

DETECTING ELEMENTS AND LUSTERNIK–SCHNIRELMANN CATEGORY OF 3-MANIFOLDS

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ABSTRACT. In this paper, we give a new simplified calculation of the Lusternik–Schnirelmann category of closed 3-manifolds. We also describe when 3-manifolds have detecting elements and prove that 3-manifolds satisfy the equality of the Ganea conjecture.

1. INTRODUCTION

The *Lusternik–Schnirelmann category* of a space X , denoted $\text{cat}(X)$, is defined to be the minimal integer k such that there exists an open covering $\{A_0, \dots, A_k\}$ of X with each A_i contractible to a point in X . Category, while easy to define, is notoriously difficult to compute in general. In particular, except for $K(\pi, 1)$'s, it cannot be expected that the category of a space is determined by its fundamental group. In [GoGo], however, the following interesting result was proved.

1.1. Theorem. *Let M^3 be a closed 3-dimensional manifold. Then*

$$\text{cat}(M) = \begin{cases} 1 & \text{if } \pi_1(M) = \{1\} \\ 2 & \text{if } \pi_1(M) \text{ is free .} \\ 3 & \text{otherwise} \end{cases}$$

In this paper, we will give a somewhat simplified proof of this theorem using the relatively new approximating invariant for category, *category weight*. Throughout, we use only basic results about 3-manifolds found, for instance, in [H]. *But we shall also do more.* We will prove that most 3-manifolds possess a *detecting element*; that is, an element whose category weight is equal to the category of M (see [R3]). It is known that a *detectable space* (i.e., a space possessing detecting elements) has some special properties which allow solutions of certain well-known problems ([R3]). For example, from the existence of detecting elements, we prove that closed 3-manifolds satisfy the Ganea conjecture.

1.2. Corollary. *For every closed 3-manifold M ,*

$$\text{cat}(M \times S^n) = \text{cat}(M) + 1.$$

This result is not obtainable from knowing the category alone, so the detecting element approach is a significant embellishment of Theorem 1.1. Another well-known problem is the relationship between degree 1 maps of manifolds and LS-category. For closed, 3-manifolds, we have

1.3. Corollary. *Let $f: M \rightarrow N$ be a degree 1 map of oriented 3-manifolds. Then $\text{cat}M \geq \text{cat}f = \text{cat}N$.*

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We now turn to the fundamentals of 3-manifolds.

2. PRELIMINARIES ON 3-MANIFOLDS

2.1. Definition. A 3-manifold M is *irreducible* if every embedded two-sphere $S^2 \hookrightarrow M$ bounds an embedded disk $D^3 \hookrightarrow M$.

A 3-manifold M is *prime* if $M = P \# Q$ implies that either $P = S^3$ or $Q = S^3$. Here, “=” denotes diffeomorphism and $\#$ is the connected sum.

The following two results clarify the relation between prime and irreducible manifolds.

2.2. Lemma. *If M^3 is irreducible, then it is prime.*

Proof. Suppose M is irreducible. In order to split M as $M = P \# Q$, there must be an embedded S^2 which separates M into two components (i.e. $P - D^3$ and $Q - D^3$). But any such S^2 bounds an embedded disk D^3 by irreducibility, so M can only split as $M = M' \# S^3$ (since $S^3 - D^3$ is a disk D^3). This says that M is prime. \square

2.3. Lemma. *If M is a prime 3-manifold and M is not irreducible, then M is the total space of a 2-sphere bundle over S^1 .*

Proof. See [H, Lemma 3.13] \square

The fundamental structural result about 3-manifolds is the following

2.4. Theorem (Prime Decomposition). *A 3-manifold M may be written as*

$$M = M_1 \# M_2 \# \dots \# M_k,$$

where each M_j is prime. Furthermore, such a prime decomposition is unique up to re-arrangement of summands.

Proof. See [H, Theorems 3.15 and 3.21] \square

The *Sphere theorem* says that, for an orientable 3-manifold M , $\pi_2(M) \neq 0$ implies that some element of $\pi_2(M)$ is represented by an embedding $S^2 \hookrightarrow M$. We will require the following generalization.

2.5. Theorem (The Projective Plane Theorem). *Let M be a 3-manifold with $\pi_2(M) \neq 0$. Then there exists a map $g: S^2 \rightarrow M$ with the following properties.*

- (1) *The map g is not null-homotopic.*
- (2) *The map $g: S^2 \rightarrow g(S^2)$ is a covering map.*
- (3) *$g(S^2)$ is a 2-sided submanifold (2-sphere or projective plane) in M .*

Proof. See [H, Theorem 4.12]. \square

With these preliminaries, we can prove the following important characterization.

2.6. Proposition. *Let M be a closed 3-manifold. Then,*

- (1) *If $\pi = \pi_1(M)$ is infinite and $\pi_2(M) = 0$, then $M = K(\pi, 1)$.*
- (2) *If $\pi_1(M)$ is finite, then the universal covering of M is a homotopy 3-sphere and M is orientable.*

Proof. For (1), assume that $\pi_1(M)$ is infinite. Let $p: \widetilde{M} \rightarrow M$ be the universal covering of M . Since $\pi_2(M) = 0$, we conclude that $H_2(\widetilde{M}) = 0$. Since $\pi_1(M)$ is infinite, we conclude that \widetilde{M} is not compact, and therefore $H_3(\widetilde{M}) = 0$. Hence, \widetilde{M} is acyclic. Moreover, \widetilde{M} is simply-connected, and, by the Whitehead theorem, it is therefore contractible. Hence, $M = K(\pi_1(M), 1)$.

For (2), assume that $\pi_1(M)$ is finite. Then the universal cover \widetilde{M} of M is a closed simply connected manifold. So, by Poincaré duality, $H_2(\widetilde{M}) = 0$, and hence, by the Hurewicz theorem, $\pi_2(\widetilde{M}) = 0$. Thus, $\pi_2(M) = 0$. Furthermore,

$$H_3(\widetilde{M}) = \mathbb{Z} = \pi_3(\widetilde{M}),$$

again by the Hurewicz theorem. Therefore, the generator of $\pi_3(\widetilde{M}) = \mathbb{Z}$ provides a degree 1 map $S^3 \rightarrow \widetilde{M}$ (i.e. an isomorphism on H_3). Since \widetilde{M} and S^3 are simply connected, the Whitehead theorem implies that $\widetilde{M} \simeq S^3$.

To see that M is orientable, we simply note that each $g \in \pi_1(M)$, thought of as a covering transformation on the orientable manifold \widetilde{M} , acts to preserve orientation. This is seen by supposing the opposite; namely, that g reverses orientation. Now, because $\widetilde{M} \simeq S^3$, homotopy classes of maps $\widetilde{M} \rightarrow \widetilde{M}$ are classified by degree. Since g is a homeomorphism which reverses orientation, its degree is -1 . But then the Lefschetz number of g is $L(g) = 2$, implying the existence of a fixed point and contradicting the fact that g is a covering transformation. Hence, all covering transformations preserve orientation, so $M = \widetilde{M}/\pi_1(M)$ is orientable. \square

These are the only ingredients from 3-manifold theory that we shall need. In the next section, we introduce the main technical tool, the approximating invariant category weight.

3. CATEGORY WEIGHT AND DETECTING ELEMENTS

3.1. Definition ([BG, Fe, F]). Let $f: X \rightarrow Y$ be a map of finite CW -spaces. The *Lusternik-Schnirelmann category* of f , denoted $\text{cat}(f)$, is defined to be the minimal integer k such that there exists an open covering $\{A_0, \dots, A_k\}$ of X with the property that each of the restrictions $f|_{A_i}: A_i \rightarrow Y$, $i = 0, 1, \dots, k$ is null-homotopic.

Clearly, $\text{cat}(X) = \text{cat}(1_X)$ and. Also, it is easy to see that $\text{cat}(f) \leq \text{cat}(X)$ since f is null-homotopic on any subset which is contractible in X .

3.2. Definition. The *category weight* of a non-zero cohomology class $u \in H^*(X; R)$ (for some, possibly local, coefficient ring R) is defined by

$$\text{wgt}(u) \geq k \text{ if and only if } \phi^*(u) = 0 \text{ for any } \phi: A \rightarrow X \text{ with } \text{cat}(\phi) < k.$$

3.3. Remarks. 1. The idea of category weight was suggested by Fadell and Husseini (see [FH]). In fact, they considered an invariant similar to our wgt (denoted in [FH] by cwgt), but where the defining maps $\phi: A \rightarrow X$ were required to be inclusions instead of general maps. Because of this, cwgt was not a homotopy invariant, and this made it a delicate quantity in homotopy calculations. Rudyak in [R2, R3] and Strom in [S] suggested the homotopy invariant version of category weight as defined in Definition 3.2. Rudyak called it *strict* category weight (using the notation $\text{swgt}(u)$) and Strom called it *essential* category weight (using the notation $E(u)$).

At the Mt. Holyoke conference for which these proceedings are a record, both creators agreed to adopt the notation wgt and call it simply *category weight*.

2. In fact, one can define category weight for $u \in F^*(X)$ where F is a suitable functor on the category of topological spaces (e.g. $F(X) = [X, Y]$ or F is an arbitrary cohomology theory), see [R2, R3, S]. However, Definition 3.2 is enough for our goals here.

3. There is an alternative definition of category weight which is actually more useful than the one given in Definition 3.2. Recall that the Ganea fibration $p_j: G_j(X) \rightarrow X$ is defined inductively starting with the path fibration $p_0: PX = G_0(X) \rightarrow X$ having fibre ΩX . Then given the fibration $p_i: G_i(X) \rightarrow X$ with fibre $F_i = *^{(i+1)}\Omega X$, the fibration p_{i+1} is constructed by taking the cofibre Z of the inclusion $F_i \rightarrow G_i(X)$ and extending p_i to a map $Z \rightarrow X$ (which is possible since the composition $F_i \rightarrow G_i(X) \xrightarrow{p_i} X$ is null-homotopic. Finally, convert the map $Z \rightarrow X$ to a fibration $p_{i+1}: G_{i+1}(X) \simeq Z \rightarrow X$. Then it is known that $\text{cat}(X) = k$ if and only if k is the least integer such that $p_k: G_k(X) \rightarrow X$ has a section, [G, Sv]. It can also be shown that, for a cohomology class $u \in H^*(X; R)$, $\text{wgt}(u) = k$ if and only if k is the greatest integer such that $p_{k-1}^*(u) = 0$, [R3, S]. We shall use this below in giving a proof of Proposition 3.4 (4).

3.4. Proposition ([R3, S]). *Category weight has the following properties.*

- (1) $1 \leq \text{wgt}(u) \leq \text{cat}(X)$, for all $u \in \tilde{H}^*(X; R)$, $u \neq 0$.
- (2) For every $f: Y \rightarrow X$ and $u \in H^*(X; R)$ with $f^*(u) \neq 0$ we have $\text{cat}(f) \geq \text{wgt}(u)$ and $\text{wgt}(f^*(u)) \geq \text{wgt}(u)$.
- (3) $\text{wgt}(u \cup v) \geq \text{wgt}(u) + \text{wgt}(v)$.
- (4) For every $u \in H^s(K(\pi, 1); R)$, $u \neq 0$, we have $\text{wgt}(u) \geq s$.

Proof. We will only prove (4) since the other results are proven in the references cited. If $X = K(\pi, 1)$, then ΩX has the homotopy type of a discrete set of points and, consequently, $F_1 = \Omega X * \Omega X$ is, up to homotopy, a wedge of circles. Also, $G_0(X) = PX \simeq *$, so the cofibre of $\Omega X \rightarrow G_0(X)$ has the type of a wedge of circles. Then $G_1(X)$ has the homotopy type of a 1-dimensional space. Similarly, it is easy to see that $G_k(X)$ has the homotopy type of a k -dimensional space. If $u \in H^s(K(\pi, 1); R)$, then $p_{s-1}^*(u) = 0$ since $G_{s-1}(X)$ is s -dimensional. By the equivalent definition of wgt given in Remark 3.3 (3), we see that $\text{wgt}(u) \geq s$. \square

3.5. Definition. We say that $u \in H^*(X; R)$ is a *detecting element* for X if $\text{wgt}(u) = \text{cat}(X)$. We say that a space X is *detectable* if it possesses a detecting element.

Recall that the *cup-length* of a space X with respect to a ring R is defined as

$$\text{cl}_R(X) = \max\{k \mid u_1 \cup \cdots \cup u_k \neq 0 \text{ for some } u_i \in \tilde{H}^*(X; R)\}.$$

3.6. Lemma. *If $\text{cat}(X) = \text{cl}_R(X)$ for some ring R then the space X is detectable.*

Proof. It is well known that $\text{cat}(X) \geq \text{cl}_R(X)$ for every R . Now, let $\text{cat}(X) = k$ and suppose that there are $u_1, \dots, u_k \in \tilde{H}^*(X; R)$ with $u_1 \cup \cdots \cup u_k \neq 0$. Then, using the first and third properties of Proposition 3.4, we conclude that $\text{wgt}(u_1 \cup \cdots \cup u_k) = k$. Thus, $u_1 \cup \cdots \cup u_k$ is a detecting element for X . \square

4. BASIC SPECIAL CASES

First, recall that $\text{cat}(X) \leq \dim(X)$ for every connected CW -space X . In particular, $\text{cat}(M) \leq 3$ for every (connected) 3-manifold M . We also notice that, by Lemma 3.6, a space X is detectable whenever $\text{cat}(X) = \text{cl}_R(X)$ for some R . Here is a first step in understanding the category of 3-manifolds.

4.1. Proposition. *If M is a 3-manifold with finite fundamental group of order $d > 1$, then $\text{cat}(M) = 3$, and every non-zero element of $H^3(M; \mathbb{Z}/d)$ is a detecting element for M . Moreover, if d is even, then every non-zero element of $H^3(M; \mathbb{Z}/2)$ is a detecting element for M as well.*

Proof. Since $\pi_1(M)$ is finite, $\pi_2(M) = 0$ because, by Proposition 2.6, the universal cover is a homotopy sphere. Hence, there is the Hopf exact sequence

$$\pi_3(M) \xrightarrow{h} H_3(M) \xrightarrow{q} H_3(\pi) \rightarrow 0$$

where h is the Hurewicz homomorphism (e.g. see [Br, Theorem II.5.2]). Since, by Proposition 2.6, the d -fold universal covering $\widetilde{M} \rightarrow M$ is a d -sheeted covering, M is orientable and \widetilde{M} is a homotopy sphere, we conclude that h has the form

$$\pi_3(M) = \mathbb{Z} \rightarrow \mathbb{Z} = H_3(M), \quad a \mapsto d \cdot a.$$

Hence, $H_3(\pi) = \mathbb{Z}/d$. Also consider the induced homomorphism $\text{Hom}(H_3(\pi); \mathbb{Z}/d) \rightarrow \text{Hom}(H_3(M); \mathbb{Z}/d)$. It is certainly injective since $H_3(M) \rightarrow H_3(\pi)$ is surjective. However, it is also true that, for any $\phi \in \text{Hom}(H_3(M); \mathbb{Z}/d)$, $\text{Im}(h) = d\mathbb{Z} \subseteq \text{Ker}(\phi)$, so there exists $\bar{\phi} \in \text{Hom}(H_3(\pi); \mathbb{Z}/d)$ with $\bar{\phi} \mapsto \phi$. Thus, we have an isomorphism $\text{Hom}(H_3(\pi); \mathbb{Z}/d) \xrightarrow{\cong} \text{Hom}(H_3(M); \mathbb{Z}/d)$.

Now consider the diagram

$$\begin{array}{ccc} H^3(\pi; \mathbb{Z}/d) & \xrightarrow{q^*} & H^3(M; \mathbb{Z}/d) \\ \downarrow & & \downarrow \\ \text{Hom}(H_3(\pi); \mathbb{Z}/d) & \xrightarrow{q^*} & \text{Hom}(H_3(M); \mathbb{Z}/d). \end{array}$$

By Proposition 3.4, (4), a non-zero element of $H^3(\pi; \mathbb{Z}/d)$ has category weight at least 3. The right arrow is an isomorphism because $H_2(M)$ is free abelian since M is orientable. The bottom arrow is an isomorphism by the argument above. Finally, the left arrow is a surjection by the Universal Coefficient Formula. Therefore, the top arrow is a surjection as well. In particular, by Proposition 3.4 (2), every non-zero element of $H^3(M; \mathbb{Z}/d)$ has category weight at least 3. But $\text{cat}(M) \leq \dim(M) = 3$, so $\text{cat}(M) = 3$, and every non-zero element of $H^3(M; \mathbb{Z}/d)$ is a detecting element for M . \square

4.2. Remark. Using the approach as in Proposition 4.1, it is also possible to prove the following result originally due to Krasnoselski [Kra] and, in fact, re-proved in [GoGo]:

For a free action of the finite group G on a homotopy sphere $S \simeq S^{2n+1}$,

$$\text{cat}(S/G) = 2n + 1 = \dim(S/G).$$

Here is another basic result which follows from the characterization of prime non-irreducible 3-manifolds.

4.3. Proposition. *Let M be a prime 3-manifold which is not irreducible. Then $\text{cat}(M) = 2 = \text{cl}_{\mathbb{Z}/2}(M)$, and M is detectable.*

Proof. In view of Lemma 2.3, M is the total space of a 2-sphere bundle over S^1 . So, M is either $S^1 \times S^2$ or the mapping torus of the map

$$r: S^2 \rightarrow S^2, \quad r(x) = -x$$

where S^2 is regarded as the set of unit vectors in \mathbb{R}^3 . It is easy to see that, in both of the cases, $M = (S^1 \vee S^2) \cup e^3$ where e^3 is a 3-cell attached to the wedge $S^1 \vee S^2$. Thus, because a wedge of spheres has category one and a mapping cone can increase category by at most one, we obtain $\text{cat}(M) \leq 2$.

Futhermore, because $\pi_1(M) = \mathbb{Z}$, we conclude that $H_1(M; \mathbb{Z}/2) = \mathbb{Z}/2$. So, because of Poincaré duality (with $\mathbb{Z}/2$ -coefficients), we have $\text{cl}_{\mathbb{Z}/2}(M) \geq 2$. Thus, $\text{cl}_{\mathbb{Z}/2}(M) = 2 = \text{cat}(M)$, and M is detectable. \square

The next two results treat the case of infinite fundamental group, excluding the S^2 -bundles over S^1 .

4.4. Proposition. *If M is a 3-manifold with $\pi_1(M)$ infinite and $\pi_2(M) = 0$, then $\text{cat}(M) = 3$ and M is detectable.*

Proof. By Proposition 2.6, $M = K(\pi_1(M), 1)$, so, by Proposition 3.4, every non-zero element of $H^3(M; R)$ has category weight 3. (Notice that, for example, $H^3(M; \mathbb{Z}/2) \neq 0$). Thus, because $\text{cat}(M) \leq \dim(M) = 3$, each of these elements is a detecting element. \square

4.5. Proposition. *If M is an irreducible 3-manifold with $\pi_1(M)$ infinite and $\pi_2(M) \neq 0$, then $\text{cat}(M) = 3 = \text{cl}_{\mathbb{Z}/2}(M)$. In particular, M is detectable. Furthermore, M is non-orientable.*

Proof. Consider a map $g: S^2 \rightarrow M$ as in Theorem 2.5. Since M is irreducible, we conclude that $g(S^2)$ is a 2-sided projective plane in M . Let $i: \mathbb{R}P^2 \rightarrow M$ be the corresponding embedding, and let $[\mathbb{R}P^2] \in H_2(\mathbb{R}P^2; \mathbb{Z}/2)$ denote the fundamental class modulo 2 of $\mathbb{R}P^2$.

Let w_k and \overline{w}_k denote the k -th Stiefel–Whitney class of M and $\mathbb{R}P^2$, respectively. Since the 1-dimensional normal bundle of i is trivial, we conclude that $i^*w_k = \overline{w}_k$. We can now compute the Kronecker products

$$\langle w_2, i_*[\mathbb{R}P^2] \rangle = \langle i^*w_2, [\mathbb{R}P^2] \rangle = \langle \overline{w}_2, [\mathbb{R}P^2] \rangle = 1,$$

and so $i_*[\mathbb{R}P^2] \neq 0 \in H_2(M; \mathbb{Z}/2)$. Now, since $\langle \overline{w}_1^2, [\mathbb{R}P^2] \rangle = 1$, we conclude that $i^*w_1^2 = \overline{w}_1^2 \neq 0$, and so $w_1^2 \neq 0$. So, by Poincaré duality, there exists $x \in H^1(M; \mathbb{Z}/2)$ with $xw_1^2 \neq 0$. Thus, $\text{cl}_{\mathbb{Z}/2}(M) = 3$. \square

We also need the following fact which, in a sense, is a converse of Lemma 2.3.

4.6. Corollary. *If M is a closed 3-manifold with non-trivial free fundamental group, then M is not irreducible.*

Proof. Notice that $\pi_2(M) \neq 0$. Indeed, if $\pi_2(M) = 0$ then, by Proposition 2.6 and the hypothesis that $\pi_1(M)$ is free,

$$M = K(\pi_1(M), 1) = \vee S^1.$$

But this is wrong since a wedge of circles has vanishing homology above degree 1 for any coefficients.

Now, if M is irreducible then, by Proposition 4.5, $\text{cl}_{\mathbb{Z}/2}(M) = 3$. But this is impossible. Indeed, let $f: M \rightarrow K(\pi_1(M), 1) = \vee S^1$ be a map which induces an isomorphism of fundamental groups. Then

$$f^*: H^1(K(\pi_1(M), 1); \mathbb{Z}/2) \rightarrow H^1(M; \mathbb{Z}/2)$$

is an isomorphism. Thus, $x \cup y = 0$ for all $x, y \in H^1(M; \mathbb{Z}/2)$, and so $\text{cl}_{\mathbb{Z}/2}(M) < 3$. This is a contradiction. \square

4.7. Remark. If $\pi_1(M) = \mathbb{Z}$ then $M = P \# \Sigma$ where Σ is a homotopy sphere and P is prime. So, $\pi_1(P) = \mathbb{Z}$. But P is not irreducible by Corollary 4.6, so, because of Lemma 2.3, $\pi_2(P) = \mathbb{Z}$. In other words, $\pi_2(M) = \mathbb{Z}$ whenever $\pi_1(M) = \mathbb{Z}$. Actually, the following general fact holds: for every closed 3-manifold M , the group $\pi_1(M)$ completely determines $\pi_2(M)$, see e.g. [R1].

5. DETECTABILITY OF 3-MANIFOLDS

5.1. Proposition. *Let M^3 be a closed 3-manifold with $\pi_1(M)$ free and non-trivial. Then $\text{cat}(M) = 2$, and M is detectable.*

Proof. Write $M = M_1 \# \dots \# M_k$ with each M_j prime. Because $\pi_1(M) = \pi_1(M_1) * \dots * \pi_1(M_k)$ is free, each $\pi^j = \pi_1(M_j)$ must be free (where we agree that the trivial group is free). If M_j is irreducible with $\pi^j \neq \{1\}$, then this contradicts Corollary 4.6. Therefore, all such M_j are non-irreducible primes; that is, the M_j are the manifolds considered in Proposition 4.3. Because of Lemma 2.3, these are the total spaces of S^2 -bundles over S^1 . There are only two such manifolds: one orientable and one non-orientable, and we denote both of them by $S^1 \times S^2$. Of course, the M_j with $\pi^j = \{1\}$ are homotopy spheres Σ_j . The key point now is that, for $M = P \# Q$ with $P = \#_k (S^1 \times S^2)$ and $Q = \#_j \Sigma_j$, $M - D^3$ deformation retracts onto the 2-skeleton $\vee_k (S^1 \vee S^2)$. Because of Proposition 4.3, $\text{cat}(S^1 \times S^2) = 2$. This handles the “trivial” case where the connected sum degenerates to a single summand. Now suppose $M = \#_j M_j = P \# Q$, where M_j is either a homotopy sphere or $S^1 \times S^2$ and $P = \#_{j_t} M_{j_t}$, $Q = \#_{j_s} M_{j_s}$ arbitrarily split M . If we remove a disk from a 3-manifold N , then the inclusion $S^2 \hookrightarrow N - D^3$ is the inclusion of a subcomplex; so therefore a cofibration. Thus, the pushout diagram

$$\begin{array}{ccc} S^2 & \longrightarrow & P - D^3 \\ \downarrow & & \downarrow \\ Q - D^3 & \longrightarrow & P \# Q = M \end{array}$$

is a homotopy pushout as well. But then we may apply the standard estimate for the category of a double mapping cylinder (see [Har]) to obtain

$$\begin{aligned} \text{cat}(M) &\leq \text{cat}(S^2) + \max\{\text{cat}(P - D^3), \text{cat}(Q - D^3)\} \\ &= 1 + \max\{\text{cat}(\vee_{j_t} (S^1 \vee S^2)), \text{cat}(\vee_{j_s} (S^1 \vee S^2))\} \\ &= 1 + 1 \\ &= 2. \end{aligned}$$

Of course, cup-length then shows that $\text{cat}(M) = 2$ and this completes the proof. \square

5.2. Theorem. *Let M be a 3-manifold whose fundamental group is non-trivial and not a free group. Then $\text{cat}(M) = 3$. Further, M is detectable unless it is non-orientable of the form $P \# Q$, where P is non-orientable and Q is prime with odd torsion. Also, in the last case, the orientable double cover of M has category 3.*

Proof. The case of finite π_1 is considered in Proposition 4.1. So, we assume that $\pi_1(M)$ is infinite. We represent M as a connected sum $M = N \# P$, where P is prime and $\pi_1(P) \neq \{1\}$. Furthermore, we can always assume that $\pi_1(P) \neq \mathbb{Z}$, and therefore P is irreducible in view of Corollary 4.6. Now, because of the results of §4, P possesses a detecting element $u \in H^3(P; R)$ for suitable R .

Now suppose that M is orientable. Then there is a map $f: M \rightarrow P$ of degree 1. (In greater detail, $M = (N \setminus D) \cup (P \setminus D)$ where D is a 3-disk, and $f: M \rightarrow P$ maps $N \setminus D$ to the disk D in P and is the identity on $P \setminus D$.) Then $f^*: H^3(P; R) \rightarrow H^3(M; R)$ is an isomorphism for every coefficient ring (group) R . Now, for the detecting element u above, $f^*(u) \neq 0$, and, therefore, $\text{wgt}(f^*(u)) = 3$. Thus, $f^*(u)$ is a detecting element for M .

Now, if M is not orientable, then let $\overline{M} \rightarrow M$ be its orientable double cover (which also is a closed 3-manifold). If $\pi_1(M)$ has odd torsion, then so does $\pi_1(\overline{M})$. Because \overline{M} is orientable, the argument above says that $\text{cat}(\overline{M}) = 3$. But because \overline{M} covers M , we know that $\text{cat}(\overline{M}) \leq \text{cat}(M)$. Therefore, $\text{cat}(M) = 3$. If, on the other hand, there is a prime component of M with non-free fundamental group having no odd torsion, then this component has a detecting element in 3-dimensional $\mathbb{Z}/2$ -cohomology. Therefore, M has a detecting element in $\mathbb{Z}/2$ -cohomology as well and $\text{cat}(M) = 3$.

Now, if $\pi_1(M)$ has odd torsion, then this occurs in individual prime components. So, M may not have a detecting element only if we can write $M = P \# Q$, where P is non-orientable and Q is a prime manifold having odd torsion. □

For completeness, note that $\text{cat}(\Sigma) = 1$ for every simply connected 3-manifold (= homotopy sphere) Σ , and, therefore, every non-zero element $u \in H^3(\Sigma)$ is a detecting element. Therefore, we now have proved Theorem 1.1 and augmented it by showing that most closed 3-manifolds possess detecting elements. The significance of this will be apparent in §6.

5.3. Remark. In fact, if we allow local coefficients, then all 3-manifolds with non-trivial and non-free fundamental groups have detecting elements. More specifically, by [Ber], $\text{cat}(X) = n = \dim(X)$ if and only if a certain element $u \in H^1(X; I(\pi))$ has $u^n \neq 0$ in $H^n(X; I(\pi) \otimes \dots \otimes I(\pi))$. Here, $\pi = \pi_1(X)$ and $I(\pi)$ is the augmentation ideal in the group ring $\mathbb{Z}\pi$. Since u^n is a cup product (with local coefficients), it is a detecting element.

6. TWO APPLICATIONS

A prime motivating problem in the study of Lusternik-Schnirelmann category has been the *Ganea conjecture*; $\text{cat}(X \times S^n) = \text{cat}(X) + 1$. We now know that the conjecture is not true in general, so it is even more interesting to understand when it *is* valid. For 3-manifolds, we have the following.

6.1. Corollary. *For every closed 3-manifold M ,*

$$\text{cat}(M \times S^n) = \text{cat}(M) + 1.$$

That is, the Ganea conjecture holds for M .

Proof. First, suppose that M is detectable. Then the equality follows from the general result [R3, Corollary 2.3], but the argument in this case is easy. Let $u \in H^*(M; R)$ have $\text{wgt}(u) = \text{cat}(M)$ and let $v \in H^n(S^n; R)$ be non-trivial, where, by the results above, we can always take $R = \mathbb{Z}$ or $R = \mathbb{Z}/d$. Let $\tilde{u} = p_M^*(u)$ and $\tilde{v} = p_{S^n}^*(v)$, where $p_M: M \times S^n \rightarrow M$ and $p_{S^n}: M \times S^n \rightarrow S^n$ are the respective projections. Clearly, $\tilde{u} \neq 0$ and $\tilde{v} \neq 0$ since the compositions

$$M \xrightarrow{\text{incl}} M \times S^n \xrightarrow{p_M} M$$

$$S^n \xrightarrow{\text{incl}} M \times S^n \xrightarrow{p_{S^n}} S^n$$

are the respective identity maps. By Proposition 3.4 (2), $\text{wgt}(\tilde{u}) \geq \text{wgt}(u) = \text{cat}(M)$ and $\text{wgt}(\tilde{v}) \geq \text{wgt}(v) = 1$. Then the Künneth theorem says that $0 \neq \tilde{u} \cup \tilde{v} \in H^*(M \times S^n; R)$ and (using Proposition 3.4 (3) and the product inequality $\text{cat}(X \times Y) \leq \text{cat}(X) + \text{cat}(Y)$)

$$\text{cat}(M) + 1 \geq \text{cat}(M \times S^n) \geq \text{wgt}(\tilde{u} \cup \tilde{v}) \geq \text{wgt}(\tilde{u}) + \text{wgt}(\tilde{v}) \geq \text{cat}(M) + 1.$$

Hence, $\text{cat}(M \times S^n) = \text{cat}(M) + 1$.

Now, suppose that M is not detectable. Then, by Theorem 5.2, the oriented double cover \overline{M} of M is detectable, and $\text{cat}(\overline{M}) = 3$. Therefore, in view of what we said above, $\text{cat}(\overline{M} \times S^n) = 4$. But $\overline{M} \times S^n$ covers $M \times S^n$, and so $\text{cat}(M \times S^n) \geq 4$. On the other hand,

$$\text{cat}(M \times S^n) \leq \text{cat}(M) + 1 = 4$$

for general reasons. Thus, $\text{cat}(M \times S^n) = 4$. □

6.2. Corollary. *Let $f: M \rightarrow N$ be a degree 1 map of oriented 3-manifolds. Then $\text{cat}M \geq \text{cat}f = \text{cat}N$.*

Proof. Let $u \in H^3(N; A)$ be a detecting element for N . (Recall that orientable 3-manifolds always have detecting elements.) Since $\deg(f) = 1$, we conclude that $f^*(u) \neq 0$. So, $\text{cat}(f) \geq \text{wgt}(f^*(u))$ by Proposition 3.4 (2). Thus

$$\text{cat}(M) \geq \text{cat}(f) \geq \text{wgt}(f^*(u)) = \text{cat}(N).$$

Of course, $\text{cat}(f) = \text{cat}(N)$ holds since $\text{cat}(f) \leq \text{cat}(N)$ for general reasons. □

6.3. Corollary. *Let $f: M \rightarrow N$ be a degree 1 map of oriented 3-manifolds. If $\pi_1(M)$ is free, then $\pi_1(N)$ is.*

Proof. By Corollary 6.2, $\text{cat}(N) \leq 2$, and so $\pi_1(N)$ is free by Theorem 5.2 □

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