The rational homotopy Lie algebra of function spaces

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Abstract

The rational homotopy Lie algebra of function spaces is fully described.

1 Introduction

Starting with the work of Thom [13] and followed by that of Haefliger [7], the rational homotopy type of function spaces has been extensively studied. However, there is no explicit and complete description of the homotopy Lie algebra structure of such spaces, and only particular cases are known:

Denote by $\mathcal{F}(X,Y)$ (respec. $\mathcal{F}_*(X,Y)$) the space of free (respec. based) maps from X to Y. From now on, X and Y are assumed to be nilpotent complexes with X finite and Y of finite type over $\mathbb{Q}$. In this way the components of both $\mathcal{F}(X,Y)$ and $\mathcal{F}_*(X,Y)$ are nilpotent CW-complexes of finite type over $\mathbb{Q}$ and can be rationalized in the classical sense.

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If $\dim X < \operatorname{conn} Y$ (so that $\mathcal{F}(X,Y)$ is connected) M. Vigué [14] showed that the homotopy Lie algebra $\pi_*\mathcal{F}(X,Y)_{\mathbb{Q}}$ (respec. $\pi_*\mathcal{F}_*(X,Y)_{\mathbb{Q}}$) is isomorphic as Lie algebra to $H^*(X;\mathbb{Q})\otimes\pi_*(Y_{\mathbb{Q}})$ (respec. $H^+(X;\mathbb{Q})\otimes\pi_*(Y_{\mathbb{Q}})$). Later on, Y. Félix [4] used essential properties of this homotopy Lie algebra to show, among other deep results, that the Lusternik Schnirelmann category of the mentioned component is often infinite.

Following the Brown–Szczarba approach to the Haefliger model of function spaces [2], we first obtain a natural description of its rational homotopy groups in terms of derivations. Then, we give a full and explicit description of the homotopy Lie algebra structure of $\mathcal{F}(X,Y)_{\mathbb{Q}}$ and $\mathcal{F}_*(X,Y)$. Let us be more precise:

Let $(\Lambda V,d)$ be a Sullivan model, non necessarily minimal, of Y, i.e., a cofibrant replacement of a commutative differential graded algebra (CDGA henceforth) homotopy equivalent to $C^*(Y;\mathbb{Q})$, and let B be a finite dimensional CDGA of the homotopy type of $C^*(X;\mathbb{Q})$. Then, there is a model of $\mathcal{F}(X;Y)$ of the form $(\Lambda(V\otimes B_*),\widetilde{d})$ (see next section for proper definitions and details). By a model of a non connected space, or a map between them, we mean a $\mathbb{Z}$-graded CDGA, or a CDGA morphism, whose simplicial realization has the homotopy type of the singular simplicial approximation of the chosen space or map.

Moreover, given a map $f\colon X\to Y$, there is a standard procedure [11] to produce a Sullivan model $(\Lambda S_\phi,\overline{d})$ (in fact, the Haefliger model) of the nilpotent space $\mathcal{F}(X,Y;f)$, the path component of $\mathcal{F}(X,Y)$ containing f.

Our first result is that the space of the indecomposables of this model $\big(S_\phi,Q(\overline{d})\big)$ is isomorphic as differential vector spaces to $(\operatorname{Der}(\Lambda V,B;\phi),\delta)$, the ϕ-derivations from ΛV to B, being $\phi\colon\Lambda V\to B$ a model of f. As the homotopy groups of a nilpotent space are isomorphic to (the dual of) the homology of the indecomposables of a given model, we obtain:

Theorem 1. *For $n\geq 1$:*

(i) $\pi_n(\mathcal{F}(X,Y;f)_{\mathbb{Q}})\cong H_n(Der(\Lambda V,B;\phi),\delta)$*;*

(ii) $\pi_n(\mathcal{F}_*(X,Y;f)_{\mathbb{Q}})\cong H_n(Der(\Lambda V,B_+;\phi),\delta)$.

This result, together with the naturality of the process followed, includes (see Corollary 6) [9, Thm.2.1] and [10, Thm.1] as particular cases.

Then we proceed to fully and explicitly describe the Lie bracket on $\pi_*\mathcal{F}(X,Y;f)_{\mathbb{Q}}$ and $\pi_*\mathcal{F}_*(X,Y;f)_{\mathbb{Q}}$ in terms of derivation:

Theorem 2. *the differential linear map of degree* 1

$$[\ ,\ ]\colon Der_*(\Lambda V,B;\phi)\otimes Der_*(\Lambda V,B;\phi)\longrightarrow Der_*(\Lambda V,B;\phi),$$

defined by

$$[\varphi,\psi](v)=(-1)^{|\varphi|+|\psi|-1}\sum\big(\sum_{i\neq j}\varepsilon_{ij}\,\phi(v_1\dots\hat{v}_i\dots\hat{v}_j\dots v_k)\varphi(v_i)\psi(v_j)\big),$$

in which $dv=\sum v_1\dots v_k$ *and* ε_{ij} *is the sign defined in Lemma 5 below, induces the Whitehead product in homology. Moreover, the restriction to*

$$[\ ,\ ]\colon Der_*(\Lambda V,B_+;\phi)\otimes Der_*(\Lambda V,B_+;\phi)\longrightarrow Der_*(\Lambda V,B_+;\phi),$$

also induces the Lie bracket in $\pi_*\mathcal{F}_*(X,Y;f)_{\mathbb{Q}}$.

A similar result gives also an explicit description of higher order Whitehead products (see Theorem 12).

As an immediate application we generalize the result of Vigué stated above: If we denote by $*\colon X\to Y$ the constant map, $\pi_n(\mathcal{F}(X,Y;*)_{\mathbb{Q}})$ (respec. $\pi_n(\mathcal{F}_*(X,Y;*)_{\mathbb{Q}})$) is isomorphic as Lie algebras to $H^*(X;\mathbb{Q})\otimes\pi_*(Y_{\mathbb{Q}})$ (respec. $H^+(X;\mathbb{Q})\otimes\pi_*(Y_{\mathbb{Q}})$).

Finally, from Theorem 2 we generalize [6] and [8, Thm. 1.2]. For a given space Y, denote by $\operatorname{dl}Y$ the least n (or infinite) for which there is a non trivial whitehead product of order n in $\pi_*(Y_{\mathbb{Q}})$ (see §3 for more about this invariant).

Theorem 3. *If* $cat_0X<dlY_{\mathbb{Q}}$*, then* $\mathcal{F}_*(X,Y;f)_{\mathbb{Q}}$ *is an* H *space for all* f*. Equivalently, its rational cohomology algebra is free.*

When f is the constant map we may replace cat_0X by the rational cup length of X recovering the main result in [6].

2 Basics of rational homotopy theory of function spaces

We shall be using known results on rational homotopy theory for which [5] is a very good and standard reference. We now recall some specific facts on the rational homotopy type of a function space $\mathcal{F}(X,Y)$ starting by its Brown-Szczarba model. Consider $A=(\Lambda V,d)\xrightarrow{\simeq}A_{PL}(Y)$ a Sullivan model, non necessarily minimal, of Y and $B \xrightarrow{\simeq} A_{PL}(X)$ a quasi-isomorphism with B a connected finite dimensional CDGA. Let $B_* = \hom(B, \mathbb{Q})$ be the differential graded coalgebra dual of B, and consider the $\mathbb{Z}$-graded CDGA $\Lambda(A \otimes B_*)$ with the natural differential induced by the one on A and by the dual of the differential of B. Now, consider the differential ideal $I \subset \Lambda(A \otimes B_*)$ generated by $1 \otimes 1^* - 1$ and by the elements of the form

$$a_1 a_2 \otimes \beta - \sum_j (-1)^{|a_2||\beta_j{}'|}(a_1 \otimes \beta_j{}')(a_2 \otimes \beta_j{}''),$$

$a_1, a_2 \in A,\ \beta \in B$, and $\Delta\beta = \sum_j \beta_j{}' \otimes \beta_j{}''$. Then, the composition

$$\rho\colon \Lambda(V \otimes B_*) \subset \Lambda(A \otimes B_*) \longrightarrow \Lambda(A \otimes B_*)/I$$

is an isomorphism of graded algebras [2, Thm.1.2], and therefore, considering on $\Lambda(V \otimes B_*)$ the differential $\widetilde{d} = \rho^{-1} d\rho$, ρ is also an isomorphism of CDGA's. Then, $(\Lambda(V \otimes B_*), \widetilde{d})$ is a model of $\mathcal{F}(X, Y_{\mathbb{Q}})$ [2, Thm.1.3]. In other words, $S_*\mathcal{F}(X, Y_{\mathbb{Q}})$ and the simplicial realization of $(\Lambda(V \otimes B_*), \widetilde{d})$ are homotopy equivalent.

In order to explicitly determine $\widetilde{d}$ on $v \otimes \beta \in V \otimes B_*$, calculate $(dv) \otimes \beta + (-1)^{|v|} v \otimes d\beta$ and then use the relations which generate the ideal I to express $(dv) \otimes \beta$ as an element of $\Lambda(V \otimes B_*)$.

We now explain how to obtain Sullivan models (in fact the Haefliger models) of the different components of $\mathcal{F}(X, Y)$ [2, 11]. For this we need some algebraic tools: let $(\Lambda W, d)$ be a CDGA in which W is $\mathbb{Z}$-graded, and let $u\colon \Lambda W \longrightarrow \mathbb{Q}$ be an augmentation. Given $\Phi = \alpha \cdot \Psi$, $\alpha \in (\Lambda^+ W^0)$ and $\Psi \in \Lambda(W^{\neq 0})$, we denote by Φ/u the element $u(\alpha)\Psi$. Define a linear map $\partial\colon W^0 \to W^1$ as follows: given $w \in W^0$, write $dw = \Phi_0 + \Phi_1 + \Phi_2$, with $\Phi_0 \in (\Lambda^+ W^{<0}) \cdot (\Lambda W)$, $\Phi_1 \in (\Lambda^+ W^0) \cdot W^1$, $\Phi_2 \in W^1$, and define $\partial(w) = \Phi_1/u + \Phi_2$.

call $\overline{W}^1$ a complement of the image of this map, $W^1 = \partial W^0 \oplus \overline{W}^1$, and define the CDGA $\Lambda(\overline{W}^1 \oplus W^{\geq 2}, \overline{d})$ as follows:

given $w \in \Lambda(\overline{W}^1 \oplus W^{\geq 2})$ write $dw = \Phi_0 + \Phi_1 + \Phi_2 + \Phi_3$, in which $\Phi_0 \in \Lambda^+ W^{<0} \cdot \Lambda W$, $\Phi_1 \in \Lambda^+(\partial W^0) \cdot \Lambda W^{\geq 0}$, $\Phi_2 \in (\Lambda^+ W^0) \cdot (\Lambda \overline{W}^1 \oplus W^{\geq 2})$ and $\Phi_3 \in \Lambda \overline{W}^1 \oplus W^{\geq 2}$. Define $\overline{d} w = \Phi_2/u + \Phi_3$.

Note that if we have in W a basis $\{w_i\}$ for which $dw_i \in \Lambda W_{<i}$, then the image of this basis in $\Lambda(\overline{W}^1 \oplus W^{\geq 2}, \overline{d})$ makes it a Sullivan model. However, even when d is decomposable in ΛW, $\overline{d}$ might not be, i.e., $\Lambda(\overline{W}^1 \oplus W^{\geq 2}, \overline{d})$

is not necessarily minimal. This depends on u. In fact, as we just remarked, for each $w \in W$, Φ_2/u could be the linear part of $\overline{d}w$.

Next, consider $(\Lambda(V\otimes B_*), \widetilde{d})$ the model of the function space $\mathcal{F}(X,Y)$ and let $\phi\colon (\Lambda V, d) \to B$ be a model of a given map $f\colon X \to Y$. The morphism ϕ clearly induces a natural augmentation which shall be denoted also by $\phi\colon (\Lambda(V \otimes B_*), \widetilde{d}) \to \mathbb{Q}$. Applying the process above to this particular case yields a CDGA $(\Lambda S_\phi, \overline{d}) = (\Lambda \overline{V \otimes B}_*^1 \oplus (V \otimes B_*)^{\geq 2}, \overline{d})$ which turns out to be a Sullivan model of $\mathcal{F}(X,Y;f)$. Moreover, the CDGA morphism

$$\omega_0\colon (\Lambda V, d) \longrightarrow (\Lambda S_\phi, \overline{d}) = (\Lambda \overline{V \otimes B}_*^1 \oplus (V \otimes B_*)^{\geq 2}, \overline{d}),$$

$\omega_0(v) = v \otimes 1^*$ if $v \in V^{\geq 2}$, or its projection over $\overline{V \otimes B_*}^1$ if $v \in V^1$, is a Sullivan model of the evaluation at the base point $\omega_0\colon \mathcal{F}(X,Y;f) \to Y$ [11, Cor. 22].

While $\omega_0(v)$ could vanish if $|v| = 1$, when $(\Lambda V, d)$ is 1-connected,

$$\omega_0\colon (\Lambda V, d) \longrightarrow (\Lambda S_\phi, \overline{d}),$$

is a KS-extension or a relative Sullivan algebra. The fibre, which is of the form

$$(\Lambda(S_\phi/V), \overline{d}) \cong (\Lambda(\overline{V \otimes B}_+^1 \oplus (V \otimes B_+)^{\geq 2}), \overline{d}),$$

is a Sullivan model of the fibre of $\omega_0\colon \mathcal{F}(X,Y;f) \to Y$, i.e., of $\mathcal{F}^*(X,Y;f)$.

Finally, we set some notation: for any pair V, B of $\mathbb{Z}$-graded vector spaces, denote by $\mathcal{L}(V,B) = \{\mathcal{L}_n(V,B)\}_{n\geq 0}$ the graded vector space of its homomorphisms. In particular, the dual of a given object (except for B_*) shall be denoted by $\mathcal{L}(-,\mathbb{Q})$. There is a natural isomorphism

$$\Theta\colon \mathcal{L}(V,B) \xrightarrow{\cong} \mathcal{L}(V \otimes B_*, \mathbb{Q}), \quad \Theta(\theta)(v \otimes \beta) = (-1)^{|\beta|(|v|+|\theta|)}\beta(\theta(v)).$$

Given a CDGA morphism $\phi\colon A \to B$, call $(Der(A,B;\phi), \delta)$ the differential graded vector space where $Der_n(A,B;\phi)$ are the ϕ-derivations of degree n, i.e., linear maps $\theta\colon A^* \to B^{*-n}$ for which $\theta(ab) = \theta(a)\phi(b) + (-1)^{n|b|}\phi(a)\theta(b)$. The differential is defined as usual $\delta\theta = d\circ\theta + (-1)^{n+1}\theta\circ d$. Note that when $A = \Lambda V$, $Der(\Lambda V, B;\phi) \cong \mathcal{L}(V,B)$ as graded vector spaces via the identification $\theta \mapsto \theta|_V$. We shall denote also by

$$\Theta\colon Der(\Lambda V, B; \phi) \xrightarrow{\cong} \mathcal{L}(V \otimes B_*, \mathbb{Q})$$

the isomorphism above under this identification.

3 Rational homotopy groups of function spaces

In this section we prove Theorem 1 and extract some consequences. Consider the Sullivan model $(\Lambda S_\phi, \overline{d})$ of $\mathcal{F}(X,Y;f)$ and recall that, as for any other nilpotent space, $\pi_*\mathcal{F}(X,Y;f)_{\mathbb{Q}}$ is naturally isomorphic to the dual of $H^*\big(S_\phi, Q(\overline{d})\big)$, being $S_\phi \cong Q(\Lambda S_\phi) = \Lambda S_\phi/(\Lambda^+ S_\phi \cdot \Lambda^+ S_\phi)$ the space of indecomposables. In other word, it is isomorphic to the dual of the homology of the following complex:

$$0 \longrightarrow \overline{V \otimes B_*}^1 \xrightarrow{Q(\overline{d})} (V \otimes B_*)^2 \xrightarrow{Q(\overline{d})} (V \otimes B_*)^3 \xrightarrow{Q(\overline{d})} \cdots$$

However, as $(V \otimes B_*)^1 = \partial (V \otimes B_*)^0 \oplus \overline{V \otimes B_*}^1$, this is exactly the homology of this slightly different complex:

$$(V \otimes B_*)^0 \xrightarrow{\partial} (V \otimes B_*)^1 \xrightarrow{0 \oplus Q(\overline{d})} (V \otimes B_*)^2 \xrightarrow{Q(\overline{d})} (V \otimes B_*)^3 \xrightarrow{Q(\overline{d})} \cdots$$

Our main result in this section is that the dual of the complex above is isomorphic to $(\mathrm{Der}\,(\Lambda V, B; \phi), \delta)$ via the map Θ defined in §2. We prove:

Theorem 4. *The following commutes:*

$$\begin{array}{ccccccc}
\mathcal{L}_0(V \otimes B_*, \mathbb{Q}) & \xleftarrow{\partial^*} & \mathcal{L}_1(V \otimes B_*, \mathbb{Q}) & \xleftarrow{\big(0 \oplus Q(\overline{d})\big)^*} & \mathcal{L}_2(V \otimes B_*, \mathbb{Q}) & \xleftarrow{Q(\overline{d})^*} & \cdots \\
\cong \uparrow \Theta & & \cong \uparrow \Theta & & \cong \uparrow \Theta & & \\
Der_0(\Lambda V, B; \phi) & \xleftarrow{\delta} & Der_1(\Lambda V, B; \phi) & \xleftarrow{\delta} & Der_2(\Lambda V, B; \phi) & \xleftarrow{\delta} & \cdots
\end{array}$$

Proof. Here, for simplicity in the notation, we write $Q(\overline{d})^*$ instead of $\mathcal{L}(Q(\overline{d}), \mathbb{Q})$. For the same purpose we shall omit signs and write just $\pm$. However, a careful use of Koszul convention leads to proper sign adjustments.

We first show that, for $n \geq 2$, the square

$$\begin{array}{ccc}
\mathcal{L}_n(V \otimes B_*, \mathbb{Q}) & \xleftarrow{Q(\overline{d})^*} & \mathcal{L}_{n+1}(V \otimes B_*, \mathbb{Q}) \\
\Theta \uparrow \cong & & \cong \uparrow \Theta \\
\mathrm{Der}_n(\Lambda V, B; \phi) & \xleftarrow{\delta} & \mathrm{Der}_{n+1}(\Lambda V, B; \phi)
\end{array}$$

commutes. On one hand, given $\theta \in \mathrm{Der}_{n+1}(\Lambda V, B; \phi)$ and $v \otimes \beta \in (V \otimes B_*)^n$,

$$(\Theta\delta\theta)(v \otimes \beta) = \pm\beta(\delta\theta(v)) = \pm\beta(d(\theta(v))) \pm \beta(\theta(dv)). \qquad (*)$$

On the other hand,

$$\big(Q(\overline{d})^*\Theta\theta\big)(v \otimes \beta) = \pm\Theta\theta(Q(\overline{d})(v \otimes \beta)) = \pm\Theta\theta(\{dv \otimes \beta\} \pm v \otimes d\beta) =$$

$$= \pm\Theta\theta(\{dv \otimes \beta\}) \pm (d\beta)(\theta(v)) = \pm\Theta\theta(\{dv \otimes \beta\}) \pm \beta(d(\theta(v))). \qquad (**)$$

Here, $\{dv \otimes \beta\}$ denotes the indecomposable part of the image of $[dv \otimes \beta]$ through the morphism:

$$A \otimes B_*/I \xrightarrow[\cong]{\rho^{-1}} \Lambda(V \otimes B_*) \to \Lambda(\overline{V \otimes B_*})^1 \oplus (V \otimes B_*^{\geq 2}) = \Lambda S_\phi.$$

To effectively compute $\{dv \otimes \beta\}$ use first the relations which generates I to write $[dv \otimes \beta]$ as an element of $\Lambda(V \otimes B_*)$. Then, cancel all elements of negative degree and their derivatives, and replace any element of degree zero by the corresponding scalar via ϕ. Finally, keep the linear part.

At the sight of $(*)$ and $(**)$, it will be enough to prove:

Lemma 5. *Given* $\Phi \in \Lambda V$, $\beta \in B_*$ *and* $\theta \in \mathit{Der}_*(\Lambda V, B; \phi)$, $(\Theta\theta)(\{\Phi \otimes \beta\}) = (-1)^{|\beta|(|\theta|+|\Phi|)}\beta(\theta(\Phi))$.

Proof. Denote by $F_B \colon B \otimes B_* \longrightarrow \mathbb{Q}$ and $F_{B\otimes B} \colon (B \otimes B) \otimes (B_* \otimes B_*) \longrightarrow \mathbb{Q}$ the maps defined respectively by $F_B(b \otimes \beta) = (-1)^{|b|}\beta(b)$ and $F_{B\otimes B}(b \otimes b' \otimes \beta \otimes \beta') = (-1)^{|b||b'|+|b|+|b'|}\beta(b)\beta'(b')$. Then, if μ is multiplication in B, it is easy to see that the following commutes:

$$\begin{array}{ccc}
(B \otimes B) \otimes B_* & \xrightarrow{\mu \otimes 1_{B_*}} & B \otimes B_* \\
{\scriptstyle 1_{B\otimes B} \otimes \triangle}\downarrow & & \downarrow{\scriptstyle F_B} \\
(B \otimes B) \otimes (B_* \otimes B_*) & \xrightarrow[F_{(B\otimes B)}]{} & \mathbb{Q}.
\end{array}$$

To prove the lemma, assume $\Phi = \Lambda^k V$ and argue by induction on k. For $k = 1$, $\Phi = v \in V$ and $\{v \otimes \beta\} = v \otimes \beta$ for which the lemma holds by definition of Θ. Assume $\Phi = \Psi \cdot v$ with $\Psi \in \Lambda^{k-1}V$. Again, to avoid excessive notation, we shall not write signs:

$$
\begin{aligned}
\beta\big(\theta(\Psi\cdot v)\big) &= \beta\big(\theta(\Psi)\phi(v)\pm\phi(\Psi)\theta(v)\big)\\
&= \pm F_B(\theta(\Psi)\phi(v)\otimes\beta)\pm F_B(\phi(\Psi)\theta(v)\otimes\beta)=\\
&= \pm F_{B\otimes B}(\theta(\Psi)\otimes\phi(v)\otimes\triangle\beta)\pm F_{B\otimes B}(\phi(\Psi)\otimes\theta(v)\otimes\triangle\beta)\\
&= (a)+(b).
\end{aligned}
$$

On the other hand

$$
(\Theta\theta)\{\Psi\cdot v\otimes\beta\}=(\Theta\theta)\{\sum_j \pm(\Psi\otimes\beta_j')(v\otimes\beta_j'')\},
$$

being $\triangle\beta=\sum_j\beta_j'\otimes\beta_j''$. By definition of $\{\,,\}$, we may keep only those summands for which one of the factors is of degree zero. Hence, the above equality becomes:

$$
(\Theta\theta)\{\sum_{|\Psi|+|\beta_j'|=0}\pm(\Psi\otimes\beta_j')(v\otimes\beta_j'')\}+(\Theta\theta)\{\sum_{|v|+|\beta_j''|=0}\pm(\Psi\otimes\beta_j')(v\otimes\beta_j'')\}
$$
$$
=(e)+(f)
$$

Using definition and induction hypothesis we get:

$$
\begin{aligned}
(f) &= \sum_{|v|+|\beta_j''|=0}\pm(\Theta\theta)\{\Psi\otimes\beta_j'\}\phi(v\otimes\beta_j'')\\
&= \sum_{|v|+|\beta_j''|=0|}\pm\beta_j'\big(\theta(\Psi)\big)\beta_j''\big(\phi(v)\big)\\
&= \sum_{|v|+|\beta_j''|=0|}\pm F_{B\otimes B}(\theta(\Psi)\otimes\phi(v)\otimes\beta_j'\otimes\beta_j'')\\
&= \pm F_{B\otimes B}(\theta(\Psi)\otimes\phi(v)\otimes\triangle\beta)=(a)
\end{aligned}
$$

Using repeatedly a similar argument one checks that $(b)=(e)$ and the proof is complete. $\square$

Finally, we see that

$$
\begin{array}{ccccc}
\mathcal{L}_0(V\otimes B_*,\mathbb{Q}) & \xleftarrow{\partial^*} & \mathcal{L}_1(V\otimes B_*,\mathbb{Q}) & \xleftarrow{\left(0\oplus Q(\overline{d})\right)^*} & \mathcal{L}_2(V\otimes B_*,\mathbb{Q})\\
\cong\big\uparrow\Theta & & \cong\big\uparrow\Theta & & \cong\big\uparrow\Theta\\
\mathrm{Der}_0(\Lambda V,B;\phi) & \xleftarrow{\delta} & \mathrm{Der}_1(\Lambda V,B;\phi) & \xleftarrow{\delta} & \mathrm{Der}_2(\Lambda V,B;\phi)
\end{array}
$$

commutes. For it note that given $v\otimes\beta \in (V\otimes B_*)^0$, $\partial(v\otimes\beta) = \{dv\otimes\beta\} + (-1)^{|v|}v\otimes d\beta$. Hence, using Lemma 5, and following exactly the above argument:

$$(\partial^*\circ\Theta\theta)(v\otimes\beta) = (-1)^{|\theta|+1}\Theta\theta(\{dv\otimes\beta\}+(-1)^{|v|}v\otimes d\beta) = (\Theta\circ\delta\theta)(v\otimes\beta),$$

which gives the commutativity of the left square. For the right square, write $w\in(V\otimes B_*)^1$ as a sum $x+v\otimes\beta$, $x\in\partial(V\otimes B_*)^0$, $v\otimes\beta\in\overline{V\otimes B_*}^1$. Then,

$$\begin{aligned}\big((0\oplus Q(\overline{d}))^*\circ\Theta\theta\big)(w) &= (-1)^{|\theta|+1}(\Theta\theta)\big(Q(\overline{d})(v\otimes\beta)\big)\\ &= (-1)^{|\theta|+1}(\Theta\theta)(\{dv\otimes\beta\}+(-1)^{|v|}v\otimes d\beta)\\ &= (\Theta\circ\delta\theta)(w).\end{aligned}$$

□

Proof of Theorem 1. Part (i) is immediate from Theorem 4. For (ii) consider the Sullivan model $(\Lambda(S_\phi/V),\overline{d})$ of $\mathcal{F}_*(X,Y;f)$ recalled in the past section, and observe that $\pi_n\big(\mathcal{F}_*(X,Y;f)_{\mathbb{Q}}\big)$ is then isomorphic to the dual of the homology of the following complex:

$$0\longrightarrow \overline{V\otimes B_+}^1 \xrightarrow{Q(\overline{d})} (V\otimes B_+)^2 \xrightarrow{Q(\overline{d})} (V\otimes B_+)^3 \xrightarrow{Q(\overline{d})} \cdots$$

To finish restrict Theorem 1 to the dual of this complex. □

We now check that the above isomorphism is natural and respects the evaluation map at the base point. Fix a map $f\colon X\to Y$ between nilpotent complexes of finite type over $\mathbb{Q}$ and let Z be a finite nilpotent complex. Let $A=(\Lambda W,d)\xrightarrow[\simeq]{\varphi} A_{PL}(X)$ and $(\Lambda V,d)\xrightarrow[\simeq]{\psi} A_{PL}(Y)$ be Sullivan models (again non necessarily minimal!) of X and Y respectively, let $C\xrightarrow[\simeq]{\nu} A_{PL}(Z)$ be a quasi-isomorphism with C connected finite dimensional, and let $\zeta\colon(\Lambda V,d)\longrightarrow(\Lambda W,d)$ be a Sullivan model for f. Define

$$\xi\colon(\Lambda(V\otimes C_*),\widetilde{d})\longrightarrow(\Lambda(W\otimes C_*),\widetilde{d}),\quad \xi(v\otimes c)=\rho^{-1}[\zeta(v)\otimes c],$$

being $(\Lambda(V\otimes C_*),\widetilde{d})$ and $(\Lambda(W\otimes C_*),\widetilde{d})$ the models of $\mathcal{F}(Z,Y)$ and $\mathcal{F}(Z,X)$ respectively, and $\rho\colon(\Lambda(W\otimes C_*),\widetilde{d})\xrightarrow{\cong}(\Lambda(A\otimes C_*),d)/I$ the CDGA isomorphism described in §2. In other words, to compute effectively $\xi(v\otimes c)$ use the relations which define I to express $\zeta(v)\otimes c$ as an element of $\Lambda(V\otimes C_*)$. For instance, if $\zeta(v) = w_1 w_2$ and $\triangle c = \sum_i c'_i \otimes c''_j$, $\xi(v \otimes c) = \sum_i (-1)^{|w_2||c'_i|}(w_1 \otimes c'_i)(w_2 \otimes c''_i)$.

Finally, let $\phi\colon (\Lambda W, d) \to C$ and $\phi \circ \zeta\colon (\Lambda V, d) \to C$ be models of $g\colon Z \to X$ and $f \circ g\colon Z \to Y$ respectively. Then [11, Thm.24], the diagram

$$\begin{array}{ccc}
(\Lambda S_\phi / W, \overline{d}) & \xleftarrow{\ \bar{\xi}\ } & (\Lambda S_{\phi\circ\zeta} / V, \overline{d}) \\
\uparrow & & \uparrow \\
(\Lambda S_\phi, \overline{d}) & \xleftarrow{\ \bar{\xi}\ } & (\Lambda S_{\phi\circ\zeta}, \overline{d}) \\
\omega_0 \uparrow & & \uparrow \omega_0 \\
(\Lambda W, d) & \xleftarrow[\ \zeta\ ]{} & (\Lambda V, d)
\end{array}$$

is a Sullivan model of

$$\begin{array}{ccc}
\mathcal{F}_*(Z, X; g) & \xrightarrow{\ (f)_*\ } & \mathcal{F}_*(Z, Y; f \circ g) \\
\downarrow & & \downarrow \\
\mathcal{F}(Z, X; g) & \xrightarrow{\ (f)_*\ } & \mathcal{F}(Z, Y; f \circ g) \\
\omega_0 \downarrow & & \downarrow \omega_0 \\
X & \xrightarrow[\ f\ ]{} & Y.
\end{array}$$

Hence, in view of Theorem 1, the following, which includes [9, Thm.2.1] and [10, Thm.1], is an easy exercise:

Corollary 6. *(1) For $n \geq 1$, $\pi_n(f_*)_{\mathbb{Q}}\colon \pi_n \mathcal{F}(Z, X; g)_{\mathbb{Q}} \to \pi_n \mathcal{F}(Z, Y; f \circ g)_{\mathbb{Q}}$ is naturally equivalent to*

$$H(\zeta_*)\colon H_n Der_*(\Lambda W, C; \phi) \longrightarrow H_n Der_*(\Lambda V, C; \phi \circ \zeta).$$

(2) Moreover,

$$\begin{array}{ccc}
\pi_n\mathcal{F}_*(Z,X;g)_{\mathbb{Q}} & \xrightarrow{\pi_n(f_*)_{\mathbb{Q}}} & \pi_n\mathcal{F}_*(Z,Y;f\circ g)_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
\pi_n\mathcal{F}(Z,X;g)_{\mathbb{Q}} & \xrightarrow{\pi_n(f_*)_{\mathbb{Q}}} & \pi_n\mathcal{F}(Z,Y;f\circ g)_{\mathbb{Q}} \\
{\scriptstyle \pi_n(\omega_0)_{\mathbb{Q}}}\downarrow & & \downarrow{\scriptstyle \pi_n(\omega_0)_{\mathbb{Q}}} \\
\pi_n(X)_{\mathbb{Q}} & \xrightarrow[\pi_n(f)_{\mathbb{Q}}]{} & \pi_n(Y)_{\mathbb{Q}}
\end{array}$$

is equivalent to

$$\begin{array}{ccc}
H_n(Der_*(\Lambda W,C_+;\phi)) & \xrightarrow{H(\zeta_*)} & H_n(Der_*(\Lambda V,C_+;\phi\circ\zeta)) \\
\downarrow & & \downarrow \\
H_n(Der_*(\Lambda W,C;\phi)) & \xrightarrow{H(\zeta_*)} & H_n(Der_*(\Lambda V,C;\phi\circ\zeta)) \\
{\scriptstyle H(\varepsilon_*)}\downarrow & & \downarrow{\scriptstyle H(\varepsilon_*)} \\
H_n(Der_*(\Lambda W,\mathbb{Q};\varepsilon)) & \xrightarrow{H(\zeta)} & H_n(Der_*(\Lambda V,\mathbb{Q};\varepsilon)).
\end{array}$$

□

Note that here $(\mathrm{Der}_*(\Lambda V,\mathbb{Q};\varepsilon),\delta) \cong \big((\mathcal{L}(V,\mathbb{Q}),Q(d)^*\big)$ and therefore $H_n(\mathrm{Der}_*(\Lambda V,\mathbb{Q};\varepsilon))$ is isomorphic to the dual of $H^*(V,Q(d))$.

4 The Lie algebra structure

This section is devoted to the proof of Theorem 2 and its first applications. For that, the following remark is essential:

Remark 7. Let $(\Lambda V,d)$ be a Sullivan model of a nilpotent space X. Recall that d can be written as the sum $d=\sum_{i\geq 1} d_i$, with $d_i(V)\subset\Lambda^i V$. The linear part $d_1=Q(d)$ induces a differential on ΛV. The differential d' induced by d on $H^*(\Lambda V,d_1)=\Lambda H^*(V,d_1)$ has no linear term and $\big(\Lambda H^*(V,d_1),d'\big)$ is the minimal model of X. The quadratic part, d'_2 is then a differential which can be identified as the Lie bracket on $\pi_*(X_{\mathbb{Q}})$ [12, II.6.(16)]. More precisely, given the natural isomorphism $\pi_*(X_{\mathbb{Q}}) \cong \mathcal{L}_*\big(H^*(\Lambda V, d_1), \mathbb{Q}\big)$ and the multilinear map

$$\langle\ ;\ ,\ \rangle : \wedge^2 H^*(\Lambda V, d_1) \times \pi_*(X_{\mathbb{Q}}) \times \pi_*(X_{\mathbb{Q}}) \longrightarrow \mathbb{Q},$$

$$\langle \alpha \wedge \beta; \gamma_0, \gamma_1\rangle = \gamma_1(\alpha)\gamma_0(\beta) + (-1)^{|\beta||\gamma_0|}\gamma_0(\alpha)\gamma_1(\beta),$$

it turns out that:

$$[\gamma_0, \gamma_1](\alpha) = (-1)^{p+q-1}\langle d_2'\alpha; \gamma_0, \gamma_1\rangle,$$

in which $\alpha \in H^*(\Lambda V, d_1)$, $\gamma_0 \in \pi_p(X_{\mathbb{Q}})$, $\gamma_1 \in \pi_q(X_{\mathbb{Q}})$.

In the same way, given the multilinear map

$$\langle\ ;\ ,\ \rangle : \wedge^j V \times V^* \times \cdots \times V^* \longrightarrow \mathbb{Q},$$

$$\langle v_1 \ldots v_j; \gamma_0, \ldots, \gamma_j\rangle = \sum_{i_1,\ldots,i_j} \delta_{i_1\ldots i_j}\gamma_1(v_{i_1})\ldots\gamma_j(v_{i_j}),$$

where $\delta_{i_1\ldots i_j}$ is the expected sign induced by the Koszul convention, the higher order Whitehead products on $\pi_*(X_{\mathbb{Q}})$ can be identified with the i-th part of d, via

$$[\gamma_1, \ldots, \gamma_j](v) = (-1)^{p_1+\cdots+p_j-1} < d_i v; \gamma_1, \ldots, \gamma_j >,$$

being each γ_i of degree p_i [1, Thm. 5.4] or [12, V.7(3)].

Consider now the component $\mathcal{F}(X, Y; f)$ of a given function space and let $(\Lambda S_\phi, \overline{d})$ be its Sullivan model defined in §2. We shall need a "quadratic" analogue of Lemma 5. Given $\Phi \in \Lambda V$ and $\beta \in B_*$, denote by $\{\Phi \otimes \beta\}_2$ the quadratic part of the image of $[\Phi \otimes \beta]$ through the morphism:

$$A \otimes B_*/I \xrightarrow[\cong]{\rho^{-1}} \Lambda(V \otimes B_*) \to \Lambda(\overline{V \otimes B_*})^1 \oplus (V \otimes B_*)^{\geq 2}.$$

To effectively compute $\{\Phi \otimes \beta\}_2$ use first the relations which generates I to write $[\Phi \otimes \beta]$ as an element of $\Lambda(V \otimes B_*)$. Then, cancel all elements of negative degree and their derivatives, and replace any element of degree zero by the corresponding scalar via ϕ. Finally, keep the quadratic part.

Lemma 8. *Let $\Phi = v_1 ... v_k \in \Lambda^k V$, $\beta \in B_*$ and $\varphi, \psi \in Der_*(\Lambda V, B; \phi)$ of strictly positive degrees. Then,*

$$\langle\{\Phi\otimes\beta\}_2; \Theta\varphi, \Theta\psi\rangle = (-1)^{|\beta|(|\varphi|+|\psi|+|\Phi|)}\sum_{i\neq j}\varepsilon_{ij}\,\beta\big(\phi(v_1...\hat{v}_i...\hat{v}_j...v_k)\varphi(v_i)\psi(v_j)\big),$$

where $\rho_0 = 0$, $\rho_j = |v_1| + \cdots + |v_j|$ *for* $j \geq 1$, *and*

$$\varepsilon_{ij} = \begin{cases} (-1)^{|\varphi|\rho_{i-1}+|\psi|\rho_{j-1}} & \text{if } i < j, \\ (-1)^{|\varphi|\rho_{i-1}+|\psi|\rho_{j-1}+|\varphi||\psi|} & \text{if } i > j. \end{cases}$$

Proof. As in Lemma 5, to be clear in presenting our argument, we shall write $\pm$ instead of proper signs, and leave to the reader the straightforward task that the equality above holds with the given signs.

We proceed by induction on k. Let $\phi = v_1 v_2$, assume $\Delta\beta = \sum_r \beta'_r \otimes \beta''_r$ and denote by Γ the sum of all terms of $\sum_r (-1)^{|\beta'_r||v_2|}(v_1 \otimes \beta'_r)(v_2 \otimes \beta''_r)$ in which at least one of the two factors is of degree 0. Then

$$\langle \{v_1 v_2 \otimes \beta\}_2; \Theta\varphi, \Theta\psi\rangle \quad = \quad \langle \sum_r \pm (v_1 \otimes \beta'_r)(v_2 \otimes \beta''_r) - \Gamma; \Theta\varphi, \Theta\psi\rangle.$$

However, as φ, ψ are of positive degree, $\langle \Gamma; \Theta\theta_g, \Theta\psi\rangle = 0$ and the formula above becomes:

$$\begin{aligned} &\sum_r \pm \langle (v_1 \otimes \beta'_r)(v_2 \otimes \beta''_r); \Theta\varphi, \Theta\psi\rangle = \\ &= \sum_r \pm \Theta\varphi(v_1 \otimes \beta'_r)\Theta\psi(v_2 \otimes \beta''_r) \pm \Theta\psi(v_1 \otimes \beta'_r)\Theta\varphi(v_2 \otimes \beta''_r) = \\ &= \sum_r \pm \beta'_r(\varphi(v_1))\beta''_r(\psi(v_2)) \pm \beta'_r(\psi(v_1))\beta''_r(\varphi(v_2)) = \\ &= \pm F_{B\otimes B}\Big(\varphi(v_1) \otimes \psi(v_2) \otimes \Delta\beta\Big) \pm F_{B\otimes B}\Big(\psi(v_1) \otimes \varphi(v_2) \otimes \Delta\beta\Big) = \\ &= \pm \beta\big(\varphi(v_1)\psi(v_2)\big) \pm \beta\big(\psi(v_1)\varphi(v_2)\big) \end{aligned}$$

which is the expected expression for $k = 2$.

Assume the lemma holds for $k-1$ and let $\Phi = v_1 ... v_k$. On one hand:

$$\begin{aligned} &\sum_{i\neq j} \pm \beta\Big(\phi(v_1 ... \hat{v}_i ... \hat{v}_j ... v_k)\varphi(v_i)\psi(v_j)\Big) = \\ &\Big[\sum_{j\neq k} \pm\beta\Big(\phi(v_1 ... \hat{v}_j ... v_{k-1})\varphi(v_k)\psi(v_j) + \sum_{i\neq k} \pm\beta\Big(\phi(v_1 ... \hat{v}_i ... v_{k-1})\varphi(v_i)\psi(v_k)\Big)\Big] + \\ &+ \sum_{\substack{i\neq j \\ i,j\neq k}} \pm\beta\Big(\phi(v_1 ... \hat{v}_i ... \hat{v}_j ... v_k)\varphi(v_i)\psi(v_j)\Big) = (I) + (II). \end{aligned}$$

On the other hand:

$$\langle \{v_1 ... v_k \otimes \beta\}_2 ; \Theta\varphi, \Theta\psi\rangle = \sum_r \pm\langle\{(v_1...v_{k-1}\otimes\beta'_r)(v_k\otimes\beta''_r)\}_2;\Theta\varphi,\Theta\psi\rangle.$$

In this formula, whenever $v_k \otimes \beta''_r$ is of degree 0, we can replace it by the scalar $\phi(v_k \otimes \beta''_r)$ resulting:

$$\sum_{|v_k\otimes\beta''_r|=0} \pm\phi(v_k\otimes\beta''_r)\langle\{v_1...v_{k-1}\otimes\beta'_r)\}_2;\Theta\varphi,\Theta\psi\rangle +$$
$$+\sum_{|v_k\otimes\beta''_r|>0} \pm\langle\{v_1\cdots v_{k-1}\otimes\beta'_r\}(v_k\otimes\beta''_r);\Theta\varphi,\Theta\psi\rangle = (II') + (I')$$

Applying induction we get:

$$\begin{aligned}
(II') &= \sum_{i\neq j,r} \pm\beta'_r\Big(\phi(v_1...\hat{v}_i...\hat{v}_j...v_{k-1})\varphi(v_i)\psi(v_j)\Big)\beta''_r(\phi(v_k)) \\
&= \sum_{i\neq j,r} \pm F_{B\otimes B}\Big(\phi(v_1...\hat{v}_i...\hat{v}_j...v_{k-1})\varphi(v_i)\psi(v_j)\otimes\phi(v_k)\otimes\beta'_r\otimes\beta''_r\Big) \\
&= \sum_{i\neq j} \pm F_{B\otimes B}\Big(\phi(v_1...\hat{v}_i...\hat{v}_j...v_{k-1})\varphi(v_i)\psi(v_j)\otimes\phi(v_k)\otimes\Delta\beta\Big) \\
&= \sum_{i\neq j} \pm F_B\Big(\phi(v_1...\hat{v}_i...\hat{v}_j...v_{k-1})\varphi(v_i)\psi(v_j)\phi(v_k))\otimes\beta\Big) \\
&= \sum_{\substack{i\neq j\\ i,j\neq k}} \pm\beta\Big(\phi(v_1...\hat{v}_i...\hat{v}_j...v_{k-1})\phi(v_k)\varphi(v_i)\psi(v_j)\Big) = (II).
\end{aligned}$$

On the other hand:

$$(I') = \sum_r \pm\Theta\varphi(\{v_1...v_{k-1}\otimes\beta'_r\})\Theta\psi(v_k\otimes\beta''_r) + \sum_r \pm\Theta\psi(\{v_1...v_{k-1}\otimes\beta'_r\})\Theta\varphi(v_k\otimes\beta''_r).$$

Applying Lemma 5 to this formula gives the following:

$$
\begin{aligned}
&= \sum_r \pm\beta'_r\Big(\varphi(v_1...v_{k-1})\Big)\beta''_r\Big(\psi(v_k)\Big) + \sum_r \pm\beta'_r\Big(\psi(v_1...v_{k-1})\Big)\beta''_r\Big(\varphi(v_k)\Big)\\
&= \sum_r \pm F_{B\otimes B}\Big(\varphi(v_1...v_{k-1})\otimes\psi(v_k)\otimes\beta'_r\otimes\beta''_r\Big) \pm\\
&\qquad \pm F_{B\otimes B}\Big(\psi(v_1...v_{k-1})\otimes\varphi(v_k)\otimes\beta'_r\otimes\beta''_r\Big)\\
&= \pm F_{B\otimes B}\Big(\varphi(v_1...v_{k-1})\otimes\psi(v_k)\otimes\Delta\beta\Big) \pm F_{B\otimes B}\Big(\psi(v_1...v_{k-1})\otimes\varphi(v_k)\otimes\Delta\beta\Big)\\
&= \pm\beta\Big(\varphi(v_1\cdots v_{k-1})\psi(v_k)\Big) \pm \beta\Big(\psi(v_1\cdots v_{k-1})\varphi(v_k)\Big).
\end{aligned}
$$

Finally, as φ and ψ are ϕ-derivations, this last equation results in

$$
\sum_{i\neq k}\pm\beta\Big(\phi(v_1...\hat{v}_i...v_{k-1})\varphi(v_i)\psi(v_k)\Big)+\sum_{j\neq k}\pm\beta\Big(\phi(v_1...\hat{v}_j...v_{k-1})\psi(v_j)\varphi(v_k)\Big) = (I)
$$

and the proof is complete. □

Proof of Theorem 2. Let $\varphi,\psi\in \mathrm{Der}\,(\Lambda V, B;\phi)$ be homogeneous derivations of positive degrees p and q respectively. In view of Theorem 1 and Remark 7, it is enough to show that, for any $v\otimes\beta\in S_\phi$

$$
\Theta[\varphi,\psi](v\otimes\beta) = (-1)^{p+q-1}\langle\overline{d}_2(v\otimes\beta);\Theta\varphi,\Theta\psi\rangle
$$

being $\overline{d}_2$, as always, the quadratic part of the differential in $(\Lambda S_\phi,\overline{d})$. But this is trivial noting that φ and ψ are of positive degree, and applying Lemma 8. Indeed:

$$
\begin{aligned}
&\quad(-1)^{p+q-1}\langle\overline{d}_2(v\otimes\beta);\Theta\varphi,\Theta\psi\rangle = (-1)^{p+q-1}\langle\{dv\otimes\beta\}_2;\Theta\varphi,\Theta\psi\rangle\\
&= (-1)^{p+q-1}\sum\langle\{v_1...v_k\otimes\beta\}_2;\Theta\varphi,\Theta\psi\rangle\\
&= (-1)^{p+q-1}\sum(-1)^{|\beta|(|p+q+|v|+1)}\sum_{i\neq j}\varepsilon_{ij}\,\beta\Big(\phi(v_1...\hat{v}_i...\hat{v}_j...v_k)\varphi(v_i)\psi(v_j)\Big)\\
&= (-1)^{|\beta|(p+q+|v|+1)}\beta\Big([\varphi,\psi](v)\Big) = \Theta[\varphi,\psi](v\otimes\beta).
\end{aligned}
$$

To finish we show that the restriction to

$$
[\ ,\ ]\colon \mathrm{Der}_*(\Lambda V, B_+;\phi)\otimes\mathrm{Der}_*(\Lambda V, B_+;\phi)\longrightarrow \mathrm{Der}_*(\Lambda V, B_+;\phi),
$$

also induces the Lie bracket in $\pi_n(\mathcal{F}_*(X,Y;f)_\mathbb{Q})$. For that note that, as the fibration

$$\mathcal{F}_*(X,Y;f) \longrightarrow \mathcal{F}(X,Y;f) \xrightarrow{\omega_0} Y$$

has a section, the exact sequence on rational homotopy induces an extension of Lie algebras

$$0 \to \pi_*\mathcal{F}_*(X,Y;f)_\mathbb{Q} \to \pi_*\mathcal{F}(X,Y;f)_\mathbb{Q} \to \pi_*Y_\mathbb{Q} \to 0.$$

Hence, the Lie bracket on $\pi_*\mathcal{F}_*(X,Y;f)_\mathbb{Q} = H_*\big(\mathrm{Der}\,(\Lambda V, B_+;\phi)\big)$ is the restriction of the one in $\pi_*\mathcal{F}(X,Y;f)_\mathbb{Q} = H_*\big(\mathrm{Der}\,(\Lambda V, B;\phi)\big)$. □

Remark 9. At the sight of the proof above, which heavily relies on Remark 7, the fact that

$$[\ ,\ ]\colon \mathrm{Der}_*(\Lambda V, B;\phi) \otimes \mathrm{Der}_*(\Lambda V, B;\phi) \longrightarrow \mathrm{Der}_*(\Lambda V, B;\phi)$$

commutes with differential automatically holds. This is far from trivial if one uses only differential homological algebra tools.

As a first and immediate application of Theorem 2 we describe the Lie algebra structure on $\pi_*\mathcal{F}(X,Y;*)_\mathbb{Q}$ and $\pi_*\mathcal{F}_*(X,Y;*)_\mathbb{Q}$ when considering the constant map $*\colon X \to Y$, recovering in particular Vigué's result [14] stated in the introduction.

Theorem 10. *$\pi_n(\mathcal{F}(X,Y;*)_\mathbb{Q})$ (respec. $\pi_n(\mathcal{F}_*(X,Y;*)_\mathbb{Q})$) is isomorphic as Lie algebra to $H^*(X;\mathbb{Q}) \otimes \pi_*(Y_\mathbb{Q})$ (respec. $H^+(X;\mathbb{Q}) \otimes \pi_*(Y_\mathbb{Q})$).*

Proof. In this case, $\phi\colon (\Lambda V, d) \to B$ annihilates V. In view of Theorem 2,

$$[\varphi,\psi](v) = (-1)^{|\varphi|+|\psi|-1}\sum_i (-1)^{|\psi||v_i'|}\varphi(v_i')\psi(v_i'') + (-1)^{|\varphi|(|v_i''|+|\psi|)}\varphi(v_i'')\psi(v_i'),$$

with $d_2 v = \sum_i v_i' v_i''$. Via the isomorphism Θ of Theorem 4, this is taken to the Lie bracket induced by $\overline{d}_2$ on $H^*(V \otimes B_*, \overline{d}_1)$. However, this is precisely the $V \otimes H^*(B)$ with the usual Lie bracket. □

We may extend Lemma 5 to calculate in $H_*\big(\mathrm{Der}\,(\Lambda V, B;\phi)\big)$ Whitehead products of higher order.

Definition 11. *Given $\varphi_1,\dots,\varphi_j \in Der_*(\Lambda V,B;\phi)$, of strictly positive degrees $p_1,\dots,p_j$, define $[\varphi_1,\dots,\varphi_j]\in Der(\Lambda V,B;\phi)$ by*

$$[\varphi_1,...,\varphi_j](v) =$$
$$(-1)^{p_1+\cdots+p_j-1}\sum\Big(\sum_{i_1,\dots,i_j}\varepsilon_{i_1\dots i_j}\,\phi(v_1\dots\hat{v}_{i_1}...\hat{v}_{i_j}...v_k)\varphi_1(v_{i_1})...\varphi_j(v_{i_j})\Big),$$

being $dv=\sum v_1\dots v_k$ and $\varepsilon_{i_1\dots i_j}$ the adequate generalization of ε_{ij}..

Then, the exact analogue of the proof of Lemma 5 shows that given $\Phi = v_1...v_k\in\Lambda^k V$ and $\beta\in B_*$,

$$\langle\{\Phi\otimes\beta\}_j;\Theta\varphi_1,\dots,\Theta\varphi_j\rangle =$$
$$(-1)^{|\beta|(p_1+\cdots+p_j+|\Phi|)}\sum_{i_1,\dots,i_j}\varepsilon_{i_1\dots i_j}\,\beta\big(\phi(v_1...\hat{v}_{i_1}...\hat{v}_{i_j}...v_k)\varphi_1(v_{i_1})\varphi_2(v_{i_2})...\varphi_j(v_{i_j})\big).$$

Again, $\{\Phi\otimes\beta\}_j$ is defined as the j-th part of the image of $[\Phi\otimes\beta]$ through the morphism:

$$A\otimes B_*/I\xrightarrow[\cong]{\rho^{-1}}\Lambda(V\otimes B_*)\to\Lambda(\overline{V\otimes B_*})^1\oplus(V\otimes B_*)^{\geq 2}.$$

Thus, as in the proof of Theorem 2, we get the following which, in view of Remark 7, describes j-order Whitehead products on $\pi_*\mathcal{F}(X,Y;f)_{\mathbb{Q}}$ and $\pi_*\mathcal{F}_*(X,Y;f)_{\mathbb{Q}}$.

Theorem 12. $\Theta[\varphi_1,\dots,\varphi_j](v\otimes\beta) = (-1)^{p_1+\cdots+p_j-1}\langle\overline{d}_i(v\otimes\beta);\Theta\varphi_1,\dots,\Theta\varphi_j\rangle.$ □

From this, we immediately deduce Theorem 3. For a given a space X, recall that $\mathrm{dl}\,X$ (dl stands for differential length) is the least n, or infinite, for which there is a non trivial whitehead product of order n on $\pi(X_{\mathbb{Q}})$. This coincides with the least n for which d_n, the n-th part of the differential of the minimal model of X is non trivial. Another geometric description of this invariant is given in [6] in terms of the Ganea spaces of X.

Proof of Theorem 3. Assume $\mathrm{cat}_0X=m$. Then, by a deep result of Cornea [3], X has a finite dimensional model B for which any product of length greater than m of nonzero elements of B^+ vanishes. Hence, for $j>m$ and for all $v\otimes\beta$, given $\varphi_1,\dots,\varphi_j\in \mathrm{Der}\,(\Lambda V,B_+;\phi)$, $[\varphi_1,\dots,\varphi_j](v\otimes\beta)\in B^{>m}=0$. However, as $\mathrm{dl}\,Y>m$, in view of Theorem 12, this implies that $\overline{d}_j$ vanishes for all $j\geq 2$. This means that the differential on the minimal model vanishes and the theorem follows. □.

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