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THE UNSTABLE PARTS FUNCTOR AND INJECTIVE OBJECTS

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Abstract. The unstable part functor Un assigns to an arbitrary module over the Steenrod algebra the largest unstable submodule. We start by showing some general properties of this functor. Then we study the functor $\mathsf{Un} \circ S^{-1}$ obtained from Un by precomposition with a localization. We show that $\mathsf{Un} \circ S^{-1}$ is an exact functor from the category of unstable noetherian modules over some unstable noetherian algebra to itself. Along the lines we describe the injective objects in this category.

1. Introduction

Let $\mathbb{F}$ be a finite field of characteristic p. Let H be a graded connected commutative unstable Noetherian $\mathbb{F}$-algebra over the Steenrod algebra of reduced powers $\mathcal{P}^*$. We denote by $\mathcal{M}_{\mathrm{H}}$ the category of $\mathrm{H} \odot \mathcal{P}^*$-modules and by $\mathcal{U}_{\mathrm{H}}$ the full subcategory of unstable $\mathrm{H} \odot \mathcal{P}^*$-modules.

In [1] Dwyer and Wilkerson introducted the functor

$$\mathsf{Un} : \mathcal{M}_{\mathrm{H}} \rightsquigarrow \mathcal{U}_{\mathrm{H}}, A \mapsto \mathsf{Un}(A)$$

that maps a $\mathrm{H} \odot \mathcal{P}^*$-module A to the largest unstable submodule, cf. [5] where the largest unstable quotient has been studied.

Consider the forgetful functor

$$\mathsf{F} : \mathcal{U}_{\mathrm{H}} \rightsquigarrow \mathcal{M}_{\mathrm{H}}, M \mapsto M$$

that forgets the property of being unstable.

Proposition 1.1. *The functor* F *is left adjoint of* Un.

Proof. Let M be an unstable $\mathrm{H} \odot \mathcal{P}^*$-module, and A an arbitrary $\mathrm{H} \odot \mathcal{P}^*$ module. We obtain a canonical map

$$\Phi : \mathrm{Hom}_{\mathcal{M}_{\mathrm{H}}}(\mathsf{F}(M), A) \longrightarrow \mathrm{Hom}_{\mathcal{U}_{\mathrm{H}}}(M, \mathsf{Un}(A)), \phi \mapsto \phi,$$

which is well-defined, because ϕ commutes with the $\mathcal{P}^*$-action. By construction Φ is bijective. Thus for any pair of maps $f : M \longrightarrow M'$ and $g : A \longrightarrow A'$ we obtain a commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{M}_{\mathrm{H}}}(\mathsf{F}(M'), A) & \xrightarrow{\mathsf{F}(f)} & \mathrm{Hom}_{\mathcal{M}_{\mathrm{H}}}(\mathsf{F}(M), A) & \xrightarrow{g_*} & \mathrm{Hom}_{\mathcal{M}_{\mathrm{H}}}(\mathsf{F}(M), A') \\ \downarrow \Phi & & \downarrow \Phi & & \downarrow \Phi \\ \mathrm{Hom}_{\mathcal{U}_{\mathrm{H}}}(M', \mathsf{Un}(A)) & \xrightarrow{f^*} & \mathrm{Hom}_{\mathcal{U}_{\mathrm{H}}}(M, \mathsf{Un}(A)) & \xrightarrow{g_*} & \mathrm{Hom}_{\mathcal{U}_{\mathrm{H}}}(M, \mathsf{Un}(A')) \end{array}$$

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with vertical isomorphisms. □

From this adjointness property we obtain some immediate corollaries on Un:

Corollary 1.2. *Consider the functor* $\mathsf{Un} : \mathcal{M}_{\mathrm{H}} \rightsquigarrow \mathcal{U}_{\mathrm{H}}$. *Then*

(1) Un *is left exact,*
(2) Un *preserves injectives, and*
(3) Un *preserves all limits.*

Proof. Since F as well as Un are additive and F is exact, the first statement follows from Theorem 2.6.1 in [14]. The second statement follows from adjointness, see, e.g., Proposition 2.3.10 loc.cit. The last statement follows from Theorem 2.6.10 loc.cit. □

We note that Un preserves finite coproducts since it preserves limits. However, Un preserves all coproducts as we show next.

Lemma 1.3. *The functor* Un *preserves all coproducts.*

Proof. Let $A = \oplus_i A_i$. Then by construction we find

$$\bigoplus_i \mathsf{Un}(A_i) \subseteq \mathsf{Un}(A) \subseteq A = \bigoplus_i A_i.$$

Let $a = \oplus_i a_i \in \mathsf{Un}(A)$ of degree d. Then

$$0 = \mathcal{P}^j(a) = \oplus_i \mathcal{P}^j(a_i)$$

for all $2j > d$ (resp. $j > d$ for $p = 2$) and thus $\mathcal{P}^j(a_i) = 0$ for all $2j > d = \deg(a_i)$ (resp. $j > d$ for $p = 2$). □

The next example shows that the functor Un is in general not exact.

Example 1.4. Let $\mathbb{F}$ have odd characteristic. Let $(\mathrm{H}\odot\mathcal{P}^*)a$ be a cyclic free module over $\mathrm{H}\odot\mathcal{P}^*$ with $\deg(a) = d$. Consider the exact sequence

$$0 \longrightarrow \sum_{2j>d} (\mathrm{H}\odot\mathcal{P}^*)(\mathcal{P}^j(a)) \hookrightarrow (\mathrm{H}\odot\mathcal{P}^*)a \longrightarrow (\mathrm{H}\odot\mathcal{P}^*)a/\Big(\sum_{2j>d} (\mathrm{H}\odot\mathcal{P}^*)\mathcal{P}^j(a)\Big) \longrightarrow 0.$$

We note that the quotient module on the right is by construction unstable. Thus upon taking unstable parts we obtain

$$0 \longrightarrow 0 \hookrightarrow 0 \longrightarrow (\mathrm{H}\odot\mathcal{P}^*)a/\Big(\sum_{2j>d} (\mathrm{H}\odot\mathcal{P}^*)\mathcal{P}^j(a)\Big).$$

Since the map on the right is no longer surjective, we have the desired example.

2. The Functor $\mathsf{Un} \circ S^{-1}$

Let $S \subseteq \mathrm{H}$ be a multiplicatively closed set. We extend the action of the Steenrod algebra to $S^{-1}\mathrm{H}$ by setting

$$(\star) \qquad \mathcal{P}(\xi)(\frac{h}{s})\mathcal{P}(\xi)(s) = \mathcal{P}(\xi)(h)$$

for all $\frac{h}{s} \in S^{-1}\mathrm{H}$, where $\mathcal{P}(\xi)$ denotes the giant Steenrod operation, cf. [13], Proposition 2.1 in [15], and Section 3.1 in [8]. The action of the Steenrod algebra on the localization $S^{-1}\mathrm{H}$ is generally no longer unstable.

Remark 2.1. It is easy to see by long division that with definition $(\star)$ the algebra $S^{-1}\mathrm{H}$ remains closed under the action of the Steenrod algebra, no matter what S is.

The localization $S^{-1}M \cong S^{-1}\mathrm{H}\otimes_{\mathrm{H}} M$ of an unstable $\mathrm{H}\odot\mathcal{P}^*$-module thus inherits a Steenrod algebra action and $S^{-1}M$ becomes an object in $\mathcal{M}_{\mathrm{H}}$.

We want to study the composite functor

$$\mathsf{Un}\circ S^{-1} : \mathcal{U}_{fg,\mathrm{H}} \rightsquigarrow \mathcal{U}_{\mathrm{H}},$$

where $\mathcal{U}_{fg,\mathrm{H}}$ denotes the category of unstable $\mathrm{H}\odot\mathcal{P}^*$-modules that are finitely generated as H-modules. This functor appears in several works, e.g., [2], [7], [11], and [16]. In particular we want to mention the following motivating results.

Denote by Un_{alg} the functor from the category of *algebras* over the Steenrod algebra to the category of unstable algebras over the Steenrod algebra by assigning to an object the largest unstable subalgebra. Let us consider the functor $\mathsf{Un}_{alg}\circ S^{-1}$ as a functor from the category $\mathcal{K}$ of unstable algebras to itself. Then for any reduced Noetherian object H we find that the unstable part of the localization

$$\mathsf{Un}_{alg}\circ S^{-1}(\mathrm{H}) = \overline{\mathrm{H}_{S^{-1}\mathrm{H}}}$$

coincides with the integral closure of H in the localization $S^{-1}\mathrm{H}$, see Proposition 1.2 in [16] for the case of integral domains H and [7] for the general case.

Remark 2.2. We note that in this case $\mathsf{Un}\circ S^{-1}(\mathrm{H}) = \mathsf{Un}_{alg}\circ S^{-1}(\mathrm{H})$ where we consider in the first expression H as a free module over itself generated by 1. This can be seen as follows: We have a inclusion of sets $\mathsf{Un}_{alg}\circ S^{-1}(\mathrm{H}) \subseteq \mathsf{Un}\circ S^{-1}(\mathrm{H})$. Take an element

$$\frac{1}{s}h \in \mathsf{Un}\circ S^{-1}(\mathrm{H}).$$

Then unstability as a module tells us that $\mathcal{P}(\xi)(s)|\mathcal{P}(\xi)(h)$. Comparing highest coefficients shows that the highest term of $\mathcal{P}(\xi)(\frac{h}{s})$ is $(\frac{h}{s})^q\xi^{\deg(h)-\deg(s)}$, where $q = |\mathbb{F}|$.

We want to mention also the following result: Consider the special case of

$$\mathsf{Un}\circ S^{-1} : \mathcal{U}_{fg,\mathbb{F}[V]} \rightsquigarrow \mathcal{U}_{\mathbb{F}[V]}$$

where $S = \mathbb{F}[V]\setminus\mathfrak{p}$ is the complement of a $\mathcal{P}^*$-invariant prime ideal. In [2] Dwyer and Wilkerson showed that this functor coincides with a certain component of Lannes's T-functor and thus inherits its properties, in particular exactness. In the next section we show that the functor

$$\mathsf{Un}\circ S^{-1} : \mathcal{U}_{fg,\mathrm{H}} \rightsquigarrow \mathcal{U}_{\mathrm{H}}$$

is exact *independent* of the choice of S or H. In Section 4 we prove more properties of $\mathsf{Un}\circ S^{-1}$: in particular we prove that $\mathsf{Un}\circ S^{-1}(M)$ remains noetherian if M is noetherian. Along the lines we describe the injective objects in $\mathcal{U}_{fg,\mathrm{H}}$. It turns out that many classical properties of injective objects in the category of *all* noetherian H-modules carry over nicely to the category of unstable noetherian H-modules.

3. Exactness

By Corollary 1.2 the functor Un is left-exact. Since localization is exact, the composite $\mathsf{Un}\circ S^{-1}$ is left-exact. Thus in order to show that $\mathsf{Un}\circ S^{-1}$ exact it is enough to show that $\mathsf{Un}\circ S^{-1}$ is exact on injective modules.

We start with an explicit calculation for the case $\mathrm{H} = \mathbb{F}[V]$.

Example 3.1. Let $V = \mathbb{F}^n$ be the n-dimensional vector space over $\mathbb{F}$. Denote by $\mathbb{F}[V]$ the symmetric algebra over the dual space V^*. We want to show that the functor

$$\mathsf{Un} \circ S^{-1} : \mathcal{U}_{fg,\mathbb{F}[V]} \rightsquigarrow \mathcal{U}_{\mathbb{F}[V]}$$

is exact. We go back to the classification of injective objects in the category $\mathcal{U}_{fg,\mathbb{F}[V]}$, see [4], and find that the indecomposable injectives look like

$$(\bullet) \qquad E[V, W, k] = \mathbb{F}[V] \oplus_{\mathbb{F}[V/W]} \mathbf{J}_{\mathbb{F}[V/W]}(k)$$

for some $W \leq V$ and $k \in \mathbb{N}_0$. By Lemma 8.5.3 in [12] the modules $E[V, W, k]$ are annihilated by some power of the prime ideal $\mathfrak{p}_W$ defined by

$$\mathfrak{p}_W = \ker(\mathbb{F}[V] \longrightarrow \mathbb{F}[W]).$$

Thus we find

$$(*) \qquad S^{-1}E[V, W, k] = \begin{cases} 0 & \text{if } S \cap \mathfrak{p} \neq \varnothing \\ S^{-1}\mathbb{F}[V] \oplus_{\mathbb{F}[V/W]} \mathbf{J}_{\mathbb{F}[V/W]}(k) & \text{otherwise.} \end{cases}$$

Since $\mathsf{Un} \circ S^{-1}$ commutes with coproducts by Lemma 1.3, and injective modules in $\mathcal{U}_{fg,\mathbb{F}[V]}$ are finite direct sums of indecomposable injectives $(\bullet)$, we are done.

The preceding example suggests that we look for the injective objects in the category $\mathcal{U}_{fg,\mathrm{H}}$. Those have been classified in [6] and thus we could proceed by an explicit calculation as above. However, their description is a bit cumbersome, so we prefer to start with a quick characterization of them. In doing so we do not refer to the original classification except that we assume that injective hulls in $\mathcal{U}_{fg,\mathrm{H}}$ exist.

Let M be a module in $\mathcal{U}_{fg,\mathrm{H}}$. Denote by $E(M)$ its injective hull in $\mathcal{U}_{fg,\mathrm{H}}$.

Lemma 3.2. *Let $\mathfrak{p} \subseteq \mathrm{H}$ be a $\mathcal{P}^*$-invariant prime ideal. Then the injective hull $E(\mathrm{H}/\mathfrak{p})$ is indecomposable.*

Proof. Assume that $E(\mathrm{H}/\mathfrak{p}) = E_1 \oplus E_2$ is decomposable as an $\mathrm{H} \odot \mathcal{P}^*$-module, where E_1, E_2 are (necessarily injective) nontrivial modules. Since

$$\mathrm{H}/\mathfrak{p} \hookrightarrow E(\mathrm{H}/\mathfrak{p}) = E_1 \oplus E_2$$

is essential we have that $E_i \cap \mathrm{H}/\mathfrak{p} \neq 0$ for $i = 1, 2$. Let $h_i \in E_i \cap \mathrm{H}/\mathfrak{p}$, $h_i \neq 0$. Then

$$h_1 h_2 \in (E_1 \cap \mathrm{H}/\mathfrak{p}) \cap (E_2 \cap \mathrm{H}/\mathfrak{p}) \subseteq \mathrm{H}/\mathfrak{p}.$$

However $\mathrm{H}/\mathfrak{p}$ is an integral domain. This is a contradiction. □

Remark 3.3. More generally, let[1] $\mathrm{H}\mathbf{t} \cong \sum \mathrm{H}/\mathfrak{p}$ be a cyclic module in $\mathcal{U}_{fg,\mathrm{H}}$, then

$$E(\mathrm{H}\mathbf{t}) \cong E(\sum \mathrm{H}/\mathfrak{p}) \cong \sum E(\mathrm{H}/\mathfrak{p})$$

is indecomposable.

Proposition 3.4. *If E is an indecomposable injective in $\mathcal{U}_{fg,\mathrm{H}}$ then $E \cong \sum E(\mathrm{H}/\mathfrak{p})$ for some $\mathcal{P}^*$-invariant prime ideal $\mathfrak{p} \subseteq \mathrm{H}$.*

[1] A $\sum$ without any indexing denotes a (possibly higher) suspension.

Proof. Since E is an object in $\mathcal{U}_{fg,\mathrm{H}}$ the set of associated prime ideals consists of $\mathcal{P}^*$-invariant prime ideals and is in particular finite, see [9]. Let $\mathfrak{p} \subseteq \mathrm{H}$ be an associated prime of E, then we find a map of $\mathrm{H} \odot \mathcal{P}^*$-modules

$$\mathrm{H}\mathbf{t} \hookrightarrow E$$

where $\mathrm{H}\mathbf{t} \cong \sum \mathrm{H}/\mathfrak{p}$, see [10]. Since E is indecomposable it follows that

$$E \cong E(\mathrm{H}\mathbf{t}) \cong E(\sum \mathrm{H}/\mathfrak{p}) \cong \sum E(\mathrm{H}/\mathfrak{p}).$$

□

Proposition 3.5. *The set of associated prime ideals of* $\sum E(\mathrm{H}/\mathfrak{p})$ *consists solely of* $\mathfrak{p}$.

Proof. Since the set of associated prime ideals of a noetherian module M and its suspensions coincide, it is enough to prove the result for $E(\mathrm{H}/\mathfrak{p})$.

We denote by $I(M)$ the injective hull of the module M in the category of H-modules.

Consider the diagram

$$\begin{array}{ccc} \mathrm{H}/\mathfrak{p} & \hookrightarrow & E(\mathrm{H}/\mathfrak{p}) \\ \cap & & \\ I(\mathrm{H}/\mathfrak{p}) & & \end{array}.$$

Since $E(\mathrm{H}/\mathfrak{p})$ is an object in $\mathcal{U}_{fg,\mathrm{H}}$ the set of its associated prime ideals consists of (finitely many) $\mathcal{P}^*$-invariant prime ideals, see [9]. Let $\mathfrak{q} \subseteq \mathrm{H}$ be in this set. Since the extension $\mathrm{H}/\mathfrak{p} \hookrightarrow E(\mathrm{H}/\mathfrak{p})$ is essential we find that

$$0 \neq \mathrm{H}/\mathfrak{p} \cap \mathrm{H}/\mathfrak{q} \subseteq \mathrm{H}/\mathfrak{p} \subseteq I(\mathrm{H}/\mathfrak{p}).$$

Since $I(\mathrm{H}/\mathfrak{p})$ is indecomposable, see, e.g., Theorem 3.3.7 in [3], we have by symmetry

$$I(\mathrm{H}/\mathfrak{p}) = I(\mathrm{H}/\mathfrak{p} \cap \mathrm{H}/\mathfrak{q}) = I(\mathrm{H}/\mathfrak{q}).$$

Thus $\mathfrak{p} = \mathfrak{q}$ by Theorem 3.3.8 in [3]. □

Proposition 3.6. $E(\mathrm{H}/\mathfrak{p})$ *is annihilated by some power of* $\mathfrak{p}$.

Proof. Let $m \in E(\mathrm{H}/\mathfrak{p})$ be a nonzero element. Thus $\mathrm{H}m \cong \mathrm{H}/\mathrm{Ann}(m) \subseteq E(\mathrm{H}/\mathfrak{p})$. By the preceding result $\mathfrak{p}$ is the only associated prime of $E(\mathrm{H}/\mathfrak{p})$. Thus $\mathrm{Ann}(m) \subseteq \mathfrak{p}$ and hence $\mathfrak{p}$ is associated to $\mathrm{H}m$. Indeed, $\mathfrak{p}$ is the unique minimal element in the support of $\mathrm{H}m$, because the modules involved are noetherian as H-modules. Thus $\mathfrak{p}$ is the radical of $\mathrm{Ann}(m)$. Since $\mathfrak{p}$ is finitely generated, $\mathfrak{p}^t$ annihilates m for some large $t \in \mathbb{N}$. Since $E(\mathrm{H}/\mathfrak{p})$ is finitely generated, $\mathfrak{p}^s$ annihilates $E(\mathrm{H}/\mathfrak{p})$ for some $s \in \mathbb{N}$. □

Remark 3.7. We note that an element $h \in \mathrm{H} \setminus \mathfrak{p}$ induces a monomorphism of H-modules

$$\mu_h : E(\mathrm{H}/\mathfrak{p}) \longrightarrow E(\mathrm{H}/\mathfrak{p}),\ m \mapsto hm$$

since $\mathfrak{p}$ is the only associated prime ideal. In other words, if $S \subseteq \mathrm{H}$ is a multiplicatively closed set with $S \cap \mathfrak{p} = \varnothing$, then the canonical map

$$E(\mathrm{H}/\mathfrak{p}) \hookrightarrow S^{-1}E(\mathrm{H}/\mathfrak{p})$$

is a monomorphism.

Proposition 3.8. *Let M be an object in $\mathcal{U}_{fg,\mathrm{H}}$. Then its injective hull is given by*

$$E(M) = \bigoplus_i E(\sum \mathrm{H}/\mathfrak{p}_i)^{\oplus n_i}$$

where the sum runs over a finite number of $\mathcal{P}^*$-invariant prime ideals $\mathfrak{p} \subseteq \mathrm{H}$, and $n_i \in \mathbb{N}$.

Proof. We proceed by induction on the length of M. By [10] an unstable module M admits a finite prime filtration of unstable $\mathrm{H} \odot \mathcal{P}^*$-modules

$$0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow M_3 \hookrightarrow \cdots \hookrightarrow M_{n-1} \hookrightarrow M_n = M$$

such that $M_i/M_{i-1} \cong \sum(\mathrm{H}/\mathfrak{p}_i)$ (as $\mathrm{H} \odot \mathcal{P}^*$-modules) for some $\mathcal{P}^*$-invariant prime ideal $\mathfrak{p}_i \subseteq \mathrm{H}$. Thus inductively we obtain

$$\begin{aligned} E(M) &\subseteq E(M_{n-1}) \oplus E(M/M_{n-1}) \\ &= E(M_{n-1}) \oplus E(\sum(\mathrm{H}/\mathfrak{p}_n)) \\ &= \cdots \\ &\subseteq E(\sum(\mathrm{H}/\mathfrak{p}_1)) \oplus \cdots \oplus E(\sum(\mathrm{H}/\mathfrak{p}_n)) \end{aligned}$$

a direct sum of indecomposable injectives. Thus $E(M)$ is the direct sum of indecomposable injectives $E(\sum(\mathrm{H}/\mathfrak{p}_i))$ for certain $i \in \{1, \ldots, n\}$. □

Remark 3.9. Indeed, the set of prime ideals $\mathfrak{p}_i$ appearing in the injective hull of a noetherian module M coincides with the set of prime ideals in H associated to M.

This characterization enables us to prove the general theorem.

Theorem 3.10. *Let* H *be an unstable Noetherian* $\mathbb{F}$*-algebra over the Steenrod algebra. Then the composite functor*

$$\mathsf{Un} \circ S^{-1} : \mathcal{U}_{fg,\mathrm{H}} \rightsquigarrow \mathcal{U}_{\mathrm{H}}$$

is exact.

Proof. Since our functor is left-exact it is enough to show that it is exact on injectives. By our above characterization of the injective indecomposable modules we find that

$$S^{-1}E(\sum(\mathrm{H}/\mathfrak{p})) = \begin{cases} 0 & \text{if } S \cap \mathfrak{p} \neq \varnothing \\ S^{-1}E(\sum(\mathrm{H}/\mathfrak{p})) & \text{otherwise,} \end{cases}$$

cf. Equation $(*)$ above and Theorem 3.3.8 in [3]. Thus an exact sequence

$$0 \longrightarrow \bigoplus_i E(\sum(\mathrm{H}/\mathfrak{p}_i))^{\oplus b_i} \hookrightarrow \bigoplus_i E(\sum(\mathrm{H}/\mathfrak{p}_i))^{\oplus a_i} \longrightarrow \bigoplus_i E(\sum(\mathrm{H}/\mathfrak{p}_i))^{\oplus c_i} \longrightarrow 0$$

with necessarily $a_i = b_i + c_i$ yields an exact sequence

$$\begin{aligned} 0 \longrightarrow \bigoplus_{i, S\cap\mathfrak{p}_i=\varnothing} S^{-1}E(\sum(\mathrm{H}/\mathfrak{p}_i))^{\oplus b_i} &\hookrightarrow \bigoplus_{i, S\cap\mathfrak{p}_i=\varnothing} S^{-1}E(\sum(\mathrm{H}/\mathfrak{p}_i))^{\oplus a_i} \\ &\longrightarrow \bigoplus_{i, S\cap\mathfrak{p}_i=\varnothing} S^{-1}E(\sum(\mathrm{H}/\mathfrak{p}_i))^{\oplus c_i} \longrightarrow 0. \end{aligned}$$

Since Un commutes with coproducts we obtain that

$$0 \longrightarrow \bigoplus_{i, S\cap\mathfrak{p}_i=\varnothing} \mathsf{Un}\circ S^{-1}E(\sum(\mathrm{H}/\mathfrak{p}_i))^{\oplus b_i} \hookrightarrow \bigoplus_{i, S\cap\mathfrak{p}_i=\varnothing} \mathsf{Un}\circ S^{-1}E(\sum(\mathrm{H}/\mathfrak{p}_i))^{\oplus a_i} \longrightarrow \bigoplus_{i, S\cap\mathfrak{p}_i=\varnothing} \mathsf{Un}\circ S^{-1}E(\sum(\mathrm{H}/\mathfrak{p}_i))^{\oplus c_i} \longrightarrow 0.$$

remains exact. □

4. Further Results and Corollaries

By Dwyer and Wilkerson's result $\mathsf{Un}\circ S^{-1}(M)$ is noetherian for any module M in $\mathcal{U}_{fg,\mathbb{F}[V]}$ when $S = \mathbb{F}[V]\setminus\mathfrak{p}$ is the complement of a $\mathcal{P}^*$-invariant prime ideal. This remains true for any H and any S as we see next.

Theorem 4.1. *Let* H *be an unstable graded connected commutative noetherian algebra. Let* M *be an object in* $\mathcal{U}_{fg,\mathrm{H}}$. *Let* $S \subseteq H$ *be a multiplicatively closed set. Then* $\mathsf{Un}\circ S^{-1}(M)$ *remains noetherian (as an* H*-module).*

Proof. We proceed by induction on the length of a prime filtration

$$0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow \cdots \hookrightarrow M_n = M.$$

If $n = 1$, then $M = \mathrm{H}\mathbf{t} \cong \sum \mathrm{H}/\mathfrak{p}$. Since the Steenrod algebra acts trivially on $\mathbf{t}$, see the proofs of Theorems 3.2 and 4.1 in [10], we have that

$$\mathsf{Un}\circ S^{-1}(M) = \mathsf{Un}\circ S^{-1}(\mathrm{H}\mathbf{t}) = \mathsf{Un}\circ S^{-1}(\mathrm{H})\mathbf{t} = \begin{cases} \overline{\mathrm{H}_{S^{-1}\mathrm{H}}}\mathbf{t} & \text{if } S\cap\mathfrak{p} = \varnothing \\ 0 & \text{otherwise,} \end{cases}$$

where $\overline{\mathrm{H}_{S^{-1}\mathrm{H}}}$ denotes the integral closure of H in $S^{-1}\mathrm{H}$. Since H is noetherian, so is $\overline{\mathrm{H}_{S^{-1}\mathrm{H}}}$, see [7], proving the induction start.

To complete the proof, note that for $n > 1$ we have a short excat sequence

$$0 \longrightarrow M_{n-1} \hookrightarrow M_n \twoheadrightarrow M_n/M_{n-1} \longrightarrow 0$$

with $M_n/M_{n-1} \cong \sum \mathrm{H}/\mathfrak{p}$. By Theorem 3.10 this yields an exact sequence

$$0 \longrightarrow \mathsf{Un}\circ S^{-1}M_{n-1} \hookrightarrow \mathsf{Un}\circ S^{-1}M_n \twoheadrightarrow \mathsf{Un}\circ S^{-1}M_n/M_{n-1} \longrightarrow 0$$

where by induction the two outer modules are noetherian. Thus $\mathsf{Un}\circ S^{-1}M_n$ is noetherian. □

We want to close with a few results on $\mathsf{Un}\circ S^{-1}$ and injective objects.

Proposition 4.2. *Let* M *be an object in* $\mathcal{U}_{fg,\mathrm{H}}$. *Let* S *be a multiplicatively closet subset of* H *such that it contains no zero divisors on* M. *Then the extension*

$$M \hookrightarrow \mathsf{Un}\circ S^{-1}M$$

is essential in the category $\mathcal{U}_{fg,\mathrm{H}}$.

Proof. Observe that the additional assumption on S guarantees that the canonical map $M \longrightarrow S^{-1}M$ is an inclusion. Furthermore, by the preceding result, $\mathsf{Un}\circ S^{-1}M$ is noetherian and thus we have an extension

$$M \hookrightarrow \mathsf{Un}\circ S^{-1}M$$

in the category $\mathcal{U}_{fg,\mathrm{H}}$. Let $N \subseteq \mathsf{Un} \circ S^{-1}M$ be an unstable submodule. Let $n \in N$. By construction we can write n as

$$n = \sum_s \frac{h_s}{s} m_s$$

with $s \in S$, $h_s \in \mathrm{H}$, and $m_s \in M$. Then

$$(\prod_s s)n \in N \cap M.$$

In particular $N \cap M \neq 0$. □

Corollary 4.3. *Let M be an object in $\mathcal{U}_{fg,\mathrm{H}}$. Let S be a multiplicatively closet subset of* H *such that it contains no zero divisors on M. Then*

$$E(M) = E(\mathsf{Un} \circ S^{-1}M).$$

In particular, if $M = E$ is injecive we have

$$E = \mathsf{Un} \circ S^{-1}E.$$

Proof. This is immediate from the preceding Proposition 4.2. □

Corollary 4.4. *Let E be an injecive module in $\mathcal{U}_{fg,\mathrm{H}}$. Then the module* $\mathsf{Un} \circ S^{-1}E$ *is always injective, independent of the choice of S.*

Proof. Since E is a finite coproduct of indecomposable injectives, and $\mathsf{Un} \circ S^{-1}$ commutes with coproducts, it is enough to show this result for $E = E(\mathrm{H}/\mathfrak{p})$. By the preceding corollary $\mathsf{Un} \circ S^{-1}E(\mathrm{H}/\mathfrak{p})$ is injective for $S \cap \mathfrak{p} = \varnothing$. If $S \cap \mathfrak{p} \neq \varnothing$, it is zero. □

Remark 4.5. We note that

$$S^{-1}I(\mathrm{H}/\mathfrak{p}) = \begin{cases} I(\mathrm{H}/\mathfrak{p}) & \text{if } S \cap \mathfrak{p} = \varnothing \\ 0 & \text{otherwise,} \end{cases}$$

see, e.g., Theorem 3.3.8 in [3]. Thus the preceding two corollaries reflect nicely this property for the category $\mathcal{U}_{fg,\mathrm{H}}$.

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