

The Noether Map II

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SUMMARY : *Let $\rho : G \hookrightarrow \text{GL}(n, \mathbb{F})$ be a faithful representation of a finite group G . In this paper we proceed with the study of the image of the associated Noether map*

$$\eta_G^{\rho} : \mathbb{F}[V(G)]^G \rightarrow \mathbb{F}[V]^G.$$

In [8] it has been shown that the Noether map is surjective if V is a projective $\mathbb{F}G$ -module. This paper deals with the converse. The converse is in general not true: we illustrate this with an example. However, for p -groups (where p is the characteristic of the ground field \mathbb{F}) as well as for permutation representations of any group the surjectivity of the Noether map implies the projectivity of V .

Let $\rho : G \hookrightarrow \text{GL}(n, \mathbb{F})$ be a faithful representation of a finite group G of order d over a field \mathbb{F} . The representation ρ induces naturally an action of G on the vector space $V = \mathbb{F}^n$ of dimension n and hence on the ring of polynomial functions $\mathbb{F}[V] = \mathbb{F}[x_1, \dots, x_n]$. Our interest is focused on the subring of invariants

$$\mathbb{F}[V]^G = \{f \in \mathbb{F}[V] \mid gf = f \ \forall g \in G\},$$

which is a graded connected Noetherian commutative algebra. Denote by $\mathbb{F}G$ the group algebra. Let

$$V(G) = \mathbb{F}G \otimes V$$

be the induced module. The group G acts on $V(G)$ by left multiplication on the first component. We obtain a G -equivariant surjection

$$(\star) \quad V(G) \longrightarrow V, (g, v) \mapsto gv.$$

Let us choose a basis e_1, \dots, e_n for V . Let x_1, \dots, x_n be the standard dual basis for V^* , and set $G = \{g_1, \dots, g_d\}$. Then $V(G)$ can be written as

$$V(G) = \text{span}_{\mathbb{F}}\{e_{ij} \mid i = 1, \dots, n, j = 1, \dots, d\},$$

and the map (\star) translates into

$$V(G) \longrightarrow V, e_{ij} \mapsto g_j e_i.$$

Similarly, we have

$$V(G)^* = \text{span}_{\mathbb{F}}\{x_{ij} \mid i = 1, \dots, n, j = 1, \dots, d\}$$

with

$$V(G)^* \longrightarrow V^*, x_{ij} \mapsto g_j x_i.$$

We obtain a surjective G -equivariant map between the rings of polynomial functions

$$\eta_G : \mathbb{F}[V(G)] \longrightarrow \mathbb{F}[V].$$

The group G acts on $\mathbb{F}[V(G)]$ by permuting the basis elements x_{ij} . By restriction to the induced ring of invariants, we obtain the classical Noether map, cf. Section 4.2 in [9],

$$\eta_G^G : \mathbb{F}[V(G)]^G \longrightarrow \mathbb{F}[V]^G.$$

We note that $V(G)$ is the n -fold regular representation of G . Thus $\mathbb{F}[V(G)]^G$ are the n -fold vector invariants of the regular representation of G .

In the classical nonmodular case, where $p \nmid d$, the map η_G^G is surjective, see Proposition 4.2.2 in [9]. This has been generalized in the sense that the Noether map is surjective if V is a projective $\mathbb{F}G$ -module, see Proposition 3.1 in [8]. The converse may fail as we illustrate with the next example.

EXAMPLE: Let $GL(2, \mathbb{F}_3)$ be the general linear group of 2×2 matrices with entries from the field with three elements. By Corollary 9.14 in [4] the top Dickson class $\mathbf{d}_{2,0}$ is in the image of the transfer. Hence it is in the image of the Noether map. In order to see that also the other Dickson class $\mathbf{d}_{2,1}$ is in the image of the Noether map, we note that $GL(2, \mathbb{F}_3)$ contains a subgroup H of order 6 generated by

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix},$$

where $\lambda \in \mathbb{F}^\times$. Denote these six elements by h_1, \dots, h_6 . Then the stabilizer subgroup of the monomial

$$(h_1 \otimes x_1) \cdots (h_6 \otimes x_1) \in \mathbb{F}[V(G)]$$

is H . Direct computation yields

$$\eta_G^G(o((h_1 \otimes x_1) \cdots (h_6 \otimes x_1))) = -\mathbf{d}_{2,1}.$$

In the next section we prove that whenever G is a p -group or ρ is a permutation representation the Noether map is surjective if and only if V is a projective $\mathbb{F}G$ -module.

Before we proceed we present a general characterization:

PROPOSITION: *V is projective if and only if*

$$\eta_G^G : \mathbb{F}[\text{End}(V)(G)]^G \longrightarrow \mathbb{F}[\text{End}(V)]^G$$

is surjective.

PROOF: V is projective if and only if $\text{End}(V)$ is projective by [2]. Thus the Noether map on that vector space is surjective by Proposition 3.1 in [8]. Conversely, if η_G^G is surjective, then it is surjective in degree one. Hence the transfer map is surjective in degree one by Corollary 1.2 below. In particular, the identity on V is in the image of the transfer. Thus V is projective by the Higman criterion, see, e.g., Proposition 3.6.4 in [3]. \square

§1. p -Groups and Permutation Representations

In this section we want to show that the converse Proposition 3.1 in [8] is true in the case of p -groups P and in the case of permutation representations.

LEMMA 1.1: *Let P be a cyclic p -group, and let \mathbb{F} have characteristic p . Then*

$$\text{Im}(\text{Tr}^P)_{(1)} = \mathbb{F}[V]_{(1)}^P$$

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if and only if V is the k -fold regular representation of P for some $k \in \mathbb{N}$.

PROOF: Since the transfer is additive it suffices to consider indecomposable modules only.

Let the order of the group be p^s . Then up to isomorphism there are exactly p^s indecomposable $\mathbb{F}P$ -modules V_1, \dots, V_{p^s} with $\dim_{\mathbb{F}} V_i = i$. The action of P on V_i is afforded by the matrix consisting of one Jordan block with 1's on the diagonal and superdiagonal. Note that $V_i^P = V_1$ for all i .

Set $\Delta = g - 1$ where $g \in P$ is a generator. Then

$$\Delta(V_i^*) = \begin{cases} V_{i-1}^* & \text{for } i = 2, \dots, p^s \\ 0 & \text{for } i = 1. \end{cases}$$

Since, $\text{Tr}^P = \Delta^{p^s-1}$, we obtain

$$\text{Tr}^P(V_i^*) = \Delta^{p^s-1}(V_i^*) = \begin{cases} 0 & \text{for } i = 1, \dots, p^s - 1 \\ V_1^* & \text{for } i = p^s \end{cases}$$

as desired. \square

We obtain the following corollary that we note here for later reference.

COROLLARY 1.2: *Let $\rho : G \hookrightarrow \text{GL}(n, \mathbb{F})$ be a faithful representation of a finite group. Let $i \in \mathbb{F}^\times$. Then*

$$\text{Im}(\eta_G^G |_{(i)}) = \text{Im}(\text{Tr}^G |_{(i)}).$$

PROOF: By construction we obtain a commutative diagram as follows

$$\begin{array}{ccc} \mathbb{F}[V(G)]^G |_{(i)} & \xrightarrow{\eta_G^G |_{(i)}} & \mathbb{F}[V]^G |_{(i)} \\ \uparrow \text{Tr}^G |_{(i)} & & \uparrow \text{Tr}^G |_{(i)} \\ \mathbb{F}[V(G)] |_{(i)} & \xrightarrow{\eta_G |_{(i)}} & \mathbb{F}[V] |_{(i)}. \end{array}$$

By Theorem 3.2 [7] and the remark following it the transfer map on the left

$$\text{Tr}^G |_{(i)} : \mathbb{F}[V(G)] |_{(i)} \longrightarrow \mathbb{F}[V(G)]^G |_{(i)}$$

is surjective. By construction the lower map $\eta_G |_{(i)}$ is surjective. Thus the result follows. \square

THEOREM 1.3: *Let $\rho : P \hookrightarrow \text{GL}(n, \mathbb{F})$ be a representation of a p -group over a field \mathbb{F} of characteristic p . Then the following are equivalent:*

- (1) *The Noether map is surjective.*
- (2) *The Noether map is surjective in degree one.*
- (3) *V is a projective $\mathbb{F}P$ -module.*

PROOF: The implication (1) \Rightarrow (2) is trivial. The implication (3) \Rightarrow (1) was proven in Proposition 3.1 in [8]. Thus we need to show that V is projective if $\eta_P^P|_{(1)}$ is surjective.

Consider the short exact sequence of $\mathbb{F}P$ -modules

$$(*) \quad 0 \longrightarrow K^* \longrightarrow V(P)^* \xrightarrow{\eta_P^P|_{(1)}} V^* \longrightarrow 0.$$

The module $V(P)$ is free and therefore cohomologically trivial. Thus the long exact cohomology sequence breaks up into

$$0 \longrightarrow (K^*)^P \longrightarrow (V(P)^*)^P \xrightarrow{\eta_P^P|_{(1)}} (V^*)^P \longrightarrow H^1(P, K^*) \longrightarrow 0$$

and

$$H^i(P, V^*) \cong H^{i+1}(P, K^*) \quad \forall i \geq 1.$$

Since $\eta_P^P|_{(1)}$ is surjective by assumption, we obtain

$$H^1(P, K^*) = 0.$$

Thus K^* is a projective $\mathbb{F}P$ -module (see, e.g., Proposition 4.4.11 in [10]). Since P is finite and K^* finitely generated, this implies that K^* is injective, see Corollary 2.7 in [5]. Thus the sequence (*) splits and V^* is projective as desired. \square

We illustrate this result with an example.

EXAMPLE 1: Let \mathbb{F} be the field with q elements of characteristic p . Let $P \leq \text{GL}(n, \mathbb{F})$ be a p -Sylow subgroup of the general linear group. With assume without loss of generality that P consists of all upper triangular matrices with 1's on the diagonal. Then

$$\mathbb{F}[V(P)]_{(1)}^P = \text{span}_{\mathbb{F}}\{o(x_{i1}) = \sum_{j=1}^{|P|} x_{ij} \mid i = 1, \dots, n\}.$$

Thus

$$\begin{aligned} \eta_P^P(o(x_{i1})) &= \sum_{j=1}^{|P|} g_j x_i \\ &= \sum_{(a_{i+1}, \dots, a_n) \in \mathbb{F}^{n-i}} (x_i + a_{i+1}x_{i+1} + \dots + a_n x_n) \\ &= q^{\frac{n(n-1)}{2} - (n-i)} (q^{n-i} x_i + q^{n-i-1} \left(\sum_{a_{i+1} \in \mathbb{F}} a_{i+1} x_{i+1} + \dots + \sum_{a_n \in \mathbb{F}} a_n x_n \right)). \end{aligned}$$

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$$= q^{\frac{n(n-1)}{2}} x_i + q^{\frac{n(n-1)}{2}-1} \left(\sum_{a_{j+1} \in \mathbb{F}} a_{j+1} x_{j+1} + \cdots + \sum_{a_n \in \mathbb{F}} a_n x_n \right).$$

If $n \leq 1$ then P is the trivial group. Therefore V is $\mathbb{F}P$ -projective and the Noether map is surjective.

If $n \geq 2$ then the factor $q^{\frac{n(n-1)}{2}}$ vanishes. The factor $q^{\frac{n(n-1)}{2}-1}$ is nonzero if and only if $n = 2$. Thus we proceed by having a closer look at the two-dimensional case: We have by the above calculations

$$\begin{aligned} \eta_P^p(o(x_{11})) &= \sum_{j=1}^{|P|} g_j x_1 = \sum_{a_2 \in \mathbb{F}} (x_1 + a_2 x_2) = \left(\sum_{a_2 \in \mathbb{F}} a_2 \right) x_2, \\ \eta_P^p(o(x_{21})) &= \sum_{j=1}^{|P|} g_j x_2 = 0 \end{aligned}$$

If p is odd then for every nonzero $a_2 \in \mathbb{F}$ there exists a negative $-a_2 \neq a_2$. Therefore

$$\sum_{a_2 \in \mathbb{F}} a_2 = 0.$$

If $p = 2$ then

$$\left(\sum_{a_2 \in \mathbb{F}} a_2 \right) x_2 = \begin{cases} x_2 & \text{if } q = 2 \\ 0 & \text{if } q > 2. \end{cases}$$

Thus we have that the Noether map is surjective if and only if $n = 2 = p = q$. Explicitly we find

$$\eta_P^p(o(x_{11})) = x_2 \quad \text{and} \quad \eta_P^p(o(x_{11}x_{12})) = x_1^2 + x_1x_2.$$

Note that in this case

$$\text{Syl}_2(\text{GL}(2, \mathbb{F}_2)) \cong \mathbb{Z}/2$$

and our representation is projective.

Before proceeding to permutation representations, we want to mention two corollaries.

COROLLARY 1.4: *Let $\rho : G \hookrightarrow \text{GL}(n, \mathbb{F})$ be a faithful representation of a finite group. Assume that the rings of invariants of G and its p -Sylow subgroup coincide in degree one. Then the Noether map is surjective if and only if V is $\mathbb{F}G$ -projective.*

PROOF: If η_G^G is surjective, then it is surjective in degree one. Hence η_G^p is surjective in degree one by assumption. Therefore η_P^p is surjective in degree one by Proposition 2.1 in [8]. Thus V is projective by Theorem

1.3. The converse was shown in Proposition 3.1 in [8]. \square

COROLLARY 1.5: *Let $G = H \times P$ be a direct product a p -group P and a p' -group H . Assume that P is a cyclic p -group. Consider a faithful representation ρ of G over a field \mathbb{F} of characteristic p such that V is indecomposable as an $\mathbb{F}P$ -module. Then the Noether map is surjective if and only if V is $\mathbb{F}G$ -projective.*

PROOF: If V is $\mathbb{F}G$ -projective then the Noether map η_G^G is surjective by Proposition 3.1 in [8].

To prove the converse, let η_G^G be surjective. By Proposition 2.1 in [8] it is enough to show that the relative Noether map η_G^P is surjective. We proceed by contradiction and assume that η_G^P is not surjective. Then, by Proposition 2.1 in [8], the map η_P^P is not surjective. Hence V is not a projective $\mathbb{F}P$ -module by Theorem 1.3.

Let σ be a generator for P . The isomorphism type of a P -module is determined by the Jordan canonical form of σ . Up to isomorphism there are $|P|$ indecomposable P modules $V_1, V_2, \dots, V_{|P|}$, where $\dim V_i = i$ and σ acts on V_i by a $i \times i$ matrix consisting of a single Jordan block with ones on the diagonal and superdiagonal. Moreover $V_{|P|}$ is the only indecomposable module which is projective. Thus by assumption we have that $V = V_n$ for $1 \leq n < |P|$.

Let x_1, x_2, \dots, x_n be the basis of V such that

$$\sigma x_i = \begin{cases} x_1 & \text{if } i = 1 \\ x_{i-1} + x_i & \text{otherwise.} \end{cases}$$

Since the action of P commutes with the action of H and the action of H is nonmodular, it follows that $V = V_n$ is a direct sum of copies of isomorphic eigen spaces for H , and the variables x_1, x_2, \dots, x_n may be taken as eigen vectors. Let $\mathbf{N} = \prod_{g \in P} g(x_n)$ be the norm of x_n . Since p and $|H|$ are relatively prime, there exists positive integer m such that $m|P| \equiv -1 \pmod{|H|}$. Consider the polynomial $x_1 \mathbf{N}^m$. This polynomial is P -invariant since both x_1 and \mathbf{N} are. Let $h \in H$. Then

$$h(x_1 \mathbf{N}^m) = \lambda_h x_1 \lambda_h^{m|P|} \mathbf{N}^m = x_1 \mathbf{N}^m.$$

It follows that $x_1 \mathbf{N}^m$ is G -invariant.

Next we want to see that $x_1 \mathbf{N}^m$ is not in the image of Tr^P . Since V is not projective, the fixed point x_1 is not in the image of Tr^P . The degree-one-component $\mathbb{F}[V]_{(1)}$ is a direct summand in $\mathbb{F}[V]_{m|P|+1}$ by multiplication by \mathbf{N} , [6]. Thus the invariant $x_1 \mathbf{N}^m$ is not in the image of Tr^P either. However, if a G -invariant polynomial is not in the image of Tr^P then it

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is not in the image of Tr^G .

Since the degree of the polynomial $x_1 \mathbf{N}^m$ is relatively prime to p , we have that it is not in the image of η_G^G by Corollary 1.2. This is a contradiction. \square

COROLLARY 1.6: *Let $P \cong \mathbb{Z}/p$ and let V be an indecomposable P -module. Then the Noether map η_P^P is surjective in degrees divisible by p .*

PROOF: As above denote by $V = V_n$ the indecomposable $\mathbb{F}\mathbb{Z}/p$ -modules and x_1, x_2, \dots, x_n be the basis for V on which \mathbb{Z}/p acts through a single Jordan block of dimension n . We note that

$$\mathbb{F}[V] = B \oplus \mathbf{N}\mathbb{F}[V]$$

as $\mathbb{F}P$ -modules, where B consists of the polynomials of x_n -degree less than p , [6].

We proceed by induction on the degree. The decomposition

$$\mathbb{F}[V]_{(p)}^P = B_{(p)}^P \bigoplus \mathbf{N}\mathbb{F}[V]^P$$

yields that any invariant in degree p is a direct summand of a fixed point of a free module and the polynomial \mathbf{N} . Since fixed points of free modules and \mathbf{N} are in the image of η_P^P , the result follows for degree p .

Using the decomposition for degree kp we have that

$$\mathbb{F}[V]_{(kp)}^P = B_{(kp)}^P \bigoplus \mathbf{N}\mathbb{F}[V]_{((k-1)p)}^P.$$

Since η_P^P is an algebra map, and $\mathbb{F}[V]_{((k-1)p)}^P$ is in the image of η_P^P by induction, the result follows. \square

We turn to permutation representations.

THEOREM 1.7: *Let $\rho : G \hookrightarrow \text{GL}(n, \mathbb{F})$ be a permutation representation of a finite group of order d . Then the Noether map η_G^G is surjective if and only if $V = \mathbb{F}^n$ is projective.*

PROOF: By Proposition 3.1 in [8] we know that η_G^G is surjective if V is projective as $\mathbb{F}G$ -module.

We show that the converse is also true as follows:

Let η_G^G be surjective, then its restriction to degree one, $\eta_G^G|_{(1)}$, is also surjective:

$$\eta_G^G|_{(1)} : (V(G)^*)^G \rightarrow (V^*)^G.$$

We note that $(V(G)^*)^G$ has an \mathbb{F} -basis consisting of

$$o(x_{ij}) = \sum_{j=1}^d x_{ij} \quad \text{for } i = 1, \dots, n.$$

Therefore, the image under the Noether map is spanned by

$$\eta_G^G \left(\sum_{j=1}^d x_{ij} \right) = k_i o(x_i) = |\text{Stab}_G(x_i)| \text{Tr}^G(x_i) \quad \text{for } i = 1, \dots, n,$$

where

$$k_i = |\text{Stab}_G(x_i)|$$

is the order of the stabilizer of x_i in G . Since ρ is a permutation representation, $(V^*)^G$ is spanned by the orbit sums of x_1, \dots, x_n . It follows that k_i 's are not zero, since the Noether map is surjective. Hence

$$|\text{Stab}_G(x_i)| \not\equiv 0 \pmod{p}.$$

In other words, no element in a p -Sylow subgroup P of G fixes x_i , $i = 1, \dots, n$. Therefore

$$(\boxtimes) \quad o^P(x_i) = \text{Tr}^P(x_i) = \eta_P^P |_{(1)}(x_{i1}),$$

where $o^P(-)$ denotes the orbit sum under the action of P , and g_1 is the identity element. Since $(V^*)^P$ is also spanned by the orbit sums of the x_i 's, we found in (\boxtimes) that $\eta_P^P |_{(1)}$ is surjective. Therefore, η_P^P is surjective by Proposition 1.3. Hence V^* is a projective $\mathbb{F}P$ -module, by the same Proposition 1.3. Since P is a p -Sylow subgroup of G , the module V^* is projective as a $\mathbb{F}G$ -module, see Corollary 3 on Page 66 of [1]. \square

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