vector space on all length zero flags, $[P]$ or $[L]$, of non-trivial and proper subspaces of $V$. The Steinberg module $\mathrm{St}(V)$ over $\mathbf{F}_2$ for $V$ is the $2^3 = 8$-dimensional kernel of the linear map $d\colon \mathbf{F}_2[1] \to \mathbf{F}_2[0]$ given by $d[P > L] = [P] + [L]$. Define $f_1 = \overline{f}_1 | \mathrm{St}(V)\colon \mathrm{St}(V) \to V$ as the restriction to $\mathrm{St}(V)$ of the linear map $\overline{f}_1\colon \mathbf{F}_2[1] \to V$ with values

$$\overline{f}_1[P > L] = \begin{cases} L & P \cap P^\perp = 0 \\ 0 & \text{otherwise} \end{cases}$$

on the basis vectors.

Let $E = \mathbf{F}_2 e_1 + \mathbf{F}_2 e_2 + \mathbf{F}_2 e_3 + \mathbf{F}_2 e_4$ be a 4-dimensional vector space over $\mathbf{F}_2$ with basis $\{e_1, e_2, e_3, e_4\}$ and (degenerate) symplectic inner product matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $\mathbf{F}_2[2]$ be the 315-dimensional $\mathbf{F}_2$-vector space on all length two flags $[V > P > L]$ and $\mathbf{F}_2[1]$ the also 315-dimensional $\mathbf{F}_2$-vector space on all length one flags, $[P > L]$ or $[V > L]$ or $[V > P]$, of non-trivial, proper subspaces of $E$. The Steinberg module $\mathrm{St}(E)$ over $\mathbf{F}_2$ for $E$ is the $2^6 = 64$-dimensional kernel of the linear map $d\colon \mathbf{F}_2[2] \to \mathbf{F}_2[1]$ given by $d[V > P > L] = [P > L] + [V > L] + [V > P]$. Define $F_1 = \overline{F}_1|\,\mathrm{St}(E)\colon \mathrm{St}(E) \to E$ as the restriction to $\mathrm{St}(E)$ of the linear map $\overline{F}_1\colon \mathbf{F}_2[2] \to E$ with values

$$\overline{F}_1[V > P > L] = \begin{cases} L & P \cap P^{\perp} = 0, V \cap V^{\perp} = \mathbf{F}_2 e_3 \\ 0 & \text{otherwise} \end{cases} \tag{6.6}$$

on the basis elements. Define $F_2 = \overline{F}_2|\,\mathrm{St}(E)\colon \mathrm{St}(E) \to E$ similarly but replace the condition $V \cap V^{\perp} = \mathbf{F}_2 e_3$ by $V \cap V^{\perp} = \mathbf{F}_2 e_4$. The linear maps $F_1$ and $F_2$ are $\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & 1\end{pmatrix}$-equivariant because this group preserves the symplectic inner product on $E$ and preserves $V^{\perp} = \mathbf{F}_2\langle e_3, e_4\rangle$ pointwise.

6.7. **Lemma.** *Let $f_1$ and $F_1, F_2$ be the linear maps defined above.*

(1) *The vector $f_1$ is a basis vector for*

$$\mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & 1\end{pmatrix}}(\mathrm{St}(V), V) \cong \mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & 1\end{pmatrix}}(\mathrm{St}(V), V/V^{\perp}) \cong \mathbf{F}_2$$

(2) *The set $\{F_1, F_2\}$ is a basis for*

$$\mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & 1\end{pmatrix}}(\mathrm{St}(E), E) \cong \mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & 1\end{pmatrix}}(\mathrm{St}(E), E/E^{\perp}) \cong \mathbf{F}_2^2$$

*The sum $F_1 + F_2$ is the linear map defined as in (6.6) but with condition $V \cap V^{\perp} = \mathbf{F}_2 e_3$ replaced by $V \cap V^{\perp} = \mathbf{F}_2(e_3 + e_4)$.*

*Proof.* This can be directly verified by machine computation. □

6.8. **Proposition.** *The differentials in the cochain complex 5.14 are given as follows:*

(1) *Let $H$ be the $(2, 0)$ object and $V$ a $(3, 1)$ object of $\mathbf{A}(\mathrm{PGL}(2m, \mathbf{C}))$. The $V$-component of the coboundary map*

$$\delta^1_V\colon \mathrm{Hom}_{\mathrm{Sp}(2)}(\mathrm{St}(H), H) \to \mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & 1\end{pmatrix}}(\mathrm{St}(V), V)$$

*is an isomorphism of 1-dimensional $\mathbf{F}_2$-vector spaces.*

(2) *Let $V$ be the $(4, 2)$ object of $\mathbf{A}(\mathrm{PGL}(2m, \mathbf{C}))$ corresponding (5.12, 5.7) to the two dimensional toral object $(1, i-1, m-i, 0)$ of $\mathbf{A}(\mathrm{PGL}(m, \mathbf{C}))$, $1 < i \le m/2$, $m \ge 4$. Then $\delta^2_E(x_i) = (x_1 + x_i)F_1 + (x_1 + x_{i-1})F_2$ where*

$$\delta^2_E\colon \prod\nolimits_{1\le i\le m/2} \mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & 1\end{pmatrix}}(\mathrm{St}(V_i), V_i) \to \mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & 1\end{pmatrix}}(\mathrm{St}(E), E)$$

*is the $E$-component of the coboundary map and $(x_i) \in \prod_{1\le i\le m/2} \mathrm{Hom}_{\mathrm{Sp}(3,1)}(\mathrm{St}(V), V)$.*

*Proof.* 1. The non-zero vector in $\mathrm{Hom}_{\mathrm{Sp}(2)}(\mathrm{St}(H), H)$ is the restriction to $\mathrm{St}(H) \subseteq \mathbf{F}_2^3$ of the linear map $\mathbf{F}_2[0] = \mathbf{F}_2^3 \to H$ that takes a basis vector $[L]$ in $\mathbf{F}_2^3$ to $L \in H$. In the composition

$$\mathrm{St}(V) \to \bigoplus_{V>P} \mathrm{St}(P) \to \bigoplus_{V>P} P \xrightarrow{+} V$$

the middle maps $\mathrm{St}(P) \to P$ equal the map just described if $P < V$ is non-toral, $P \cap P^{\perp} = 0$, and are trivial if $P < V$ is toral, $P \cap P^{\perp} = P$. This is precisely the map $f_1$.

2. For any non-toral three dimensional subspace $V$ of $E$ we have either

- $V \cap V^{\perp} = \mathbf{F}_2 e_3$, and then $V = V_i$, or,
- $V \cap V^{\perp} = \mathbf{F}_2 e_4$, and then $V = V_{i-1}$, or,
- $V \cap V^{\perp} = \mathbf{F}_2(e_3 + e_4)$, and then $V = V_1$,

and thus the composite linear map

$$\mathrm{St}(E) \to \bigoplus_{E>V} \mathrm{St}(V) \xrightarrow{\bigoplus x_i} \bigoplus V \xrightarrow{+} E$$

equals $x_iF_1 + x_{i-1}F_2 + x_1(F_1+F_2) = (x_1+x_i)F_1 + (x_{i-1}+x_1)F_2$. □

*Proof of Lemma* 6.5. Since we already know that these higher limits vanish when $n+1$ is odd (5.4) we can assume that $n+1 = 2m$ is even.

1. In Oliver's cochain complex 5.14, the coboundary map $\delta^1$ is injective and $\ker\delta^2$ is 1-dimensional by 6.8 when $m \geq 4$. See 6.3 for the case $m = 1$. For $m = 2$ and $m = 3$, the cochain complexes 5.14 reduce to

$$0 \to \mathrm{Hom}_{\mathrm{Sp}(2)}(\mathrm{St}(H), H) \xrightarrow{\delta^1} \mathrm{Hom}_{\mathrm{Sp}(3,1)}(\mathrm{St}(V), V/V^\perp) \to 0$$

$$0 \to \mathrm{Hom}_{\mathrm{Sp}(2)}(\mathrm{St}(H), H) \xrightarrow{\delta^1} \mathrm{Hom}_{\mathrm{Sp}(3,1)}(\mathrm{St}(V), V/V^\perp) \to \mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0 \\ * & \mathrm{GL}(E^\perp)\end{pmatrix}}(\mathrm{St}(E), E/E^\perp) = 0$$

with two non-trivial groups, both 1-dimensional $\mathbf{F}_2$-vector spaces, and with just one differential $\delta^1$ which is an isomorphism (6.8). Thus the higher limits vanish in these cases as well.

2. Oliver's cochain complex for computing these higher limits over $\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))$ involve the $\mathbf{Z}_2$-modules

$$\mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0 \\ * & \mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(E^\perp\end{pmatrix}}(\mathrm{St}(E), \pi_2(BZC_{\mathrm{PGL}(2m,\mathbf{C})}(E))), \quad \dim_{\mathbf{F}_2} E = 3, 4,$$

that are submodules of finite products of $\mathbf{Z}_2$-modules of the form

$$\mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0 \\ * & 1\end{pmatrix}}(\mathrm{St}(E), \mathbf{Z}_2), \quad \dim_{\mathbf{F}_2} E = 3, 4,$$

where the action on $\mathbf{Z}_2$ is trivial. According to the computer program magma, these latter modules are trivial. □

*Proof of Theorem* 1.1. By induction over $n$ using 3.3 and 4.6. The start of the induction is provided by 6.3. Use (2.7) to compute the automorphism group. □

*Proof of Corollary* 1.2. The connected 2-compact group $\mathrm{GL}(n,\mathbf{C})$ is uniquely $N$-determined because (3.2, 4.3) its adjoint form $\mathrm{PGL}(n,\mathbf{C})$ is (1.1). Since the maximal torus normalizer for $\mathrm{GL}(n,\mathbf{C})$ is a split extension, we get (2.7) that $\mathrm{Aut}(\mathrm{GL}(n,\mathbf{C}))$ is isomorphic to $Z(\Sigma_n)\backslash\mathrm{Aut}_{\mathbf{Z}_2\Sigma_n}(\mathbf{Z}_2^n)$. □

This finishes the discussion of the 2-compact groups in the $A$-family. The relevance of these are that they occur as centralizers of elementary abelian subgroups of many other 2-compact groups. Here is a result illustrating this.

6.9. **Theorem.** [34, 1.3] *The simple* 2*-compact group* $\mathrm{G}_2$ *is uniquely* $N$*-determined and its automorphism group* $\mathrm{Aut}(\mathrm{G}_2)$ *equals* $\mathbf{Z}^\times\backslash\mathbf{Z}_2^\times \times C_2$.

*Proof.* The Quillen category $\mathbf{A}(\mathrm{G}_2)$ is equivalent to the category $\mathbf{A}(\mathrm{GL}(V), V)$ of all non-trivial subspaces of $V = \mathbf{F}_2^3$ [12, 6.1] [10, 1.6] [9, 5.3] and the value of centralizer functor $BC_{\mathrm{G}_2}$ on the three isomorphism classes of objects $L$, $P$, $V$ is $\mathrm{SL}(4,\mathbf{R})$, $T \rtimes \mathbf{Z}/2$, $V$. The rank one centralizer, $\mathrm{SL}(4,\mathbf{R}) = \mathrm{SL}(2,\mathbf{C})\circ\mathrm{SL}(2,\mathbf{C})$, is uniquely $N$-determined (6.3, 3.2, 3.4, 4.3, 4.4). Condition 4.6.(2) is satisfied because $H^1(W(X); \check{T}(X)) = 0$ for $X = \mathrm{G}_2$, $\mathrm{SL}(4,\mathbf{R})$ [13], 4.6.(1) and 4.6.(3) because the only rank two object in $\mathrm{G}_2$ is toral and its centralizer is a 2-compact toral group. The functor $\pi_1(BZC_{\mathrm{G}_2})$ is the identity functor and $\pi_2(BZC_{\mathrm{G}_2})$ the zero functor so the obstruction groups vanish. Now 3.3 and 4.6 show that $\mathrm{G}_2$ is uniquely $N$-determined. The short exact sequence (2.7) can be used to calculate the automorphism group. We have $\mathrm{Aut}(\mathrm{G}_2) = W(\mathrm{G}_2)\backslash N_{GL(2,Z_2)}(W(\mathrm{G}_2))$ as the extension class $e(\mathrm{G}_2) = 0$ [3]. Using the description of the root system from [2, VI.4.13] with short root $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and long root $\alpha_2 = 2\varepsilon - \varepsilon_2 - \varepsilon_3$ generating the integral lattice in $\mathbf{Z}_2^3$ one finds that

$$N_{GL(2,\mathbf{Z}_2)}(W(\mathrm{G}_2)) = \left\langle \mathbf{Z}_2^\times, A, W(\mathrm{G}_2)\right\rangle, \quad A = \sqrt{-3}\begin{pmatrix}0 & 3\\ 1 & 0\end{pmatrix}$$

and therefore $\mathrm{Aut}(\mathrm{G}_2) = \mathbf{Z}_2^\times/\mathbf{Z}^\times \times C_2$ where the cyclic group of order two is generated by the exotic automorphism $A$ interchanging the two roots. □

## 7. Miscellaneous

This section contains auxiliary results that are used at various places in the main argument of this paper.

7.1. **The 2-compact toral groups O(2) and Pin(2).** Let $\mathbf{H} = \{a + bj | a, b \in \mathbf{C}\}$, where $j^2 = -1$ and $ja = \overline{a}j$ for $a \in \mathbf{C}$, be the quaternion algebra. The normalizer of $\mathbf{C}^\times$ in $\mathbf{H}^\times$ is the Lie group $N_{\mathbf{H}^\times}(\mathbf{C}^\times) = \langle \mathbf{C}^\times, j\rangle$ generated by the multiplicative Lie group $\mathbf{C}^\times$ and $j$. The short exact sequence

$$1 \to \mathbf{C}^\times \to N_{\mathbf{H}^\times}(\mathbf{C}^\times) \to \langle j\rangle \,/\, \langle -1\rangle \to 1$$

does not split for all elements of $j\mathbf{C}^\times$ have order 4. Its discrete approximation $\check{\mathrm{Pin}}(2) = \check{N}_{\mathbf{H}^\times}(\mathbf{C}^\times) = \langle \mathbf{Z}/2^\infty, j\rangle \subseteq \langle \mathbf{C}^\times, j\rangle \subseteq \mathbf{H}^\times$, the non-split extension

$$1 \to \mathbf{Z}/2^\infty \to \check{N}_{\mathbf{H}^\times}(\mathbf{C}^\times) \to \mathbf{Z}/2 \to 1$$

of $\mathbf{Z}/2$ by $\mathbf{Z}/2^\infty$, is the discrete approximation to 2-compact toral group Pin(2). The semi-direct product $\check{\mathrm{O}}(2) = \mathbf{Z}/2^\infty \rtimes \mathbf{Z}/2$ is the discrete approximation to the 2-compact toral group O(2) or to $\mathrm{GL}(2, \mathbf{R})$.

7.2. **Type $A_n$, $n \geq 1$.** (Cf. [20, 19, 13]) The discrete maximal torus normalizer for the center-less 2-compact group $\mathrm{PGL}(n+1, \mathbf{C}) = \mathrm{GL}(n+1, \mathbf{C})/\mathbf{C}^\times$ is the extended 2-discrete toral group

$$\check{N}(\mathrm{PGL}(n+1,\mathbf{C})) = \check{\mathrm{U}}(1)^{n+1}/\check{\mathrm{U}}(1) \rtimes \Sigma_{n+1} = \check{T} \rtimes \Sigma_{n+1}$$

where $\check{U}(1) = \mathbf{Z}/2^\infty$ is a discrete 2-torus of rank 1. In the coefficient sequence for $\check{U}(1) \to \check{U}(1)^{n+1} \to \check{T}$ we have $H^*(\Sigma_{n+1}; \check{U}(1)^{n+1}) \cong H^*(\Sigma_n; \check{U}(1))$ by Shapiro so that

$$H^i(W; \check{T}) \cong \ker\left(H^{i+1}(\Sigma_{n+1}; \mathbf{Z}/2^\infty) \xrightarrow{\mathrm{res}} H^{i+1}(\Sigma_n; \mathbf{Z}/2^\infty)\right)$$

is trivial for $n + 1 > 2(i + 1)$ by [30, 5.8, 6.7]. For small values of $i$ we have

$$H^0(W; \check{T}) = \begin{cases} 0 & n \neq 1 \\ \mathbf{Z}/2 & n = 1 \end{cases} \quad \text{and} \quad H^1(W; \check{T}) = \begin{cases} 0 & n \neq 3 \\ \mathbf{Z}/2 & n = 3 \end{cases}$$

as can be seen by using that the Schur multiplier $H_2(\Sigma_n; \mathbf{Z})$ is of order 2 for $n \geq 4$ and trivial for $1 \leq n \leq 3$ [16, V.25.12]. Thus the center $ZN(\mathrm{PGL}(n+1, \mathbf{C}))$ of the maximal torus normalizer is trivial for $n > 1$ but cyclic of order 2 for $n = 1$. For $n = 3$, the crossed homomorphism $\Sigma_4 \to \check{\mathrm{U}}(1)^4/\check{\mathrm{U}}(1)$ whose values on the three generators $(12), (23), (34) \in \Sigma_4$ [16, I.19.7] are the columns of the matrix

$$\begin{pmatrix} -1 & +1 & +1 \\ -1 & -1 & -1 \\ +1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

is not principal.

7.3. **Type $B_n$, $n \geq 2$.** (Cf. [20, 19, 13]) The discrete maximal torus normalizer for the center-less 2-compact group $\mathrm{SL}(2n+1, \mathbf{R})$ is the extended 2-discrete torus

$$\check{N}(\mathrm{SL}(2n+1,\mathbf{R})) = \check{\mathrm{O}}(2) \wr \Sigma_n = (\mathbf{Z}/2^\infty \rtimes \mathbf{Z}/2) \wr \Sigma_n = (\mathbf{Z}/2^\infty)^n \rtimes (\mathbf{Z}/2 \wr \Sigma_n)$$

where $\mathbf{Z}/2$ acts on $\mathbf{Z}/2^\infty$ by sign. There is an isomorphism

$$H^1(\mathbf{Z}/2 \wr \Sigma_n; (\mathbf{Z}/2^\infty)^n) \cong \mathrm{Hom}(\Sigma_{n-1}, \mathbf{Z}/2) \oplus \mathrm{Hom}(\Sigma_n, \mathbf{Z}/2) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$$

that to the pair $(v, \chi) \in \mathbf{Z}/2 \oplus \mathrm{Hom}(\Sigma_n, \mathbf{Z}/2)$ associates the derivation $D(v, \chi)$ given by

$$D(v, \chi)(\varepsilon_i, \sigma) = (v + \chi(\sigma), \ldots, v + \chi(\sigma), \chi(\sigma), v + \chi(\sigma), \ldots, v + \chi(\sigma))$$

where $\varepsilon_i$ is the $i$th canonical basis vector for $(\mathbf{Z}/2)^n$ and $\chi(\sigma)$ is in the $i$th coordinate. To see this, use the exact sequence from the Lyndon–Hochschild–Serre spectral sequence

$$0 \to H^1(\Sigma_n; (\mathbf{Z}/2)^n) \to H^1(\mathbf{Z}/2 \wr \Sigma_n; (\mathbf{Z}/2^\infty)^n) \to H^1((\mathbf{Z}/2)^n; (\mathbf{Z}/2^\infty)^n)^{\Sigma_n}$$

where $H^1(\Sigma_n; (\mathbf{Z}/2)^n) \cong \mathbf{Z}/2$ $(n \geq 3)$ and also the third term is of order 2 as, in general,

$$H^*(G^n; M^n) = H^*(G^n; M)^n = H^*(G^{n-1}; H^*(G; M)) = \cdots = H^*(G; \ldots; H^*(G; M)\cdots)^n$$

for a group $G$ and a $G$-module $M$. This gives

$$H^0(W; \check{T}) = \mathbf{Z}/2, \qquad H^1(W; \check{T}) = \begin{cases} \mathbf{Z}/2 & n = 2 \\ (\mathbf{Z}/2)^2 & n \geq 3 \end{cases}$$

in this case. The computation of $H^0(W; \check{T})$ uses that the center of the maximal torus normalizer

$$Z(\check{N}(\mathrm{SL}(2n+1, \mathbf{R}))) = Z(\check{\mathrm{O}}(2) \wr \Sigma_n) = Z\check{\mathrm{O}}(2) = \mathbf{Z}/2$$

is cyclic of order two for all $n \geq 2$ (whereas $\check{Z}\mathrm{SL}(2n+1, \mathbf{R}) = 0$).

7.4. **The center of a semi-direct product.** Let $G \rtimes \Sigma$ be the semi-direct product for the action $\Sigma \to \mathrm{Aut}(G)$ of the group $\Sigma$ on the group $G$. Let $G^\Sigma = \{g \in G | \Sigma g = g\}$ and $\Sigma_G = \{\sigma \in \Sigma | \sigma(g) = g \text{ for all } g \in G\}$.

7.5. **Lemma.** *The center* $Z(G \rtimes \Sigma) = G^\Sigma \times_{\mathrm{Aut}(G)} Z(\Sigma)$ *of* $G \rtimes \Sigma$ *is the pull-back*

$$\begin{tikzcd} Z(G \rtimes \Sigma) \arrow[r] \arrow[d] & Z(\Sigma) \arrow[d] \\ G^\Sigma \arrow[r] & \mathrm{Aut}(G) \end{tikzcd}$$

*of the action map restricted to the center of* $\Sigma$ *along the map* $G^\Sigma \to \mathrm{Aut}(G)$ *given by inner automorphisms.*

*Proof.* Suppose that $(g, \sigma) \in G \times \Sigma$ is in the center of $G \rtimes \Sigma$. Since

$$(g, \sigma) \cdot (1, \tau) = (g, \sigma\tau) = (1, \tau) \cdot (g, \sigma) = (\tau(g), \tau\sigma)$$

for all $\tau \in \Sigma$, $g$ is fixed by $\Sigma$ and $\sigma$ is central in $\Sigma$. Moreover, from

$$(g, \sigma) \cdot (h, 1) = (g \cdot \sigma(h), \sigma) = (h, 1) \cdot (g, \sigma) = (hg, \sigma)$$

we see that $\sigma(h) = h^g$ for all $h \in G$. □

7.6. **Corollary.** *If* $G$ *is abelian,* $Z(G \rtimes \Sigma) = G^\Sigma \times Z(\Sigma)_G$ *is a direct product.*

*Proof.* The bottom horizontal homomorphism $G^\Sigma \to \mathrm{Aut}(G)$ is trivial. □

7.7. **Corollary.** *Let* $G$ *be a group and* $Z \neq G$ *a central subgroup. Let the cyclic group* $C_p$ *of prime order* $p$ *act on* $G^p/Z$ *by cyclic permutation. Then*

$$Z(G)/Z \times \{z \in Z | z^p = 1\} \cong Z(G^p/Z \rtimes C_p)$$

*via the isomorphism that takes the element* $z \in Z$ *of order* $p$ *to* $(1, z, \ldots, z^{p-1})Z \in G^p/Z$ *and is the diagonal on* $Z(G)/Z$.

*Proof.* Observe that

$$G/Z \times \{z \in Z | z^p = 1\} \xrightarrow{\cong} \left(G^p/Z\right)^{C_p}$$

via the isomorphism that takes $(gZ, z)$ to $g(1, z, \ldots, z^{p-1})Z$. To see this, consider an element $(g_1, \ldots, g_p)Z$ which is fixed by $C_p$. Then $(g_1, g_2, \ldots, g_p)Z = (g_p, g_1, \ldots, g_{p-1})Z$ so there exists an element $z \in Z$ so that $g_2 = g_1 z, g_3 = g_2 z = g_1 z^2, \ldots, g_p = g_1 z^{p-1}, g_1 = g_1 z^p$. Therefore, $z^p = 1$ and $(g_1, g_2, \ldots, g_p) = g_1(1, z, \ldots, z^{p-1})$.

Thus $Z(G^p/Z \rtimes C_p)$ is the pull back of the group homomorphisms

$$G/Z \times \{z \in Z | z^p = 1\} \xrightarrow{\varphi} \mathrm{Aut}(G^p/Z) \leftarrow C_p$$

where $\varphi(gZ, z)((g_1, \ldots, g_p)Z) = (g_1^g, \ldots, g_p^g)Z$. Let $((gZ, z), \sigma)$ be an element of the pull back. Assume that $\sigma$ is non-trivial. Since $p$ is a prime number, $\sigma$ has no fixed points. The equation

$$\forall g_1, \ldots, g_p \in G\colon (g_1^g, \ldots, g_p^g)Z = (g_{\sigma(1)}, \ldots, g_{\sigma(p)})Z$$

shows that $g_1^g Z = g_{\sigma(1)}Z$. This is impossible unless $\sigma$ is the identity since otherwise we can find a $g_1 \in Z$ and a $g_{\sigma(1)} \notin Z$. Thus the permutation $\sigma$ must be the identity. The requirement for $((gZ, z), 1)$ to be in the pull back is that

$$\forall (g_1, \dots, g_p) \in G^p \exists u \in Z \colon (g_1^g, g_2^g, \dots, g_p^g) = (g_1 u, g_2 u, \dots, g_p u)$$

which implies that $[g_1, g] = u = [g_2, g]$ for all $g_1, g_2 \in G$. If we take $g_1 = 1$ to be the identity, we see that $g$ must be central. $\square$

7.8. **Action in Lie case.** Let $\nu \colon V \to G$ be a monomorphism of a non-trivial elementary abelian $p$-group to a compact Lie group $G$. There is a canonical map $\mathrm{B}C_G(\nu(V)) \to \mathrm{map}(\mathrm{B}V, \mathrm{B}G)_{\mathrm{B}\nu}$ from the classifying space of the Lie theoretic centralizer of $\nu(V)$ to the mapping space component containing $\mathrm{B}\nu$. Write $c_g$ for conjugation with $g \in G$.

7.9. **Lemma.** *Suppose that $\nu\alpha = c_g \nu$ for some element $g \in G$ and some automorphism $\alpha \in \mathrm{GL}(V)$. Then conjugation by $g$ takes $C_G(\nu(V))$ to $C_G(c_g\nu(V)) = C_G(\nu\alpha(V)) = C_G(\nu(V))$ and the diagram*

$$\begin{tikzcd}
\mathrm{B}C_G(\nu(V)) \arrow[r] & \mathrm{map}(\mathrm{B}V, \mathrm{B}G)_{\mathrm{B}\nu} \arrow[d, "(\mathrm{B}\alpha)^*", "\cong"'] \\
\mathrm{B}C_G(\nu(V)) \arrow[u, "\mathrm{B}c_g", "\cong"'] \arrow[r] & \mathrm{map}(\mathrm{B}V, \mathrm{B}G)_{\mathrm{B}\nu}
\end{tikzcd}$$

*is homotopy commutative.*

*Proof.* The commutative diagram of Lie group morphisms

$$\begin{tikzcd}
V \times C_G(\nu(V)) \arrow[r, "\nu\times 1"] \arrow[d, "\alpha\times c_g"'] & \nu(V) \times C_G(\nu(V)) \arrow[r, "\mathrm{mult}"] & G \arrow[d, equal] \\
V \times C_G(\nu(V)) \arrow[r, "\nu\times 1"'] & \nu(V) \times C_G(\nu(V)) \arrow[r, "\mathrm{mult}"'] & G
\end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd}
\mathrm{B}V \times \mathrm{B}C_G(\nu(V)) \arrow[r, "\mathrm{B}(\mathrm{mult}\circ(\nu\times 1))"] \arrow[d, "\mathrm{B}\alpha\times \mathrm{B}c_g"'] & \mathrm{B}G \arrow[d, equal] \\
\mathrm{B}V \times \mathrm{B}C_G(\nu(V)) \arrow[r, "\mathrm{B}(\mathrm{mult}\circ(\nu\times 1))"] & \mathrm{B}G
\end{tikzcd}$$

of classifying spaces. Taking adjoints, we obtain the homotopy commutative diagram

$$\begin{tikzcd}
\mathrm{B}C_G(\nu(V)) \arrow[r] & \mathrm{map}(\mathrm{B}V, \mathrm{B}G)_{\mathrm{B}\nu} \arrow[d, "(\mathrm{B}\alpha)^*"] \\
\mathrm{B}C_G(\nu(V)) \arrow[u, "\mathrm{B}c_g"] \arrow[r] & \mathrm{map}(\mathrm{B}V, \mathrm{B}G)_{\mathrm{B}\nu}
\end{tikzcd}$$

as claimed. $\square$

7.10. **Corollary.** *Suppose that $\mu \colon V \to N(G)$ is a monomorphism and that $\mu\alpha = c_n\mu$ for some $\alpha \in \mathrm{GL}(V)$ and $n \in N(G)$. Then*

$$w^{-1} = \pi_2((\mathrm{B}\alpha)^*) \colon \pi_2(\mathrm{B}T(G))^{\pi_0(\mu)(V)} \to \pi_2(\mathrm{B}T(G))^{\pi_0(\mu)(V)}$$

*where $w \in W(G)$ is the image of $n \in N(G)$.*

*Proof.* There is a commutative diagram

$$\begin{tikzcd}
\pi_2(\mathrm{B}T) \arrow[r, equal] & \pi_2(\mathrm{B}N(G)) & \pi_2(\mathrm{B}C_{N(G)}(V, \mu)) \arrow[l, hook'] \arrow[r, "\cong"] & \pi_2(\mathrm{map}(\mathrm{B}V, \mathrm{B}N), \mathrm{B}\mu) \arrow[d, "\pi_2((\mathrm{B}\alpha)^*)"] \\
\pi_2(\mathrm{B}T) \arrow[u, "w"] \arrow[r, equal] & \pi_2(\mathrm{B}N(G)) \arrow[u, "\pi_2(\mathrm{B}c_n)"'] & \pi_2(\mathrm{B}C_{N(G)}(V, \mu)) \arrow[u, "\pi_2(\mathrm{B}c_n)"'] \arrow[l, hook'] \arrow[r, "\cong"] & \pi_2(\mathrm{map}(\mathrm{B}V, \mathrm{B}N), \mathrm{B}\mu)
\end{tikzcd}$$

where $\pi_2(\mathrm{B}C_{N(G)}(V,\mu)) = \pi_2(\mathrm{B}T(G))^{\pi_0(\mu)(V)}$ denotes the fixed point group for the group action $\pi_0(\mu)\colon V \to W(G) \subseteq \mathrm{Aut}(\pi_2(\mathrm{B}T(G)))$. Since $\mathrm{B}c_n\colon \mathrm{B}N \to \mathrm{B}N$ is freely homotopic to the identity along the loop $w \in \pi_1(\mathrm{B}N)$ its effect on the $\mathbf{Z}_p[\pi_1(\mathrm{B}N)]$-module $\pi_2(\mathrm{B}N)$ is multiplication by $w$. $\square$

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