AN E^2-TYPE CLOSED MODEL CATEGORY FOR BISIMPLICIAL GROUPS

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ABSTRACT. A closed model category structure on the category of bisimplicial groups is defined in which the weak equivalences are isomorphisms on bigraded homotopy groups $\pi_{k,l}$ and at the same time isomorphisms on the E^2 term of the Quillen spectral sequence. There is an analogue of the spiral exact sequence of Dwyer-Kan-Stover [4]. This structure is considered as a convenient setting for a study of the relation between bigraded homotopy and hyperhomology.

CONTENTS

1. Introduction and notation

1.1. Definition and properties of bigraded homotopy groups

Let G be a bisimplicial group, and let $c_v\pi_0^v G$ be a vertical constant extension of the π_0 groups of its vertical terms. Let

$$1 \to \Omega_v G \to E_v G \to I_v G \to \{1\}, \quad \{1\} \to I_v G \to G \to c\pi_0^v G \to 1$$

be exact sequences obtained by taking the contractible path group $E(\_)$ for every vertical term $G_{k,\bullet}$. In this sequence the kernel $\Omega_v G$ is the vertical loop group of G, and $I_v G = G_{(1)}$ the identity component of G. Iterating this construction of Quillen gives the exact couple of [6]:

$$\begin{array}{ccc} \pi_{p-1}(\Omega_v^{q+1}G) & \longrightarrow & \pi_p(\Omega_v^q G) \\ & \nwarrow \qquad \swarrow & \\ & \pi_p{}^h\pi_q^v G & \end{array}$$

In what follows we denote $\pi_{p,q}G \overset{def}{=} \pi_p \Delta \Omega_v^q G$, and call these groups *the bigraded homotopy groups* of G. In this notation Quillen's exact couple becomes

$$\begin{array}{ccc} \pi_{p-1,q+1}G & \longrightarrow & \pi_{p,q}G \\ & \nwarrow \qquad \swarrow & \\ & \pi_p{}^h\pi_q^v G & \end{array}$$

and as a long exact sequence it can be rewritten:

$$\cdots \to \pi_{p-1,q+1}G \to \pi_{p,q}G \to \pi_p{}^h\pi_q^v G \longrightarrow \pi_{p-2,q+1}G \to \pi_{p-1,q}G \to \cdots$$

Following [3] this sequence is called the spiral exact sequence.

Rewriting the exact sequence for $p = 2$ and keeping in mind the connection between the vertical identity components $I_v G$ and the vertical loops $\Omega_v G$, we get

$$\ldots \to \pi_1 \Delta I_v \Omega_v^l G \to \pi_1 \Delta \Omega_v^l G \to \pi_1^h \pi_l^v G \to \pi_0 \Delta I_v \Omega_v^l G \to \pi_0 \Delta \Omega_v^l G \to \pi_0^h \pi_l^v G \to \{1\}$$

Hence, since $\pi_1 \Delta I_v \Omega_v^l G \cong \pi_0 \Delta \Omega_v^{l+1} G$,

$$\ldots \to \pi_{0,l+1}G \to \pi_{1,l}G \to \pi_1^h \pi_l^v G \to \pi_0 \Delta I_v \Omega_v^l G \to \pi_{0,l}G \to \pi_0^h \pi_l^v G \to \{1\}$$

where $l \geq 0$.

From $\pi_0^v I_v \Omega_v^l G = 0$ for all vertical terms it follows that $\pi_0 \Delta I_v \Omega_v^l G = 0$ for all $l \geq 0$.

Hence $\pi_0 \Delta \Omega_v^l G \cong \pi_0^h \pi_l^v G$. In other words, $\pi_{0,l} G \cong \pi_0^h \pi_l^v G$ -the 0-th column of bigraded groups coincides with the 0-th column of the $\mathbf{E^2}$-term. The spiral exact sequence ends as:

$$\ldots \to \pi_{1,l+1} G \to \pi_{2,l} G \to \pi_2^h \pi_l^v G \to \pi_{0,l+1} G \to \pi_{1,l} G \to \pi_1^h \pi_l^v G \to 0$$

The following lemma follows from the above and from the definitions:

Lemma 1.1. For any bisimplicial group G

(1). $\pi_{*,0} G = \pi_* \Delta G$ (the homotopy of the diagonal of G)
(2). $\pi_{p,q}(G \times H) = \pi_{p,q} G \times \pi_{p,q} H$
(3). $\pi_{p,q} G$ are abelian unless $p = q = 0$
(4). $\pi_{0,l} G \cong \pi_0^h \pi_l^v G$
(5). $\pi_{1,l} G \to \pi_1^h \pi_l^v G$ is onto

Theorem 1.2. Let $f : G \to H$ be a homomorphism of bisimplicial groups, and let f induce isomorphisms on all fringe bigraded groups $\pi_{-1,*}$. Then f induces an isomorphism of the E^2-terms of the Quillen spectral sequence if and only if it induces isomorphisms of the bigraded homotopy groups

$$\pi_{p,q} f : \pi_{p,q} G \xrightarrow{\cong} \pi_{p,q} H$$

for all $(p,q) \geq 0$.

Proof. One direction follows from the spiral exact sequence and the five lemma. For the other we need to show that if

$$\pi_p{}^h \pi_q^v f : \pi_p{}^h \pi_q^v G \stackrel{\simeq}{\to} \pi_p{}^h \pi_q^v H$$

then

$$\pi_{p,q} f : \pi_{p,q} G \stackrel{\simeq}{\to} \pi_{p,q} H$$

This follows from the convergence of Quillen spectral sequence for groups $\Omega_v^q G$, $\Omega_v^q H$. □

Remark 1.3. In particular, this implies that Reedy equivalencies induce isomorphisms on all bigraded homotopy groups, since they are also weak equivalencies on vertical identity components.

1.2. Constant Extensions

Let $c_v G_\bullet$ be a constant vertical extension of a simplicial group $G_\bullet$. Then $\Omega_v^q c_v G_\bullet = \{1\}$ for $q > 0$ and we have in the zeroth row

$$\pi_{p,0}(c_v G_\bullet) = \pi_p G$$

and for positive q

$$\pi_{p,q}(c_v G_\bullet) = 0, \qquad q > 0$$

Let $c_h G_\bullet$ be a constant horizontal extension of a simplicial group $G_\bullet$. Then

$$\Omega_v^q(c_h G_\bullet) = c_h(\Omega^q G_\bullet)$$

and

$$\Delta\Omega_v^q(c_h G_\bullet) = \Omega^q G_\bullet$$

After taking iterated vertical loops and then vertical identity components, the group remains horizontally constant, so its diagonal is isomorphic to a vertical term, which is now zero-connected. So

$$\pi_{-1,l}(c_h G_\bullet) = 0$$

$$\pi_{p,q}(c_h G_\bullet) = \pi_p \Omega^q G_\bullet = \pi_{p+q} G_\bullet$$

and homomorphisms of bigraded homotopy groups of bidegree $(1,-1)$ are isomorphisms.

1.3. Notation and preliminaries

Since much of the notation is self-explanatory, such as π_0^v, π_0^h for the 0-th homotopy groups of vertical or horizontal terms, it is introduced as a table below with short explanations. For a bisimplicial group G and a simplicial set K the operations $G \otimes K$ and G^K are horizontal (see the section on simplicial structure) and because of this the external tensor product of a simplicial group G and a simplicial set K is denoted $K \underset{\sim}{\otimes} G$, rather than $G \underset{\sim}{\otimes} K$. Let $\Delta^{\bullet}_{\bullet\bullet}$ be the bisimplicial cosimplicial object which is $\Delta[n] \underset{\sim}{\times} \Delta[n]$ in cosimplicial degree n with cosimplicial operations induced by those of $\underset{\sim}{\Delta}$. Then the diagonal of a bisimplicial group $\Delta G \cong Hom(\Delta^{\bullet}_{\bullet\bullet}, G)$ with the group structure induced by that of G, and the left adjoint to diagonal $L\Gamma = \Gamma \otimes \Delta^{\bullet}_{\bullet\bullet}$ where Γ is a simplicial group and the colimits are in the category of groups, similarly for bisimplicial and pointed bisimplicial sets (this recovers Lemma B.8 of [2]).

The object $c\mathbb{Z} \otimes \Delta^{\bullet}_{\bullet\bullet}$ and all of its vertical suspensions allow to recover the closed model category structure as a localization of the Reedy structure with respect to an object, or a set of such. However, this set of objects does not belong to the category of simplicial groups, compare with the definitions of the model structure in [5].

The iterated loops of a simplicial group G are obtained by repeated use of a functor $\underline{\mathcal{H}om}(S^1, G)$ and because of this in the construction of generators of cofibrations there appear smash products with $\overline{S}^k = S^1 \wedge \ldots \wedge S^1$ which is weakly equivalent but not isomorphic to the ordinary k-sphere S^k.

When notation consistent with [6] is needed $I_v H = H_{(1)}$ is used for the identity component of a simplicial group.

$\underset{\sim}{\times}$	external product of simplicial sets [2], which induces:
$\underset{\sim}{\wedge}$	external smash product of pointed simplicial sets
$\underset{\sim}{\ltimes}$	external half-smash product of pointed and unpointed simplicial sets
$(S^0 \underset{\sim}{\wedge} S^0)$	constantly extended simplicial S^0
$K \underset{\sim}{\otimes} G$	external tensor product of a simplicial set with a simplicial group
$K \otimes G$	tensor product of a simplicial set K with a bisimplicial group G, see the section on simplicial structure
$G \otimes K$	ordinary tensor product of a bisimplicial group G and a bisimplicial set K
$\pi_{k,l} G_{\bullet\bullet}$	bigraded homotopy groups of $G_{\bullet\bullet}$, or
$\pi_{k,l} G$	when it is clear G is bisimplicial;
$j \nearrow \varphi$	j has the Left Lifting Property with respect to φ
$(\underset{\sim}{\Delta} \underset{\sim}{\times} \underset{\sim}{\Delta})$	the standard double cosimplicial object
$\underline{\underline{\mathcal{H}om}}(K, G)$	$= \mathcal{H}om(K \rtimes (\underset{\sim}{\Delta} \underset{\sim}{\times} \underset{\sim}{\Delta}), G)$, bisimplicial $\mathcal{H}om$-group with group structure induced by that of G, where K is a pointed bisimplicial set
$\Omega^l_v G$	$= \underline{\underline{\mathcal{H}om}}((S^0 \underset{\sim}{\wedge} S^1), G)$ l-th vertical loops of G, the group structure induced by that of G
FK	the extension of the pointed free functor (usually from pointed bisimplicial sets to bisimplicial groups)
$F(S^0 \underset{\sim}{\wedge} S^0)$	$= cZ$, infinite cyclic group, constant in both indices
$\Sigma^l_v G$	$= G \otimes (S^0 \underset{\sim}{\wedge} S^l)$, l-th vertical suspension of G, see Lemma 2.2
$I_v G = G^v_{(1)}$	vertical identity components of G
$(\Delta G)_{(1)} = \Delta(G_{(1)})$	The identity component of the diagonal of G
$G^v_{(1)} = I_v G$	vertical identity components subgroup of G

Acknowledgement I wish to thank Jie Wu for his comments.

2. The closed model category structure

2.1. The existence of a closed model category

Theorem 2.1. There exists a closed simplicial model category structure on the category $\mathcal{G}roups_{\bullet\bullet}$ of bisimplicial groups in which

(1). The fibrations are homomorphisms of bisimplicial groups that
  (a). are Reedy fibrations;
  (b). induce fibrations, as simplicial groups , on the diagonals of vertical q - loops for all $q > 0$.
(ii) The weak equivalences are homomorphisms of bisimplicial groups that induce isomorphisms on all bigraded homotopy groups;
(iii) The cofibrations are defined by the LLP (the left lifting property).

Equivalently:

(1). $\mathcal{F}ib_{\bullet\bullet} = \{ \varphi \mid \Delta\Omega_v^q\varphi \in \mathcal{F}ib_\bullet,\ \varphi_{k,\bullet} \in \mathcal{F}ib_\bullet \}$

(2). $\mathcal{W}_{\bullet\bullet} = \{ f : G \to H \mid \pi_{k,l}f : \pi_{k,l}G \xrightarrow{\cong} \pi_{k,l}H \}$

(3). $\mathcal{C}of_{\bullet\bullet} = \{ j \mid j \nearrow \mathcal{F}ib_{\bullet\bullet} \cap \mathcal{W}_{\bullet\bullet} \}$

The generators of cofibrations are:

$$\Sigma_v^l FLi[n] = FLi[n] \otimes (S^0 \underset{\sim}{\wedge} S^l) = c\mathbb{Z} \otimes (Li[n] \ltimes (S^0 \underset{\sim}{\wedge} S^l))$$

The generators of trivial cofibrations are:

(1). $\Sigma_v^l FLi^j[n] = c\mathbb{Z} \otimes (Li^j[n] \ltimes (S^0 \underset{\sim}{\wedge} S^l))$

(2). $F(\Delta[m] \underset{\sim}{\times} i^j[n])$

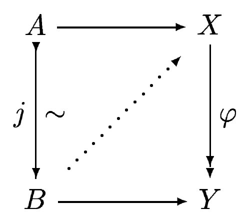
Proof. CM1, CM2, CM3 and CM4(I) follow immediately from the definitions.

Let φ be a fibration. The generators of trivial cofibrations (1) are determined from a sequence of lifting conditions related to each other by adjunctions:

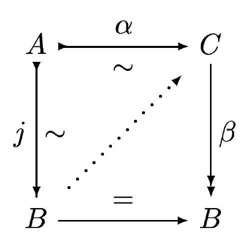
$i^j[k] \nearrow \Delta\Omega_v\varphi$
$i^j[k] \nearrow \Delta\underline{\underline{\mathcal{H}om}}(S^0 \underset{\sim}{\wedge} S^1, \varphi)$ (by the definition of vertical loops)
$Li^j[k] \nearrow \underline{\underline{\mathcal{H}om}}(S^0 \underset{\sim}{\wedge} S^1, \varphi)$ (L is left adjoint to diagonal)
$Li^j[k] \nearrow \mathcal{H}om((S^0 \underset{\sim}{\wedge} S^1) \rtimes (\underset{\sim}{\Delta} \underset{\sim}{\times} \underset{\sim}{\Delta}), \varphi)$ (see Notation)
$Li^j[k] \times (\underset{\sim}{\Delta} \underset{\sim}{\times} \underset{\sim}{\Delta}) \nearrow \mathcal{H}om((S^0 \underset{\sim}{\wedge} S^1), \varphi)$

$Li^j[k] \nearrow \mathcal{H}om((S^0 \underset{\sim}{\wedge} S^1), \varphi)$
$Li^j[k] \rtimes (S^0 \underset{\sim}{\wedge} S^1) \nearrow \varphi$
$Li^j[k] \rtimes (S^0 \underset{\sim}{\wedge} S^1) \nearrow \text{Forgetful}(\varphi)$
$F(Li^j[k] \rtimes (S^0 \underset{\sim}{\wedge} S^1)) \nearrow \varphi$ (F is the free functor, pointed version)
$c\mathbb{Z} \otimes (Li^j[k] \rtimes (S^0 \underset{\sim}{\wedge} S^1)) \nearrow \varphi$, or $FLi^j[k] \otimes (S^0 \underset{\sim}{\wedge} S^1) \nearrow \varphi$

CM4(II) Consider the diagram:



in which φ is a fibration and j is a trivial cofibration. Factor j using the small object argument with the generators (2):



From Lemma 2.2 it follows that α is an equivalence and so β is a trivial fibration. The existence of a lifting implies that j is a retract of α and since $\alpha \nearrow \varphi$ by the definition of fibrations the lifting $j \nearrow \varphi$ exists.

CM5(I) Factor the given homomorphism using both sets of generators in the small object argument .

CM5(II) Factor the given homomorphism using only the generators of trivial cofibrations. □

Lemma 2.2. The generators of trivial cofibrations are $\mathbf{E^2}$- equivalences.

Proof. For type (1), we need to show that

$$\Sigma^l_v FLi^j[n] = FLi^j[n] \otimes (S^0 \underset{\sim}{\wedge} S^l) = c\mathbb{Z} \otimes (Li^j[n] \ltimes (S^0 \underset{\sim}{\wedge} S^l))$$

are equivalences for all l. Recall from [2] that

$$L\Lambda^j[n] = \Lambda^j[n] \underset{\sim}{\times} \Lambda^j[n], \quad L\Delta[n] = \Delta[n] \underset{\sim}{\times} \Delta[n]$$

In the following diagram the vertical maps are induced by choosing a vertex in $\Lambda^j[n]$ and taking retractions. For each horizontal degree they are homotopy equivalencies of simplicial sets, whose disjoint unions comprise the vertical terms:

$$\begin{array}{ccc}
\Delta[n] \underset{\sim}{\times} \Lambda^j[n] \coprod_{\Lambda^j[n]\underset{\sim}{\times}\Lambda^j[n]} \Lambda^j[n] \underset{\sim}{\times} \Delta[n] & \xrightarrow{Li^j[n]} & \Delta[n] \underset{\sim}{\times} \Delta[n] \\
\downarrow & & \downarrow \\
\Delta[n] \underset{\sim}{\times} (pt) \coprod_{\Lambda^j[n]\underset{\sim}{\times}(pt)} \Lambda^j[n] \underset{\sim}{\times} (pt) & \xrightarrow{\cong} & \Delta[n] \underset{\sim}{\times} (pt)
\end{array}$$

After taking vertical half-smash products with S^l and applying the pointed free functor, the vertical maps remain Reedy equivalencies, hence $\mathbf{E^2}$-equivalencies. Thus in the diagram below the top horizontal map is an equivalence:

$$\begin{array}{ccc}
\Delta[n] \underset{\sim}{\otimes} F(\Lambda^j[n] \ltimes S^l) \coprod_{\Lambda^j[n]\underset{\sim}{\otimes}F(\Lambda^j[n]\wedge S^l)} \Lambda^j[n] \underset{\sim}{\otimes} F(\Delta[n] \ltimes S^l) & \longrightarrow & \Delta[n] \underset{\sim}{\otimes} F(\Delta[n] \ltimes S^l) \\
\downarrow & & \downarrow \\
\Delta[n] \underset{\sim}{\otimes} FS^l & \xrightarrow{\cong} & \Delta[n] \underset{\sim}{\otimes} FS^l
\end{array}$$

Rewriting the top horizontal line as a homomorphism from

$$F(\Delta[n] \underset{\sim}{\times} \Lambda^j[n]) \otimes (S^0 \underset{\sim}{\wedge} S^l) \coprod_{F(\Lambda^j[n]\underset{\sim}{\times}\Lambda^j[n])\otimes(S^0\underset{\sim}{\wedge}S^l)} F(\Lambda^j[n] \underset{\sim}{\times} \Delta[n]) \otimes (S^0 \underset{\sim}{\wedge} S^l)$$

to

$$F(\Delta[n] \underset{\sim}{\times} \Delta[n]) \otimes (S^0 \underset{\sim}{\wedge} S^l)$$

we see that it is $FLi^j[n] \otimes (S^0 \underset{\sim}{\wedge} S^l)$, one of the generators of trivial cofibrations. As for type (2), these generators are Reedy equivalences and hence equivalences. □

Remark 2.3. Taking $l = 0$ for generators of trivial cofibrations of type (1) gives

$$FLi^j[n] : F(\Lambda^j[n] \underset{\sim}{\times} \Lambda^j[n]) \to F(\Delta[n] \underset{\sim}{\times} \Delta[n])$$

Choosing a vertex in $\Lambda^j[n]$ gives a retraction of this generator onto $F(i^j[n] \underset{\sim}{\times} Id(pt)) \cong i^j[n] \underset{\sim}{\otimes} Id(c\mathbb{Z})$, which is a constant vertical extension of $i^j[n]$. Since the retract inherits the LLP with respect to fibrations, by adjunction to taking the vertical degree zero, it implies that an $\mathbf{E^2}$ fibration $\varphi : G \twoheadrightarrow H$ induces a fibration of simplicial groups $\varphi_{\bullet,0} : G_{\bullet,0} \twoheadrightarrow H_{\bullet,0}$.

3. A criterion of $\mathbf{E^2}$ fibrations for bisimplicial groups

The next theorem provides a characterization of fibrations:

Theorem 3.1. A Reedy fibration $\varphi : G \to H$ of bisimplicial groups is an $\mathbf{E^2}$ fibration if and only if the induced homomorphism

$$\pi_k^v \varphi : \pi_k^v G \to \pi_k^v H$$

is a fibration of simplicial groups for all $k \geq 0$.

Proof. "Only if": we have a diagram with exact rows

$$\begin{array}{ccccccccc} \{1\} & \longrightarrow & \Delta G^v_{(1)} & \longrightarrow & \Delta G_{(1)} & \longrightarrow & (\pi_0^v G)_{(1)} & \longrightarrow & \{1\} \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \{1\} & \longrightarrow & \Delta H^v_{(1)} & \longrightarrow & \Delta H_{(1)} & \longrightarrow & (\pi_0^v H)_{(1)} & \longrightarrow & \{1\} \end{array}$$

Let $\varphi : G \to H$ induce fibrations on all diagonals. Then $\Delta\varphi_{(1)}$ is onto and consequently φ induces surjections of identity components $(\pi_0^v G)_{(1)} \twoheadrightarrow (\pi_0^v H)_{(1)}$. Replacing φ with its l-fold vertical loops $\Omega_v^l \varphi$ proves "only if".

In order to prove "if" consider the same diagram. The left vertical morphism is onto, and likewise after iterated vertical looping, since φ is a Reedy fibration. Hence if the right terms surject, so do the extended groups at the center. It remains valid after iterated vertical looping, proving "if". □

Lemma 3.2. If $\varphi : G \to H$ is a fibration, then $\varphi_{\bullet,0} : G_{\bullet,0} \to H_{\bullet,0}$ is a fibration of simplicial groups.

Proof. Since the vertical constant extension is the left adjoint of the 0-row functor, we need to show that

$$(i^j[n] \underset{\sim}{\times} (pt)) \nearrow \varphi$$

Since φ is a fibration, $Li^j[n] = i^j[n] \underset{\sim}{\otimes} i^j[n] \nearrow \varphi$. It remains to note that $i^j[n] \underset{\sim}{\times} (pt)$ is a retract of $i^j[n] \underset{\sim}{\times} i^j[n]$. □

Remark 3.3. A Reedy fibration does not need to be an $\mathbf{E^2}$ fibration. Let the pointed interval be $I = \Delta[1]/_{(0)\sim *}$ and consider $j : \{1\} \to c_v FI$. Since the vertical terms of $c_v FI$ are constant, j is a Reedy fibration. But its diagonal $\Delta j : \{1\} \to FI$ is not a fibration of simplicial groups since it is not onto for components of identity.

Lemma 3.4. For a bisimplicial group G, the natural homomorphism $\Delta G \to \pi_0^v G$ induced by the adjunction homomorphism $i_v G : G \to c_v \pi_0^v G$ is a fibration of simplicial groups, inducing an isomorphism: $\pi_0 \Delta G \xrightarrow{\cong} \pi_0^h \pi_0^v G$

Proof. Since the homomorphism is surjective on each vertical term, its restriction to diagonals is also surjective, and the isomorphism on π_0 implies it is onto on identity components. □

3.1. The horizontal long exact sequence of a fibration

The diagonals of iterated vertical loops of a fibration again form a fibration, resulting in a homotopy long exact sequence for each row of bigraded homotopy groups. If in the following short exact sequence φ is a fibration of bisimplicial groups : $\{1\} \longrightarrow K \longrightarrow G \overset{\varphi}{\twoheadrightarrow} H \longrightarrow \{1\}$, then the following sequence of simplicial groups is also a fibration sequence:

$$\{1\} \longrightarrow \Delta\Omega_v^l K \longrightarrow \Delta\Omega_v^l G \xrightarrow{\Delta\Omega_v^l \varphi} \Delta\Omega_v^l H$$

It gives rise to a homotopy sequence

$$\ldots \to \pi_k \Delta\Omega_v^l K \to \pi_k \Delta\Omega_v^l G \to \pi_k \Delta\Omega_v^l H \to \pi_{k-1} \Delta\Omega_v^l K \to \ldots$$

which can be rewritten as

$$\ldots \to \pi_{k,l} K \to \pi_{k,l} G \to \pi_{k,l} H \to \pi_{k-1,l} K \to \ldots$$

giving long exact sequences for each horizontal row of bigraded homotopy groups. It ends as:

$$\ldots \to \pi_{0,l} K \to \pi_{0,l} G \to \pi_{0,l} H$$

Remark 3.5. Note that if φ is a trivial fibration this does not imply that its restriction to the 0-th row $\varphi_{\bullet,0}$ is a trivial fibration. For example let $G = c_h FI$- a constant horizontal extension of a free group generated by the (vertical) interval - and let $H = \{1\}$. Then G is homotopically trivial but its 0-th row is a constant free group on two generators, (and so is its π_0).

4. The simplicial structure

Definition 4.1. Let G be a bisimplicial group and K an unpointed simplicial set. For each $k \geq 0,\ l \geq 0$ let

$$(G \otimes K)_{n,\bullet} = \coprod_{K_n} G_{n,\bullet} \quad (G^K)_{n,\bullet} = \prod_{K_n} G_{n,\bullet}$$

with horizontal simplicial operations induced by those on G and K and define

$$\underline{\mathcal{H}om}(G,H) \overset{def}{=} \mathcal{H}om(G \otimes \underset{\sim}{\Delta}, H) \cong \mathcal{H}om(G, H^{\underset{\sim}{\Delta}})$$

Remark 4.2. This means that simplicial operations are in the horizontal direction, compare with the simplicial structure of [3]. Conditions such as

$$\underline{\mathcal{H}om}(G,H)_0 = \mathcal{H}om(G \otimes \Delta[0], H) = \mathcal{H}om(G \otimes (pt), H) \cong \mathcal{H}om(G,H)$$

follow from those for simplicial groups, similarly for compositions. The constructions are functorial and agree with relations between the free functor and the simplicial tensoring of [8] , 1.2.

Theorem 4.3. The functors $(\_\,) \otimes K,\ (\_\,)^K$ are compatible with the closed model category structure on $\mathcal{G}roups_{\bullet\bullet}$ in the sense of [7], Part II, 2.2

Proof. We check the condition SM7(a) of [7], Part II, 2.2. Let $\varphi : G \twoheadrightarrow H$ be a fibration of bisimplicial groups and let $n \geq 0$.

(i) First, we need to check that the following is a fibration:

$$\varphi^{i[n]} : G^{\Delta[n]} \longrightarrow G^{\dot{\Delta}[n]} \underset{H^{\dot{\Delta}[n]}}{\times} H^{\Delta[n]}$$

It is convenient to use the definition of fibrations rather than the criterion in Th.3.4. It is given that the diagonal of any iterated vertical loops of φ is a fibration of simplicial groups. Since for any fixed first index n the n-th horizontal row of $G^K_{\bullet\bullet}$ consists of products $\prod_{|K_n|} G_{n,\bullet}$, the functors $(\_)^K$ and $\Delta(\_)$ commute: $\Delta(G^K) = (\Delta G)^K$

Hence we can first take the diagonal and use the fact that the construction above preserves fibrations of simplicial groups. The construction also commutes with taking iterated vertical loops, etc.

(ii) Let now φ be a trivial fibration. We need to show that the diagonal of iterated vertical loops of

$$\varphi^{i[n]} : G^{\Delta[n]} \longrightarrow G^{\dot{\Delta}[n]} \underset{H^{\dot{\Delta}[n]}}{\times} H^{\Delta[n]}$$

is a trivial fibration. This reduces to the analogous property of simplicial groups in the same way as above.

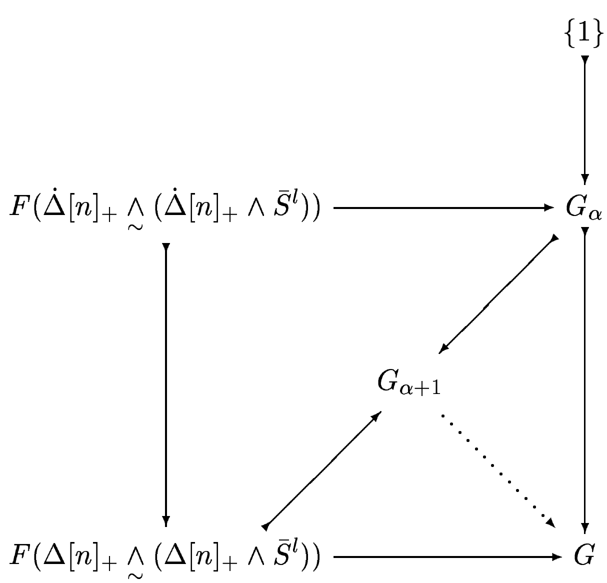
(iii) Finally we need to show that for any generator of trivial cofibrations of simplicial sets $i^j[n] : \Lambda^j[n] \to \Delta[n]$ and $\varphi : G \to H$ the homomorphism $\varphi^{i^j[n]}$ is a trivial fibration. This is seen by an argument similar to the previous two.

□

A pointed version of the simplicial structure is defined as usual: for a pointed simplicial set K define $G \otimes K$ as the cokernel in the diagram induced by the inclusion of the basepoint in K, and dually for G^K. To simplify the notation the use of the subscript is avoided and the type of a simplicial operation should be clear from the context.

5. Attachment of Cells and Simplicial Resolutions

Let G be a bisimplicial group (an important example is $c_h G$, a constant horizontal extension of a simplicial group G). The stage $\alpha + 1$ in the small object argument construction of a factorization $\{1\} \rightarrowtail G_{cof} \overset{\sim}{\twoheadrightarrow} G$ is obtained by taking a pushout of a generator (for now, of type (1)),
$F\Sigma_v^l Li[n] = F\Sigma_v^l (i[n] \underset{\sim}{\times} i[n]) = F(i[n]_+ \underset{\sim}{\wedge} (i[n]_+ \wedge \bar{S}^l))$:



Let $\alpha = 0$, $(c_h G)_0 = \{1\}$. Then the top horizontal map is trivial and the resulting first stage is free on wedges of "thickened" l-spheres, on in horizontal degree n and one for each of $\Delta[n]$-s horizontal degeneracies in higher degrees. In case $l = 0$ the summands are homotopy equivalent to infinite cyclic groups. The horizontal degeneracies of the pushout take summands to summands.

Now consider the cases of other generators. If it is J_l, then each vertical term of the source is contractible, and vertical terms of the target free on wedges of spheres. Adjoining the first type of generators is similiar to the diagram above, and the last one does not involve suspensions and does not change the homotopy type of vertical terms.

The sequential colimits of the small object argument preserve this structure.

Recall the definition of Π-algebras in the category of simplicial groups , see [1], [9]. Let the small category Π have as objects, for any non-negatively graded pointed set K, the free products $\coprod_{i\in K-\{*\}} F\bar{S}^{|i|}$, where F is the pointed version of the free group functor. Π-algebras are functors $\Pi^{opp} \mapsto \mathbf{Gr}Sets_*$ whose values on coproducts in Π are naturally isomorphic to products of their values on free factors $F\bar{S}^{|i|}$. As usual the free functor and the forgetful functor between the categories of Π-algebras and the category $\mathbf{Gr}Sets_*$ form an adjoint pair.

The considerations above sum up in the following Lemma:

Lemma 5.1. The homotopy groups of the vertical terms of a cofibrant approximation G_{cof} of a bisimplicial group G, with simplicial structure induced by the horizontal simplicial operations, form a free simplicial Π-algebra.

6. Properness

Theorem 6.1. The $\mathbf{E^2}$ closed model category on $\mathcal{G}roups_{\bullet\bullet}$ is right proper.

Proof. Let φ be a fibration and f an equivalence. We need to show that in the diagram below F is an equivalence.

$$\begin{array}{ccc} G & \xrightarrow[F]{\overset{?}{\sim}} & X \\ {\scriptstyle\Phi}\big\downarrow & \textit{pull back} & \big\downarrow{\scriptstyle\varphi} \\ H & \xrightarrow[f]{\sim} & Y \end{array}$$

Since both the vertical loops and the diagonal functors commute with pullbacks we obtain

$$\begin{array}{ccc}
\Delta\Omega_v^l G & \xrightarrow[\Delta\Omega_v^l F]{\sim} & \Delta\Omega_v^l X \\
{\scriptstyle \Delta\Omega_v^l \Phi}\downarrow & \textit{pullback} \qquad \textit{square} & \downarrow{\scriptstyle \Delta\Omega_v^l \varphi} \\
\Delta\Omega_v^l H & \xrightarrow[\Delta\Omega_v^l f]{\sim} & \Delta\Omega_v^l Y
\end{array}$$

The functor $\Delta\Omega_v^l(\_)$ preserves equivalences and fibrations, hence the top horizontal homomorphism is an equivalence of simplicial groups . This implies that F, the pullback of f along φ, induces isomorphisms of bigraded groups for $p \geq 0$.

$\square$

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