

THE INTEGRAL COHOMOLOGY RING OF $E_8/T^1 \cdot E_7$

MASAKI NAKAGAWA*

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ABSTRACT. We determine the integral cohomology ring of the homogeneous space $E_8/T^1 \cdot E_7$ by the Borel presentation and a method due to Toda. Then using the Gysin exact sequence associated with the circle bundle $S^1 \rightarrow E_8/E_7 \rightarrow E_8/T^1 \cdot E_7$, we also determine the integral cohomology ring of E_8/E_7 .

1. INTRODUCTION

Let G be a compact connected Lie group and H a centralizer of a toral subgroup. Then the homogeneous space G/H is called a *generalized flag manifold* and plays an important role in the modern mathematics such as algebraic topology, differential geometry and algebraic geometry. In fact, G/H admits a complex structure, Kähler structure and symplectic structure. In algebraic topology, it is a classical problem to determine the integral cohomology ring $H^*(G/H; \mathbb{Z})$ of G/H . Since the Chow ring of G/H , $A(G/H)$ is canonically isomorphic to $H^*(G/H; \mathbb{Z})$ ([10]), the determination of $H^*(G/H; \mathbb{Z})$ is of fundamental importance.

There are two descriptions of the cohomology ring $H^*(G/H; \mathbb{Z})$; One is the Borel presentation which uses the invariants of Weyl groups and the spectral sequence technique ([2]). The other is the Schubert presentation which uses the so called *Schubert calculus* ([1]). Using the Borel presentation of rational cohomology and the results on mod p cohomology $H^*(G; \mathbb{Z}/p\mathbb{Z})$ for all primes p , Toda initiated the research for computing the integral cohomology ring of G/H in [17]. Along the lines of his idea, the integral cohomology rings of various flag manifolds have been computed explicitly ([18], [19], [12], [11], [15]). On the other hand, recently H. Duan developed extensively a multiplicative formula of Schubert classes which is a generalization of the Littlewood-Richardson rule of Grassmann manifold ([8]). Furthermore, H. Duan

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and X. Zhao also computed the integral cohomology rings of the above flag manifolds independently of Toda's method ([9]). Until recently none of these two methods have been successful in computing the integral cohomology rings of homogeneous spaces of the exceptional Lie group E_8 . The exceptional Lie group E_8 contains a closed connected subgroup $T^1 \cdot E_7$ whose local type is $T^1 \times E_7$, where T^1 is a certain one dimensional torus (see [12], §2). It is obtained as a centralizer of a certain one dimensional torus (see 2.1). Hence the homogeneous space $E_8/T^1 \cdot E_7$ is a generalized flag manifold. In this paper, using the above method due to Borel and Toda, we compute the integral cohomology ring of $E_8/T^1 \cdot E_7$ explicitly.

The motivation of the current work is to study the cohomology of the irreducible symmetric space $EIX = E_8/S^3 \cdot E_7$ (see [12], §1), as well as the integral cohomology ring of the full flag manifold E_8/T , where T is a maximal torus in E_8 , and the Chow ring of the corresponding complex algebraic group ([13], [14]). In fact, we will use Theorem 3.8 to compute $H^*(E_8/T; \mathbb{Z})$, which is the only remaining case among G/T 's for G simple, in our forthcoming paper([16]). Moreover, the homogeneous space $E_8/T^1 \cdot E_7$ is a generating variety of E_8 in the sense of Bott ([5]) and its integral cohomology ring is needed to compute the Pontrjagin ring $H_*(\Omega E_8; \mathbb{Z})$, where ΩE_8 denotes the based loop space of E_8 .

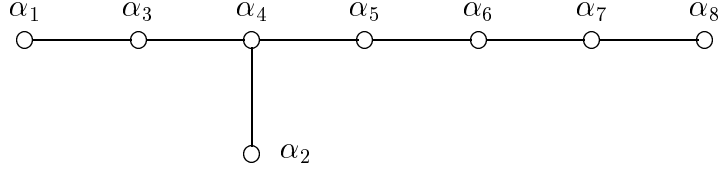
The paper is organized as follows: In §2, we compute the rings of invariants of the Weyl groups of E_8 and $T^1 \cdot E_7$. In §3, using these results and the Borel presentation of rational cohomology ring, we compute the rational cohomology ring of $E_8/T^1 \cdot E_7$. Then, by investigating the integral cohomology ring of E_8/T in low degrees, we determine the integral cohomology ring of $E_8/T^1 \cdot E_7$ explicitly. Furthermore, using the Gysin exact sequence associated with the circle bundle $S^1 \rightarrow E_8/E_7 \rightarrow E_8/T^1 \cdot E_7$, we also determine the integral cohomology ring of E_8/E_7 .

Throughout this paper, $\sigma_i(x_1, \dots, x_n)$ denotes the i -th elementary symmetric function in variables x_1, \dots, x_n .

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2. THE RATIONAL INVARIANT SUBALGEBRAS OF THE WEYL GROUPS

2.1. Notations. Let T be a maximal torus of E_8 . According to [7], the Dynkin diagram of E_8 is as follows:



where α_i 's are the simple roots. As usual we may regard each root as an element of $H^1(T; \mathbb{Z}) \cong H^2(BT; \mathbb{Z})$.

Let C_8 be the centralizer of a one dimensional torus determined by $\alpha_i = 0$ ($i \neq 8$). Then as shown in ([12], §2),

$$C_8 = T^1 \cdot E_7 \quad \text{and} \quad T^1 \cap E_7 \cong \mathbb{Z}/2\mathbb{Z}.$$

The Weyl groups $W(\cdot)$ of E_8, C_8 are respectively given as follows:

$$W(E_8) = \langle s_i \ (1 \leq i \leq 8) \rangle, \quad W(C_8) = \langle s_i \ (i \neq 8) \rangle,$$

where s_i denotes the simple reflection corresponding to the simple root α_i .

Let $\{\omega_i\}_{1 \leq i \leq 8}$ be the fundamental weights corresponding to the system of the simple roots $\{\alpha_i\}_{1 \leq i \leq 8}$. We also regard each weight as an element of $H^2(BT; \mathbb{Z})$ and then $\{\omega_i\}_{1 \leq i \leq 8}$ forms a basis of $H^2(BT; \mathbb{Z})$. The action of s_i 's on $\{\omega_i\}_{1 \leq i \leq 8}$ is given as follows:

$$s_i(\omega_k) = \begin{cases} \omega_i - \sum_{j=1}^8 \frac{2(\alpha_i | \alpha_j)}{(\alpha_j | \alpha_j)} \omega_j & \text{if } k = i, \\ \omega_k & \text{if } k \neq i. \end{cases}$$

Next we define the elements

$$\begin{aligned} t_8 &= \omega_8, \\ t_7 &= s_8(t_8) = \omega_7 - \omega_8, \\ t_6 &= s_7(t_7) = \omega_6 - \omega_7, \\ t_5 &= s_6(t_6) = \omega_5 - \omega_6, \\ t_4 &= s_5(t_5) = \omega_4 - \omega_5, \\ t_3 &= s_4(t_4) = \omega_2 + \omega_3 - \omega_4, \\ t_2 &= s_3(t_3) = \omega_1 + \omega_2 - \omega_3, \\ t_1 &= s_1(t_2) = -\omega_1 + \omega_2, \\ c_i &= \sigma_i(t_1, \dots, t_8), \\ t &= \omega_2 = \frac{1}{3}c_1. \end{aligned}$$

Then t and t_i 's span $H^2(BT; \mathbb{Z})$, since each ω_i is an integral linear combination of t and t_i 's and we have the following isomorphism:

$$H^*(BT; \mathbb{Z}) = \mathbb{Z}[t_1, \dots, t_8, t]/(c_1 - 3t).$$

Furthermore, the action of s_i 's on $\{t_i\}_{1 \leq i \leq 8}$ and t is given by TABLE 1, where blanks indicate the trivial action.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
t_1	t_2	$t - t_2 - t_3$						
t_2	t_1	$t - t_1 - t_3$	t_3					
t_3		$t - t_1 - t_2$	t_2	t_4				
t_4				t_3	t_5			
t_5					t_4	t_6		
t_6						t_5	t_7	
t_7							t_6	t_8
t_8								t_7
t		$2t - t_1 - t_2 - t_3$						

TABLE 1

Putting

$$\begin{aligned} u &= t_8, \\ \tau &= t - \frac{3}{2}u, \\ \tau_i &= t_i - \frac{1}{2}u \quad (1 \leq i \leq 7), \end{aligned}$$

we have

$$H^*(BT; \mathbb{Q}) = \mathbb{Q}[u, \tau, \tau_1, \dots, \tau_7]/(\bar{c}_1 - 3\tau)$$

for $\bar{c}_1 = \tau_1 + \dots + \tau_7$. The action of s_i 's on $\{\tau_i\}_{1 \leq i \leq 7}$ and τ is given by TABLE 2, where blanks also indicate the trivial action.

Since $E_7 \cap T = T'$ is a maximal torus of E_7 , we have a commutative diagram of natural maps:

$$(2.1) \quad \begin{array}{ccccc} E_7/T' & \xrightarrow{\sim} & C_8/T & \xrightarrow{i} & E_8/T \\ \downarrow & & \downarrow & & \downarrow \iota_0 \\ BT' & \xrightarrow{g} & BT & \xrightarrow{=} & BT. \end{array}$$

Since E_8 is 2-connected, $\iota_0^* : H^2(BT; \mathbb{Z}) \xrightarrow{\sim} H^2(E_8/T; \mathbb{Z})$. Under this isomorphism, we denote the ι_0^* -images of t_i ($1 \leq i \leq 8$), t by the same letters. Thus we have the generators t_i ($1 \leq i \leq 8$), $t \in H^2(E_8/T; \mathbb{Z})$

	s_1	s_2	s_3	s_4	s_5	s_6	s_7
τ_1	τ_2	$\tau - \tau_2 - \tau_3$					
τ_2	τ_1	$\tau - \tau_1 - \tau_3$	τ_3				
τ_3		$\tau - \tau_1 - \tau_2$	τ_2	τ_4			
τ_4				τ_3	τ_5		
τ_5					τ_4	τ_6	
τ_6						τ_5	τ_7
τ_7							τ_6
τ		$-\tau + \tau_4 + \tau_5 + \tau_6 + \tau_7$					
u							

TABLE 2

with a relation $c_1 = 3t$. We denote the generators t_i ($1 \leq i \leq 7$), x in [19] by t'_i ($1 \leq i \leq 7$), t' . Then, by a similar arguments to that in ([19], §1), we have

$$(2.2) \quad g^*(t_i) = t'_i \ (1 \leq i \leq 7), \quad g^*(t_8) = 0, \quad g^*(t) = t'.$$

2.2. Invariant subalgebra of $W(C_8)$. We recall the rational invariant forms for E_7 given in [19]. We put

$$x'_i = 2t'_i - t' \ (1 \leq i \leq 7), \quad x'_8 = t'.$$

Then the set

$$S' = \{\pm (x'_i + x'_j) \ (1 \leq i < j \leq 8)\} \subset H^2(BT'; \mathbb{Q})$$

is invariant under the action of $W(E_7)$. Thus we have $W(E_7)$ -invariant forms

$$(2.3) \quad I'_n = \sum_{y \in S'} y^n \in H^{2n}(BT'; \mathbb{Q})^{W(E_7)}.$$

Then direct computation using the same method as in ([19], §2) yields the following results:

$$(2.4) \quad \begin{aligned} I'_2 &= -2^5 \cdot 3(c'_2 - 4t'^2), \\ I'_6 &\equiv 2^8 \cdot 3^2(c'_3{}^2 + 8c'_6) \pmod{(t', \mathfrak{a}'_6)}, \\ I'_8 &\equiv 2^{12} \cdot 5(2c'_4{}^2 - 3c'_3c'_5) \pmod{(t', \mathfrak{a}'_8)}, \\ I'_{10} &\equiv 2^{12} \cdot 3^2 \cdot 5 \cdot 7(c'_5{}^2 - 4c'_3c'_7) \pmod{(t', \mathfrak{a}'_{10})}, \\ I'_{12} &\equiv 2^{15} \cdot 3^2 \cdot 5(-54c'_6{}^2 + 18c'_5c'_7 - c'_3c'_4c'_5) \pmod{(t', \mathfrak{a}'_{12})}, \\ I'_{14} &\equiv 2^{16} \cdot 3 \cdot 7 \cdot 11 \cdot 29(2c'_7{}^2 + 2c'_3c'_4c'_7 - c'_3c'_5c'_6) \pmod{(t', \mathfrak{a}'_{14})}, \\ I'_{18} &\equiv 2^{21} \cdot 5 \cdot 1229(-126c'_5c'_6c'_7 - 5c'_3c'_4c'_5c'_6) \pmod{(t', \mathfrak{a}'_{18})}, \end{aligned}$$

where $c'_i = \sigma_i(t'_1, \dots, t'_7)$ and \mathfrak{a}'_i denotes the ideal of $H^*(BT'; \mathbb{Q})$ generated by I'_j for $j < i$ with $j \in \{2, 6, 8, 10, 12, 14, 18\}$. We also recall the following result:

Proposition 2.1 ([19], Lemma 2.1). *The rational invariant subalgebra of the Weyl group $W(E_7)$ is given as follows:*

$$H^*(BT'; \mathbb{Q})^{W(E_7)} = \mathbb{Q}[I'_2, I'_6, I'_8, I'_{10}, I'_{12}, I'_{14}, I'_{18}].$$

TABLE 2 shows that the action of $W(C_8)$ on $\tau, \tau_1, \dots, \tau_7$ is the same as that of $W(E_7)$ on t', t'_1, \dots, t'_7 . Therefore we have

Lemma 2.2. *If we represent*

$$I'_n = \psi_n(t', t'_1, \dots, t'_7) \in H^{2n}(BT'; \mathbb{Q})^{W(E_7)},$$

the rational invariant subalgebra of the Weyl group $W(C_8)$ is given as follows:

$$H^*(BT; \mathbb{Q})^{W(C_8)} = \mathbb{Q}[u, J_2, J_6, J_8, J_{10}, J_{12}, J_{14}, J_{18}],$$

where

$$J_n = \psi_n(\tau, \tau_1, \dots, \tau_7) \in H^{2n}(BT; \mathbb{Q})^{W(C_8)}.$$

2.3. Invariant subalgebra of $W(E_8)$. We put

$$\xi_i = 2t_i - \frac{2}{3}t \quad (1 \leq i \leq 8), \quad \xi_9 = -\frac{2}{3}t.$$

Then we see from TABLE 1 that the set

$$S = \{\pm (\xi_i - \xi_j) \ (1 \leq i < j \leq 9), \pm (\xi_i + \xi_j + \xi_k) \ (1 \leq i < j < k \leq 9)\}$$

is invariant under the action of $W(E_8)$. In fact, S is an orbit of $2\omega_8$ under the action of $W(E_8)$. Thus we have $W(E_8)$ -invariant forms

$$I_n = \sum_{y \in S} y^n \in H^{2n}(BT; \mathbb{Q})^{W(E_8)}.$$

Let us compute I_n 's in the following way; We put

$$s_n = \xi_1^n + \dots + \xi_9^n, \quad d_n = \sigma_n(\xi_1, \dots, \xi_9).$$

Then s_n 's and d_n 's are related to each other by the Newton formula:

$$(2.5) \quad s_n = \sum_{i=1}^{n-1} (-1)^{i-1} s_{n-i} d_i + (-1)^{n-1} n d_n \quad (d_n = 0 \text{ for } n > 9).$$

Note that

$$s_0 = 9, \quad s_1 = d_1 = \xi_1 + \dots + \xi_9 = 0.$$

Then

$$\begin{aligned}
\sum_n \frac{I_n}{n!} &= \sum_{i < j} e^{\xi_i - \xi_j} + \sum_{i < j} e^{-\xi_i + \xi_j} \sum_{i < j < k} e^{\xi_i + \xi_j + \xi_k} + \sum_{i < j < k} e^{-\xi_i - \xi_j - \xi_k} \\
&= \left(\sum_i e^{\xi_i} \right) \left(\sum_j e^{-\xi_j} \right) - 9 + \frac{1}{3} \left(\sum_i e^{3\xi_i} + \sum_i e^{-3\xi_i} \right) \\
&\quad - \frac{1}{2} \left\{ \left(\sum_i e^{\xi_i} \right) \left(\sum_i e^{2\xi_i} \right) + \left(\sum_i e^{-\xi_i} \right) \left(\sum_i e^{-2\xi_i} \right) \right\} \\
&\quad + \frac{1}{6} \left\{ \left(\sum_i e^{\xi_i} \right)^3 + \left(\sum_i e^{-\xi_i} \right)^3 \right\} \\
&= \left(\sum_n \frac{s_n}{n!} \right) \left(\sum_m \frac{(-1)^m s_m}{m!} \right) - 9 + \frac{1}{3} \left(\sum_n \frac{3^n s_n}{n!} + \sum_n \frac{(-1)^n 3^n s_n}{n!} \right) \\
&\quad - \frac{1}{2} \left\{ \left(\sum_n \frac{s_n}{n!} \right) \left(\sum_m \frac{2^m s_m}{m!} \right) + \left(\sum_n \frac{(-1)^n s_n}{n!} \right) \left(\sum_m \frac{(-1)^m 2^m s_m}{m!} \right) \right\} \\
&\quad + \frac{1}{6} \left\{ \left(\sum_n \frac{s_n}{n!} \right)^3 + \left(\sum_n \frac{(-1)^n s_n}{n!} \right)^3 \right\}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
(2.6) \quad I_n &= \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} s_i s_{n-i} + 2 \cdot 3^{n-1} s_n \\
&\quad - \sum_{i=0}^n \binom{n}{i} 2^{n-i} s_i s_{n-i} + \frac{1}{3} \sum_{i=0}^n \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} s_i s_j s_{n-i-j}
\end{aligned}$$

for n even. On the other hand, since $\xi_i = 2t_i - \frac{2}{3}t$ ($1 \leq i \leq 8$) and $\xi_9 = -\frac{2}{3}t$, we have

$$\begin{aligned}
\sum_{n=0}^9 d_n &= \prod_{i=1}^9 (1 + \xi_i) = \left(1 - \frac{2}{3}t \right) \prod_{i=1}^8 \left(1 - \frac{2}{3}t + 2t_i \right) \\
&= \left(1 - \frac{2}{3}t \right) \sum_{i=0}^8 \left(1 - \frac{2}{3}t \right)^{8-i} 2^i c_i
\end{aligned}$$

and therefore

$$(2.7) \quad d_n = \sum_{i=0}^n \left\{ \binom{8-i}{n-i} + \binom{8-i}{n-i-1} \right\} \left(-\frac{2}{3}t \right)^{n-i} 2^i c_i$$

for $1 \leq n \leq 9$. Using (2.6), (2.5) and (2.7), we can compute I_n 's explicitly and obtain the following results:

Lemma 2.3.

$$\begin{aligned} \text{(i)} \quad I_2 &= -2^5 \cdot 3 \cdot 5(c_2 - 4t^2), \\ \text{(ii)} \quad I_8 &\equiv 2^{14} \cdot 3 \cdot 5 \{ -18c_8 - 3c_3c_5 + 2c_4^2 + t(12c_7 - 3c_3c_4) \\ &\quad + t^2(-6c_6 + 3c_3^2) + 12t^3c_5 + 2t^4c_4 - 12t^5c_3 + 14t^8 \} \pmod{\tilde{\mathfrak{a}}_8}, \\ \text{(iii)} \quad I_{12} &\equiv 2^{18} \cdot 3^5 \cdot 7 \left(c_6^2 - \frac{5}{3}c_5c_7 + \frac{5}{54}c_3c_4c_5 - \frac{1}{6}c_3^2c_6 + \frac{1}{24}c_3^4 \right) \\ &\quad \pmod{(t, c_8, \tilde{\mathfrak{a}}_{12})}, \\ \text{(iv)} \quad I_{14} &\equiv 2^{20} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \left(c_7^2 - \frac{1}{2}c_3c_5c_6 + \frac{1}{3}c_3c_4c_7 + \frac{1}{6}c_4c_5^2 \right) \\ &\quad \pmod{(t, c_8, \tilde{\mathfrak{a}}_{14})}, \\ \text{(v)} \quad I_{18} &\equiv -2^{23} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \left(c_3^6 - 7c_3^4c_6 + \frac{29}{9}c_3^3c_4c_5 + 182c_3^2c_5c_7 \right. \\ &\quad \left. + 75c_3c_5^3 - \frac{476}{3}c_3c_4c_5c_6 - 24c_5c_6c_7 \right) \pmod{(t, c_8, \tilde{\mathfrak{a}}_{18})}, \\ \text{(vi)} \quad I_{20} &\equiv 2^{27} \cdot 3^3 \cdot 5^2 \cdot 11 \cdot 17 \cdot 41 \left(\frac{1}{144}c_5^4 - \frac{1}{18}c_3c_5^2c_7 - \frac{1}{54}c_3^2c_4c_5^2 \right. \\ &\quad \left. - \frac{1}{27}c_3^3c_4c_7 + \frac{1}{18}c_3^3c_5c_6 \right) \pmod{(t, c_8, \tilde{\mathfrak{a}}_{20})}, \\ \text{(vii)} \quad I_{24} &\equiv 2^{32} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 19 \cdot 199 \left(\frac{31}{8640}c_3^5c_4c_5 + \frac{1}{480}c_3^4c_5c_7 \right. \\ &\quad + \frac{337}{25920}c_3^3c_5^3 - \frac{71}{4320}c_3^3c_4c_5c_6 + \frac{31}{240}c_3^2c_5c_6c_7 + \frac{31}{480}c_3c_5^3c_6 \\ &\quad \left. - \frac{22}{135}c_3c_4c_5^2c_7 - \frac{1}{120}c_4c_5^4 \right) \pmod{(t, c_8, \tilde{\mathfrak{a}}_{20})}, \\ \text{(viii)} \quad I_{30} &\equiv 2^{38} \cdot 3^4 \cdot 5^5 \cdot 7 \cdot 11 \cdot 13 \cdot 61 \left(-\frac{599}{51480}c_3^5c_4c_5c_6 + \frac{47}{34560}c_3^5c_5^3 \right. \\ &\quad + \frac{1519}{25920}c_3^4c_5c_6c_7 + \frac{6293}{7290}c_3^3c_4c_5^2c_7 - \frac{32537}{25920}c_3^3c_5^3c_6 + \frac{189919}{466560}c_3^2c_4c_5^4 \\ &\quad \left. + \frac{2012}{1215}c_3c_4c_5^2c_6c_7 - \frac{16693}{25920}c_3c_5^4c_7 - \frac{223}{6480}c_4c_5^4c_6 - \frac{1}{1728}c_5^6 \right) \\ &\quad \pmod{(t, c_8, \tilde{\mathfrak{a}}_{20})}, \end{aligned}$$

where $c_i = \sigma_i(t_1, \dots, t_8)$ and $\tilde{\mathfrak{a}}_i$ denotes the ideal of $H^*(BT; \mathbb{Q})$ generated by I_j for $j < i$ with $j \in \{2, 8, 12, 14, 18, 20, 24, 30\}$.

Next consider the following elements of $H^*(BT; \mathbb{Q})$:

$$(2.8) \quad \begin{aligned} \tilde{I}_{20} &= 9u^{20} + 45u^{14}v + 12u^{12}w + 60u^8v^2 + 30u^4vw + 10u^2v^3 + 3w^2, \\ \tilde{I}_{24} &= 11u^{24} + 60u^{18}v + 21u^{14}w + 105u^{12}v^2 + 60u^8vw + 60u^6v^3 + 9u^4w^2 \\ &\quad + 30u^2v^2w + 5v^4, \\ \tilde{I}_{30} &= -9u^{30} - 24u^{24}v - 12u^{20}w + 36u^{14}vw - 40u^{12}v^3 - 12u^{10}w^2 + 120u^8v^2w \\ &\quad - 140u^6v^4 + 24u^4vw^2 - 40u^2v^3w - 16v^5 - 8w^3, \end{aligned}$$

where

$$(2.9) \quad \begin{aligned} u &= t_8, \\ v &= \frac{1}{46080}J_6 - \frac{273}{640}u^6, \\ w &= \frac{1}{15482880}J_{10} - \frac{55}{24}u^4v - \frac{666919}{645120}u^{10}. \end{aligned}$$

Remark 2.4. We have chosen the elements u, v, w so that they become the generators of the integral cohomology ring of $E_8/T^1 \cdot E_7$ (see 3.3).

In order to find the relations among J_n 's, I_n 's and $\tilde{I}_{20}, \tilde{I}_{24}, \tilde{I}_{30}$, we consider the ring of invariants:

$$A = H^*(BT; \mathbb{Q})^{\langle s_1, s_3, \dots, s_7 \rangle}.$$

Then A is a subalgebra of $H^*(BT; \mathbb{Q})$ containing $H^*(BT; \mathbb{Q})^{W(C_8)}$. More explicitly, we have

$$(2.10) \quad A = \mathbb{Q}[u, c_1, c_2, \dots, c_7].$$

In fact, we can show (2.10) as follows; Putting

$$\tilde{c}_i = \sigma_i(t_1, \dots, t_7),$$

we have

$$c_n = \tilde{c}_n + u\tilde{c}_{n-1} \quad (1 \leq n \leq 8),$$

since

$$\sum_{n=0}^8 c_n = \prod_{i=1}^8 (1 + t_i) = (1 + u) \prod_{i=1}^7 (1 + t_i) = (1 + u) \sum_{n=0}^7 \tilde{c}_n.$$

Conversely, one can express

$$\tilde{c}_n = c_n - uc_{n-1} + u^2c_{n-2} - \dots + (-1)^n u^n \quad (1 \leq n \leq 7).$$

In particular, the following relation holds:

$$(2.11) \quad c_8 = uc_7 - u^2c_6 + u^3c_5 - u^4c_4 + u^5c_3 - u^6c_2 + u^7c_1 - u^8.$$

Therefore, by TABLE 1, we have

$$\begin{aligned}
A &= H^*(BT; \mathbb{Q})^{\langle s_1, s_3, \dots, s_7 \rangle} \\
&= \mathbb{Q}[t_1, t_2, \dots, t_7, u]^{\langle s_1, s_3, \dots, s_7 \rangle} \\
&= \mathbb{Q}[u, \tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_7] \\
&= \mathbb{Q}[u, c_1, c_2, \dots, c_8] / (c_8 - uc_7 + u^2c_6 - \dots + u^8) \\
&= \mathbb{Q}[u, c_1, c_2, \dots, c_7],
\end{aligned}$$

which has shown (2.10).

Denote by

$$\mathfrak{a}_i \subset A \quad (\text{resp. } \mathfrak{b}_i \subset H^*(BT; \mathbb{Q})^{W(C_8)}),$$

the ideal of A (resp. of $H^*(BT; \mathbb{Q})^{W(C_8)}$) generated by I_j 's for $j < i$ where $j \in \{2, 8, 12, 14, 18, 20, 24, 30\}$.

The remainder of this section is devoted to proving the next lemma:

Lemma 2.5. *In $H^*(BT; \mathbb{Q})^{W(C_8)} = \mathbb{Q}[u, J_2, J_6, J_8, J_{10}, J_{12}, J_{14}, J_{18}]$, we have*

$$\begin{aligned}
\text{(i)} \quad I_2 &= 5J_2 + 120u^5, & I_8 &= 2^2 \cdot 3J_8 + (\text{decomp.}), \\
I_{12} &= -2^2 \cdot 7J_{12} + (\text{decomp.}), & I_{14} &= \frac{2^3 \cdot 3 \cdot 5}{29}J_{14} + (\text{decomp.}), \\
I_{18} &= -\frac{2^4 \cdot 3 \cdot 5^2 \cdot 7 \cdot 13}{1229}J_{18} + (\text{decomp.}),
\end{aligned}$$

where (decomp.) means decomposable elements.

$$\begin{aligned}
\text{(ii)} \quad I_{20} &\equiv 2^{27} \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 17 \cdot 41 \tilde{I}_{20} \pmod{\mathfrak{b}_{20}}, \\
I_{24} &\equiv 2^{32} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 199 \tilde{I}_{24} \pmod{\mathfrak{b}_{20}}, \\
I_{30} &\equiv 2^{35} \cdot 3^4 \cdot 5^5 \cdot 7 \cdot 11 \cdot 13 \cdot 61 \tilde{I}_{30} \pmod{\mathfrak{b}_{20}}.
\end{aligned}$$

Suppose Lemma 2.5 for the moment. Then we have the following:

Lemma 2.6. *The rational invariant subalgebra of the Weyl group $W(E_8)$ is given as follows:*

$$H^*(BT; \mathbb{Q})^{W(E_8)} = \mathbb{Q}[I_2, I_8, I_{12}, I_{14}, I_{18}, I_{20}, I_{24}, I_{30}].$$

Proof. By Lemma 2.5, I_i is not a polynomial of I_j 's for $j < i$ where $i = 2, 8, 12, 14, 18, 20, 24, 30$. Since

$$H^*(BT; \mathbb{Q})^{W(E_8)} \cong H^*(BE_8; \mathbb{Q}) = \mathbb{Q}[y_4, y_{16}, y_{24}, y_{28}, y_{36}, y_{40}, y_{48}, y_{60}]$$

with $\deg(y_i) = i$, we have the required result. \square

Proof of Lemma 2.5 (i). Since $\tau_i = t_i - \frac{1}{2}u$ ($1 \leq i \leq 7$) and $\tau = t - \frac{3}{2}u$, we have

$$\tau_i \equiv t_i \pmod{u}, \quad \tau \equiv t \pmod{u}.$$

Therefore, putting

$$\bar{c}_i = \sigma_i(\tau_1, \dots, \tau_7),$$

we obtain

$$c_n \equiv \bar{c}_n \pmod{u}, \quad c_8 \equiv 0 \pmod{u}.$$

Then, in view of (2.4) and Lemma 2.2, we have

$$(2.12) \quad \begin{aligned} J_2 &= -2^5 \cdot 3(\bar{c}_2 - 4\tau^2) \\ &\equiv -2^5 \cdot 3c_2 \pmod{(t, u)}, \\ J_6 &\equiv 2^8 \cdot 3^2(\bar{c}_3^2 + 8\bar{c}_6) \pmod{(\tau, \bar{\mathfrak{a}}_6)} \\ &\equiv 2^8 \cdot 3^2(c_3^2 + 8c_6) \pmod{(t, u, \bar{\mathfrak{a}}_6)}, \\ J_8 &\equiv 2^{12} \cdot 5(2\bar{c}_4^2 - 3\bar{c}_3\bar{c}_5) \pmod{(\tau, \bar{\mathfrak{a}}_8)} \\ &\equiv 2^{12} \cdot 5(2c_4^2 - 3c_3c_5) \pmod{(t, u, \bar{\mathfrak{a}}_8)}, \\ J_{10} &\equiv 2^{12} \cdot 3^2 \cdot 5 \cdot 7(\bar{c}_5^2 - 4\bar{c}_3\bar{c}_7) \pmod{(\tau, \bar{\mathfrak{a}}_{10})} \\ &\equiv 2^{12} \cdot 3^2 \cdot 5 \cdot 7(c_5^2 - 4c_3c_7) \pmod{(t, u, \bar{\mathfrak{a}}_{10})}, \\ J_{12} &\equiv 2^{15} \cdot 3^2 \cdot 5(-54\bar{c}_6^2 + 18\bar{c}_5\bar{c}_7 - \bar{c}_3\bar{c}_4\bar{c}_5) \pmod{(\tau, \bar{\mathfrak{a}}_{12})} \\ &\equiv 2^{15} \cdot 3^2 \cdot 5(-54c_6^2 + 18c_5c_7 - c_3c_4c_5) \pmod{(t, u, \bar{\mathfrak{a}}_{12})}, \\ J_{14} &\equiv 2^{16} \cdot 3 \cdot 7 \cdot 11 \cdot 29(2\bar{c}_7^2 + 2\bar{c}_3\bar{c}_4\bar{c}_7 - \bar{c}_3\bar{c}_5\bar{c}_6) \pmod{(\tau, \bar{\mathfrak{a}}_{14})} \\ &\equiv 2^{16} \cdot 3 \cdot 7 \cdot 11 \cdot 29(2c_7^2 + 2c_3c_4c_7 - c_3c_5c_6) \pmod{(t, u, \bar{\mathfrak{a}}_{14})}, \\ J_{18} &\equiv 2^{21} \cdot 5 \cdot 1229(-126\bar{c}_5\bar{c}_6\bar{c}_7 - 5\bar{c}_3\bar{c}_4\bar{c}_5\bar{c}_6) \pmod{(\tau, \bar{\mathfrak{a}}_{18})} \\ &\equiv 2^{21} \cdot 5 \cdot 1229(-126c_5c_6c_7 - 5c_3c_4c_5c_6) \pmod{(t, u, \bar{\mathfrak{a}}_{18})}, \end{aligned}$$

where $\bar{\mathfrak{a}}_i$ denotes the ideal of $H^*(BT; \mathbb{Q})^{W(C_8)}$ generated by J_j 's for $j < i$ with $j \in \{2, 6, 8, 10, 12, 14, 18\}$.

Now we prove the last formula of (i): For degree reasons, in $H^*(BT; \mathbb{Q})^{W(C_8)} = \mathbb{Q}[u, J_2, J_6, J_8, J_{10}, J_{12}, J_{14}, J_{18}]$, we can put

$$(2.13) \quad I_{18} = \alpha_{18}J_{18} + (\text{decomp.})$$

for some $\alpha_{18} \in \mathbb{Q}$. On the other hand, by using (2.10) and (2.12), we have

$$\begin{aligned}
A/(t, u, \bar{\mathbf{a}}_{18}) &= A/(t, u, J_2, J_6, J_8, J_{10}, J_{12}, J_{14}) \\
&= \mathbb{Q}[u, c_1, c_2, c_3, c_4, c_5, c_6, c_7] \\
&\quad \Big/ \left(\begin{array}{l} t, u, c_2, c_3^2 + 8c_6, c_4^2 - \frac{3}{2}c_3c_5, c_5^2 - 4c_3c_7, \\ c_6^2 - \frac{1}{3}c_5c_7 + \frac{1}{54}c_3c_4c_5, c_7^2 + c_3c_4c_7 - \frac{1}{2}c_3c_5c_6 \end{array} \right) \\
&= \mathbb{Q}[c_3, c_4, c_5, c_6, c_7] \\
&\quad \Big/ \left(\begin{array}{l} c_3^2 + 8c_6, c_4^2 - \frac{3}{2}c_3c_5, c_5^2 - 4c_3c_7, \\ c_6^2 - \frac{1}{3}c_5c_7 + \frac{1}{54}c_3c_4c_5, c_7^2 + c_3c_4c_7 - \frac{1}{2}c_3c_5c_6 \end{array} \right).
\end{aligned}$$

We consider (2.13) in the ring $A/(t, u, \bar{\mathbf{a}}_{18})$. Then, by Lemma 2.3 (v) and (2.12), we have

$$\begin{aligned}
I_{18} &\equiv -2^{25} \cdot 3 \cdot 5^3 \cdot 7 \cdot 13(-126c_5c_6c_7 - 5c_3c_4c_5c_6), \\
J_{18} &\equiv 2^{21} \cdot 5 \cdot 1229(-126c_5c_6c_7 - 5c_3c_4c_5c_6).
\end{aligned}$$

Therefore $\alpha_{18} = -\frac{2^4 \cdot 3 \cdot 5^2 \cdot 7 \cdot 13}{1229}$. Similar tedious computation gives the other formulas. \square

Before proceeding the proof of Lemma 2.5 (ii), we need the following lemma:

Lemma 2.7. *Explicit forms of $W(C_8)$ -invariants J_6, J_{10} are given as follows:*

$$\begin{aligned}
J_6 &= 2^8 \cdot 3 \{ 24\bar{c}_6 + 3\bar{c}_3^2 - 4\bar{c}_2\bar{c}_4 - 2\bar{c}_2^3 + (-12\bar{c}_5 - 6\bar{c}_2\bar{c}_3)\tau + (31\bar{c}_2^2 + 16\bar{c}_4)\tau^2 \\
&\quad + 12\bar{c}_3\tau^3 - 136\bar{c}_2\tau^4 + 188\tau^6 \}, \\
J_{10} &= 2^{12} \cdot 3 \{ 105\bar{c}_5^2 - 420\bar{c}_3\bar{c}_7 + 90\bar{c}_2\bar{c}_3\bar{c}_5 - 60\bar{c}_2\bar{c}_4^2 + 300\bar{c}_2^2\bar{c}_6 + 15\bar{c}_2^2\bar{c}_3^2 \\
&\quad - 20\bar{c}_2^3\bar{c}_4 - 2\bar{c}_2^5 + (-30\bar{c}_2^3\bar{c}_3 - 330\bar{c}_2^2\bar{c}_5 + 480\bar{c}_2\bar{c}_7 + 90\bar{c}_2\bar{c}_3\bar{c}_4 - 210\bar{c}_4\bar{c}_5)\tau \\
&\quad + (270\bar{c}_2^2\bar{c}_4 - 210\bar{c}_2\bar{c}_3^2 - 150\bar{c}_3\bar{c}_5 - 2220\bar{c}_2\bar{c}_6 + 75\bar{c}_2^4 + 345\bar{c}_4^2)\tau^2 \\
&\quad + (480\bar{c}_2^2\bar{c}_3 - 570\bar{c}_3\bar{c}_4 + 2070\bar{c}_2\bar{c}_5 - 660\bar{c}_7)\tau^3 \\
&\quad + (-1050\bar{c}_2\bar{c}_4 + 4080\bar{c}_6 - 950\bar{c}_2^3 + 705\bar{c}_3^2)\tau^4 + (-2250\bar{c}_2\bar{c}_3 - 3420\bar{c}_5)\tau^5 \\
&\quad + (1580\bar{c}_4 + 5165\bar{c}_2^2)\tau^6 + 2820\bar{c}_3\tau^7 - 12360\bar{c}_2\tau^8 + 10868\tau^{10} \},
\end{aligned}$$

where $\bar{c}_i = \sigma_i(\tau_1, \dots, \tau_7)$.

Proof. Since J_n has the same expression as I'_n replacing t', c'_i with τ, \bar{c}_i (Lemma 2.2), we have to compute the $W(E_7)$ -invariant forms I'_6, I'_{10} explicitly. But this can be done from the data in ([19], §2). \square

We can rewrite J_6, J_{10} in terms of t, u, c_i ($2 \leq i \leq 7$); Since $u = t_8$ and $\tau_i = t_i - \frac{1}{2}u$ ($1 \leq i \leq 7$), we have

$$\begin{aligned} \left(1 + \frac{1}{2}u\right) \sum_{n=0}^7 \bar{c}_n &= \left(1 + \frac{1}{2}u\right) \prod_{i=1}^7 (1 + \tau_i) = \left(1 + \frac{1}{2}u\right) \prod_{i=1}^7 \left(1 - \frac{1}{2}u + t_i\right) \\ &= \prod_{i=1}^8 \left(1 - \frac{1}{2}u + t_i\right) = \sum_{i=0}^8 \left(1 - \frac{1}{2}u\right)^{8-i} c_i \end{aligned}$$

and hence

$$\bar{c}_n + \frac{1}{2}u\bar{c}_{n-1} = \sum_{i=0}^n \binom{8-i}{n-i} \left(-\frac{1}{2}\right)^{n-i} c_i u^{n-i} \quad (1 \leq n \leq 7).$$

From this, we obtain

(2.14)

$$\begin{aligned} \bar{c}_1 &= 3t - \frac{9}{2}u, \\ \bar{c}_2 &= c_2 - 12tu + \frac{37}{4}u^2, \\ \bar{c}_3 &= c_3 - \frac{7}{2}c_2u + \frac{87}{4}tu^2 - \frac{93}{8}u^3, \\ \bar{c}_4 &= c_4 - 3c_3u + \frac{11}{2}c_2u^2 - 24tu^3 + \frac{163}{16}u^4, \\ \bar{c}_5 &= c_5 - \frac{5}{2}c_4u + 4c_3u^2 - \frac{21}{4}c_2u^3 + \frac{297}{16}tu^4 - \frac{219}{32}u^5, \\ \bar{c}_6 &= c_6 - 2c_5u + \frac{11}{4}c_4u^2 - \frac{13}{4}c_3u^3 + \frac{57}{16}c_2u^4 - \frac{45}{4}tu^5 + \frac{247}{64}u^6, \\ \bar{c}_7 &= c_7 - \frac{3}{2}c_6u + \frac{7}{4}c_5u^2 - \frac{15}{8}c_4u^3 + \frac{31}{16}c_3u^4 - \frac{63}{32}c_2u^5 + \frac{381}{64}tu^6 - \frac{255}{128}u^7. \end{aligned}$$

On the other hand, by Lemma 2.3 (ii) and (2.11), we have

$$\begin{aligned} I_8 &\equiv 2^{14} \cdot 3 \cdot 5(2c_4^2 - 3c_3c_5) \pmod{(t, c_8, \mathfrak{a}_8)}, \\ c_8 &= uc_7 - u^2c_6 + u^3c_5 - u^4c_4 + u^5c_3 - u^6c_2 + u^7c_1 - u^8 \\ &\equiv uc_7 - u^2c_6 + u^3c_5 - u^4c_4 + u^5c_3 - u^8 \pmod{(t, \mathfrak{a}_8)}, \end{aligned}$$

and hence

$$\begin{aligned} c_4^2 &\equiv \frac{3}{2}c_3c_5 \pmod{(t, c_8, \mathfrak{a}_{12})}, \\ u^8 &\equiv uc_7 - u^2c_6 + u^3c_5 - u^4c_4 + u^5c_3 \pmod{(t, c_8, \mathfrak{a}_8)}. \end{aligned}$$

Therefore, by (2.9) and Lemma 2.7, we obtain
(2.15)

$$\begin{aligned} v &= \frac{1}{46080}J_6 - \frac{273}{640}u^6 \\ &\equiv \frac{2}{5}c_6 + \frac{1}{20}c_3^2 - \frac{1}{2}c_5u + \frac{1}{3}c_4u^2 - \frac{1}{2}c_3u^3 \pmod{(t, \mathfrak{a}_8)}, \\ w &= \frac{1}{15482880}J_{10} - \frac{55}{24}u^4v - \frac{666919}{645120}u^{10} \\ &\equiv \frac{1}{12}c_5^2 - \frac{1}{3}c_3c_7 + \left(\frac{1}{2}c_3c_6 - \frac{1}{6}c_4c_5\right)u - \frac{1}{6}c_3c_5u^2 + \left(-c_7 + \frac{1}{3}c_3c_4\right)u^3 \\ &\quad - \frac{1}{2}c_3^2u^4 + \frac{1}{3}c_4u^6 + \frac{1}{2}c_3u^7 \pmod{(t, c_8, \mathfrak{a}_{12})}. \end{aligned}$$

Under these preparations, we will prove Lemma 2.5 (ii).

Proof of Lemma 2.5 (ii). First note that

$$\begin{aligned} H^*(BT; \mathbb{Q})^{W(C_8)} &= \mathbb{Q}[u, J_2, J_6, J_8, J_{10}, J_{12}, J_{14}, J_{18}] \\ &= \mathbb{Q}[u, I_2, v, I_8, w, I_{12}, I_{14}, I_{18}] \end{aligned}$$

by (i). Since $I_{20} \in H^*(BT; \mathbb{Q})^{W(E_8)} \subset H^*(BT; \mathbb{Q})^{W(C_8)}$, we can put

$$\begin{aligned} (*) \quad I_{20} &\equiv 2^{27} \cdot 3^3 \cdot 5^2 \cdot 11 \cdot 17 \cdot 41(\lambda_1u^{20} + \lambda_2u^{14}v + \lambda_3u^8v^2 + \lambda_4u^2v^3 \\ &\quad + \lambda_5u^{10}w + \lambda_6u^4vw + \lambda_7w^2) \pmod{\mathfrak{b}_{20}} \end{aligned}$$

for some $\lambda_i \in \mathbb{Q}$.

In order to determine the coefficients λ_i , we need the following lemma, which is directly verified by making use of (2.10) and Lemma 2.3.

Lemma 2.8.

$$\begin{aligned} A/(t, c_8, \mathfrak{a}_{20}) &= A/(t, I_2, c_8, I_8, I_{12}, I_{14}, I_{18}) \\ &= \mathbb{Q}[u, c_3, c_4, c_5, c_6, c_7]/J, \end{aligned}$$

where J is the ideal generated by

$$\begin{aligned}
& uc_7 - u^2c_6 + u^3c_5 - u^4c_4 + u^5c_3 - u^8, \\
& c_4^2 - \frac{3}{2}c_3c_5, \\
& c_6^2 - \frac{5}{3}c_5c_7 + \frac{5}{54}c_3c_4c_5 - \frac{1}{6}c_3^2c_6 + \frac{1}{24}c_3^4, \\
& c_7^2 - \frac{1}{2}c_3c_5c_6 + \frac{1}{3}c_3c_4c_7 + \frac{1}{6}c_4c_5^2, \\
& c_3^6 - 7c_3^4c_6 + \frac{29}{9}c_3^3c_4c_5 + 182c_3^2c_5c_7 + 75c_3c_5^3 - \frac{476}{3}c_3c_4c_5c_6 - 24c_5c_6c_7.
\end{aligned}$$

In particular, $A/(t, c_8, \mathfrak{a}_{20})$ has a basis $\{u^i c_3^j c_4^k c_5^l c_6^m c_7^n \mid (0 \leq i \leq 7, 0 \leq j \leq 5, 0 \leq l, 0 \leq k, m, n \leq 1)\}$ as a \mathbb{Q} -vector space.

Now we consider the relation $(*)$ in the ring $A/(t, c_8, \mathfrak{a}_{20})$. By Lemma 2.3 (vi), we have

$$I_{20} \equiv 2^{27} \cdot 3^3 \cdot 5^2 \cdot 11 \cdot 17 \cdot 41 \left(\frac{1}{144}c_5^4 - \frac{1}{18}c_3c_5^2c_7 - \frac{1}{54}c_3^2c_4c_5^2 - \frac{1}{27}c_3^3c_4c_7 + \frac{1}{18}c_3^3c_5c_6 \right).$$

On the other hand, using (2.15) and Lemma 2.8, we can rewrite each monomial in the right hand side of $(*)$. For example, we have

$$\begin{aligned}
w^2 &\equiv \frac{1}{144}c_5^4 - \frac{1}{18}c_3c_5^2c_7 - \frac{1}{54}c_3^2c_4c_5^2 + \frac{1}{18}c_3^3c_5c_6 - \frac{1}{27}c_3^3c_4c_7 \\
&\quad - \frac{1}{12}u^7c_3^2c_7 + \frac{1}{12}u^7c_3c_5^2 + \frac{1}{24}u^6c_3^3c_5 + \frac{1}{3}u^6c_3c_5c_6 - \frac{5}{9}u^6c_3c_4c_7 \\
&\quad + \dots
\end{aligned}$$

Then using the second half of Lemma 2.8, the coefficients in $(*)$ are obtained as follows:

$$\lambda_1 = 3, \lambda_2 = 15, \lambda_3 = 20, \lambda_4 = \frac{10}{3}, \lambda_5 = 4, \lambda_6 = 10, \lambda_7 = 1.$$

Thus we have obtained

$$\begin{aligned}
I_{20} &\equiv 2^{27} \cdot 3^3 \cdot 5^2 \cdot 11 \cdot 17 \cdot 41 \left(3u^{20} + 15u^{14}v + 20u^8v^2 + \frac{10}{3}u^2v^3 + 4u^{10}w \right. \\
&\quad \left. + 10u^4vw + w^2 \right) \\
&\equiv 2^{27} \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 17 \cdot 41 (9u^{20} + 45u^{14}v + 60u^8v^2 + 10u^2v^3 + 12u^{10}w \\
&\quad + 30u^4vw + 3w^2) \\
&\equiv 2^{27} \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 17 \cdot 41 \tilde{I}_{20} \pmod{\mathfrak{b}_{20}}.
\end{aligned}$$

Putting

$$(**) \quad I_{24} \equiv 2^{32} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 19 \cdot 199 \left(\mu_1 u^{24} + \mu_2 u^{18} v + \mu_3 u^{12} v^2 + \mu_4 u^6 v^3 \right. \\ \left. + \mu_5 v^4 + \mu_6 u^{14} w + \mu_7 u^8 v w + \mu_8 u^2 v^2 w + \mu_9 u^4 w^2 \right) \pmod{\mathfrak{b}_{2\circ}}$$

for some $\mu_i \in \mathbb{Q}$, we will proceed quite similarly. By Lemma 2.3 (vii), we have

$$I_{24} \equiv 2^{32} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 19 \cdot 199 \left(\frac{31}{8640} c_3^5 c_4 c_5 + \frac{1}{480} c_3^4 c_5 c_7 + \frac{337}{25920} c_3^3 c_5^3 \right. \\ \left. - \frac{71}{4320} c_3^3 c_4 c_5 c_6 + \frac{31}{240} c_3^3 c_5 c_6 c_7 + \frac{31}{480} c_3 c_5^3 c_6 - \frac{22}{135} c_3 c_4 c_5^2 c_7 - \frac{1}{120} c_4 c_5^4 \right).$$

On the other hand, in $A/(t, c_8, \mathfrak{a}_{2\circ})$, we have, for example,

$$v^4 \equiv \frac{31}{8640} c_3^5 c_4 c_5 + \frac{1}{480} c_3^4 c_5 c_7 + \frac{337}{25920} c_3^3 c_5^3 - \frac{71}{4320} c_3^3 c_4 c_5 c_6 + \frac{31}{240} c_3^3 c_5 c_6 c_7 \\ + \frac{31}{480} c_3 c_5^3 c_6 - \frac{22}{135} c_3 c_4 c_5^2 c_7 - \frac{1}{120} c_4 c_5^4 + \frac{11}{16} u^7 c_3^4 c_5 - u^7 c_3^2 c_5 c_6 \\ - \frac{11}{18} u^7 c_3 c_4 c_5^2 + \frac{9}{160} u^6 c_3^4 c_6 - \frac{619}{480} u^6 c_3^3 c_4 c_5 + \dots$$

Then using the second half of Lemma 2.8, the coefficients in $(**)$ are obtained as follows:

$$\mu_1 = \frac{11}{5}, \mu_2 = 12, \mu_3 = 21, \mu_4 = 12, \mu_5 = 1, \mu_6 = \frac{21}{5}, \mu_7 = 12, \\ \mu_8 = 6, \mu_9 = \frac{9}{5}.$$

Thus we have obtained

$$I_{24} \equiv 2^{32} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 19 \cdot 199 \left(\frac{11}{5} u^{24} + 12 u^{18} v + 21 u^{12} v^2 + 12 u^6 v^3 \right. \\ \left. + v^4 + \frac{21}{5} u^{14} w + 12 u^8 v w + 6 u^2 v^2 w + \frac{9}{5} u^4 w^2 \right) \\ \equiv 2^{32} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 199 (11 u^{24} + 60 u^{18} v + 105 u^{12} v^2 + 60 u^6 v^3 \\ + 5 v^4 + 21 u^{14} w + 60 u^8 v w + 30 u^2 v^2 w + 9 u^4 w^2) \\ \equiv 2^{32} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 199 \tilde{I}_{24} \pmod{\mathfrak{b}_{2\circ}}.$$

Finally, we can also put

$$(***) \quad I_{30} \equiv 2^{38} \cdot 3^4 \cdot 5^5 \cdot 7 \cdot 11 \cdot 13 \cdot 61 (\nu_1 u^{30} + \nu_2 u^{24} v + \nu_3 u^{18} v^2 + \nu_4 u^{12} v^3 \\ + \nu_5 u^6 v^4 + \nu_6 v^5 + \nu_7 u^{20} w + \nu_8 u^{14} v w + \nu_9 u^8 v^2 w + \nu_{10} u^2 v^3 w \\ + \nu_{11} u^{10} w^2 + \nu_{12} u^{14} v w^2 + \nu_{13} w^3) \pmod{\mathfrak{b}_{2\circ}}$$

for some $\nu_i \in \mathbb{Q}$. Then, by Lemma 2.3 (viii), we have

$$\begin{aligned} I_{30} \equiv & 2^{38} \cdot 3^4 \cdot 5^5 \cdot 7 \cdot 11 \cdot 13 \cdot 61 \left(-\frac{599}{51840} c_3^5 c_4 c_5 c_6 + \frac{47}{34560} c_3^5 c_5^3 + \frac{1519}{25920} c_3^4 c_5 c_6 c_7 \right. \\ & + \frac{6293}{7290} c_3^3 c_4 c_5^2 c_7 - \frac{32537}{25920} c_3^3 c_5^3 c_6 + \frac{189919}{466560} c_3^2 c_4 c_5^4 + \frac{2012}{1215} c_3 c_4 c_5^2 c_6 c_7 - \frac{16693}{25920} c_3 c_5^4 c_7 \\ & \left. - \frac{223}{6480} c_4 c_5^4 c_6 - \frac{1}{1728} c_5^6 \right). \end{aligned}$$

On the other hand, in $A/(t, c_8, \mathfrak{a}_{20})$, we have, for example,

$$\begin{aligned} v^5 \equiv & \frac{31}{11520} c_3^5 c_4 c_5 c_6 - \frac{47}{69120} c_3^5 c_5^3 + \frac{1}{640} c_3^4 c_5 c_6 c_7 + \frac{1993}{5760} c_3^3 c_5^3 c_6 - \frac{91}{360} c_3^3 c_4 c_5^2 c_7 \\ & - \frac{1279}{11520} c_3^2 c_4 c_5^4 - \frac{49}{135} c_3 c_4 c_5^2 c_6 c_7 + \frac{7}{1440} c_4 c_5^4 c_6 + \frac{299}{1920} c_3 c_5^4 c_7 + \cdots, \\ w^3 \equiv & \frac{1}{162} c_3^5 c_4 c_5 c_6 - \frac{5}{81} c_3^4 c_5 c_6 c_7 + \frac{365}{648} c_3^3 c_5^3 c_6 - \frac{1043}{2916} c_3^3 c_4 c_5^2 c_7 - \frac{1079}{5832} c_3^2 c_4 c_5^4 \\ & - \frac{226}{243} c_3 c_4 c_5^2 c_6 c_7 + \frac{2}{81} c_4 c_5^4 c_6 + \frac{431}{1296} c_3 c_5^4 c_7 + \frac{1}{1728} c_5^6 + \cdots. \end{aligned}$$

Then using the second half of Lemma 2.8, the coefficients in (***) are obtained as follows:

$$\begin{aligned} \nu_1 = -\frac{9}{8}, \nu_2 = -3, \nu_3 = 0, \nu_4 = -5, \nu_5 = -\frac{35}{2}, \nu_6 = -2, \nu_7 = -\frac{3}{2} \\ \nu_8 = \frac{9}{2}, \nu_9 = 15, \nu_{10} = -5, \nu_{11} = -\frac{3}{2}, \nu_{12} = 3, \nu_{13} = -1. \end{aligned}$$

Thus we have obtained

$$\begin{aligned} I_{30} \equiv & 2^{38} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 61 \left(-\frac{9}{8} u^{30} - 3u^{24}v - 5u^{12}v^3 - \frac{35}{2} u^6 v^4 - 2v^5 \right. \\ & \left. - \frac{3}{2} u^{20}w + \frac{9}{2} u^{14}vw + 15u^8v^2w - 5u^2v^3w - \frac{3}{2} u^{10}w^2 + 3u^4vw^2 - w^3 \right) \\ \equiv & 2^{35} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 61 (-9u^{30} - 24u^{24}v - 40u^{12}v^3 - 140u^6v^4 - 16v^5 \\ & - 12u^{20}w + 36u^{14}vw + 120u^8v^2w - 40u^2v^3w - 12u^{10}w^2 + 24u^4vw^2 - 8w^3) \\ \equiv & 2^{35} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 61 \tilde{I}_{30} \pmod{\mathfrak{b}_{20}}. \end{aligned}$$

Consequently, we have established Lemma 2.5. \square

3. COHOMOLOGY OF $E_8/T^1 \cdot E_7$

3.1. Rational cohomology ring of $E_8/T^1 \cdot E_7$. With the above results, we will compute the rational cohomology ring of E_8/C_8 . First of all we review the classical results of Borel ([2]); Let G be a compact

connected Lie group, H a closed connected subgroup of G of maximal rank and T a common maximal torus. Consider the fibration

$$G/H \xrightarrow{\iota} BH \xrightarrow{\rho} BG.$$

Since $H^*(BG; \mathbb{Q})$ is a polynomial ring generated by elements of even degrees and $H^*(G/H; \mathbb{Q})$ has vanishing odd dimensional part (Hirsch formula [2]), the rational cohomology spectral sequence for this fibration collapses. In particular, we have the following description of the rational cohomology ring of G/H :

$$\begin{aligned} H^*(G/H; \mathbb{Q}) &\xleftarrow{\iota^*} H^*(BH; \mathbb{Q}) / (\rho^* H^+(BG; \mathbb{Q})) \\ &\cong H^*(BT; \mathbb{Q})^{W(H)} / (H^+(BT; \mathbb{Q})^{W(G)}), \end{aligned}$$

where $H^+ = \bigoplus_{i>0} H^i$ and $(\)$ means the ideal generated by the elements in parenthesis.

We apply this result to the fibration:

$$E_8/C_8 \xrightarrow{\iota} BC_8 \xrightarrow{\rho} BE_8.$$

Then using Lemmas 2.2, 2.6, 2.5 and (2.9), we have

$$\begin{aligned} H^*(E_8/C_8; \mathbb{Q}) &\cong H^*(BT; \mathbb{Q})^{W(C_8)} / (H^+(BT; \mathbb{Q})^{W(E_8)}) \\ &\cong \mathbb{Q}[u, J_2, J_6, J_8, J_{10}, J_{12}, J_{14}, J_{18}] / (I_2, I_8, I_{12}, I_{14}, I_{18}, I_{20}, I_{24}, I_{30}) \\ &\cong \mathbb{Q}[u, I_2, v, I_8, w, I_{12}, I_{14}, I_{18}] / (I_2, I_8, I_{12}, I_{14}, I_{18}, I_{20}, I_{24}, I_{30}) \\ &\cong \mathbb{Q}[u, v, w] / (\tilde{I}_{20}, \tilde{I}_{24}, \tilde{I}_{30}). \end{aligned}$$

Thus we have obtained

Lemma 3.1. *The rational cohomology ring of $E_8/T^1 \cdot E_7$ is given as follows:*

$$H^*(E_8/T^1 \cdot E_7; \mathbb{Q}) = \mathbb{Q}[u, v, w] / (\tilde{I}_{20}, \tilde{I}_{24}, \tilde{I}_{30}),$$

where $\deg u = 2$, $\deg v = 12$, $\deg w = 20$ and $\tilde{I}_{20}, \tilde{I}_{24}$ and \tilde{I}_{30} are given by (2.8).

3.2. Integral cohomology ring of E_8/T in low degrees. Consider the fibration

$$E_7/T' \cong C_8/T \xrightarrow{i} E_8/T \xrightarrow{p} E_8/C_8.$$

Since $H^*(E_8/C_8; \mathbb{Z})$ and $H^*(E_7/T'; \mathbb{Z})$ have no torsion and vanishing odd dimensional part by Bott [4], the Serre spectral sequence for the above fibration collapses and the following sequence

$$\mathbb{Z} \rightarrow H^*(E_8/C_8; \mathbb{Z}) \xrightarrow{p^*} H^*(E_8/T; \mathbb{Z}) \xrightarrow{i^*} H^*(C_8/T; \mathbb{Z}) \cong H^*(E_7/T'; \mathbb{Z}) \rightarrow \mathbb{Z}$$

is co-exact; that is,

p^* is injective, i^* is surjective and

$\text{Ker } i^* = (p^*H^+(E_8/C_8; \mathbb{Z}))$, the ideal generated by $p^*H^+(E_8/C_8; \mathbb{Z})$.

Therefore we will obtain some information about the generators of $H^*(E_8/C_8; \mathbb{Z})$ by considering $\text{Ker } i^*$. In order to investigate $\text{Ker } i^*$, we will determine $H^*(E_8/T; \mathbb{Z})$ up to degrees ≤ 36 . First we need a simple lemma.

Lemma 3.2. *For the elements t and $c_i = \sigma_i(t_1, \dots, t_8) \in H^*(BT; \mathbb{Z})$, we have*

- (i) $Sq^2(c_2) \equiv c_3 + tc_2$,
 $Sq^4(c_3) \equiv c_5 + tc_4 + c_2c_3$,
 $Sq^8(c_5 + tc_4) \equiv tc_8 + c_2c_7 + c_3c_6 + c_4c_5 + tc_4^2 + t^2c_7 + t^3c_6 + t^2c_2c_5$
 $+ t^2c_3c_4$,
 $Sq^{14}(c_8 + c_4^2 + t^2c_6 + t^4c_4 + t^8) \equiv (c_8 + t^2c_6 + t^4c_4 + t^8)(c_7 + tc_6) \pmod{2}$.
- (ii) $\mathcal{P}^1(c_2 - t^2) \equiv c_4 + c_2^2 + t^4$,
 $\mathcal{P}^3(c_4 - t^4) \equiv c_5^2 + 2c_4c_6 + 2c_3c_7 + 2c_2c_8 + c_3^2c_4 + c_2c_4^2 + c_2^2c_6 + 2c_2c_3c_5$
 $+ 2t^{10} \pmod{3}$.
- (iii) $\mathcal{P}^1(c_2 + t^2) \equiv c_6 + 2c_3^2 + 4c_2c_4 + 2c_2^3 + 2tc_5 + tc_2c_3 + 4t^2c_4 + 4t^2c_2^2$
 $+ 3t^3c_3 + t^4c_2 + 2t^6 \pmod{5}$.

Proof. (i) follows immediately from the Wu formula: $Sq^{2i-2}(c_i) \equiv \sum_{j=0}^{i-1} c_{2i-1-j}c_j$ and $c_1 = 3t \equiv t \pmod{2}$.

(ii) Put

$$p_i = t_1^i + \dots + t_8^i \quad (\text{power sum}).$$

Then p_i 's and c_i 's are related to each other by the Newton formula:

$$p_n = \sum_{i=1}^{n-1} (-1)^{i-1} p_{n-i} c_i + (-1)^{n-1} n c_n.$$

In particular, considering with mod 3 coefficients, we have

$$p_1 \equiv c_1 \equiv 0,$$

$$p_2 \equiv c_2,$$

$$p_4 \equiv 2c_4 + 2c_2^2,$$

$$p_{10} \equiv 2c_5^2 + c_4c_6 + c_3c_7 + c_2c_8 + 2c_3^2c_4 + 2c_2c_4^2 + 2c_2^2c_6 + c_2^3c_4 + c_2c_3c_5 + c_2^5.$$

On the other hand, we have

$$\begin{aligned} \mathcal{P}^1(p_2) &\equiv \mathcal{P}^1\left(\sum_i t_i^2\right) \equiv \sum_i \mathcal{P}^1(t_i^2) \equiv \sum_i 2t_i \mathcal{P}^1(t_i) \equiv \sum_i 2t_i \cdot t_i^3 \\ &\equiv \sum_i 2t_i^4 \equiv 2p_4, \end{aligned}$$

$$\begin{aligned} \mathcal{P}^3(p_4) &\equiv \mathcal{P}^3\left(\sum_i t_i^4\right) \equiv \sum_i \mathcal{P}^3(t_i^4) \equiv \sum_i (2\mathcal{P}^3(t_i^2)t_i^2 + 2\mathcal{P}^2(t_i^2)\mathcal{P}^1(t_i^2)) \\ &\equiv \sum_i (2t_i^6 \cdot 2t_i^4) \equiv \sum_i t_i^{10} \equiv p_{10}. \end{aligned}$$

Using these facts, we have easily the required results.

(iii) Similar computation yields the required results. \square

Lemma 3.3. *The integral cohomology ring of E_8/T for degrees ≤ 36 is given as follows:*

$$\begin{aligned} H^*(E_8/T; \mathbb{Z}) &= \mathbb{Z}[t_1, \dots, t_8, t, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_9, \gamma_{10}, \gamma_{15}] \\ &\quad / (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_8, \rho_9, \rho_{10}, \rho_{12}, \rho_{14}, \rho_{15}, \rho_{18}), \end{aligned}$$

where $t_1, \dots, t_8, t \in H^2$ are as in §2, $\gamma_i \in H^{2i}$ ($i = 3, 4, 5, 6, 9, 10, 15$) and

$$\begin{aligned} \rho_1 &= c_1 - 3t, \\ \rho_2 &= c_2 - 4t^2, \\ \rho_3 &= c_3 - 2\gamma_3, \\ \rho_4 &= c_4 + 2t^4 - 3\gamma_4, \\ \rho_5 &= c_5 - 3t\gamma_4 + 2t^2\gamma_3 - 2\gamma_5, \\ \rho_6 &= c_6 - 2\gamma_3^2 - t\gamma_5 + t^2\gamma_4 - 2t^6 - 5\gamma_6, \\ \rho_8 &= -3c_8 + 3\gamma_4^2 - 2\gamma_3\gamma_5 + t(2c_7 - 6\gamma_3\gamma_4) + t^2(2\gamma_3^2 - 5\gamma_6) + 3t^3\gamma_5 \\ &\quad + 4t^4\gamma_4 - 6t^5\gamma_3 + t^8, \\ \rho_9 &= 2c_6\gamma_3 + tc_8 + t^2c_7 - 3t^3c_6 - 2\gamma_9, \\ \rho_{10} &= \gamma_5^2 - 2c_7\gamma_3 - t^2c_8 + 3t^3c_7 - 3\gamma_{10}, \end{aligned}$$

$$\begin{aligned}
\rho_{12} &= 15\gamma_6^2 + 2\gamma_3\gamma_4\gamma_5 - 2c_7\gamma_5 + 2\gamma_3^4 + 10\gamma_3^2\gamma_6 - 3c_8\gamma_4 - 2\gamma_4^3 \\
&\quad + t(c_8\gamma_3 - 2\gamma_3^2\gamma_5 + 4c_7\gamma_4 + 6\gamma_3\gamma_4^2) + t^2(3\gamma_{10} - 25\gamma_4\gamma_6 - c_7\gamma_3 - 16\gamma_3^2\gamma_4) \\
&\quad + t^3(25\gamma_3\gamma_6 - 3\gamma_4\gamma_5 + 10\gamma_3^3) + t^4(3c_8 + 3\gamma_3\gamma_5 + 5\gamma_4^2) + t^5(-3c_7 - 5\gamma_3\gamma_4) \\
&\quad + 4t^6\gamma_3^2 - 7t^8\gamma_4 + 4t^9\gamma_3, \\
\rho_{14} &= c_7^2 - 3c_8\gamma_6 + 6\gamma_4\gamma_{10} - 4c_8\gamma_3^2 + 6c_7\gamma_3\gamma_4 - 6\gamma_3^2\gamma_4^2 - 12\gamma_4^2\gamma_6 - 2\gamma_3\gamma_5\gamma_6 \\
&\quad + t(24\gamma_3\gamma_4\gamma_6 - 8c_7\gamma_3^2 - 8c_7\gamma_6 + 4c_8\gamma_5 - 6\gamma_3\gamma_{10} + 12\gamma_3^3\gamma_4) \\
&\quad + t^2(-2\gamma_3\gamma_4\gamma_5 + 6\gamma_4^3 + 2\gamma_3^2\gamma_6 + 20\gamma_6^2 - 4\gamma_3^4 - c_7\gamma_5) \\
&\quad + t^3(-12\gamma_3\gamma_4^2 + 8c_8\gamma_3 - 5c_7\gamma_4 + 3\gamma_5\gamma_6) + t^4(3\gamma_{10} - 26\gamma_4\gamma_6 + 6c_7\gamma_3 - 4\gamma_3^2\gamma_4) \\
&\quad + t^5(24\gamma_3\gamma_6 + 3\gamma_4\gamma_5 + 12\gamma_3^3) + t^6(-6c_8 + 2\gamma_4^2) - 4t^7c_7 + t^8(6\gamma_6 - 6\gamma_3^2) \\
&\quad - 6t^{10}\gamma_4 + 12t^{11}\gamma_3 - 2t^{14}, \\
\rho_{15} &= (c_8 - t^2c_6 + 2t^3\gamma_5 + 3t^4\gamma_4 - t^8)(c_7 - 3tc_6) - 2(\gamma_3^2 + c_6)(\gamma_9 - c_6\gamma_3) - 2\gamma_{15}, \\
\rho_{18} &= -\gamma_9^2 - 27c_8\gamma_{10} - 18\gamma_4^2\gamma_{10} + 4\gamma_3^3\gamma_9 + 10\gamma_3\gamma_6\gamma_9 + 6\gamma_3\gamma_5\gamma_{10} - 6\gamma_3\gamma_4\gamma_5\gamma_6 \\
&\quad - 18c_7\gamma_3\gamma_4^2 + 9c_8\gamma_4\gamma_6 + 3c_8\gamma_3^2\gamma_4 + 18\gamma_3^2\gamma_4^3 + 36\gamma_4^3\gamma_6 + 6c_7^2\gamma_4 + 6c_7\gamma_3^2\gamma_5 \\
&\quad - 6\gamma_3^3\gamma_4\gamma_5 + 6c_7\gamma_5\gamma_6 - 4\gamma_3^6 - 30\gamma_6^3 - 26\gamma_3^4\gamma_6 - 55\gamma_3^2\gamma_6^2 - 27c_7c_8\gamma_3 \\
&\quad + t(2\gamma_3\gamma_5\gamma_9 - 72c_7\gamma_4\gamma_6 + 24c_8\gamma_4\gamma_5 + 12c_7\gamma_3^2\gamma_4 + 12c_7\gamma_{10} + c_8\gamma_9 + 6c_7^2\gamma_3 \\
&\quad - 8c_8\gamma_3\gamma_6 + 36\gamma_3\gamma_4\gamma_{10} - 108\gamma_3\gamma_4^2\gamma_6 - 4\gamma_3^2\gamma_5\gamma_6 - 5c_8\gamma_3^3 + 2\gamma_3^4\gamma_5 - 54\gamma_3^3\gamma_4^2) \\
&\quad + t^2(88\gamma_3^4\gamma_4 - 7c_8^2 + c_7\gamma_9 - 45\gamma_3^2\gamma_{10} - 2\gamma_3\gamma_4\gamma_9 - 10c_8\gamma_3\gamma_5 + 28c_7\gamma_3\gamma_6 - 9c_7\gamma_4\gamma_5 \\
&\quad + 225\gamma_4\gamma_6^2 - 18\gamma_4^4 - 27c_8\gamma_4^2 + 283\gamma_3^2\gamma_4\gamma_6 - 39\gamma_6\gamma_{10} + 12\gamma_3\gamma_4^2\gamma_5 - 19c_7\gamma_3^3) \\
&\quad + t^3(-9\gamma_5\gamma_{10} - 150\gamma_3\gamma_6^2 - 165\gamma_3^3\gamma_6 - 6\gamma_3^2\gamma_9 - 10c_7\gamma_3\gamma_5 + 63c_7\gamma_4^2 + 43c_7c_8 \\
&\quad + 9\gamma_4\gamma_5\gamma_6 - \gamma_3^2\gamma_4\gamma_5 + 54\gamma_3\gamma_4^3 - 15\gamma_6\gamma_9 + 46c_8\gamma_3\gamma_4 - 42\gamma_3^5) \\
&\quad + t^4(-9c_8\gamma_6 + 6\gamma_4\gamma_{10} - 16c_7^2 - 103\gamma_3^2\gamma_4^2 - 3\gamma_5\gamma_9 + 36\gamma_3\gamma_5\gamma_6 - 135\gamma_4^2\gamma_6 \\
&\quad - 119c_7\gamma_3\gamma_4 + 9\gamma_3^3\gamma_5 - 19c_8\gamma_3^2) \\
&\quad + t^5(39c_7\gamma_6 - 18\gamma_4^2\gamma_5 + 117c_7\gamma_3^2 + 195\gamma_3\gamma_4\gamma_6 - 12c_8\gamma_5 + 3\gamma_4\gamma_9 + 36\gamma_3\gamma_{10} + 87\gamma_3^3\gamma_4) \\
&\quad + t^6(33\gamma_4^3 + 4\gamma_3\gamma_9 + 18c_7\gamma_5 + 3\gamma_3\gamma_4\gamma_5 - 32\gamma_3^4 - 62\gamma_3^2\gamma_6) \\
&\quad + t^7(-87\gamma_3\gamma_4^2 - 16\gamma_3^2\gamma_5 - 6c_7\gamma_4 - 45\gamma_5\gamma_6 + 4c_8\gamma_3) \\
&\quad + t^8(-39\gamma_{10} + 115\gamma_3^2\gamma_4 - 77c_7\gamma_3 + 81\gamma_4\gamma_6) \\
&\quad + t^9(-6\gamma_9 + 18\gamma_4\gamma_5 - 30\gamma_3^3 - 57\gamma_3\gamma_6) \\
&\quad + t^{10}(-9c_8 - 27\gamma_4^2 + 9\gamma_3\gamma_5) \\
&\quad + t^{11}(33\gamma_3\gamma_4 + 48c_7) \\
&\quad - 34t^{12}\gamma_3^2 - 18t^{13}\gamma_5 + 9t^{14}\gamma_4 + 12t^{15}\gamma_3 - 6t^{18}.
\end{aligned}$$

Proof. According to Toda ([17], Proposition 3.2), one can give the general description of $H^*(E_8/T; \mathbb{Z})$ as follows:

$$H^*(E_8/T; \mathbb{Z}) = \mathbb{Z}[t_1, \dots, t_8, t, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_9, \gamma_{10}, \gamma_{15}] \\ /(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_8, \rho_9, \rho_{10}, \rho_{12}, \rho_{14}, \rho_{15}, \rho_{18}, \rho_{20}, \rho_{24}, \rho_{30}),$$

where $t_1, \dots, t_8, t \in H^2$ as above and

$$\begin{aligned} \rho_1 &= c_1 - 3t, \\ \rho_i &= \delta_i - 2\gamma_i \quad (i = 3, 5, 9, 15), \\ \rho_i &= \delta_i - 3\gamma_i \quad (i = 4, 10), \\ \rho_6 &= \delta_6 - 5\gamma_6. \end{aligned}$$

Here δ_i ($i = 3, 4, 5, 6, 9, 10, 15$) is an arbitrary element satisfying

$$\begin{aligned} \delta_3 &\equiv Sq^2(\rho_2), \quad \delta_5 \equiv Sq^4(\delta_3), \quad \delta_9 \equiv Sq^8(\delta_5), \quad \delta_{15} \equiv Sq^{14}(\rho_8) \pmod{2}, \\ \delta_4 &\equiv \mathcal{P}^1(\rho_2), \quad \delta_{10} \equiv \mathcal{P}^3(\delta_4) \pmod{3}, \\ \delta_6 &\equiv \mathcal{P}^1(\rho_2) \pmod{5}. \end{aligned}$$

Other relation ρ_j ($j = 2, 8, 12, 14, 18, 20, 24, 30$) is determined by the maximality of the integer n_j in

$$(3.1) \quad n_j \cdot \rho_j \equiv \iota_0^*(I_j) \pmod{(\rho_i; i < j)}.$$

Now let us determine the generators and the relations explicitly;

(1) In view of Lemma 2.3 (i) and (3.1), we can take

$$\rho_2 = c_2 - 4t^2.$$

(2) By Lemma 3.2 (i), we have

$$\delta_3 \equiv Sq^2(\rho_2) \equiv Sq^2(c_2) \equiv c_3 + tc_2 \equiv c_3 \pmod{(2, \rho_1, \rho_2)}$$

and we can take $\delta_3 = c_3$ so that

$$\rho_3 = c_3 - 2\gamma_3.$$

(3) By Lemma 3.2 (ii), we have

$$\delta_4 \equiv \mathcal{P}^1(\rho_2) \equiv \mathcal{P}^1(c_2 - 4t^2) \equiv c_4 + c_2^2 + t^4 \equiv c_4 + 2t^4 \pmod{(3, \rho_1, \rho_2)}$$

and we can take $\delta_4 = c_4 + 2t^4$ so that

$$\rho_4 = c_4 + 2t^4 - 3\gamma_4.$$

(4) By Lemma 3.2 (i), we have

$$\begin{aligned} \delta_5 &\equiv Sq^4(\delta_3) \equiv Sq^4(c_3) \equiv c_5 + tc_4 + c_2c_3 \equiv c_5 + tc_4 \pmod{(2, \rho_1, \rho_2, \rho_3, \rho_4)} \\ &\equiv c_5 - 3t\gamma_4 + 2t^2\gamma_3 \pmod{(2, \rho_1, \rho_2, \rho_3, \rho_4)} \end{aligned}$$

and we can take $\delta_5 = c_5 - 3t\gamma_4 + 2t^2\gamma_3$ so that

$$\rho_5 = c_5 - 3t\gamma_4 + 2t^2\gamma_3 - 2\gamma_5.$$

(5) By Lemma 3.2 (iii), we have

$$\begin{aligned}
\delta_6 &\equiv \mathcal{P}^1(\rho_2) \equiv \mathcal{P}^1(c_2 - 4t^2) \equiv c_6 + 2c_3^2 + 4c_2c_4 + 2c_2^3 + 2tc_5 + tc_2c_3 \\
&\quad + 4t^2c_4 + 4t^2c_2^2 + 3t^3c_3 + t^4c_2 + 2t^6 \pmod{(5, \rho_1)} \\
&\equiv c_6 + 3\gamma_3^2 + 4t\gamma_5 + t^2\gamma_4 + 3t^6 \pmod{(5, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5)} \\
&\equiv c_6 - 2\gamma_3^2 - t\gamma_5 + t^2\gamma_4 - 2t^6 \pmod{(5, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5)}
\end{aligned}$$

and we can take $\delta_6 = c_6 - 2\gamma_3^2 - t\gamma_5 + t^2\gamma_4 - 2t^6$ so that

$$\rho_6 = c_6 - 2\gamma_3^2 - t\gamma_5 + t^2\gamma_4 - 2t^6 - 5\gamma_6.$$

(6) By Lemma 2.3 (ii), we have

$$\begin{aligned}
I_8 &\equiv 2^{14} \cdot 3 \cdot 5 \{-18c_8 - 3c_3c_5 + 2c_4^2 + t(12c_7 - 3c_3c_4) + t^2(-6c_6 + 3c_3^2) \\
&\quad + 12t^3c_5 + 2t^4c_4 - 12t^5c_3 + 14t^8\} \pmod{(I_2)} \\
&\equiv 2^{15} \cdot 3^2 \cdot 5 \{-3c_8 + 3\gamma_4^2 - 2\gamma_3\gamma_5 + t(2c_7 - 6\gamma_3\gamma_4) + t^2(2\gamma_3^2 - 5\gamma_6) \\
&\quad + 3t^3\gamma_5 + 4t^4\gamma_4 - 6t^5\gamma_3 + t^8\} \pmod{(\rho_i; i < 8)}.
\end{aligned}$$

Hence, by (3.1), we have

$$2^{15} \cdot 3^2 \cdot 5 \rho_8 \equiv \iota_0^*(I_8) \pmod{(\rho_i; i < 8)}$$

and it follows the form of ρ_8 .

(7) By Lemma 3.2 (i), we have

$$\begin{aligned}
\delta_9 &\equiv Sq^8(\delta_5) \equiv Sq^8(c_5 + tc_4) \equiv tc_8 + c_2c_7 + c_3c_6 + c_4c_5 + tc_4^2 + t^2c_7 \\
&\quad + t^3c_6 + t^2c_2c_5 + t^2c_3c_4 \pmod{(2, \rho_1)} \\
&\equiv tc_8 + t^2c_7 + t^3c_6 \pmod{(2, \rho_i; i < 9)} \\
&\equiv tc_8 + t^2c_7 - 3t^3c_6 + 2c_6\gamma_3 \pmod{(2, \rho_i; i < 9)}
\end{aligned}$$

and we can take $\delta_9 = tc_8 + t^2c_7 - 3t^3c_6 + 2c_6\gamma_3$ so that

$$\rho_9 = tc_8 + t^2c_7 - 3t^3c_6 + 2c_6\gamma_3 - 2\gamma_9.$$

(8) By Lemma 3.2 (ii), we have

$$\begin{aligned}
\delta_{10} &\equiv \mathcal{P}^3(\delta_4) \equiv \mathcal{P}^3(c_4 + 2t^4) \equiv c_5^2 + 2c_4c_6 + 2c_3c_7 + 2c_2c_8 + c_3^2c_4 \\
&\quad + c_2c_4^2 + c_2^2c_6 + 2c_2c_3c_5 + 2t^{10} \pmod{(3, \rho_1)} \\
&\equiv \gamma_5^2 + c_7\gamma_3 + 2t^2c_8 \pmod{(3, \rho_i; i < 10)} \\
&\equiv \gamma_5^2 - 2c_7\gamma_3 + 3t^3c_7 - t^2c_8 \pmod{(3, \rho_i; i < 10)}
\end{aligned}$$

and we can take $\delta_{10} = \gamma_5^2 - 2c_7\gamma_3 + 3t^3c_7 - t^2c_8$ so that

$$\rho_{10} = \gamma_5^2 - 2c_7\gamma_3 + 3t^3c_7 - t^2c_8.$$

(9) By (2.6), (2.5) and (2.7), we obtain

$$\begin{aligned}
I_{12} &\equiv 2^{18} \cdot 3^4 \cdot 5 \cdot 7 \left\{ \frac{3}{5}c_6^2 - c_5c_7 - c_4c_8 + \frac{1}{6}c_3c_4c_5 - \frac{2}{27}c_4^3 - \frac{1}{10}c_3^2c_6 + \frac{1}{40}c_3^4 \right. \\
&\quad + t \left(\frac{1}{2}c_3c_8 + \frac{7}{3}c_4c_7 - \frac{3}{5}c_5c_6 - \frac{1}{5}c_3^2c_5 + \frac{1}{6}c_3c_4^2 \right) \\
&\quad + t^2 \left(\frac{2}{5}c_5^2 - \frac{5}{2}c_3c_7 - \frac{2}{3}c_4c_6 - \frac{1}{6}c_3^2c_4 \right) \\
&\quad + t^3 \left(\frac{19}{10}c_3c_6 - \frac{2}{3}c_4c_5 - \frac{1}{5}c_3^3 \right) + t^4 \left(-\frac{1}{9}c_4^2 + \frac{19}{30}c_3c_5 \right) + t^5 \left(\frac{14}{3}c_7 + \frac{1}{2}c_3c_4 \right) \\
&\quad \left. + t^6 \left(-\frac{56}{15}c_6 + \frac{23}{30}c_3^2 \right) - \frac{2}{15}t^7c_5 - \frac{5}{9}t^8c_4 - \frac{22}{15}t^9c_3 + \frac{154}{135}t^{12} \right\} \pmod{(I_2)} \\
&\equiv 2^{18} \cdot 3^4 \cdot 5 \cdot 7 \{ 15\gamma_6^2 + 2\gamma_3\gamma_4\gamma_5 - 2c_7\gamma_5 + 2\gamma_3^4 + 10\gamma_3^2\gamma_6 - 3c_8\gamma_4 - 2\gamma_4^3 \\
&\quad + t(c_8\gamma_3 - 2\gamma_3^2\gamma_5 + 4c_7\gamma_4 + 6\gamma_3\gamma_4^2) + t^2(3\gamma_{10} - 25\gamma_4\gamma_6 - c_7\gamma_3 - 16\gamma_3^2\gamma_4) \\
&\quad + t^3(25\gamma_3\gamma_6 - 3\gamma_4\gamma_5 + 10\gamma_3^3) + t^4(3c_8 + 3\gamma_3\gamma_5 + 5\gamma_4^2) + t^5(-3c_7 - 5\gamma_3\gamma_4) \\
&\quad + 4t^6\gamma_3^2 - 7t^8\gamma_4 + 4t^9\gamma_3 \} \pmod{(\rho_i; i < 12)}.
\end{aligned}$$

Hence, by (3.1), we have

$$2^{18} \cdot 3^4 \cdot 5 \cdot 7 \rho_{12} \equiv \iota_0^*(I_{12}) \pmod{(\rho_i; i < 12)}$$

and it follows the form of ρ_{12} . Quite similarly, we have

$$2^{20} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \rho_{14} \equiv \iota_0^*(I_{14}) \pmod{(\rho_i; i < 14)},$$

$$2^{26} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13 \rho_{18} \equiv \iota_0^*(I_{18}) \pmod{(\rho_i; i < 18)}$$

and it follows the forms of ρ_{14} and ρ_{18} .

(10) Finally, we will determine the relation ρ_{15} . Since

$$\begin{aligned}
\rho_8 &= -3c_8 + 3\gamma_4^2 - 2\gamma_3\gamma_5 + t(2c_7 - 6\gamma_3\gamma_4) + t^2(2\gamma_3^2 - 5\gamma_6) + 3t^3\gamma_5 \\
&\quad + 4t^4\gamma_4 - 6t^5\gamma_3 + t^8 \\
&\equiv c_8 + c_4^2 + t^2c_6 + t^4c_4 + t^8 \pmod{(2, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6)},
\end{aligned}$$

we have, by Lemma 3.2 (i),

$$\begin{aligned}
\delta_{15} &\equiv Sq^{14}(\rho_8) \\
&\equiv Sq^{14}(c_8 + c_4^2 + t^2c_6 + t^4c_4 + t^8) \\
&\equiv (c_8 + t^2c_6 + t^4c_4 + t^8)(c_7 + tc_6) \pmod{(2, \rho_1)} \\
&\equiv (c_8 - t^2c_6 + 2t^3\gamma_5 + 3t^4\gamma_4 - t^8)(c_7 - tc_6) - 2(c_6 + \gamma_3^2)(\gamma_9 - c_6\gamma_3) \\
&\qquad\qquad\qquad \pmod{(2, \rho_i; i < 15)}
\end{aligned}$$

and we can take

$$\delta_{15} = (c_8 - t^2 c_6 + 2t^3 \gamma_5 + 3t^4 \gamma_4 - t^8)(c_7 - t c_6) - 2(c_6 + \gamma_3^2)(\gamma_9 - c_6 \gamma_3)$$

so that

$$\rho_{15} = (c_8 - t^2 c_6 + 2t^3 \gamma_5 + 3t^4 \gamma_4 - t^8)(c_7 - t c_6) - 2(c_6 + \gamma_3^2)(\gamma_9 - c_6 \gamma_3) - 2\gamma_{15}.$$

Consequently, we have established the lemma. \square

In order to determine $\text{Ker } i^*$, we need the result on $H^*(E_7/T'; \mathbb{Z})$ in ([15], Theorem 5.9), which can be restated as follows:

Theorem 3.4. *The integral cohomology ring of E_7/T' is given as follows:*

$$H^*(E_7/T'; \mathbb{Z}) = \mathbb{Z}[t'_1, \dots, t'_7, t', \gamma'_3, \gamma'_4, \gamma'_5, \gamma'_9] \\ /(\rho'_1, \rho'_2, \rho'_3, \rho'_4, \rho'_5, \rho'_6, \rho'_8, \rho'_9, \rho'_{10}, \rho'_{12}, \rho'_{14}, \rho'_{18}),$$

where $t'_1, \dots, t'_7, t' \in H^2$ are as in §2, $\gamma'_i \in H^{2i}$ ($i = 3, 4, 5, 9$) and

$$\begin{aligned} \rho'_1 &= c'_1 - 3t', \\ \rho'_2 &= c'_2 - 4t'^2, \\ \rho'_3 &= c'_3 - 2\gamma'_3, \\ \rho'_4 &= c'_4 + 2t'^4 - 3\gamma'_4, \\ \rho'_5 &= c'_5 - 3t'\gamma'_4 + 2t'^2\gamma'_3 - 2\gamma'_5, \\ \rho'_6 &= \gamma'_3{}^2 + 2c'_6 - 2t'\gamma'_5 - 3t'^2\gamma'_4 + t'^6, \\ \rho'_8 &= 3\gamma'_4{}^2 - 2\gamma'_3\gamma'_5 + t'(2c'_7 - 6\gamma'_3\gamma'_4) - 9t'^2c'_6 + 12t'^3\gamma'_5 + 15t'^4\gamma'_4 - 6t'^5\gamma'_3 - t'^8, \\ \rho'_9 &= 2c'_6\gamma'_3 + t'^2c'_7 - 3t'^3c'_6 - 2\gamma'_9, \\ \rho'_{10} &= \gamma'_5{}^2 - 2c'_7\gamma'_3 + 3t'^3c'_7, \\ \rho'_{12} &= 3c'_6{}^2 - 2\gamma'_4{}^3 - 2c'_7\gamma'_5 + 2\gamma'_3\gamma'_4\gamma'_5 + t'(4c'_7\gamma'_4 - 2c'_6\gamma'_5 + 6\gamma'_3\gamma'_4{}^2) \\ &\quad + t'^2(-3c'_7\gamma'_3 + 3c'_6\gamma'_4) + t'^3(-12\gamma'_4\gamma'_5 + 5c'_6\gamma'_3) + t'^4(-2\gamma'_3\gamma'_5 - 15\gamma'_4{}^2) \\ &\quad - 10t'^6c'_6 + 12t'^7\gamma'_5 + 19t'^8\gamma'_4 - 6t'^9\gamma'_3 - 2t'^{12}, \\ \rho'_{14} &= c'_7{}^2 + 6c'_7\gamma'_3\gamma'_4 - 2c'_6\gamma'_3\gamma'_5 - t'^2c'_7\gamma'_5 + t'^3(-9c'_7\gamma'_4 + 3c'_6\gamma'_5) - 6t'^4c'_7\gamma'_3 \\ &\quad + 9t'^7c'_7, \\ \rho'_{18} &= -\gamma'_9{}^2 + 2c'_6c'_7\gamma'_5 + 6c'_7\gamma'_3\gamma'_4{}^2 - 2c'_7{}^2\gamma'_4 - 2c'_6\gamma'_3\gamma'_4\gamma'_5 + 2c'_6\gamma'_3\gamma'_9 \\ &\quad + t'(-6c'_7{}^2\gamma'_3 + 24c'_6c'_7\gamma'_4) + t'^2(-25c'_7\gamma'_4\gamma'_5 + c'_7\gamma'_9 - 18c'_6c'_7\gamma'_3) \\ &\quad + t'^3(-45c'_7\gamma'_4{}^2 + 20c'_7\gamma'_3\gamma'_5 + 3c'_6\gamma'_4\gamma'_5 - 3c'_6\gamma'_9) \end{aligned}$$

$$\begin{aligned}
& + t^4(11c_7'^2 + 2c_6'\gamma_3'\gamma_5' + 48c_7'\gamma_3'\gamma_4') + 51t^{15}c_6'c_7' - 53t^{16}c_7'\gamma_5' \\
& + t^7(-69c_7'\gamma_4' - 3c_6'\gamma_5') + 16t^{18}c_7'\gamma_3' + 15t^{11}c_7'.
\end{aligned}$$

Remark 3.5. Using the result in [15], we expressed the relations ρ_{12}' , ρ_{14}' and ρ_{18}' in terms of the generators $t_1', \dots, t_7', t', \gamma_3', \gamma_4', \gamma_5', \gamma_9'$.

Corollary 3.6. For the induced homomorphism

$$i^* : H^*(E_8/T; \mathbb{Z}) \longrightarrow H^*(C_8/T; \mathbb{Z}) \cong H^*(E_7/T'; \mathbb{Z}),$$

we obtain that

$$\text{Ker } i^* = (u, \tilde{\gamma}_6, \gamma_{10}, \gamma_{15}),$$

where $\tilde{\gamma}_6 = 2\gamma_6 + \gamma_3^2 - t^2\gamma_4 + t^6$.

Proof. By (2.2), we have

$$(3.2) \quad i^*(t_i) = t_i' \quad (1 \leq i \leq 7), \quad i^*(t_8) = 0, \quad i^*(t) = t'$$

and therefore

$$(3.3) \quad i^*(c_n) = c_n' \quad (1 \leq n \leq 7), \quad i^*(c_8) = 0.$$

Then it is verified directly that

$$\begin{aligned}
(3.4) \quad i^*(\gamma_i) &= \gamma_i' \quad (i = 3, 4, 5, 9), \\
i^*(\gamma_6) &= c_6' - t'\gamma_5' - t'^2\gamma_4', \\
i^*(\gamma_{10}) &= 0, \\
i^*(\gamma_{15}) &= 0.
\end{aligned}$$

Now we put

$$I = (u, \tilde{\gamma}_6, \gamma_{10}, \gamma_{15}),$$

the ideal of $H^*(E_8/T; \mathbb{Z})$ generated by the elements in the parenthesis.

Using (3.2), (3.4) and Theorem 3.4, we see that I is contained in $\text{Ker } i^*$.

Hence there is an induced map

$$H^*(E_8/T; \mathbb{Z})/I \longrightarrow H^*(C_8/T; \mathbb{Z}) \cong H^*(E_7/T'; \mathbb{Z}).$$

Then, by Lemma 3.3, we have

$$\begin{aligned}
H^*(E_8/T; \mathbb{Z})/I &= \mathbb{Z}[t_1, \dots, t_8, t, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_9, \gamma_{10}, \gamma_{15}] \\
&\quad / (u, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \tilde{\gamma}_6, \rho_8, \rho_9, \rho_{10}, \gamma_{10}, \rho_{12}, \rho_{14}, \rho_{15}, \gamma_{15}, \rho_{18}) \\
&= \mathbb{Z}[t_1, \dots, t_7, t, \gamma_3, \gamma_4, \gamma_5, \gamma_9] \\
&\quad / (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \tilde{\gamma}_6, \rho_8, \rho_9, \rho_{10}, \rho_{12}, \rho_{14}, \rho_{18})
\end{aligned}$$

for degrees ≤ 36 . Then it follows from Lemma 3.3, Theorem 3.4, (3.2),

(3.3) and (3.4) that

$$\begin{aligned}
i^*(\rho_i) &\equiv \rho_i' \quad (i = 1, 2, 3, 4, 5, 8, 9, 10, 12, 14, 18), \\
i^*(\tilde{\gamma}_6) &\equiv \rho_6'.
\end{aligned}$$

Therefore this map induces an isomorphism and the assertion follows. \square

3.3. Generators of $H^*(E_8/T^1 \cdot E_7; \mathbb{Z})$. From Corollary 3.6, we see that $H^*(E_8/C_8; \mathbb{Z})$ is generated by some four elements $\tilde{u} \in H^2$, $\tilde{v} \in H^{12}$, $\tilde{w} \in H^{20}$ and $\tilde{x} \in H^{30}$ such that

$$(3.5) \quad (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{x}) = (u, \tilde{\gamma}_6, \gamma_{10}, \gamma_{15})$$

as ideals. So our next task is to describe these generators in the ring $H^*(E_8/T; \mathbb{Z})$. Hereafter we identify $H^*(E_8/C_8; \mathbb{Z})$ with the subalgebra $\text{Im } p^*$ of $H^*(E_8/T; \mathbb{Z})$.

Firstly, by Lemma 2.7 and (2.14), we have

$$(3.6) \quad \begin{aligned} \tilde{J}_6 &= \frac{1}{2^{10} \cdot 3^2 \cdot 5} J_6 \\ &\equiv \frac{2}{5} c_6 + \frac{1}{20} c_3^2 + c_5 \left(-\frac{1}{5} t - \frac{1}{2} u \right) + c_4 \left(\frac{1}{2} t u + \frac{1}{3} u^2 \right) \\ &\quad + c_3 \left(-\frac{1}{5} t^3 - \frac{1}{2} t^2 u - \frac{1}{2} u^3 \right) + \frac{1}{5} t^6 + t^5 u - \frac{1}{3} t^4 u^2 + t^3 u^3 + t^2 u^4 - t u^5 \\ &\quad + \frac{273}{640} u^6 \pmod{(I_2)}; \end{aligned}$$

$$(3.7) \quad \begin{aligned} \tilde{J}_{10} &= \frac{1}{2^{14} \cdot 3^3 \cdot 5 \cdot 7} J_{10} \\ &\equiv \frac{1}{12} c_5^2 - \frac{1}{3} c_3 c_7 + \frac{1}{2} c_3 c_6 u + c_4 c_5 \left(-\frac{1}{6} t - \frac{1}{6} u \right) + c_3 c_5 \left(\frac{1}{6} t^2 - \frac{23}{84} u^2 \right) \\ &\quad + c_4^2 \left(\frac{1}{12} t^2 + \frac{1}{6} t u + \frac{1}{14} u^2 \right) + c_3^2 \left(\frac{1}{12} t^4 + \frac{23}{84} t^2 u^2 - \frac{37}{96} u^4 \right) \\ &\quad + c_3 c_4 \left(-\frac{1}{6} t^3 - \frac{1}{6} t^2 u - \frac{23}{84} t u^2 + \frac{1}{3} u^3 \right) \\ &\quad + c_7 \left(t^3 + \frac{1}{6} t^2 u - \frac{17}{42} t u^2 + \frac{5}{14} u^3 \right) + c_6 \left(-\frac{3}{2} t^3 u - \frac{3}{14} t^2 u^2 + \frac{1}{2} t u^3 - \frac{37}{84} u^4 \right) \\ &\quad + c_5 \left(-\frac{1}{3} t^5 + \frac{1}{6} t^4 u + \frac{23}{21} t^3 u^2 - \frac{1}{3} t^2 u^3 + \frac{1}{24} t u^4 + \frac{71}{336} u^5 \right) \\ &\quad + c_4 \left(\frac{1}{3} t^6 + \frac{1}{6} t^5 u + \frac{17}{42} t^4 u^2 - \frac{2}{3} t^3 u^3 - \frac{1}{3} t^2 u^4 + \frac{5}{16} t u^5 - \frac{131}{504} u^6 \right) \\ &\quad + c_3 \left(-\frac{1}{3} t^7 + \frac{1}{6} t^6 u - \frac{23}{21} t^5 u^2 - \frac{1}{3} t^4 u^3 + \frac{37}{24} t^3 u^4 + \frac{73}{48} t^2 u^5 - \frac{3}{2} t u^6 + \frac{239}{336} u^7 \right) \end{aligned}$$

$$\begin{aligned}
& +\frac{1}{3}t^{10} - \frac{1}{3}t^9u + \frac{7}{6}t^8u^2 + \frac{2}{3}t^7u^3 - \frac{7}{8}t^6u^4 - \frac{35}{8}t^5u^5 + \frac{233}{72}t^4u^6 + \frac{7}{24}t^3u^7 \\
& - \frac{129}{56}t^2u^8 + \frac{215}{168}tu^9 - \frac{208601}{645120}u^{10} \pmod{(I_2)}.
\end{aligned}$$

On the other hand, by Lemma 3.3, we have

$$\begin{aligned}
(3.8) \quad c_3 &= 2\gamma_3, \\
c_4 &= 3\gamma_4 - 2t^4, \\
c_5 &= 2\gamma_5 + 3t\gamma_4 - 2t^2\gamma_3, \\
c_6 &= 5\gamma_6 + 2\gamma_3^2 + t\gamma_5 - t^2\gamma_4 + 2t^6
\end{aligned}$$

in $H^*(E_8/T; \mathbb{Z}) \hookrightarrow H^*(E_8/T; \mathbb{Q})$. Substituting (3.8) into (3.6), we have

$$\begin{aligned}
(3.9) \quad \tilde{J}_6 &= \frac{1}{2^{10} \cdot 3^2 \cdot 5} J_6 \\
&= 2\gamma_6 + \gamma_3^2 - u\gamma_5 + \gamma_4(-t^2 + u^2) - u^3\gamma_3 + t^6 - t^4u^2 \\
&\quad + t^3u^3 + t^2u^4 - tu^5 + \frac{273}{640}u^6.
\end{aligned}$$

Similarly, substituting (3.8) into (3.7) and using the relations ρ_8, ρ_9 and ρ_{10} , we have

$$\begin{aligned}
(3.10) \quad \tilde{J}_{10} &= \frac{1}{2^{14} \cdot 3^3 \cdot 5 \cdot 7} J_{10} \\
&= \gamma_{10} + u\gamma_9 - u^3c_7 - u\gamma_4\gamma_5 + 2u^2\gamma_4^2 - 2u^2\gamma_3\gamma_5 + \gamma_3\gamma_4(-6tu^2 + 2u^3) \\
&\quad + \gamma_3^2 \left(2t^2u^2 + 2tu^3 + \frac{7}{24}u^4 \right) + \gamma_6 \left(-5t^2u^2 + 5tu^3 + \frac{55}{12}u^4 \right) \\
&\quad + \gamma_5 \left(t^4u + 3t^3u^2 + t^2u^3 - \frac{55}{24}u^5 \right) \\
&\quad + \gamma_4 \left(6t^4u^2 - 3t^3u^3 - \frac{103}{24}t^2u^4 - tu^5 + \frac{79}{24}u^6 \right) \\
&\quad + \gamma_3 \left(-6t^5u^2 - 2t^4u^3 + 4t^3u^4 + 6t^2u^5 - 4tu^6 - \frac{31}{24}u^7 \right) \\
&\quad + 4t^7u^3 + \frac{55}{24}t^6u^4 - 6t^5u^5 - \frac{7}{24}t^4u^6 + \frac{79}{24}t^3u^7 + \frac{31}{24}t^2u^8 - \frac{55}{24}tu^9 \\
&\quad + \frac{666919}{645120}u^{10}.
\end{aligned}$$

Now let us determine our generators $\tilde{u}, \tilde{v}, \tilde{w}$ and \tilde{x} ; Obviously, we can take $u = t_8$ as our generator \tilde{u} .

Next, since $H^*(E_8/C_8; \mathbb{Q})$ is generated by u, J_6 and J_{10} (see 3.1), we can put

$$\tilde{v} = \alpha \tilde{J}_6 + \beta u^6$$

for some $\alpha, \beta \in \mathbb{Q}$. On the other hand, by (3.5), we can express

$$\tilde{v} = \tilde{\gamma}_6 + f$$

for some element $f \in (u) \cap H^{12}(E_8/T; \mathbb{Z})$. Then using (3.9), we have

$$\begin{aligned} 2\gamma_6 + \gamma_3^2 - t^2\gamma_4 + t^6 + f &= \alpha(2\gamma_6 + \gamma_3^2 - t^2\gamma_4 + t^6) \\ &+ \alpha(-u\gamma_5 + u^2\gamma_4 - u^3\gamma_3 - t^4u^2 + t^3u^3 + t^2u^4 - tu^4) + \left(\frac{273}{640}\alpha + \beta\right)u^6, \end{aligned}$$

and we can take $\alpha = 1, \beta = -\frac{273}{640}$. Thus we see that

$$\begin{aligned} (3.11) \quad v &= \frac{1}{2^{10} \cdot 3^2 \cdot 5} J_6 - \frac{273}{640} u^6 \\ &= 2\gamma_6 + \gamma_3^2 - u\gamma_5 + \gamma_4(-t^2 + u^2) - u^3\gamma_3 + t^6 - t^4u^2 + t^3u^3 + t^2u^4 - tu^5 \end{aligned}$$

can be chosen as our generator \tilde{v} .

Similarly, we can put

$$\tilde{w} = \lambda \tilde{J}_{10} + \mu u^4 v + \nu u^{10}$$

for some $\lambda, \mu, \nu \in \mathbb{Q}$. On the other hand, by (3.5), we can express

$$\tilde{w} = \gamma_{10} + g$$

for some element $g \in (u, \tilde{\gamma}_6) \cap H^{20}(E_8/T; \mathbb{Z})$. Then using (3.10), we can take $\lambda = 1$ and hence

$$\begin{aligned} \gamma_{10} + g &= \gamma_{10} + u\gamma_9 - u^3c_7 - u\gamma_4\gamma_5 + 2u^2\gamma_4^2 - 2u^2\gamma_3\gamma_5 + \gamma_3\gamma_4(-6tu^2 + 2u^3) \\ &+ \gamma_3^2 \left\{ 2t^2u^2 + 2tu^3 + \left(\mu + \frac{7}{24}\right)u^4 \right\} \\ &+ \gamma_6 \left\{ -5t^2u^2 + 5tu^3 + \left(2\mu + \frac{55}{12}\right)u^4 \right\} \\ &+ \gamma_5 \left\{ t^4u + 3t^3u^2 + t^2u^3 + \left(-\mu - \frac{55}{24}\right)u^5 \right\} \\ &+ \gamma_4 \left\{ 6t^4u^2 - 3t^3u^3 + \left(-\mu - \frac{103}{24}\right)t^2u^4 - tu^5 + \left(\mu + \frac{79}{24}\right)u^6 \right\} \\ &+ \gamma_3 \left\{ -6t^5u^2 - 2t^4u^3 + 4t^3u^4 + 6t^2u^5 - 4tu^6 + \left(-\mu - \frac{31}{24}\right)u^7 \right\} \end{aligned}$$

$$\begin{aligned}
& + 4t^7u^3 + \left(\mu + \frac{55}{24}\right)t^6u^4 - 6t^5u^5 + \left(-\mu - \frac{7}{24}\right)t^4u^6 + \left(\mu + \frac{79}{24}\right)t^3u^7 \\
& + \left(\mu + \frac{31}{24}\right)t^2u^8 + \left(-\mu - \frac{55}{24}\right)tu^9 + \left(\nu + \frac{666919}{645120}\right)u^{10},
\end{aligned}$$

and we can take $\mu = -\frac{55}{24}$, $\nu = -\frac{666919}{645120}$. Thus we see that

$$\begin{aligned}
(3.12) \quad w &= \frac{1}{2^{14} \cdot 3^3 \cdot 5 \cdot 7} J_{10} - \frac{55}{24}u^4v - \frac{666919}{645120}u^{10} \\
&= \gamma_{10} + u\gamma_9 - u^3c_7 - u\gamma_4\gamma_5 + 2u^2\gamma_4^2 - 2u^2\gamma_3\gamma_5 + \gamma_3\gamma_4(-6tu^2 + 2u^3) \\
&+ \gamma_3^2(2t^2u^2 + 2tu^3 - 2u^4) + \gamma_6(-5t^2u^2 + 5tu^3) + \gamma_5(t^4u + 3t^3u^2 + t^2u^3) \\
&+ \gamma_4(6t^4u^2 - 3t^3u^3 - 2t^2u^4 - tu^5 + u^6) \\
&+ \gamma_3(-6t^5u^2 - 2t^4u^3 + 4t^3u^4 + 6t^2u^5 - 4tu^6 + u^7) \\
&+ 4t^7u^3 - 6t^5u^5 + 2t^4u^6 + t^3u^7 - t^2u^8
\end{aligned}$$

can be chosen as our generator \tilde{w} .

Finally, we have to find an element x of degree 30 such that $x \equiv \gamma_{15} \pmod{(u, v, w)}$ in $H^*(E_8/T; \mathbb{Z})$. Consider the element

$$x = \frac{1}{2}u^{15} \text{ in } H^{30}(E_8/T; \mathbb{Q}).$$

In fact, it can be shown that x is an integral cohomology class and is contained in $\text{Im } p^*$. Furthermore, we can check directly that

$$x \equiv \gamma_{15} \pmod{(u, v, w)}$$

(see appendix). Hence the element x can be chosen as our generator \tilde{x} .

Remark 3.7. *We can describe the element x explicitly in the ring $H^*(E_8/T; \mathbb{Z})$. But it is too lengthy to write down here, so we will give the explicit form of x in the appendix.*

3.4. Integral cohomology ring of $E_8/T^1 \cdot E_7$. Using the element x , we can rewrite \tilde{I}_{30} :

$$\begin{aligned}
\tilde{I}_{30} &= -9u^{30} - 24u^{24}v - 12u^{20}w + 36u^{14}vw - 40u^{12}v^3 - 12u^{10}w^2 + 120u^8v^2w \\
&- 140u^6v^4 + 24u^4vw^2 - 40u^2v^3w - 16v^5 - 8w^3 \\
&= -36x^2 - 48u^9vx - 24u^5wx + 36u^{14}vw - 40u^{12}v^3 - 12u^{10}w^2 + 120u^8v^2w \\
&- 140u^6v^4 + 24u^4vw^2 - 40u^2v^3w - 16v^5 - 8w^3 \\
&= 4(-9x^2 - 12u^9vx - 6u^5wx + 9u^{14}vw - 10u^{12}v^3 - 3u^{10}w^2 + 30u^8v^2w \\
&- 35u^6v^4 + 6u^4vw^2 - 10u^2v^3w - 4v^5 - 2w^3).
\end{aligned}$$

Therefore, in view of Lemma 3.1, we obtain the following main result:

Theorem 3.8. *The integral cohomology ring of $E_8/T^1 \cdot E_7$ is given as follows:*

$$H^*(E_8/T^1 \cdot E_7; \mathbb{Z}) = \mathbb{Z}[u, v, w, x]/(r_{15}, r_{20}, r_{24}, r_{30}),$$

where $\deg u = 2, \deg v = 12, \deg w = 20, \deg x = 30$ and

$$r_{15} = u^{15} - 2x,$$

$$r_{20} = 9u^{20} + 45u^{14}v + 12u^{12}w + 60u^8v^2 + 30u^4vw + 10u^2v^3 + 3w^2,$$

$$r_{24} = 11u^{24} + 60u^{18}v + 21u^{14}w + 105u^{12}v^2 + 60u^8vw + 60u^6v^3 + 9u^4w^2 \\ + 30u^2v^2w + 5v^4,$$

$$r_{30} = -9x^2 - 12u^9vx - 6u^5wx + 9u^{14}vw - 10u^{12}v^3 - 3u^{10}w^2 + 30u^8v^2w \\ - 35u^6v^4 + 6u^4vw^2 - 10u^2v^3w - 4v^5 - 2w^3.$$

In order to determine the integral cohomology ring of E_8/E_7 , we consider the Gysin exact sequence associated with the following circle bundle

$$(3.13) \quad S^1 \longrightarrow E_8/E_7 \xrightarrow{\pi} E_8/T^1 \cdot E_7,$$

where π is the natural projection. In this case, it reduces to the following short exact sequences:

$$(3.14) \quad 0 \longrightarrow H^{\text{odd}}(E_8/E_7; \mathbb{Z}) \longrightarrow H^*(E_8/T^1 \cdot E_7; \mathbb{Z}) \\ \xrightarrow{\times u} H^*(E_8/T^1 \cdot E_7; \mathbb{Z}) \xrightarrow{\pi^*} H^{\text{even}}(E_8/E_7; \mathbb{Z}) \longrightarrow 0,$$

where $H^{\text{even}} = \bigoplus_{i \geq 0} H^{2i}$ and $H^{\text{odd}} = \bigoplus_{i \geq 0} H^{2i+1}$.

From the exactness of (3.14), it follows that $H^{\text{even}}(E_8/E_7; \mathbb{Z})$ is isomorphic to $H^*(E_8/T^1 \cdot E_7; \mathbb{Z})/(u)$. Define the elements z_i ($i = 12, 20, 30$) of $H^*(E_8/E_7; \mathbb{Z})$ as follows:

$$z_{12} = \pi^*(v), \quad z_{20} = \pi^*(w), \quad z_{30} = \pi^*(x).$$

Then, by Theorem 3.8, we obtain

$$H^{\text{even}}(E_8/E_7; \mathbb{Z}) = \mathbb{Z}[z_{12}, z_{20}, z_{30}]/(2z_{30}, 3z_{20}^2, 5z_{12}^4, 4z_{12}^5 + 2z_{20}^3 + 9z_{30}^2). \\ = \mathbb{Z}[z_{12}, z_{20}, z_{30}]/(2z_{30}, 3z_{20}^2, 5z_{12}^4, z_{12}^5 + z_{20}^3 + z_{30}^2).$$

By Poincaré duality there exist elements $z_i \in H^i(E_8/E_7; \mathbb{Z})$ ($i = 59, 71, 79, 83, 91, 95, 103, 115$) such that

$$z_{12}^3 z_{20} z_{59} = z_{12}^2 z_{20} z_{71} = z_{12}^3 z_{79} = z_{12} z_{20} z_{83} = z_{12}^2 z_{91} = z_{20} z_{95} = z_{12} z_{103} = z_{115}.$$

Then it is not hard to show that

$$\begin{aligned}
z_{71} &= z_{12}z_{59}, \\
z_{79} &= z_{20}z_{59}, \\
z_{83} &= z_{12}^2z_{59}, \\
z_{91} &= z_{12}z_{20}z_{59}, \\
z_{103} &= z_{12}^2z_{20}z_{59}, \\
z_{115} &= z_{12}^3z_{20}z_{59}.
\end{aligned}$$

Summing up the results, we obtain the following:

Corollary 3.9. *The structure of $H^*(E_8/E_7; \mathbb{Z})$ is given by the following table:*

nontrivial $H^k(E_8/E_7; \mathbb{Z})$	basis elements
$H^0 = \mathbb{Z}$	1
$H^{12} = \mathbb{Z}$	z_{12}
$H^{20} = \mathbb{Z}$	z_{20}
$H^{24} = \mathbb{Z}$	z_{12}^2
$H^{30} = \mathbb{Z}_2$	z_{30}
$H^{32} = \mathbb{Z}$	$z_{12}z_{20}$
$H^{36} = \mathbb{Z}$	z_{12}^3
$H^{40} = \mathbb{Z}_3$	z_{20}^2
$H^{42} = \mathbb{Z}_2$	$z_{12}z_{30}$
$H^{44} = \mathbb{Z}$	$z_{12}^2z_{20}$
$H^{48} = \mathbb{Z}_5$	z_{12}^4
$H^{50} = \mathbb{Z}_2$	$z_{20}z_{30}$
$H^{52} = \mathbb{Z}_3$	$z_{12}z_{20}^2$
$H^{54} = \mathbb{Z}_2$	$z_{12}^2z_{30}$
$H^{56} = \mathbb{Z}$	$z_{12}^3z_{20}$
$H^{59} = \mathbb{Z}$	z_{59}
$H^{62} = \mathbb{Z}_2$	$z_{12}z_{20}z_{30}$
$H^{64} = \mathbb{Z}_3$	$z_{12}^2z_{20}^2$
$H^{66} = \mathbb{Z}_2$	$z_{12}^3z_{30}$
$H^{68} = \mathbb{Z}_5$	$z_{12}^4z_{20}$
$H^{71} = \mathbb{Z}$	$z_{12}z_{59}$
$H^{74} = \mathbb{Z}_2$	$z_{12}^2z_{20}z_{30}$
$H^{76} = \mathbb{Z}_3$	$z_{12}^3z_{20}^2$
$H^{79} = \mathbb{Z}$	$z_{20}z_{59}$
$H^{83} = \mathbb{Z}$	$z_{12}^2z_{59}$

nontrivial $H^k(E_8/E_7; \mathbb{Z})$	basis elements
$H^{86} = \mathbb{Z}_2$	$z_{12}^3 z_{20} z_{30}$
$H^{91} = \mathbb{Z}$	$z_{12} z_{20} z_{59}$
$H^{95} = \mathbb{Z}$	$z_{12}^3 z_{59}$
$H^{103} = \mathbb{Z}$	$z_{12}^2 z_{20} z_{59}$
$H^{115} = \mathbb{Z}$	$z_{12}^3 z_{20} z_{59}$

4. APPENDIX

In 3.3, we defined the element x as a rational cohomology class given by

$$x = \frac{1}{2}u^{15} \quad \text{in} \quad H^{30}(E_8/T; \mathbb{Q}).$$

We need to show that x is in fact an integral cohomology class. By Lemma 3.3, the following relations hold in $H^*(E_8/T; \mathbb{Z})$:

$$(4.1) \quad \begin{aligned} c_1 &= 3t, \\ c_2 &= 4t^2, \\ c_3 &= 2\gamma_3, \\ c_4 &= 3\gamma_4 - 2t^4, \\ c_5 &= 2\gamma_5 + 3t\gamma_4 - 2t^2\gamma_3, \\ c_6 &= 5\gamma_6 + 2\gamma_3^2 + t\gamma_5 - t^2\gamma_4 + 2t^6. \end{aligned}$$

Note that, by (2.11) and (4.1), the following relation holds:

$$(4.2) \quad \begin{aligned} c_8 &= uc_7 - 5u^2\gamma_6 - 2u^2\gamma_3^2 + (-tu^2 + 2u^3)\gamma_5 + (t^2u^2 + 3tu^3 - 3u^4)\gamma_4 \\ &\quad + (-2t^2u^3 + 2u^5)\gamma_3 - 2t^6u^2 + 2t^4u^4 - 4t^2u^6 + 3tu^7 - u^8. \end{aligned}$$

Using (4.2), we can rewrite the higher relations $\rho_8, \rho_9, \rho_{10}, \rho_{12}, \rho_{14}$ and ρ_{15} . For example,

$$\begin{aligned} \rho_8 &= -3c_8 + 3\gamma_4^2 - 2\gamma_3\gamma_5 + t(2c_7 - 6\gamma_3\gamma_4) + t^2(2\gamma_3^2 - 5\gamma_6) + 3t^3\gamma_5 \\ &\quad + 4t^4\gamma_4 - 6t^5\gamma_3 + t^8 \\ &= 3\gamma_4^2 - 2\gamma_3\gamma_5 + (2t - 3u)c_7 - 6t\gamma_3\gamma_4 + (-5t^2 + 15u^6)\gamma_6 \\ &\quad + (2t^2 + 6u^2)\gamma_3^2 + (3t^3 + 3tu^2 - 6u^3)\gamma_5 + (4t^4 - 3t^2u^2 - 9tu^3 + 9u^4)\gamma_4 \\ &\quad + (-6t^5 + 6t^2u^3 - 6u^5)\gamma_3 + t^8 + 6t^6u^2 - 6t^4u^4 + 12t^2u^6 - 9tu^7 + 3u^8. \end{aligned}$$

Using the relations ρ_i ($i = 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 15$), we rewrite the element $x = \frac{1}{2}u^{15}$ as follows:

$$\begin{aligned}
x &= \frac{1}{2}u^{15} \\
&= \frac{1}{2} \{ u^{15} - \rho_{15} + u\rho_{14} - u^3\rho_{12} + (t^4u - t^2u^3)\rho_{10} \\
&\quad + (\gamma_6 + t\gamma_5 + u\gamma_5 + u^2\gamma_4 + u^3\gamma_3 + t^6 + t^4u^2 + t^3u^3 + t^2u^4 + tu^5 + u^6)\rho_9 \\
&\quad - 39u^7\rho_8 \} + (-tu\gamma_3 + 2t^3u^2 - 5tu^4)\rho_{10} \\
&= \gamma_{15} - 20\gamma_3\gamma_6^2 + 3\gamma_3^2\gamma_9 - 23\gamma_3^3\gamma_6 - 6\gamma_3^5 + 4\gamma_6\gamma_9 \\
&\quad + 3u\gamma_4\gamma_{10} - u\gamma_5\gamma_9 - 3u\gamma_3^2\gamma_4^2 + 3uc_7\gamma_3\gamma_4 - 6u\gamma_4^2\gamma_6 + (-3t + 2u)\gamma_3^3\gamma_5 \\
&\quad + (-4t + 4u)\gamma_3\gamma_5\gamma_6 \\
&\quad + (-t^2 - u^2)\gamma_4\gamma_9 + (t^2 + tu - u^2)c_7\gamma_3^2 + (9t^2 + 12tu + 5u^2)\gamma_3\gamma_4\gamma_6 \\
&\quad + (5t^2 + 6tu + 2u^2)\gamma_3^3\gamma_4 + (3t^2 + 4tu + u^2)c_7\gamma_6 \\
&\quad + (-6t^3 - 2t^2u - 6tu^2 + 5u^3)\gamma_3^4 - u^3\gamma_3\gamma_9 + (3t^2u + u^3)\gamma_4^3 + (2t^2u + 3tu^2)c_7\gamma_5 \\
&\quad + (-45t^3 + 10t^2u - 40tu^2)\gamma_6^2 + (t^3 - 2t^2u + tu^2 - u^3)\gamma_3\gamma_4\gamma_5 \\
&\quad + (-33t^3 + t^2u - 31tu^2 + 13u^3)\gamma_3^2\gamma_6 \\
&\quad + (-2t^4 - 4t^3u - 3tu^3 + 3u^4)c_7\gamma_4 + (-9t^4 - 6t^3u - 18t^2u^2 + 5tu^3 - 3u^4)\gamma_5\gamma_6 \\
&\quad + (-3t^4 - 3t^3u - 7t^2u^2 + 5tu^3 - 4u^4)\gamma_3^2\gamma_5 + (-t^4 - 6t^3u - t^2u^2 - 3tu^3)\gamma_3\gamma_4^2 \\
&\quad + (-3t^4u - 6t^3u^2 + 3t^2u^3 + 15tu^4)\gamma_{10} + (-3t^4u + t^3u^2 + 5t^2u^3 + 10tu^4 - u^5)c_7\gamma_3 \\
&\quad + (15t^5 - 2t^4u + 3t^3u^2 + 14t^2u^3 - 16tu^4 + 3u^5)\gamma_3^2\gamma_4 \\
&\quad + (39t^5 - 13t^4u + 8t^3u^2 + 35t^2u^3 - 31tu^4 - 3u^5)\gamma_4\gamma_6 \\
&\quad + (t^6 - t^4u^2 - t^3u^3 - t^2u^4 - tu^5 - u^6)\gamma_9 \\
&\quad + (-13t^6 + 12t^5u + 5t^4u^2 - 56t^3u^3 + 8t^2u^4 + 21tu^5 + 2u^6)\gamma_3\gamma_6 \\
&\quad + (6t^6 + 3t^5u + 2t^4u^2 + 7t^3u^3 + t^2u^4 - 8tu^5 + 3u^6)\gamma_4\gamma_5 \\
&\quad + (-8t^6 + 6t^5u + 2t^4u^2 - 22t^3u^3 + 6t^2u^4 + 8tu^5 - 2u^6)\gamma_3^3 \\
&\quad + (-6t^7 + t^6u - 7t^4u^3 + 5t^3u^4 + 3t^2u^5 + 3tu^6 - 63u^7)\gamma_4^2 \\
&\quad + (-t^7 + 2t^6u + t^5u^2 - 11t^4u^3 + 6t^3u^4 + 5t^2u^5 + 6tu^6 + 39u^7)\gamma_3\gamma_5 \\
&\quad + (2t^8 + 6t^7u + 3t^6u^2 - 4t^5u^3 - 15t^4u^4 + 6t^3u^5 + 3t^2u^6 - 40tu^7 + 59u^8)c_7 \\
&\quad + (3t^8 + t^6u^2 + 11t^5u^3 + 14t^4u^4 - 20t^3t^5 - 4t^2u^6 + 118tu^7 + 3u^8)\gamma_3\gamma_4 \\
&\quad + (-48t^9 + 3t^8u - 41t^7u^2 + 18t^6u^3 + 16t^5u^4 - 13t^4u^5 - 67t^3u^6 + 125t^2u^7 \\
&\quad - 15tu^8 - 291u^9)\gamma_6
\end{aligned}$$

$$\begin{aligned}
& + (-18t^9 - 3t^8u - 16t^7u^2 + 10t^6u^3 - 4t^5u^4 - 8t^4u^5 - 16t^3u^6 - 23t^2u^7 - 10tu^8 \\
& - 115u^9)\gamma_3^2 \\
& + (-6t^{10} - 3t^9u - 9t^8u^2 + 5t^7u^3 - 5t^6u^4 - 14t^4u^6 - 52t^3u^7 + 6t^2u^8 - 60tu^9 \\
& + 117u^{10})\gamma_5 \\
& + (18t^{11} - 3t^{10}u + 5t^9u^2 + 11t^8u^3 - 28t^7u^4 + 8t^6u^5 + 20t^5u^6 - 64t^4u^7 - 15t^3u^8 \\
& + 54t^2u^9 + 178tu^{10} - 177u^{11})\gamma_4 \\
& + (-2t^{12} + 6t^{11}u + 2t^{10}u^2 - 20t^9u^3 + 11t^8u^4 + 22t^7u^5 - 8t^6u^6 + 83t^5u^7 \\
& + 15t^4u^8 + 5t^3u^9 - 116t^2u^{10} + tu^{11} + 117u^{12})\gamma_3 \\
& - 12t^{15} - t^{14}u - 10t^{13}u^2 + 6t^{12}u^3 + 7t^{11}u^4 - 13t^{10}u^5 - 31t^9u^6 + 9t^8u^7 - t^7u^8 \\
& - 118t^6u^9 - 18t^5u^{10} + 131t^4u^{11} - 6t^3u^{12} - 233t^2u^{13} + 175tu^{14} - 58u^{15},
\end{aligned}$$

which has shown that x is an integral cohomology class.

Next, we have to show that

$$x \pmod{(u, v, w)} \equiv \gamma_{15}.$$

By (3.11) and (3.12), we have

$$\begin{aligned}
v & \equiv 2\gamma_6 + \gamma_3^2 - t^2\gamma_4 + t^6 \pmod{(u)}, \\
w & \equiv \gamma_{10} \pmod{(u, v)}.
\end{aligned}$$

Therefore, in the ring $H^*(E_8/T; \mathbb{Z})/(u, v, w)$, the following relations hold:

$$\begin{aligned}
(4.3) \quad u & = 0, \\
\gamma_3^2 & = -2\gamma_6 + t^2\gamma_4 - t^6, \\
\gamma_{10} & = 0.
\end{aligned}$$

On the other hand, we determined the ring $H^*(E_8/T; \mathbb{Z})$ up to degrees ≤ 36 (Lemma 3.3). Taking (4.3) into account, we can show directly that $x = \gamma_{15}$ in the ring $H^*(E_8/T; \mathbb{Z})/(u, v, w)$.

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DEPARTMENT OF GENERAL EDUCATION
TAKAMATSU NATIONAL COLLEGE OF TECHNOLOGY
TAKAMATSU 761-8058, JAPAN
E-mail address: nakagawa@takamatsu-nct.ac.jp