HEISENBERG GROUPS AND ALGEBRAIC TOPOLOGY

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ABSTRACT. We study the Madsen-Tillmann spectrum $\mathbb{C}P^{\infty}_{-1}$ as a quotient of the Mahowald pro-object $\mathbb{C}P^{\infty}_{-\infty}$, which is closely related to the Tate cohomology of circle actions. That theory has an associated symplectic structure, whose symmetries define the Virasoro operations on the cohomology of moduli space constructed by Kontsevich and Witten.

1. INTRODUCTION

A sphere S^n maps essentially to a sphere S^k only if $n \geq k$, and since we usually think of spaces as constructed by attaching cells, it follows that algebraic topology is in some natural sense upper-triangular, and thus not very self-dual: as in the category of modules over the mod p group ring of a p-group, its objects are built by iterated extensions from a small list of simple ones.

Representation theorists find semi-simple categories more congenial, and for related reasons, physicists are happiest in Hilbert space. This paper is concerned with some remarkable properties of the cohomology of the moduli space of Riemann surfaces discovered by physicists studying two-dimensional topological gravity (an enormous elaboration of conformal field theory), which appear at first sight quite unfamiliar. Our argument is that these new phenomena are forced by the physicists' interest in self-dual constructions, which leads to objects which are (from the point of view of classical algebraic topology) very large [1 §2].

Fortunately, equivariant homotopy theory provides us with tools to manage these constructions. The first section below is a geometric introduction to the Tate cohomology of the circle group; the conclusion is that it possesses an intrinsic symplectic module structure, which pairs positive and negative dimensions in a way very useful for applications. Section two studies operations on this (not quite cohomology) functor, and exhibits the action of an algebraic analog of the Virasoro group on it. The third section relates rational Tate cohomology of the circle to that of the infinite loopspace $Q\mathbb{C}P^{\infty}_{+}$ considered by Madsen and Tillmann in recent work on Mumford's conjecture.

I owe thanks to many people for help with the ideas in this paper, but it is essentially a collage of a lifetime's conversations with Graeme Segal, who more or less adopted me when we were both very young.

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2. Geometric Tate cohomology

2.1 Let G be a compact Lie group of dimension d. We will be concerned with a cobordism category of smooth compact G-manifolds, with the action free on the boundary: this can be regarded as a categorical cofiber for the forgetful functor from manifolds with free G-action to manifolds with unrestricted action. Under reasonable assumptions this cofiber category is closed under Cartesian products (given the diagonal action).

If E is a geometric cycle theory (eg stable homotopy, or classical homology) then the graded E-bordism group of free G-manifolds is isomorphic to $E_{*+d}(BG_+)$. On the other hand, the homotopy quotient of a G-manifold is a bundle of manifolds over the classifying space BG, and Quillen's conventions [22] associate to such a thing, a class in the graded cobordism group $E^{-*}(BG_+)$. The forgetful functor from free to unrestricted G-manifolds defines a long exact sequence

$$\cdots \to E_{*-d}(BG_+) \to E^{-*}(BG_+) \to t_G^{-*}(E) \to E_{*-d-1}(BG_+) \to \dots$$

which interprets the relative groups as the (Tate-Swan [28]) E-bordism of manifolds with G-action free on the boundary. The geometric boundary homomorphism

$$\partial_E : t_G^{-*}(E) \to E_{*-d-1}(BG_+) \to E_{*-d-1}$$

sends a manifold with G-free boundary to the quotient of that action on the boundary; it will be useful later.

Remarks:

1) $t_G(E)$ is a ring-spectrum if E is; in fact it is an E-algebra, at least in some naive sense;

2) tom Dieck stabilization [4] extends this geometric bordism theory to an equivariant theory;

3) the functor t_G sends cofibrations to cofibrations, but it lacks good limit properties: it is defined by a kind of hybrid of homology and cohomology, and Milnor's limit fails. In more modern terms [11], the construction sends a G-spectrum E to the equivariant function spectrum $[EG_+, E] \wedge \tilde{E}G_+$.

4) The eventual focus of §2 is the case when G is the circle group $\mathbb{T}$, and E is ordinary cohomology: this is closely related to cyclic cohomology [2], but I don't know enough about that subject to say anything useful.

2.2 Suppose now that E is a general complex-oriented ring-spectrum; then $E^*(B\mathbb{T}_+)$ is a formal power series ring generated by the Euler (or first Chern) class $\mathbf{e}$. If E_* is concentrated in even degrees, then the cofiber sequence above reduces for dimensional reasons to a short exact sequence

$$0 \to E^*(B\mathbb{T}_+) = E^*[[\mathbf{e}]] \to t_{\mathbb{T}}^* E = E^*((\mathbf{e})) \to E_{-*-2}(B\mathbb{T}_+) \to 0$$

with middle group the ring of formal Laurent series in $\mathbf{e}$. By a lemma of [12 §2.4] we can think of $t_{\mathbb{T}}^* E$ as the homotopy groups of a pro-object

$$S^2\mathbb{C}P^{\infty}_{-\infty} \wedge E := \{S^2 \mathrm{Th}(-k\eta) \wedge E \mid k > -\infty\}$$

in the category of spectra, constructed from the filtered vector bundle

$$\dots (k-1)\eta \subset k\eta \subset (k+1)\eta \subset \dots$$

defined by sums of copies of the tautological line bundle over $\mathbb{C}P^\infty \cong B\mathbb{T}$, as discussed in the appendix to [6] (see also [18]). More precisely: the Thom spectrum can be taken to be

$$\mathrm{Th}(-k\eta) := \lim_n S^{-2(n+1)k}\mathbb{C}P_n^{k\eta^\perp} ,$$

where $\eta^\perp$ is the orthogonal complement to the canonical line η in $\mathbb{C}^{n+1}$.]

When E is **not** complex-orientable, $t_\mathbb{T}E$ can behave very differently: the Segal conjecture for Lie groups implies that, up to a profinite completion [10],

$$t_\mathbb{T}S^0 \sim S^0 \vee S[\prod B\mathbb{T}/C]$$

where the product runs through proper subgroups C of $\mathbb{T}$ (and S denotes suspension).

In the universal complex-oriented case, the class $\mathbf{e}^{-1} \in t_\mathbb{T}^{-2}\mathrm{MU}$ is represented geometrically by the unit disk in $\mathbb{C}$ with the standard action of $\mathbb{T}$ as unit complex numbers; more generally, the unit ball in $\mathbb{C}^k$ represents $\mathbf{e}^{-k}$. The geometric boundary homomorphism sends that $\mathbb{T}$-manifold to $\mathbb{C}P_{k-1}$; this observation can be restated, using Mishchenko's logarithm, as the formula

$$\partial_E(f) = \mathrm{res}_{\mathbf{e}=0}\ f(\mathbf{e})\ d\log_{\mathrm{MU}}(\mathbf{e}) : t_\mathbb{T}^*MU \to \mathrm{MU}_{-*-2} ,$$

where the algebraic residue homomorphism

$$\mathrm{res}_{\mathbf{e}=0} : \mathrm{MU}^{*-2}((\mathbf{e})) \to \mathrm{MU}^*$$

is defined by $\mathrm{res}_{\mathbf{e}=0}\ \mathbf{e}^k\ d\mathbf{e} = \delta_{k+1,0}$, cf. [19, 21, 29].

2.3 The relative theory of manifolds with free group action on the boundary alone defines bordism groups $\tau_G^*(E)$ analogous to a truncation of Tate cohomology, with useful geometric applications. In place of the long exact sequence above, we have

$$\cdots \to E^{-*}(S^0) \to \tau_G^{-*}(E) \to E_{*-d-1}(BG_+) \to \dots$$

compatible with a natural transformation $t_G^*(E) \to \tau_G^*(E)$ which forgets the interior G-action. In our case (when E is complex-oriented), this is just the E-homology of the collapse map

$$\mathbb{C}P_{-\infty}^\infty \to \mathbb{C}P_{-1}^\infty := \mathrm{Th}(-\eta)$$

defined by the pro-spectrum in the previous paragraph.

A Riemann surface with geodesic boundary is in a natural way an orientable manifold with a free $\mathbb{T}$-action on its boundary, and a family of such things, parametrized by a space X, defines an element of

$$\tau_\mathbb{T}^{-2}\mathrm{MU}(X_+) \cong [X, \mathbb{C}P_{-1}^\infty \wedge \mathrm{MU}] .$$

The Hurewicz image of this element in ordinary cohomology is the homomorphism

$$H^*(\mathbb{C}P_{-1}^\infty, \mathbb{Z}) \to H^*(X_+, \mathbb{Z})$$

defined by the classifying map of Madsen and Tillmann, which will be considered in more detail below.

3. Automorphisms of classical Tate cohomology

3.1 There are profound analogies – and differences – among the Tate cohomology rings of the groups $\mathbb{Z}/2\mathbb{Z}, \mathbb{T}$, and $\mathrm{SU}(2)$ [3]. A property unique to the circle is the existence of the nontrivial involution $I : z \to z^{-1}$.

When E is complex oriented, the symmetric bilinear form

$$f, g \mapsto (f, g) = \partial_E(fg)$$

on the Laurent series ring $t_{\mathbb{T}}^* E$ is nondegenerate, and the involution on $\mathbb{T}$ defines a symplectic form

$$\{f, g\} = (I(f), g)$$

which restricts to zero on the subspace of elements of positive (or negative) degree. This Tate cohomology thus has an intrinsic inner product, with canonical polarization and involution.

This bilinear form extends to a generalized Kronecker pairing

$$t_{\mathbb{T}} E^*(X) \otimes_E t_{\mathbb{T}} E_*(X) \to E_{*-2}$$

which can be interpreted as a kind of Spanier-Whitehead duality between $t_{\mathbb{T}} E_*$, viewed as a pro-object as in §2.1, and the direct system $\{E^*(\mathrm{Th}(-k\eta)) \mid k > -\infty\}$ defined by the cohomology of that system. This colimit again defines a Laurent series ring, but this object is not quite its own dual: a shift of degree two intervenes, and it is most natural to think of the (non-existent) functional dual of $t_{\mathbb{T}} E$ as $S^{-2} t_{\mathbb{T}} E$. The residue map $t_{\mathbb{T}} E \to S^2 E$ can thus be understood as dual to the unit ring-morphism $E \to t_{\mathbb{T}} E$.

3.2 The Tate construction is too large to be conveniently represented, so the usual Hopf-algebraic approach to the study of its automorphisms is technically difficult. Fortunately, methods from the theory of Tannakian categories can be applied: we consider automorphisms of $t_{\mathbb{T}} E$ as E varies, and approximate the resulting group-valued functor by representable ones. There is no difficulty in carrying this out for a general complex-oriented theory E, but the result is a straightforward extension of the case of ordinary cohomology.

To start, it is clear that the group(scheme, representing the functor

$$A \mapsto \mathbb{G}_0(A) \;=\; \{g(x) = \sum_{k \geq 0} g_k x^{k+1} \in A[[x]] \mid g_0 = 1\}$$

on commutative rings A) of automorphisms of the formal line acts as multiplicative natural transformations of the cohomology-theory-valued functor $A \mapsto t_{\mathbb{T}}^* HA$, with $g \in \mathbb{G}_0(A)$ sending the Euler class $\mathbf{e}$ to $g(\mathbf{e})$. [I am treating these theories as graded by $\mathbb{Z}/2\mathbb{Z}$, with A concentrated in degree zero; but one can be more careful.]

Clearly $\mathbb{G}_0$ is represented by a polynomial Hopf algebra on generators g_k, with diagonal

$$(\Delta g)(x) = (g \otimes 1)((1 \otimes g)(x)) \; .$$

However, $\mathbb{G}_0$ is a subgroup of a larger system $\mathbb{G}$ of natural automorphisms, which is a colimit of representable functors (though not itself representable): following [16], let

$$A \mapsto \mathbb{G}(A) \;=\; \{g(x) = \sum_{k \gg -\infty} g_k x^{k+1} \in A((x)) \mid g_0 \in A^\times, g_k \in \sqrt{A} \text{ if } 0 > k\}$$

be the group of invertible **nil-Laurent** series, ie Laurent series with g_0 a unit, and g_k nilpotent for negative k. It is clear that $\mathbb{G}$ is a monoid, but in fact [20] it possesses inverses. The Lie algebra of $\mathbb{G}$ is spanned by the derivations $x^{k+1}\partial_x,\ k \in \mathbb{Z}$: it is the algebra of vector fields on the circle.

3.3 A related group-valued functor preserves the symplectic structure defined above: to describe it, I will specialize even further, and work over a field in whch two is invertible: $\mathbb{R}$, for convenience. Thus let $\check{\mathbb{G}}$ be the (ind-pro)-algebraic groupscheme defined by invertible nil-Laurent series over the field $\mathbb{R}((\sqrt{x}))$ obtained from $\mathbb{R}((x))$ by adjoining a formal square root of x, and let $\check{\mathbb{G}}_{\rm odd}$ denote the subgroup of **odd** invertible series $\check{g}(\sqrt{x}) = -\check{g}(-\sqrt{x})$. The homomorphism

$$\check{g} \mapsto g(x) := \check{g}(\sqrt{x})^2 : \check{\mathbb{G}}_{\rm odd} \to \mathbb{G}$$

is then a kind of double cover.

The functor $\check{\mathbb{G}}$ acts by symplectic automorphisms of the module $\mathbb{R}((\sqrt{x}))$, given the bilinear form

$$\langle u, v\rangle := \pi \ {\rm res}_{x=0}\ u(x)\ dv(x)$$

[27]; it is in fact a group of restricted symplectic automorphisms of this module. The Galois group of $\mathbb{R}((\sqrt{x}))/\mathbb{R}((x))$ defines a $\mathbb{Z}/2\mathbb{Z}$ - action, and the subgroup $\check{\mathbb{G}}_{\rm odd}$ preserves the subspace $\mathbb{R}((\sqrt{x}))_{odd}$ of odd power series.

Proposition The linear transformation

$$t^*_{\mathbb{T}} H\mathbb{R} \to \mathbb{R}((\sqrt{x}))_{odd}$$

defined on normalized basis elements by

$$\mathbf{e}^k \mapsto \gamma_{-k-\frac{1}{2}}(x)$$

(where $\gamma_s(x) = \Gamma(1+s)^{-1}x^s$ denotes a divided power), is a dense symplectic embedding.

Proof: We have

$$\{\mathbf{e}^k, \mathbf{e}^l\} = (-1)^k \ {\rm res}_{\mathbf{e}=0}\ \mathbf{e}^{k+l}\ d\mathbf{e} = (-1)^k \delta_{k+l+1,0}$$

while

$$\langle \gamma_s, \gamma_t\rangle = {\rm res}_{x=0}\ \gamma_s(x)\gamma_{t-1}(x)\ dx = \frac{\pi}{\Gamma(t)\Gamma(1+s)}\delta_{s+t,0}\ .$$

The assertion then follows from the duplication formula for the Gamma function.

The half-integral divided powers lie in $\mathbb{Q}((\sqrt{x}))$, aside from distracting powers of π. The remaining rational coefficients involve the characteristic 'odd' factorials of 2D topological gravity [8, 15], eg when k is positive,

$$\Gamma(k + \tfrac{1}{2}) = (2k-1)!!\ 2^{-k}\sqrt{\pi}\ .$$

4. Symmetries of the stable cohomology of the Riemann moduli space

The preceding sections describe the construction of a polarized symplectic structure on the Tate cohomology of the circle group. The algebra of symmetric functions on the Lagrangian submodule

$$H^*(\mathbb{C}P^\infty_+, \mathbb{Q}) \subset t_{\mathbb{T}} H\mathbb{Q}$$

of that cohomology can be identified with the homology of the infinite loopspace

$$Q\mathbb{C}P^\infty_+ = \lim_n \Omega^n S^n \ \mathbb{C}P^\infty_+ \ ;$$

on the other hand, this module of functions admits a canonical action of the Heisenberg group associated to its defining symplectic module [24 §9.5].

The point of this paper is that the homology of this infinite loopspace, considered in this way as a Fock representation, manifests the Virasoro representation constructed by Witten and Kontsevich on the stable cohomology of the moduli space of Riemann surfaces, identified with $H_*(Q\mathbb{C}P^\infty_+, \mathbb{Q})$ through work of Madsen, Tillman, and Weiss. Some of those results are summarized in the next two subsections; a more thorough account can be found in Michael Weiss's survey in these Proceedings. The third section below discusses their connection with representation theory.

4.1 Here is a very condensed account of one component of [17]: if $F \subset \mathbb{R}^n$ is a closed connected two-manifold embedded smoothly in a high-dimensional Euclidean space, its Pontrjagin-Thom construction $\mathbb{R}^n_+ \to F^\nu$ maps compactified Euclidean space to the Thom space of the normal bundle of the embedding. The tangent plane to F is classified by a map $\tau : F \to \mathrm{Grass}_{2,n}$ to the Grassmannian of oriented two-planes in $\mathbb{R}^n$, and the canonical two-plane bundle η over this space has a complementary $(n-2)$-plane bundle, which I will call $(n-\eta)$. The normal bundle ν is the pullback along τ of $(n-\eta)$; composing the map induced on Thom spaces with the collapse defines the map

$$\mathbb{R}^n_+ \to F^\nu \to \mathrm{Grass}_{2,n}^{(n-\eta)} \ .$$

The space $\mathrm{Emb}(F)$ of embeddings of F in $\mathbb{R}^n$ becomes highly connected as n increases, and the group $\mathrm{Diff}(F)$ of orientation-preserving diffeomorphisms of F acts freely on it, defining a compatible family

$$\mathbb{R}^n_+ \wedge_{\mathrm{Diff}} \mathrm{Emb}(F) \to \mathrm{Grass}_{2,n}^{(n-\eta)}$$

which can be interpreted as a morphism

$$B\mathrm{Diff}(F) \to \lim \ \Omega^n \mathrm{Grass}_{2,n}^{(n-\eta)} \ := \ \Omega^\infty \mathbb{C}P^\infty_{-1} \ .$$

This construction factors through a map

$$\coprod_{g \geq 0} B\mathrm{Diff}(F_g) \to \mathbb{Z} \times B\Gamma^+_\infty \to \Omega^\infty \mathbb{C}P^\infty_{-1}$$

of infinite loopspaces. Collapsing the bottom cell defines a cofibration

$$S^{-2} \to \mathbb{C}P^\infty_{-1} \to \mathbb{C}P^\infty_+$$

of spectra; the fiber of the corresponding map

$$\Omega^2 QS^0 \to \Omega^\infty \mathbb{C}P^\infty_{-1} \to Q\mathbb{C}P^\infty_+$$

of spaces has torsion homology, and the resulting composition

$$\mathbb{Z} \times B\Gamma_\infty^+ \to Q\mathbb{C}P_+^\infty \sim QS^0 \times Q\mathbb{C}P^\infty$$

is a rational homology isomorphism which identifies Mumford's polynomial algebra on classes κ_i, $i \geq 1$, with the symmetric algebra on positive powers of $\mathbf{e}$. The rational cohomology of QS^0 adds a copy of the group ring of $\mathbb{Z}$, which can be interpreted as a ring of Laurent series in a zeroth Mumford class κ_0.

The standard convention is to write b_k for the generators of $H_*\mathbb{C}P_+^\infty$ dual to $\mathbf{e}^k$, and to use the same symbols for their images in the symmetric algebra $H_*(Q\mathbb{C}P_+^\infty, \mathbb{Q})$. The Thom construction defines a map

$$\mathbb{C}P^\infty \to \mathrm{MU}$$

which extends to a ring isomorphism

$$H_*(Q\mathbb{C}P^\infty, \mathbb{Q}) \to H_*(\mathrm{MU}, \mathbb{Q}) \ ;$$

sending the b_k to classes usually denoted t_k, with $k \geq 1$; but it is convenient to extend this to allow $k = 0$.

4.2 The homomorphism

$$\lim \mathrm{MU}^{*+n-2}(\mathrm{Th}(n-\eta)) \to \mathrm{MU}^{*-2}(B\mathrm{Diff}(F))$$

defined on cobordism by the Madsen-Tillmann construction sends the Thom class to a kind of Euler class: according to Quillen, the Thom class of $n - \eta$ is its zero-section, regarded as a map between manifolds. Its image is the class defined by the fiber product

$$\begin{array}{ccc} Z_n & \longrightarrow & \mathrm{Grass}_{2,n} \\ \downarrow & & \downarrow \\ \mathbb{R}_+^n \wedge_{\mathrm{Diff}} \mathrm{Emb}(F) & \longrightarrow & \mathrm{Grass}_{2,n}^{(n-\eta)} \end{array} \ ;$$

this is the space of equivalence classes, under the action of $\mathrm{Diff}(F)$, of pairs (x, ϕ), with $x \in \phi(F) \subset \mathbb{R}^n$ a point of the surface (ie, in the zero-section of ν), and ϕ an embedding. Up to suspension, this image is thus the element

$$[Z_n \to \mathbb{R}_+^n \wedge_{\mathrm{Diff}} \mathrm{Emb}] \mapsto \mathrm{MU}^{n-2}(S^n B\mathrm{Diff}(F))$$

defined by the tautological family $F \times_{\mathrm{Diff}} E\mathrm{Diff}(F)$ of surfaces over the classifying space of the diffeomorphism group. It is primitive in the Hopf-like structure defined by gluing: in fact it is the image of

$$\sum_{k \geq 1} \kappa_k t_{k+1} \in \mathrm{MU}^{-2}(B\Gamma_\infty^+) \otimes \mathbb{Q} \ .$$

If v is a formal indeterminate of cohomological degree two, then the class

$$\Phi = \exp(v\mathrm{Th}(-\eta)) \in \mathrm{MU}_{\mathbb{Q}}^0(\Omega^\infty \mathbb{C}P_{-1}^\infty)[[v]]$$

defined by finite unordered configurations of points on the universal surface (with v a book-keeping indeterminate of cohomological degree two) is a kind of exponential transformation

$$\tilde{\Phi}_* : H_*(Q\mathbb{C}P_+^\infty, \mathbb{Q}) \to H_*(\mathrm{MU}, \mathbb{Q}[[v]]) \ .$$

From this perspective it is natural to interpret the Thom class in $\mathrm{MU}^{-2}(\mathbb{C}P^{\infty}_{-1}) \otimes \mathbb{Q}$ as the sum $\sum_{k \geq -1} t_{k+1} \mathbf{e}^k$, with $t_0 = v^{-1}\mathbf{e}$.

4.3 A class in the cohomology group

$$H^2_{\mathrm{Lie}}(V, \mathbb{R}) \cong \Lambda^2(V^*)$$

of a real vector space V defines a Heisenberg extension

$$0 \to \mathbb{T} \to H \to V \to 0 \ ;$$

the representation theory of such groups, and in particular the construction of their Fock representations, is classical [5]. What is important to us is that these are **projective** representations of V, with positive energy; such representations have very special properties.

The loop group of a circle is a key example; it possesses an intrinsic symplectic form, defined by formulae much like those of §2 [23 §5, §7b]. Diffeomorphisms of the circle act on any such loop group, and it is a deep property of positive-energy representations, that they extend to representations of the resulting semidirect product of the loop group by $\mathrm{Diff} S^1$. Therefore by restriction a positive-energy representation of a loop group automatically provides a representation of $\mathrm{Diff} S^1$. This [Segal-Sugawara [25 §13.4]] construction yields the action of Witten's Virasoro algebra on the Fock space

$$\mathrm{Symm}(H^*(\mathbb{C}P^{\infty}_{+})) \cong \mathbb{Q}[t_k \mid k \geq 0] \ .$$

In Kontsevich's model, the classes t_k are identified with the symmetric functions

$$\mathrm{Trace}\ \gamma_{k-\frac{1}{2}}(\Lambda^2) \sim -(2k-1)!!\ \mathrm{Trace}\ \Lambda^{-2k-1}$$

of a positive-definite Hermitian matrix Λ.

Note, however, that the deeper results of Kontsevich and Witten theory [31] are inaccessible in this toy model: that theory is formulated in terms of compactified moduli spaces $\overline{\mathcal{M}}_g$ of algebraic curves. The rational homology of $Q(\coprod \overline{\mathcal{M}}_g)$ (suitably interpreted, for small g) contains a fundamental class

$$\exp(\sum_{g \geq 0} [\overline{\mathcal{M}}_g] v^{3(g-1)})$$

for the moduli space of not-necessarily-connected curves. Witten's tau-function is the image of this 'highest-weight' vector under the analog of $\tilde{\Phi}$; it is killed by the subalgebra of Virasoro generated by the operators L_k with $k \geq -1$.

5. Concluding remarks

5.1 Witten has proposed a generalization of 2D topological gravity which encompasses surfaces with higher spin structures: for a closed smooth surface F an r-spin structure is roughly a complex line bundle L together with a fixed isomorphism $L^{\otimes r} \cong T_F$ of two-plane bundles, but for surfaces with nodes or marked points the necessary technicalities are formidable [14]. The group of automorphisms of such a structure is an extension of its group of diffeomorphisms by the group of rth roots of unity, and there is a natural analog of the group completion of the category defined by such surfaces. The generalized Madsen-Tillmann construction maps this loopspace to the Thom spectrum $\mathrm{Th}(-\eta^r)$, and it is reasonable to expect that this map is equivariant with respect to automorphisms of the group of roots of unity. This fits with some classical homotopy theory: if (for simplicity) $r = p$ is prime, multiplication by an integer u relatively prime to p in the H-space structure of $\mathbb{C}P^\infty$ defines a morphism

$$\mathrm{Th}(-\eta^p) \to \mathrm{Th}(-\eta^{up})$$

of spectra, and the classification of fiber-homotopy equivalences of vector bundles yields an equivalence of $\mathrm{Th}(-\eta^{up})$ with $\mathrm{Th}(-\eta^p)$ after p-completion. There is an analogous decomposition of $t_{\mathbb{T}} H\mathbb{Z}_p$ and a corresponding decomposition of the associated Fock representations [20 §2.4].

5.2 Tillmann has also studied categories of surfaces above a parameter space X; the resulting group completions have interesting connections with both Tate and quantum cohomology. When X is a compact smooth almost-complex manifold, its Hodge-deRham cohomology admits a natural action of the Lie algebra generated by the Hodge dimension operator H together with multiplication by the first Chern class (E) and its adjoin ($F = *E*$) [26]. Recently Givental [9 §8.1] has shown that earlier work of (the schools of) Eguchi, Dubrovin, and others can be reformulated in terms of structures on $t^{*,*}_{\mathbb{T}} H_{\mathrm{dg}}(X)$, given a symplectic structure generalizing that of §3. In this work, the relevant involution is

$$I_{\mathrm{Giv}} = \exp(\tfrac{1}{2}H)\exp(-E)\; I\; \exp(E)\exp(-\tfrac{1}{2}H)\ ;$$

it would be very interesting if this involution could be understood in terms of the equivariant geometry of the free loopspace of X [7].

5.3 Nothing forces us to restrict the construction of Madsen and Tillmann to two-manifolds, and I want to close with a remark about the cobordism category of smooth spin four-manifolds bounded by ordinary three-spheres. A parametrized family of such objects defines, as in §2.3, an element of the truncated equivariant cobordism group

$$\tau^{-4}_{\mathrm{SU}(2)}\mathrm{MSpin}(X_+)\ .$$

On the other hand, it is a basic fact of four-dimensional life that

$$\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)\ ,$$

so the Madsen-Tillmann spectrum for the cobordism category of such spin four-folds is the twisted desuspension

$$B\mathrm{Spin}(4)^{-\rho} = (\mathbb{H}P_\infty \times \mathbb{H}P_\infty)^{-V^* \otimes_{\mathbb{H}} V}$$

of the classifying space of the spinor group by the representation ρ defined by the tensor product of two standard rank one quaternionic modules over SU(2) [13 §1.4]. Composition with the Dirac operator defines an interesting rational homology isomorphism

$$(\mathbb{H}P_\infty \times \mathbb{H}P_\infty)^{-V^* \otimes_{\mathbb{H}} V} \to \mathbb{H}P^{-V}_\infty \wedge \mathrm{MSpin} \to \mathbb{H}P^{-V}_\infty \wedge \mathrm{kO}$$

related in low dimensions to the classification of unimodular even indefinite lattices [27, 30]. This suggests that the Tate cohomology $t^*_{\mathrm{SU}(2)}\mathrm{kO}$ may have an interesting role to play in the study of topological gravity in dimension four.

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