DEGREE BOUNDS AND THE REGULAR REPRESENTATION: APPENDIX

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Abstract. Let $G$ be a matrix group consisting of permutation matrices. Let $\mathbb{F}$ and $\mathbb{K}$ be two different fields. We show that if the polynomial invariants $\mathbb{F}[V]^G$ and $\mathbb{K}[V]^G$ are both Cohen-Macaulay, then they are simultaneously Gorenstein, complete intersections, hypersurfaces, resp. polynomial. Thus Cohen-Macaulay rings of permutation invariants are polynomial exactly when $G$ is generated by pseudo-reflections.

1. Appendix

Let $G$ be a matrix group consisting of $n \times n$-permutation matrices. As such we can consider $G$ as a subgroup of $\text{GL}(n, \mathbb{F})$ for any field $\mathbb{F}$.

Denote by $\mathbb{F}$ and $\mathbb{K}$ two different fields. In [2] it has been shown that if the polynomial invariants $\mathbb{F}[V]^G$ and $\mathbb{K}[V]^G$ are both Cohen-Macaulay, then there exists a common minimal algebra generating set (consisting of orbit sums of monomials) for both rings, see Theorem 2.8 in ibid.

In this appendix we want to prove some more immediate corollaries from that. In particular, in this appendix we are interested in the fine classification of Cohen-Macaulay rings.

Proposition 1.1. If $\mathbb{F}[V]^G$ and $\mathbb{K}[V]^G$ are both Cohen-Macaulay, then they are simultaneously Gorenstein.

Proof. This follows immediately from a result due to Stanley: If $\mathbb{F}[V]^G$ is Cohen-Macaulay, then it is Gorenstein if and only if its Poincaré series satisfies the following condition

$$P(\mathbb{F}[V]^G, \frac{1}{t}) = (-1)^n t^s P(\mathbb{F}[V]^G, t)$$

for some integer $s$, see Theorem 5.7.5 in [3]. Since the Poincaré series is independent of the ground field, see Proposition 3.2.2 in [3], the result follows. \hfill \Box

Next we turn to complete intersections.

Proposition 1.2. If $\mathbb{F}[V]^G$ and $\mathbb{K}[V]^G$ are Cohen-Macaulay, then they are simultaneously complete intersections.
Proof. As in the result above we use the fact that the Poincaré series of invariant rings of permutations of groups is independent of the ground field.

Let \( \{ f_1, \ldots, f_k \} \) be a common minimal algebra generating set for \( \mathbb{F}[V]^G \) and \( \mathbb{K}[V]^G \). Let \( \mathbb{F}[V]^G \) be a complete intersection. Then
\[
\mathbb{F}[V]^G = \mathbb{F}[f_1, \ldots, f_k]/(r_1, \ldots, r_{k-n}),
\]
where \( r_1, \ldots, r_{k-n} \) is a regular sequence. Thus
\[
P(\mathbb{F}[V]^G, t) = \frac{(1 - t^{r_1}) \cdots (1 - t^{r_{k-n}})}{(1 - t^{d_1}) \cdots (1 - t^{d_{k-n}})} = P(\mathbb{K}[V]^G, t),
\]
where \( | - | \) denotes the degree of the polynomial \( - \), see Lemma 2.2 in [1].

If both fields have the same characteristic there is nothing to show.

Assume that \( \mathbb{F} \) has characteristic zero and \( \mathbb{K} \) has finite characteristic \( p \). A relation \( r_i \) among the algebra generators \( f_1, \ldots, f_k \) valid in characteristic zero remains valid in finite characteristic. Thus the remembering map
\[
\phi : \mathbb{K}[F_1, \ldots, F_k]/(R_1, \ldots, R_{k-n}) \longrightarrow \mathbb{K}[V]^G, \quad \phi(F_i) = f_i \quad i = 1, \ldots, k,
\]
is surjective. Therefore the Poincaré series of the algebra on the left must be termwise greater or equal to the Poincaré series of the algebra on the right:
\[
P(\mathbb{K}[F_1, \ldots, F_k]/(R_1, \ldots, R_{k-n}), t) \gg P(\mathbb{K}[V]^G, t).
\]
However, by construction we have
\[
P(\mathbb{F}[V]^G, t) = P(\mathbb{K}[F_1, \ldots, F_k]/(R_1, \ldots, R_{k-n}), t) \gg P(\mathbb{K}[V]^G, t) = P(\mathbb{F}[V]^G, t).
\]
Thus the Poincaré series are equal and \( \phi \) is an isomorphism.

Finally assume that \( \mathbb{F} \) has finite characteristic \( p \) and \( \mathbb{K} \) has characteristic zero. The argument is similar to the preceding case: If \( r \) is a relation among the \( f_i \)'s modulo \( p \). Then we lift it to characteristic zero and obtain
\[
r(f_1, \ldots, f_k) = ph(f_1, \ldots, f_k)
\]
for some invariant \( h \in \mathbb{K}[V]^G \). Thus each \( r_j, j = 1, \ldots, k-n \) leads to a characteristic zero equation
\[
r_j - ph_j = 0
\]
for suitable invariants \( h_j \). Thus we obtain a surjective remembering map
\[
\psi : \mathbb{K}[F_1, \ldots, F_k]/(R_j - pH_j, j = 1, \ldots, k-n) \longrightarrow \mathbb{K}[V]^G, \quad \psi(F_i) = f_i \quad i = 1, \ldots, k.
\]
As above we can conclude that the Poincaré series of both rings are equal and \( \psi \) is an isomorphism.

\[\square\]

Corollary 1.3. If \( \mathbb{F}[V]^G \) and \( \mathbb{K}[V]^G \) are Cohen-Macaulay, then they are simultaneously hypersurfaces.

Proof. This is the preceding result specialized to \( k = n + 1 \). \[\square\]

Proposition 1.4. If \( \mathbb{F}[V]^G \) and \( \mathbb{K}[V]^G \) are Cohen-Macaulay, then they are simultaneously polynomial algebras.

Proof. If \( \mathbb{F}[V]^G \) is polynomial then any minimal algebra generating set consists of exactly \( n \) polynomials, hence \( \mathbb{K}[V]^G \) is polynomial. \[\square\]

Since rings of invariants in characteristic zero are polynomial exactly when the group \( G \) is generated by pseudo reflections we have the following result.
Corollary 1.5. Let $G \leq \text{GL}(n, \mathbb{F})$ be a permutation group. Assume that its ring of invariants, $\mathbb{F}[V]^G$, is Cohen-Macaulay. Then $\mathbb{F}[V]^G$ is polynomial if and only if $G$ is generated by transpositions.

References


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