

# AN ALGEBRAIC ANALOG OF THE VIRASORO GROUP

JACK MORAVA

ABSTRACT. The group of diffeomorphisms of a circle is not an infinite-dimensional algebraic group, though in many ways it behaves as if it were. Here we construct an algebraic model for this object, and discuss some of its representations, which appear in the Kontsevich-Witten theory of two-dimensional topological gravity through the homotopy theory of moduli spaces. [This is a version of a talk on 23 June 2001 at the Prague Conference on Quantum Groups and Integrable Systems.]

## 1. SOME FUNCTORS FROM COMMUTATIVE RINGS TO GROUPS

**1.1** A formal diffeomorphism of the line, with coefficients in a commutative ring  $A$ , is an element  $g$  of the ring  $A[[x]]$  of formal power series with coefficients in  $A$ , such that  $g(0) = 0$  and  $g'(0)$  is a unit. More precisely, the **group** of formal diffeomorphisms of the line, defined over  $A$ , is the set

$$\mathbb{G}(A) = \{g \in A[[x]] \mid g(x) = \sum_{k \geq 0} g_k x^{k+1} \text{ with } g_0 \in A^\times \}.$$

Composition  $g_0, g_1 \mapsto (g_0 \circ g_1)(x) = g_0(g_1(x))$  of formal power series makes this set into a monoid with  $e(x) = x$  as identity element, and it is an exercise in induction to show that such an invertible power series [ie with leading coefficient a unit] possesses a composition inverse in  $\mathbb{G}(A)$ . Thus  $\mathbb{G}$  defines a covariant functor from commutative rings to groups; in fact this functor is representable, in the sense that  $\mathbb{G}(A)$  is naturally isomorphic to the set of ring homomorphisms from the polynomial algebra  $\mathbb{Z}[g_k \mid k \geq 0][g_0^{-1}]$  to  $A$ . Composition endows this representing algebra with the Hopf diagonal

$$\Delta g(x) = (g \otimes 1)((1 \otimes g)(x)),$$

making  $\mathbb{G}$  into a (pro-)algebraic group [9]. The kernel of the homomorphism

$$\epsilon \mapsto 0 : \mathbb{G}(A[\epsilon]/(\epsilon^2)) \rightarrow \mathbb{G}(A)$$

can be given the structure of a Lie algebra, which is naturally isomorphic to the Lie algebra over  $A$  spanned by the differentiation operators

$$v_k = x^{k+1} \frac{d}{dx}, \quad k \geq 0.$$

satisfying  $[v_k, v_l] = (l - k)v_{k+l}$ .

**1.2** There is a closely related functor  $\tilde{\mathbb{G}}$  from commutative rings to groups, which in some ways resembles the group of diffeomorphisms of the circle. This functor is

---

*Date:* 15 June 2001.

*1991 Mathematics Subject Classification.* 81R10, 55S25.

The author was supported in part by the NSF.

quite representable - it is an ind-proalgebraic group - but it is close enough to being so to have some useful properties. In particular its Lie algebra, in the sense above, is spanned by operators  $v_k$  with  $k \in \mathbb{Z}$ : thus  $k$  can be negative as well as positive.

In the terminology of [5 §2.3] an element  $g$  of the Laurent series ring  $A((x)) := A[[x]][x^{-1}]$  is a **nil-Laurent** series if its coefficients  $g_i$  for  $i < -1$  are nilpotent. If  $\sqrt{A}$  is the radical of  $A$  (ie the ideal of nilpotent elements) and  $A_{red} := A/\sqrt{A}$  then the set of such nil-Laurent series is the inverse image of  $A_{red}[[x]]$  under the quotient homomorphism  $\rho : A((x)) \rightarrow A_{red}((x))$ , and

$$\check{\mathbb{G}}(A) = \{g \in A((x)) \mid \rho(g) \in \mathbb{G}(A_{red})\}$$

is the set of formal Laurent series  $g(x) = \sum_{k \gg -\infty} g_k x^{k+1} \in A((x))$  such that

- i)  $g_0$  is a unit, and
- ii)  $g_{-k}$  is nilpotent, if  $k \geq 1$ .

This set is closed under composition of power series, and is in fact a group: We can write  $g \in \check{\mathbb{G}}$  as a sum

$$g(x) = g_+(x) + g_-(x^{-1})$$

of an invertible formal power series  $g_+$  and a polynomial  $g_-$  in  $x^{-1}$  with nilpotent coefficients; this implies that the sum

$$g^{-1} = \sum_{k \geq 0} (-g_-)^k g_+^{-k-1} \in A((x))$$

is finite. If  $h = h_+ + h_-$  is another series of the same sort, we can thus make sense of the composition  $h_- \circ g$ , so it suffices to show that  $h_+ \circ (g_+ + g_-)$  is well-defined; but  $g_-$ , being a polynomial with nilpotent coefficients, is itself nilpotent, so this composition can be written as a finite Maclaurin expansion

$$\sum_{k \geq 0} (D_k h_+)(g_+) g_-^k \in A((x)) ,$$

where

$$D_k x^n = \frac{1}{k!} \frac{d^k x^n}{dx^k} = \binom{n}{k} x^{n-k} \text{ if } n \geq k ,$$

and is otherwise zero. To show the existence of (composition) inverses, we use the fact that

$$h \circ g \equiv h \circ g_+ \pmod{I_-((x))} ,$$

where  $I_-$  is the ideal generated by the coefficients of  $g_-$ . [I would like to thank M. Kapranov for suggesting this line of argument, which has substantially improved the result.] Because this ideal is generated by finitely many nilpotent elements, it is itself nilpotent, in the sense that  $(I_-)^n = 0$  for  $n \gg 0$ . It suffices to construct an inverse for  $g$  under the assumption that  $g_0 = 1$ , and that the rest of its coefficients lie in such a nil-ideal: for  $g_+$  has a composition inverse  $h^+$ , such that

$$(g \circ h^+)(x) \equiv x \pmod{I_-((x))} ,$$

and if  $u_0 \in A^\times$  is the coefficient of  $x$  in  $g \circ h^+$  then  $h_{(0)}(x) = h^+(u_0^{-1}x)$  is a formal series such that  $(g \circ h_{(0)})(x) \equiv x \pmod{I_-((x))}$ . Under that hypothesis, then, let  $h_{(1)}^+(x) = 2x - g(x) = x - \tilde{g}(x)$ ; then

$$g(h_{(1)}^+(x)) = x - \tilde{g}(x) + \tilde{g}(x - \tilde{g}(x)) = x - \tilde{g}'(x)\tilde{g}(x) + \dots ,$$

the dots representing further Taylor's series-style corrections, so

$$g \circ h_{(1)}^+ \equiv x \pmod{I_-^2((x))} .$$

If  $u_1$  is the coefficient of  $x$  in  $g \circ h_{(1)}^+$ , and we define  $h_{(1)}(x) = h_{(1)}^+(u_1^{-1}x)$ , then it follows that  $g \circ h_{(1)}$  is a series with all coefficients in  $I_-^2$  except that of  $x$ , which equals 1. Now we can iterate this process: induction defines a sequence

$$h = h_{(0)} \circ h_{(1)} \circ \dots .$$

of compositions which will terminate in finitely many steps, defining the promised composition inverse for  $g$ .

**1.3** If  $\phi$  is a diffeomorphism of the circle with the property that

$$\phi(\zeta z) = \zeta \phi(z)$$

(where  $z \in \mathbb{C}$  with  $|z| = 1$  and  $\zeta$  is a primitive  $n$ th root of unity), then  $z \mapsto \phi(z)^n$  is an  $n$ -fold covering, which factors through a diffeomorphism  $\Phi$  of the circle satisfying

$$\Phi(z^n) = \phi(z)^n .$$

The group of such periodic diffeomorphisms thus defines an  $n$ -fold cover of  $\text{Diff } S^1$ . The group-valued functor  $\check{\mathbb{G}}$  has similar 'covers': for simplicity let  $n = p$  be prime, e.g. two, and assume that  $A$  contains a nontrivial  $p$ th root  $\zeta$  of unity: then

$$\check{\mathbb{G}}_{1/p}(A) = \{g \in \check{\mathbb{G}}(A) \mid g(\zeta x) = \zeta g(x)\}$$

is the subgroup of nil-Laurent series  $g(x) = \sum_{k \gg -\infty} g_{kp} x^{kp+1}$  with  $g_0$  a unit, and when  $p$  is invertible in  $A$  (e.g. if  $A$  is a  $\mathbb{Q}$ -algebra) the homomorphism

$$g(x) \mapsto g(x^{1/p})^p := g^{(p)}(x) : \check{\mathbb{G}}_{1/p} \rightarrow \check{\mathbb{G}}$$

induces an isomorphism of Lie algebras. This allows us to think of the group of invertible nil-Laurent series in  $A((x))$  as a subgroup of the invertible nil-Laurent series in  $A((x^{1/p}))$ .

## 2. SOME REPRESENTATIONS OF THESE FUNCTORS

Certain standard representations of  $\text{Diff } S^1$  have analogs for  $\check{\mathbb{G}}$ ; because these are representations over the complexes, I will assume in this section that  $A$  is an algebra over a field of characteristic zero.

**2.1** The  $A$ -bilinear form

$$g, h \mapsto \langle g, h \rangle := - \sum_{k \in \mathbb{Z}} k g_{k-1} h_{-k-1} : A((x)) \times A((x)) \rightarrow A$$

is antisymmetric, and (aside from the subring of constants) is nondegenerate if  $A$  is a  $\mathbb{Q}$ -algebra; it is an algebraic analog of the symplectic pairing

$$g, h \mapsto \text{res}_{x=0} gdh$$

of [10 §1]. The set  $\text{Sp}_L(A)$  of 'Laurent-symplectic'  $A$ -linear automorphisms of  $A((x))$  which

- i) preserve the bilinear form  $\langle \cdot, \cdot \rangle$ , and
- ii) are continuous in the pro-discrete topology of  $A((x))$

defines a group-valued functor, analogous to the restricted symplectic group [10 §5]. It is classical [12] that the residue of a differential over a local formal Laurent series field is independent of the choice of uniformizer. This remains true over general commutative rings  $A$  [13], which implies that the composition

$$f \mapsto [h \mapsto h \circ f^{-1}] : \check{\mathbb{G}}(A) \rightarrow \mathrm{Sp}_L(A)$$

is a natural homomorphism between group-valued functors; thus the  $\check{\mathbb{G}}$  has a natural linear representation as automorphisms of the functor which sends  $A$  to the  $A$ -module  $A((x))$ . It is in any case elementary to see that the Lie algebra of  $\check{\mathbb{G}}$  preserves the symplectic form: if

$$x \mapsto x + \epsilon x^{n+1}$$

then

$$x^k \mapsto x^k [1 + k\epsilon x^n], \quad dx^l \mapsto l x^{l-1} [1 + (n+l)\epsilon x^n] dx$$

so  $\langle x^k, x^l \rangle$  changes under such a transformation by

$$l(n+k+l)\epsilon \operatorname{res}_{x=0} x^{n+k+l-1} dx .$$

The residue in this expression is zero unless  $n+k+l=0$ , but in that case the coefficient is zero; nothing in this argument requires that  $n$  be positive.

**2.2** The residue pairing restricts to a bilinear form on the ring  $A[x, x^{-1}]$  of Laurent polynomials, which has a canonical decomposition

$$A[x, x^{-1}] = A[x] + A[x^{-1}]$$

into Lagrangian subspaces. The symplectic form defines a Heisenberg algebra which is essentially (when  $A$  is the field of real numbers) the identity component of the loop group of the circle. The Fock representation [10 §3, 11 §9.5] associated to this decomposition is an algebra of symmetric functions on the ‘positive-frequency’ subspace  $A[x]$ . The restricted symplectic group acts as well on (a completion of) this representation, intertwining projectively with the action of the loops on the circle [10 §5, §7b; 11 §13.4]; this is simultaneously a (positive-energy) representation of the Heisenberg algebra of the bilinear form, and (an extension of) the Lie algebra of  $\check{\mathbb{G}}$ . The Segal-Sugawara construction expresses the action of the Virasoro generators as quadratic expressions in the Heisenberg group elements [14 §1.7].

**2.3** This Fock representation has an interpretation in terms of symmetric functions [4]; more generally, a certain class of twisted representations of the Virasoro algebra [2 §9.4], associated to Hall-Littlewood polynomials at roots of unity [7 III §8.12], fit naturally into this framework. For simplicity, let  $p$  be a fixed prime [e.g.  $p=2$ ] and let

$$\mathbb{C}((x)) := V_0 \subset \mathbb{C}((x^{1/p})) := V$$

be the extension of the field of formal complex Laurent series defined by adjoining a  $p$ th root of  $x$ . The Galois group of the field extension  $V/V_0$  is cyclic of order  $p$ , generated by the automorphism

$$x^{1/p} \mapsto \zeta x^{1/p} ,$$

and the bilinear form satisfies

$$\langle \zeta(g), \zeta(h) \rangle = \langle g, h \rangle ,$$

so the invariant subspace  $V_0$  is a bilinear submodule. More generally,

$$V = \bigoplus_{a \in \mathbb{Z}/p\mathbb{Z}} V_a$$

splits into orthogonal bilinear submodules  $V_a$  spanned by series of the form

$$g = \sum_{s \gg -\infty} g_s x^s$$

in which  $s = \pm(k + a/p)$  with  $k$  and  $a$  nonnegative integers,  $0 \leq a \leq p - 1$ . The restriction of the Fock representation of the group of nil-Laurent series over  $\mathbb{C}((x^{1/p}))$  to  $\tilde{\mathbb{G}}_{1/p}$  can be interpreted (using the isomorphism of §1.3) as a representation of  $\tilde{\mathbb{G}}$  on the (completed) tensor product of the rings  $S(V_a)$  of symmetric functions.

These rings have Hopf algebra structures, which are usually described in terms of exponentials: the Witt functor assigns to a commutative ring  $A$  the multiplicative group

$$\mathbb{W}(A) = (1 + xA[[x]])^\times$$

of formal series  $w(x) = \sum w_k x^k$  with constant coefficient  $w_0 = 1$ . This is naturally isomorphic to the set of ring homomorphisms from a polynomial algebra on generators  $\{w_k, k > 0\}$  to  $A$ ; the group structure endows this representing ring with the structure of a (commutative and cocommutative) Hopf algebra. The involution

$$w(T) \mapsto w^*(x) = w(-x)$$

respects the product, and so defines a  $\mathbb{Z}/2\mathbb{Z}$ -action on  $\mathbb{W}$ . The Hopf algebra of Schur  $Q$ -functions represents the kernel of the norm homomorphism

$$w \mapsto w \cdot w^* : \mathbb{W} \rightarrow \mathbb{W} ;$$

in other words it represents the functor which sends a ring to the group of power series  $q(x)$  with  $q(0) = 1$  over that ring, which satisfy the relation  $q(x)q(-x) = 1$ .

This ring is torsion-free, and we can reformulate the relation above in the universal example as the assertion that the formal logarithm  $\log q(x)$  is an odd power series in  $T$ . More generally, the group of  $p$ th roots of unity acts on  $\mathbb{W}$  by  $w(x) \mapsto w(\zeta x)$ , and

$$w(x) \mapsto \prod_{a \in \mathbb{Z}/p\mathbb{Z}} w(\zeta_p^a x) = N(w)(x) : \mathbb{W} \rightarrow \mathbb{W}$$

is the Frobenius homomorphism of Witt theory. The Hopf algebra representing its kernel can be described as an algebra of Hall-Littlewood symmetric functions evaluated at a  $p$ th root of unity. For our purposes it can most conveniently be understood in terms of power series  $w(x)$  with  $w(0) = 1$  such that the projection of  $\log w(x^{1/p})$  to  $V_0$  is zero. The polynomial algebra underlying the Fock representation thus splits (over  $\mathbb{Q}$ ) as a product of Hopf algebras, indexed by  $a \in \mathbb{Z}/p\mathbb{Z}$ ; its  $a$ th component is the Fock representation of the Heisenberg group defined by  $V_a$ .

**2.4** The primitives in these Hopf algebras acquire natural normalizations from the Heisenberg algebra: the fractional divided powers

$$\gamma_s = \frac{x^s}{\Gamma(s+1)}$$

satisfy

$$\langle \gamma_{n/p}, \gamma_{m/p} \rangle = \frac{\operatorname{res} x^{(n+m)/p-1} dx}{\Gamma(1+n/p)\Gamma(m/p)} = -\pi^{-1} \delta_{n+m,0} \sin(n\pi/p), 2$$

so the elements

$$\gamma(a)_{\pm k} := |\sin(a\pi/p)|^{-\frac{1}{2}} \gamma_{\pm(k+a/p)}, k \in \mathbb{Z}_+$$

define a normalized symplectic basis for  $V_a$  when  $a$  is **not** congruent to zero mod  $p$ .

When  $p = 2$ , this defines the Virasoro representation with  $c = 1$  and  $h = 1/16$  studied in the Kontsevich-Witten theory of two-dimensional topological gravity [1]; that theory has a conjectural generalization [3,6] in which the more general representations defined above (with  $c = 1$  and  $h = (p^2 - 1)/48$ ) play a similar role. From a geometric point of view, these representations are mysterious: they are somehow homotopy-theoretic, and do not arise in any natural way from the Lie algebra of vector fields on the circle; instead, they seem to be related to automorphisms of the cohomology of infinite-dimensional complex projective space, along the lines laid out in this paper, through work of Madsen and Tillmann [8].

#### REFERENCES

1. P. Di Francesco, C. Itzykson, J.-B. Zuber, Polynomial averages in the Kontsevich model, *CMP* 151 (1993) 193-219
2. I. Frenkel, J. Lepowsky, A. Meurman, **Vertex operators and the Monster**, Academic Press Pure and Applied Mathematics no. 134 (1988)
3. T. Jarvis, T. Kimura, A. Vaintrob, Moduli spaces of higher spin curves and integrable hierarchies, available at [math.AG/9905034](https://arxiv.org/abs/math/9905034)
4. Jing Naihuan, Vertex operators, symmetric functions, and the spin group  $\Gamma_n$ , *J. Algebra* 138 (1991) 340-398
5. M. Kapranov, E. Vasserot, Vertex algebras and the formal loop space, available at [math.AG/0107143](https://arxiv.org/abs/math/0107143)
6. E. Looijenga, Intersection theory on Deligne-Mumford compactifications (after Witten and Kontsevich), **Seminaire Bourbaki** 768, Asterisque 216 (1993)
7. I. McDonald, **Symmetric functions and Hall polynomials**, (2nd ed), OUP (1995)
8. I. Madsen, U. Tillmann, The stable mapping-class group and  $Q(\mathbb{C}P_+^\infty)$ , Aarhus preprint 14 (1999); and further work, in progress
9. J. Morava, Noetherian localisations of categories of cobordism comodules. *Ann. of Math.* (1985) 1-39.
10. G. Segal, Unitary representations of some infinite-dimensional groups, *Comm. Math. Phys.* 80 (1981) 301-342.
11. —, A. Pressley, **Loop groups**, OUP (1986)
12. J.P. Serre, **Corps Locaux**, Hermann (1968)
13. J. Tate, Residues of differentials on curves, *Ann. Sci. Ecole Norm. Sup.* 1 (1968) 149-159
14. E. Witten, On the Kontsevich model, and other models of two-dimensional gravity, preprint, IASSNS-HEP-91/24

DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND 21218  
*E-mail address:* [jack@math.jhu.edu](mailto:jack@math.jhu.edu)