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COUNTING SPECIAL MONOMIALS

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Abstract. Let $\rho : G \hookrightarrow \mathsf{GL}(n, \mathbb{F})$ be a permutation representation of a finite group G. In this paper we study the number of orbits of special monomials of G acting on the polynomials in n variables via ρ. We give formulae for several crucial families of groups, for direct sums of representations, as well as for vector invariants. In addition we give two algorithms for arbitrary permutation groups, one relying on the geometry of G acting on $V = \mathbb{F}^n$, the other relying on the representation theory of the symmetric groups.

1. Introduction

Let $\rho : G \longrightarrow \mathsf{GL}(n, \mathbb{F})$ be a faithful permutation representation of a finite group G over a field $\mathbb{F}$. The representation ρ induces a natural action of G on the vector space $V = \mathbb{F}^n$ of dimension n, thus on the dual space V^*, and hence on the ring of polynomial functions $\mathbb{F}[V]$. Our interest is focused on the subring of invariants

$$\mathbb{F}[V]^G = \{f \in \mathbb{F}[V] | gf = f \ \forall g \in G\},$$

which is a finitely generated $\mathbb{F}$-algebra by a classic result due to E. Noether, see Theorem 2.1.4 in [4].

Let $x_1, \dots, x_n$ be the basis of V^* being permuted by the G-action. Let $\mathbf{x}^I = x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{F}[V]$ be a monomial. We order the exponent sequence I in non-increasing order and summarize this in its **associated ordered partition** $\lambda(I) = (\lambda_1(I) \geq \dots \geq \lambda_n(I))$. Then $\mathbf{x}^I$ is called **special** if

(1) $\lambda_n(I) = 0$ and
(2) $\lambda_i(I) - \lambda_{i+1}(I) \leq 1$, for $i = 1, \dots, n-1$.

It is known that the ring of invariants $\mathbb{F}[V]^G$ of a permutation representation is generated by the orbit sums of special monomials and the top elementary symmetric function $s_n = x_1 \cdots x_n$, see Theorem 3.4.2 in [4]. The maximal degree of an algebra generator of $\mathbb{F}[V]^G$ in a minimal generating set is called the **Noether number**. It is denoted by $\beta(\mathbb{F}[V]^G)$. By the above mentioned theorem we find that

$$\beta(\mathbb{F}[V]^G) \leq \max \left\{ \binom{n}{2}, n \right\}$$

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for any permutation group G, see Corollary 3.4.3 in [4]. This degree bound is sharp as the defining representation of the alternating group A_n shows, see Example 1 in Section 3.4 in [4].

Degree bounds are an ongoing theme in the Invariant Theory of Finite Groups with many results for large families of group and representations, see [3] for a survey in these matters. In contrast, not much is known about the **number of algebra generators** of a ring of polynomial invariants, i.e., the **embedding degree** of the algebra $\mathbb{F}[V]^G$ denoted by $\mathrm{emb}-\deg(\mathbb{F}[V]^G)$.

In this paper we want to study the embedding degree of $\mathbb{F}[V]^G$ for permutation representations by counting the number $\mathrm{N}_{G,V}$ of G-orbits of special monomials. This gives then an upper bound on the embedding degree. We note that the trivial monomial $1 = x_1^0 \cdots x_n^0$ is a special monomial.

Proposition 1.1. *Let $\rho : G \hookrightarrow \mathsf{GL}(n, \mathbb{F})$ be a faithful permutation representation of a finite group G. Let $H \leq G$ be an arbitrary subgroup. Then the embedding degree is bounded above by the number of orbits of special monomials under the action of H:*

$$\mathrm{emb}-\deg(\mathbb{F}[V]^G) \leq \mathrm{N}_{G,V} \leq \mathrm{N}_{H,V} \leq \mathrm{N}_{\{e\},V}.$$

Proof. Obvious. □

Even though it is not expected that $\mathrm{N}_{G,V}$ ever gives a sharp upper bound on the embedding degree, the number of special G-orbits has the advantage of being independent of the ground field. (This follows because the Poincaré series of $\mathbb{F}[V]^G$ is independent of the choice of the ground field $\mathbb{F}$ whenever G acts by permutations, see Proposition 3.2.2 in [4].) In contrast, the embedding degree depends on the choice of $\mathbb{F}$ as the 3-fold regular representation of $\mathbb{Z}/2$ shows: The embedding degree of the associated ring of invariants is 9, unless the characteristic of $\mathbb{F}$ is even. In that case the embedding degree is 10. Furthermore, the study of $\mathrm{N}_{G,V}$ is interesting in its own right as this number has intriguing combinatorial properties as we will see in the following.

2. SPECIAL MONOMIALS IN $\mathbb{F}[V]$

We start by looking at the trivial group, i.e., we count the number $\mathrm{N}_{\{e\},V}$ of special monomials in the ambient polynomial ring $\mathbb{F}[V]$ itself thus obtaining a universal upper bound for $\mathrm{N}_{G,V}$ for an n-dimensional permutation representation of any group G.

For this denote by $P(n,k)$ the number of distinct maps from a set with n elements onto a set with k elements. We note that

$$P(n,k) = kP(n-1,k) + kP(n-1,k-1) = k!S(n,k) = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n,$$

where $S(n,k)$ is the Stirling number of second kind.

Proposition 2.1. *The number of special monomials in $\mathbb{F}[V]$ is given by*

$$\mathrm{N}_{\{e\},V} = \sum_{k=1}^{n} k!S(n,k) = \sum_{k=1}^{n} P(n,k).$$

Proof. Let $\mathcal{S} = \{x_1, \ldots, x_n\}$ and let $\mathcal{T} = \{0, \ldots, k-1\}$. Then $P(n,k)$ counts the number of special monomials $\mathbf{x}^I$ with k different exponents. □

We will denote $S(n) = \sum_{k=1}^{n} k!S(n,k)$. Note that $S(n)$ counts the total number of ordered partitions of a set with n elements which coincides with the number of surjective maps with a domain with n elements. One may compute $S(n)$ recursively by

$$S(1) = 1$$

$$S(n) = 1 + \sum_{i=1}^{n-1} \binom{n}{i} S(n-i),$$

see Page 146 [6]. The first few values of $S(n)$ beginning at $n = 1$ are

$$1, 3, 13, 75, 541, 4683, 545835, 7087261, \ldots.$$

This is sequence A000670 in [5].

Remark 2.2. In Lemma 2 of [1] we find a different formula for the number of special monomials in $\mathbb{F}[V]$, namely

$$\mathrm{N}_{\{e\},V} = \frac{\mathrm{d}^n}{\mathrm{d}x^n} \frac{1}{2 - \mathrm{e}^x}\bigg|_{x=0}.$$

Recall (from, e.g., Page 34 of [6]) that a generating function for the Stirling numbers of second kind is given by

$$\sum_{n \geq k} S(n,k) \frac{x^n}{n!} = \frac{1}{k!}(\mathrm{e}^x - 1)^k$$

for $k \in \mathbb{N}_0$. Thus

$$k!S(n,k) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^x - 1)^k\bigg|_{x=0}.$$

Hence we recover the generating function for $S(n)$, see Page 146 [6],

$$S(n) = \sum_{k=1}^{n} k!S(n,k) = \sum_{k=1}^{n} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^x - 1)^k\bigg|_{x=0} = \frac{\mathrm{d}^n}{\mathrm{d}x^n} \frac{1}{2 - \mathrm{e}^x}\bigg|_{x=0}.$$

Remark 2.3. We note that by Proposition 1.1 we obtain a universal bound on the embedding degree valid for any permutation group:

$$\mathrm{emb} - \deg(\mathbb{F}[V]^G) \leq \mathrm{N}_{G,V} \leq \mathrm{N}_{\{e\},V} = \sum_{k=1}^{n} P(n,k) = S(n).$$

However, this bound is never sharp: Assume it were for some group $G \leq \Sigma_n$. Then every special monomial would be an invariant. Thus the basis elements $x_1, \ldots, x_n$ are invariant. Therefore $G = \{e\}$ is the trivial group. Its ring of invariants is $\mathbb{F}[V]$ which is generated by the n forms $x_1, \ldots, x_n$ and thus has embedding dimension n.

3. Special Orbits in $\mathbb{F}[V]^G$

In this section we will derive a formula for the number of special G-orbits for an arbitrary permutation group G that makes use of the geometry of the G-action on the dual space V^*. We illustrate the result with the calculation of the number of special Σ_n-orbits.

Let $\rho : G \hookrightarrow \mathsf{GL}(n, \mathbb{F})$ be a faithful permutation representation of a finite group G. Then $\rho(G) \leq \Sigma_n$ where we identified the symmetric group with its image in the general linear group under the defining representation.

Proposition 3.1. *Denote by $\lambda_{G,V}(k)$ the number of elements $g \in G$ such that the dimension of the fixed field V^g is exactly k:*

$$\lambda_{G,V}(k) = |\{g \in G| \dim_{\mathbb{F}} V^g = k\}|.$$

Then

$$\mathrm{N}_{G,V} = \frac{1}{|G|} \sum_{k=1}^{n} \lambda_{G,V}(k) S(k).$$

Proof. For any two integers $a, b,\ a \leq b$, we set $[a, b] = \{a, a+1, a+2, \dots, b\} \subseteq \mathbb{Z}$. For $1 \leq k \leq n$ we let

$$\mathcal{P}_{n,k} = \{\psi : [1, n] \longrightarrow [0, k-1] | \psi \text{ is surjective}\}$$

be the set of surjective maps from $[1, n]$ to $[0, k-1]$. Thus $|\mathcal{P}_{n,k}| = P(n, k)$. Set $\mathcal{P}_n = \cup_{k=1}^{n} \mathcal{P}_{n,k}$. Our group G acts on V^* by permuting the variables $x_1, \dots, x_n$

$$g(x_1^{i_1} \cdots x_n^{i_n}) = x_{g(1)}^{i_1} \cdots x_{g(n)}^{i_n} = x_1^{i_{g^{-1}(1)}} \cdots x_n^{i_{g^{-1}(n)}}.$$

Thus we might as well consider G as a group permuting the set $[1, n]$. We define a G-action on the set $\mathcal{P}_n$ by

$$G \times \mathcal{P}_n \longrightarrow \mathcal{P}_n,\ (g, \psi) \mapsto \psi \circ g^{-1}.$$

Furthermore, the set $\mathcal{P}_n$ is in one-to-one correspondence with the set of special monomials in $\mathbb{F}[x_1, \dots, x_n]$ by

$$\mathcal{P}_n \longrightarrow \{\mathbf{m} = x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{F}[V] | \mathbf{m} \text{ special}\}, \psi \mapsto x_1^{\psi(1)} x_2^{\psi(2)} \cdots x_n^{\psi(n)}.$$

By construction this correspondence is G-equivariant, i.e., the diagram

$$\begin{tikzcd}
\psi \arrow[r] \arrow[d] & x_1^{\psi(1)} x_2^{\psi(2)} \cdots x_n^{\psi(n)} \arrow[d] \\
\psi \circ g^{-1} \arrow[r] & x_1^{\psi(g^{-1}(1))} x_2^{\psi(g^{-1}(2))} \cdots x_n^{\psi(g^{-1}(n))}
\end{tikzcd}$$

commutes. Thus the number of special G-orbits $\mathrm{N}_{G,V}$ is equal to the number of orbits in $\mathcal{P}_n$ under the G-action. For $g \in G$ let $\mathrm{Fix}(g)$ denote the number of fixed points of g acting on $\mathcal{P}_n$

$$\mathrm{Fix}(g) = |\{\psi \in \mathcal{P}_n | \psi \circ g^{-1} = \psi\}|.$$

Then by Burnside's Lemma

$$\mathrm{N}_{G,V} = \frac{1}{|G|} \sum_{g \in G} \mathrm{Fix}(g).$$

For a fixed $g \in G$, notice that $\psi \circ g^{-1} = \psi$ if and only if

$$(\star) \qquad \psi(g^{-1}m) = \psi(m)$$

for all $m \in [1, n]$. The number of independent equations in the System $(\star)$ is precisely the number of cycles in the disjoint cycle representation of g (considered as a permutation of $[1, n]$ and counting 1-cycles). So if $\mathrm{c}(g)$ is the number of such cycles, it follows that

$$\mathrm{Fix}(g) = \sum_{j=1}^{\mathrm{c}(g)} P(\mathrm{c}(g), j) = \sum_{j=1}^{\mathrm{c}(g)} j! S(\mathrm{c}(g), j) = S(\mathrm{c}(g)).$$

Set $\lambda_{G,V}(k) = |\{g \in G | \mathrm{c}(g) = k\}|$ to be the number of elements $g \in G$ which have exactly k disjoint cycles. Then

$$\mathrm{N}_{G,V} = \frac{1}{|G|} \sum_{g \in G} \mathrm{Fix}(g) = \frac{1}{|G|} \sum_{k=1}^{n} \lambda_{G,V}(k) S(k).$$

Note that $\lambda_{G,V}(k)$ counts the number of elements $g \in G$ such that the dimension of the fixed field V^g is exactly k. □

Example 3.2. We consider the defining representation of the symmetric group Σ_n. An element $\sigma \in \Sigma_n$ has cycle type

$$\mathrm{t}(\sigma) = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$$

if there are m_i cycles of length i in its decomposition into disjoint cycles. Thus such an element has a fixed field V^σ of dimension $m_1 + \cdots + m_n$. Furthermore, the conjugacy class of σ has

$$|C_{\Sigma_n}(\sigma)| = \frac{n!}{1^{m_1} m_1! 2^{m_2} m_2! \cdots n^{m_n} m_n!}$$

elements. Note that $1m_1 + 2m_2 + \cdots + nm_n = n$ by definition. Combining these formulas gives

$$\begin{aligned}\mathrm{N}_{\Sigma_n,V} &= \frac{1}{n!} \sum_{k=1}^{n} \left(\sum_{\sigma_{\mathrm{t}}} |C_{\Sigma_n}(\sigma_{\mathrm{t}})| \right) S(k) \\ &= \frac{1}{n!} \sum_{k=1}^{n} \left(\sum_{\substack{1m_1+2m_2+\cdots+nm_n=n \\ m_1+\cdots+m_n=k}} \frac{n!}{1^{m_1} m_1! 2^{m_2} m_2! \cdots n^{m_n} m_n!} \right) S(k).\end{aligned}$$

Indeed, the number of special Σ_n-orbit was already calculated by Göbel in Lemma 1 of [1]: The number of special Σ_n-invariant orbits is just 2^{n-1}. This coincides with the number of *descending* special monomials $x_1^{i_1} \cdots x_n^{i_n}$, $i_1 \geq i_2 \geq \cdots \geq i_n$, since every Σ_n-orbit contains exactly one descending special monomial. Thus our formula above is just equal to 2^{n-1}.

Remark 3.3. We note that

$$\mathrm{N}_{G,V} \geq \mathrm{N}_{\Sigma_n,V},$$

i.e., $\mathrm{N}_{\Sigma_n,V}$ is a universal lower bound for the number of special G-orbits.

4. Direct Products and Finite Abelian Groups

In this section we will illustrate Proposition 3.1 with a few examples of representations of (mainly abelian) groups. Moreover, we will develop a formula for the number of special orbits in direct sums of representations.

We start with an example.

Example 4.1 (Regular Representation of $\mathbb{Z}/4$)**.** Consider the cyclic group $G = \mathbb{Z}/4$ of order 4 generated by the cycle $(1234) \in \Sigma_4$. Thus our group consists of the elements $(1), (1234), (13)(24)$, and (1432). Remembering that we must be careful to count 1-cycles, we see that $\mathbb{Z}/4$ has one element with four disjoint cycles, one element with two disjoint cycles, and two elements with only one cycle. Thus

$$\begin{aligned}\mathrm{N}_{\mathbb{Z}/4,V} &= \frac{1}{4}(\lambda_{G,V}(1)S(1) + \lambda_{G,V}(2)S(2) + \lambda_{G,V}(4)S(4))\\ &= \frac{1}{4}(2S(1) + 1S(1) + 1S(4))\\ &= \frac{1}{4}(2 + 3 + 75) = 20.\end{aligned}$$

More generally, we consider the regular representation for an arbitrary finite cyclic group.

Example 4.2 (Regular Representation of $\mathbb{Z}/n$)**.** Consider the cyclic group of order n, denoted by $\mathbb{Z}/n$, generated by the cycle $(123\cdots n) \in \Sigma_n$. Since a permutation $(123\cdots n)^j$ has precisely $\gcd(n,j)$ disjoint cycles, we find that

$$\lambda_{\mathbb{Z}/n,V}(k) = \sum_{\substack{1\le j\le n\\ \gcd(n,j)=k}} 1 = \begin{cases}\phi(n/k), & \text{if } k|n\\ 0, & \text{otherwise.}\end{cases}$$

Thus $\mathrm{N}_{\mathbb{Z}/n,V} = \frac{1}{n}\sum \phi(n/d)S(d)$. The first few values, beginning at $n = 1$ are

$$1, 2, 5, 20, 109, 784, 6757, 68240, 787477, 10224812, \ldots.$$

This is sequence A019536 in [5], which counts the number of necklaces of n beads with up to n unlabeled colors.

We want to develop a formula for sums of representations. We start with two easy examples.

Example 4.3 (Elementary Abelian 2-Groups)**.** Consider the Klein-Four Group $K_4 \subseteq \Sigma_4$ consisting of the elements $(1), (12), (34), (12)(34)$. Then

$$\mathrm{N}_{K_4,V} = \frac{1}{4}(S(2) + 2S(3) + S(4)) = 26.$$

More generally, for an elementary abelian 2-group $(\mathbb{Z}/2)^m \subseteq \Sigma_{2m}$ generated by the transpositions $(2i-1, 2i)$, $i = 1, \ldots, m$, we find

$$\lambda_{(\mathbb{Z}/2)^m,V}(k) = \begin{cases}\binom{m}{k-m}, & \text{if } m \le k \le 2m,\\ 0, & \text{otherwise.}\end{cases}$$

Hence

$$\mathrm{N}_{(\mathbb{Z}/2)^m,V} = \frac{1}{2^m}\sum_{k=m}^{2m}\binom{m}{k-m}S(k).$$

The first few terms of this sequence from $n = 1$ are

$$2, 26, 818, 47834, 4488722, 617364026, 117029670578, \ldots.$$

This is sequence A059516 in [5]. It counts the number of ways n intervals of length at least zero can overlap.

Example 4.4 (Elementary Abelian p-Groups)**.** Let p be a prime number. Consider an elementary abelian p-group $(\mathbb{Z}/p)^m \subseteq \Sigma_{pm}$ generated by the cycles $(p(i-1)+1, \ldots, pi)$ for $i = 1, \ldots, m$. If $m = 1$ we find by the calculations in Example 4.2:

$$\mathrm{N}_{\mathbb{Z}/p,V} = \frac{1}{p}(\phi(p)S(1) + \phi(1)S(p)).$$

For $m = 2$ we have

$$\mathrm{N}_{\mathbb{Z}/p\times\mathbb{Z}/p,V} = \frac{1}{p^2}(1\cdot S(2p) + (p-1)2S(p+1) + (p-1)^2 S(2)).$$

In general we find

$$\mathrm{N}_{(\mathbb{Z}/p)^m,V} = \frac{1}{p^m}\sum_{k=0}^{m}(p-1)^k\binom{m}{k}S(mp-k(p-1)).$$

The first few terms of this sequence for $p = 3$ from $m = 1$ are
5, 555, 273245, 357071595, 969618315005, 4732462989457035, 37693215885154484765, ...

For $p = 5$ from $m = 1$ we have
109, 4091403, 1842421318717, 4284436559065961835, 34143676145867329865644189, ...

In general, we obtain the following result for direct sums of representations.

Proposition 4.5. *Let* $\rho : G \hookrightarrow \mathsf{GL}(n,\mathbb{F})$ *and* $\rho' : H \hookrightarrow \mathsf{GL}(m,\mathbb{F})$ *be permutation representations of finite groups* G *and* H*. Denote by* $V = \mathbb{F}^n$ *and* $W = \mathbb{F}^m$*. Then*

$$\mathrm{N}_{G\times H,V\times W} = \frac{1}{|G||H|}\sum_{k=1}^{n+m}\left(\sum_{l=1}^{k-1}\lambda_{G,V}(l)\lambda_{H,W}(k-l)\right)S(k).$$

Proof. We note that

$$\lambda_{G\times H,V\oplus W}(k) = \sum_{l=1}^{k-1}\lambda_{G,V}(l)\lambda_{H,W}(k-l)$$

and thus we are done by Proposition 3.1. □

We illustrate this with the computation for arbitrary finite abelian groups.

Example 4.6 (Finite Abelian Groups)**.** Let $G = \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_m \leq \Sigma_{n_1+\cdots+n_m}$ be generated by the m disjoint cycles

$$(1,\dots,n_1),(n_1+1,\dots,n_1+n_2),\ \dots,(n_1+\cdots+n_{m-1}+1,\dots,n_1+n_2+\cdots+n_m).$$

Let $g = (g_1,\dots,g_m) \in \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_m$. Let V_i be the corresponding subspace of dimension n_i. Then

$$\dim_{\mathbb{F}} V^g = \dim_{\mathbb{F}} V_1^{g_1} + \cdots + \dim_{\mathbb{F}} V_m^{g_m}.$$

We obtain by the preceding proposition

$$\lambda_{G,V}(k) = \sum_{l_1+\cdots+l_m=k}\lambda_{\mathbb{Z}/n_1,V_1}(l_1)\cdots\lambda_{\mathbb{Z}/n_m,V_m}(l_m).$$

$\mathrm{N}_{\mathbb{Z}/n,V} = \frac{1}{n}\sum\phi(n/d)S(d)$. Thus

$$\begin{aligned}\mathrm{N}_{G,V} &= \frac{1}{n_1\cdots n_m}\sum_k\sum_{l_1+\cdots+l_m=k}\phi(n_1/l_1)\cdots\phi(n_m/l_m)S(k)\\ &= \frac{1}{n_1\cdots n_m}\sum_{l_1|n_1}\cdots\sum_{l_m|n_m}\phi(n_1/l_1)\cdots\phi(n_m/l_m)S(l_1+\cdots+l_m).\end{aligned}$$

The following tables gives some values of $N_{G,V}$ for $G = \mathbb{Z}/n_1 \times \mathbb{Z}/n_2$.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$\mathbb{Z}/1\times\mathbb{Z}/n$	3	8	27	140	939
$\mathbb{Z}/2\times\mathbb{Z}/n$	8	26	108	668	5204
$\mathbb{Z}/3\times\mathbb{Z}/n$	27	108	555	4092	37035
$\mathbb{Z}/4\times\mathbb{Z}/n$	140	668	4092	34844	357308
$\mathbb{Z}/5\times\mathbb{Z}/n$	939	5204	37035	357308	4091403
$\mathbb{Z}/6\times\mathbb{Z}/n$	7900	49496	399332	4289444	54115732
$\mathbb{Z}/7\times\mathbb{Z}/n$	77979	545228	4920939	58243388	802679403
$\mathbb{Z}/8\times\mathbb{Z}/n$	885980	6833768	68202372	881517284	13172347508

We close this section with a nonabelian example.

Example 4.7 (The Dihedral Group D_{2n} of Order $2n$)**.** For $n \geq 3$ let D_{2n} denote the dihedral group of order $2n$. It has a natural permutation representation of dimension n as D_{2n} can be considered as the permutation group of the vertices of a regular n-gon. As such it is generated by a rotation $(1, 2, \dots, n)$ and a reflection $(1, n)(2, n-1)\cdots(\frac{n}{2}, \frac{n+2}{2})$ for n even, resp. $(1, n)(2, n-1)\cdots(\frac{n-1}{2}, \frac{n+1}{2})$ for odd n. See Example 2 in Section 5.8 of [4] for the special case of $n = 5$. We can partition D_{2n} as $D_{2n} = \mathbb{Z}/n \sqcup R_n$ where R_n consists of elements of order 2. Having already deduced the necessary information about the disjoint cycle representations of elements of $\mathbb{Z}/n$ in Example 4.2, it remain only to do the same for elements in R_n. But since these are elements of order 2, the required information can be deduced by counting the number of fixed points of each elements of R_n, which will depend on the parity of n. The end result is

$$\mathrm{N}_{D_{2n},V} = \begin{cases} \frac{1}{2}\mathrm{N}_{\mathbb{Z}/n,V} + \frac{1}{2}S(\frac{n+1}{2}), & \text{for } n \text{ odd,} \\ \frac{1}{2}\mathrm{N}_{\mathbb{Z}/n,V} + \frac{1}{4}S(\frac{n}{2}) + \frac{1}{4}S(\frac{n+2}{2}), & \text{for } n \text{ even} \end{cases}$$

with $\mathrm{N}_{\mathbb{Z}/n,V}$ from the previous example. The first few terms, from $n = 1$ are

$$1, 2, 4, 14, 61, 414, 3416, 34274, 394009, 5113712, \dots.$$

This is Sequence A019537 in [5].

5. Regular Representations and Vector Invariants

If G acts diagonally on m copies of V then

$$\lambda_{G,V^{\oplus m}}(k) = \lambda_{G,V}\left(\frac{k}{m}\right)$$

We want to find the function $\lambda_{G,V}$ for the regular representation of an arbitrary finite group G, $\mathbb{F}G$. If $g \in G$, then $\langle g\rangle \leq G$ is a cyclic subgroup. Thus $\mathbb{F}G$ restricted to $\langle g\rangle$ gives the $|G : \langle g\rangle|$-fold regular representation of $\langle g\rangle$. Hence

$$\lambda_{G,V}(k) = \left|\left\{g \in G \mid |g| = \frac{|G|}{k}\right\}\right|.$$

Thus

$$\mathrm{N}_{G,V} = \frac{1}{|G|}\sum_{k||G|}\lambda_{G,V}(k)S(k) = \frac{1}{|G|}\sum_{k||G|}\left|\left\{g \in G||g| = \frac{|G|}{k}\right\}\right| S(k)$$

for the regular representation. Thus for the m-fold regular representation we get

$$\begin{aligned}\mathrm{N}_{G,V} &= \frac{1}{|G|}\sum_{k=1}^{|G|m} \lambda_{G,V^{\oplus m}}(k)S(k)\\ &= \frac{1}{|G|}\sum_{k=1}^{|G|m} \lambda_{G,V}\left(\frac{k}{m}\right)S(k)\\ &= \frac{1}{|G|}\sum_{l=1}^{|G|} \lambda_{G,V}(l)S(ml).\end{aligned}$$

More generally we find for arbitrary vector invariants:

Proposition 5.1. *Let $\rho : G \hookrightarrow \mathsf{GL}(n,\mathbb{F})$ be a faithful representation of a finite group G. Let $V = \mathbb{F}^n$, and set $\rho_1 = \rho$. Then the m-fold vector invariants of ρ are given by the diagonal action of G on m copies of V. We find*

$$\mathrm{N}_{G,V^{\oplus m}} = \frac{1}{|G|}\sum_{k=1}^{|G|} \lambda_{G,V}(k)S(mk).$$

Proof. This is a straightforward generalization of the case of vector invariants of the regular representation which we discussed above. $\square$

6. Special Orbits in $\mathbb{F}[V]^G$: an Algorithm

In this section we show how to use the results from the preceding sections to create another algorithm that counts the number of orbits of special monomials for an arbitrary permutation group which relies on the representation theory of the symmetric group.

Let $\rho : \Sigma_n \hookrightarrow \mathsf{GL}(n,\mathbb{F})$ be the defining representation of Σ_n. Let $V = \mathbb{F}^n$. Then we define the function

$$\mu : \Sigma_n \longrightarrow \mathbb{N},\ g \mapsto \dim V^g.$$

Then for any subgroup $G \leq \Sigma_n$ we obtain

$$\lambda_{G,V}(k) = |\mu^{-1}(k) \cap G|.$$

Lemma 6.1. *Note that this implies*

$$|G|\mathrm{N}_{G,V} \leq \mathrm{N}_{\Sigma_n,V}|\Sigma_n|,$$

since λ is nonnegative. Thus we obtain a relative bound for the number of special orbits

$$\mathrm{N}_{G,V} \leq \mathrm{N}_{\Sigma_n,V}|\Sigma_n : G| = \frac{2^{n-1}n!}{|G|}.$$

Proof. We simply compute

$$|G|\mathrm{N}_{G,V} = \sum_{k=1}^{n} \lambda_{G,V}(k)S(k) \leq \sum_{k=1}^{n} \lambda_{\Sigma_n,V}(k)S(k) = |\Sigma_n|\mathrm{N}_{\Sigma_n,V}.$$

$\square$

We observe that μ is a class function. Even more, μ is constant on the sets of elements in Σ_n consisting of the same *number* of disjoint cycles. Thus we can write

$$\mathrm{N}_{G,V} = \frac{1}{|G|}\sum_g |C_g(G)| S(\mathrm{c}(g)) = \frac{1}{|G|}\sum_g |C_g(\Sigma_n)\cap G| S(\mathrm{c}(g))$$

where the sum runs over a set of representatives of the conjugacy classes, $C_g(G)$ is the conjugacy class of $g \in G$, and $\mathrm{c}(g)$ is the number of disjoint cycles in g. The disadvantage of this formula is that it requires precise knowledge about the embedding $G \hookrightarrow \Sigma_n$, because one needs to find the values of $|C_g(\Sigma_n)\cap G|$. Therefore, we propose the following:

Proposition 6.2.

$$\mathrm{N}_{G,V} = \frac{1}{|G|}\sum_{g\in G} S(\dim V^g) = \frac{1}{|G|}\sum_{g\in G} S(\mu(g)).$$

Proof. Follows immediately from Proposition 3.1. □

Since our calculations are characteristic free, we might as well assume we are working over the complex numbers, thus we can write the class function μ as a linear combination of the characters of the full symmetric group. Let $p_i(n)$ be the number of partitions of n into i nonnegative integers. Then the total number of conjugacy classes in Σ_n is equal

$$C = \sum_{l=1}^{n} p_l(n)$$

(i.e., the total number of partitions into nonnegative integers). Thus the character table of Σ_n is a $C \times C$ complex invertible matrix, call it M. Thus if we write the function μ as a linear combination of the irreducible characters χ_i of Σ_n

$$\mu = a_1\chi_1 + \cdots + a_C\chi_C$$

we find

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_C \end{bmatrix} = \mathsf{M}^{-1} \begin{bmatrix} n \\ n-1 \\ \vdots \\ 1 \end{bmatrix}$$

where in the column vector on the right each i appears $p_i(n)$ times. Thus once we have the character table of Σ_n, we can compute the function μ and hence $\mathrm{N}_{G,V}$ for any subgroup $G \leq \Sigma_n$. We will study this in more detail in a succeeding paper, [2]. For now, we want to illustrate this process with the calculations for $n = 2, \ldots, 6$.

Example 6.3 (Σ_2)**.** The following table contains in character table of Σ_2 and, in the last row, the values of the function μ on the conjugacy classes:

	(1)	(12)
χ_1	1	1
χ_2	1	-1
μ	2	1

We obtain the two linear equations

$$a_1 + a_2 = 2 \quad \text{and} \quad a_1 - a_2 = 1.$$

Thus $a_1 = 3/2$ and $a_2 = 1/2$.

Example 6.4 (Σ_3)**.** The following table contains in character table of Σ_3 and, in the last row, the values of the function μ on the conjugacy classes:

	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0
μ	3	2	1

Direct computation yields
$a_1 = 5/3$, $a_2 = 2/3$, and $a_3 = 1/3$.

Example 6.5 (Σ_4)**.** The following table contains in character table of Σ_4 and, in the last row, the values of the function μ on the conjugacy classes:

	(1)	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	-1	1
μ	4	3	2	2	1

Direct computation yields

$$a_1 = 25/12, \quad a_2 = 1/12, \quad a_3 = 1/6, \quad a_4 = 3/4, \quad \text{and} \quad a_5 = -1/4.$$

Example 6.6 (Σ_5)**.** The following table contains in character table of Σ_5 and, in the last row, the values of the function μ on the conjugacy classes:

	(1)	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1
χ_3	4	2	1	0	0	-1	-1
χ_4	4	-2	1	0	0	1	-1
χ_5	6	0	0	-2	0	0	1
χ_6	5	1	-1	1	-1	1	0
χ_7	5	-1	-1	1	1	-1	0
μ	5	4	3	3	2	2	1

Direct computation yields

$$a_1 = 137/60, \quad a_2 = -1/20, \quad a_3 = 4/5, \quad a_4 = 2/15,$$
$$a_5 = -3/10, \quad a_6 = 1/4, \quad \text{and} \quad a_7 = -1/12.$$

Example 6.7 (Σ_6)**.** The following table contains in character table of Σ_6 and, in the last row, the values of the function μ on the conjugacy classes:

	(1)	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)	(12)(34)(56)	(123)(456)	(1234)(56)	(123456)
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1	-1	1	1	-1
χ_3	5	3	2	1	1	0	0	-1	-1	-1	-1
χ_4	5	-3	2	1	-1	0	0	1	-1	-1	1
χ_5	10	2	1	-2	0	-1	0	-2	1	0	1
χ_6	10	-2	1	-2	0	1	0	2	1	0	-1
χ_7	9	3	0	1	-1	0	-1	3	0	1	0
χ_8	9	-3	0	1	1	0	-1	-3	0	1	0
χ_9	5	1	-1	1	-1	1	0	3	2	-1	0
χ_{10}	5	-1	-1	1	1	-1	0	-3	2	-1	0
χ_{11}	16	0	-2	0	0	0	1	0	-2	0	0
μ	6	5	4	4	3	3	2	3	2	2	1

Direct computation yields

$$\begin{aligned} & a_1 = 289/120, \quad a_2 = 3/40, \quad a_3 = 7/8, \quad a_4 = -1/8, \quad a_5 = -1/4, \\ & a_6 = 1/12, \quad a_7 = 7/40, \quad a_8 = 7/40, \quad a_9 = 5/24, \quad a_{10} = -1/8, \\ & \text{and} \quad a_{11} = -2/15. \end{aligned}$$

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