MISLIN GENUS OF MAPS

JIANZHONG PAN AND MOO HA WOO

ABSTRACT. In this paper, we prove that the Mislin genus of a (co-)H-map between (co-)H-spaces under certain natural conditions is a finite abelian group which generalizes results in Zabrodsky[9] , McGibbon[6] and Hurvitz[5]

1. INTRODUCTION

Let X be a nilpotent connected CW complex of finite type and $X_{(p)}$ be the p-localization. The Mislin genus[7] of X is the set $G(X)$ of the homotopy types of nilpotent connected CW complexes Y of finite type such that $Y_{(p)} \simeq X_{(p)}$ for all prime p. In general $G(X)$ is not trivial. The first general result about Mislin genus is that of Wilkerson:

Theorem 1.1. [8] *Let X be a 1-connected CW-complex of finite type. If $H_n(X, \mathbb{Z}) = 0$ for n sufficiently large or $\pi_n(X) = 0$ for n sufficiently large , then $G(X)$ is finite.*

In general it is difficult to determine the set $G(X)$. An exceptional case is the following result of Zabrodsky[9][10].

1991 *Mathematics Subject Classification.* 55D99.

Key words and phrases. Mislin genus, H-space , co-H-space.

The first author is partially supported by the NSFC projects 19701032 , 10071087 and ZD9603 of Chinese Academy of Science and Brain Pool program of KOSEF and the second author wishes to acknowledge the financial supports of the Korea Research Foundation made in the program year of (2000).

Theorem 1.2. *Let X be a 1-connected rational H-space (i.e., space of finite type such that its rationalization is an H-space) with only finitely number of nontrivial homotopy groups. Then the following is an exact sequence*

$$[X, X]_{\hat{t}} \xrightarrow{d} (\mathbb{Z}_{\hat{t}}^*/\pm 1)^l \xrightarrow{\xi} G(X) \to 0$$

where $\hat{t}$ is a certain positive integer depending on X , l is the number of integers k with $QH^k(X;\mathbb{Q}) \neq 0$ and $[X, X]_{\hat{t}}$ is the set of homotopy classes of the self-maps of X which are $\hat{t}$-equivalences.

Remark 1.3. $QH^*(X;\mathbb{Q})$ in the above Theorem is the indecomposable module of the algebra $H^*(X, \mathbb{Q})$ over $\mathbb{Q}$.

Recently McGibbon [6] generalized this to the case of connected nilpotent rational H-space and to the case of 1-connected rational co-H-space.

On the other hand Hurvitz[5] introduced the genus of maps as follows

Definition 1.4. Let $f : X \to Y$ be a map between two nilpotent connected CW complexes X, Y. The *genus* $G(f)$ of f is defined to be the set of equivalence classes of maps $f' : X' \to Y'$ such that for each prime p there exist homotopy equivalences $h_p : X'_{(p)} \to X_{(p)}$ and $k_p : Y'_{(p)} \to Y_{(p)}$ satisfying $f_{(p)}h_p \simeq k_p f'_{(p)}$, where two maps $f'_i : X'_i \to Y'_i$, $i = 1, 2$, are *equivalent* if there exist homotopy equivalences $g : X'_1 \to X'_2$ and $h : Y'_1 \to Y'_2$ such that $f'_2 g \simeq h f'_1$.

Let $[f, f]_t$ be the set of equivalence classes of pairs (h, k) of t-equivalences $h : X \to X$, $k : Y \to Y$ satisfying $kf \sim fh$ where t is a positive integer. One of the main results in Hurvitz's paper is the following extension of Zabrodsky's result.

Theorem 1.5. *Let $f : X \to Y$ be an H-map between H-spaces such that $H^*(X, \mathbb{Q})$ and $H^*(Y, \mathbb{Q})$ are primitively generated and $H^*(f, \mathbb{Q})$ is either a monomorphism , an epimorphism , an isomorphism or zero. Then $G(f)$ admits an abelian group structure and there exist integers k and $\hat{t}$ (depending on X, Y and f) and an exact sequence*

$$[f, f]_{\hat{t}} \stackrel{\alpha'}{\to} [\mathbb{Z}^*_{\hat{t}}/\pm 1]^k \stackrel{\hat{\xi}}{\to} G(f) \to 0$$

The main result in this paper is the following

Theorem 1.6. *Let $f : X \to Y$ be an H-map between H-spaces such that $H^*(X, \mathbb{Q})$ and $H^*(Y, \mathbb{Q})$ are primitively generated . Then $G(f)$ admits an abelian group structure and there exist integers k and $\hat{t}$ (depending on X, Y and f) and an exact sequence*

$$[f, f]_{\hat{t}} \stackrel{\alpha'}{\to} [\mathbb{Z}^*_{\hat{t}}/\pm 1]^k \stackrel{\hat{\xi}}{\to} G(f) \to 0$$

where

$$k = \begin{cases} l(X) & \text{if } H^*(f, \mathbb{Q}) \text{ is an isomorphism} \\ l(X) + l(Y) & \text{otherwise} \end{cases}$$

and $l(X)$ is the number of i such that $\pi_i(X) \bigotimes \mathbb{Q} \neq 0$ while $\hat{t}$ will be defined in Section 2.

Remark 1.7. Obviously the restriction on $H^*(f, \mathbb{Q})$ in Hurvitz's result has been removed .

It is not clear if Hurvitz's original approach can be extended to get the dual result . Combining Hurvitz's and McGibbon's approaches we are able to extend the above result to the dual case as follows:

Theorem 1.8. *Let $f : X \to Y$ be a co-H-map between co-H-spaces with coprimitive generated homotopy Lie algebras . Then $G(f)$ admits* *an abelian group structure and there exist integers k and $\hat{t}$ (depending on X, Y and f) and an exact sequence*

$$[f,f]_{\hat{t}} \xrightarrow{\alpha'} [\mathbb{Z}_{\hat{t}}^*/\pm 1]^k \xrightarrow{\hat{\xi}} G(f) \to 0$$

where

$$k = \begin{cases} l(X) & \text{if } \pi_*(f)\bigotimes \mathbb{Q} \text{ is an isomorphism} \\ l(X)+l(Y) & \text{otherwise} \end{cases}$$

and $l(X)$ is the number of i such that $H_i(X,\mathbb{Q}) \neq 0$ while $\hat{t}$ will be defined in Section 3.

Remark 1.9. There are similar results for $G_Y(f)$ and $G^X(f)$ as defined in [5]. We will omit these quite clear extensions.

Remark 1.10. The analogue of rational (co-)H-space for map should be rational (co-)H-map between rational (co-)H-spaces. Although we are unable to extend the main results in this paper to the general case until now, we do believe that it is true at least for rational (co-)H-map between rational (co-)H-spaces with (co-)primitively generated rational (homotopy Lie algebras) cohomology rings.

All spaces considered in this paper are based and of the homotopy type of 1-connected CW complexes of finite type. For simplicity 0 will be included into the set of primes. All spaces are assumed to have finite number of nontrivial homotopy groups when we are dealing with genus of H-map while spaces are assumed to have finite number of nontrivial homology groups when we are concerned with genus of co-H-map.

$\mathbb{Z}_t$ is the ring of *modt* integers and $\mathbb{Z}_t^*$ is the group of units in $\mathbb{Z}_t$. A map is said to be *t-equivalence* if it is p-equivalence for all prime p dividing t.

2. Genus of maps between H-spaces

First we fix some notations. Let (X, μ) be an H-space. We shall denote by $+$ the operation on $[Z, X]$ induced by μ , by ϕ_n the n-th power map

$$\phi_n = \mu(\mu \times 1) \cdots (\mu \times 1 \times \cdots \times 1) \circ (\Delta \times 1 \times \cdots \times 1) \cdots \Delta$$

where Δ denotes the diagonal map.

In general the product μ is not assumed to have an inverse. Nevertheless we have the following elementary but important result

Lemma 2.1. *If (X, μ) is an H-space, then $[W, X]$ is an algebraic loop for any space W, i.e., for any two maps $f, g \in [W, X]$ there exists a unique $D_{f,g} \in [W, X]$ such that $D_{f,g} + g = f$.*

Lemma 2.2. *Let $h : X_1 \to X_2$ be an H-map and $g : W_1 \to W_2$ be a map. For any two maps f_1,$f_2 \in [W_2, X_1]$, we have*

$$hD_{f_1,f_2} = D_{hf_1,hf_2} \text{ and } (D_{f_1,f_2})g = D_{f_1g,f_2g}.$$

Remark 2.3. For the proof of the lemmas above, see [9]

For simplicity, denote $D_{f,g}$ by $f - g$. Define

$$\bar{\mu} = \mu - \pi_1 - \pi_2$$

where π_1, π_2 are projections from $X \times X$ to its factors. Then for any coefficient $\mathbb{A}$, define

$$PH^*(X, \mathbb{A}) = ker H^*(\bar{\mu}, \mathbb{A})$$

Now let X be a rational H-space with only finitely nontrivial homotopy groups. Denote by $K(X)$ the generalized Eilenberg-MacLane space with homotopy groups $\pi_*(X)/torsion$. Let $\bar{\Delta} : X \to X \bigwedge X$

denote the reduced diagonal map. Define $PH_*(X,\mathbb{Z})$ to be kernel of $H_*(\bar{\Delta},\mathbb{Z})$. Let $\sigma_n : \pi_n(X) \to PH_*(X,\mathbb{Z})$ be the obvious quotient map of the Hurewicz homomorphism. Since X is a rational H-space , the map σ_n has a finite kernel and cokernel for each n. Define

$$t_n(X) = exp(\text{coker}\sigma_{n+1})exp(\text{ker}\sigma_n)$$

$$t(X) = \prod_{n \le N(X)} t_n(X)$$

where , for a given finite abelian group G, $expG$ is the smallest integer $n \ge 1$ such that $ng = 0$ for all $g \in G$ and $N(X)$ is the least integer such that $\pi_n(X) = 0$ for every $n > N(X)$.

Let $s_n(X) = exp(torsion(H^n(X,\mathbb{Z})))$ when $QH^n(X,\mathbb{Q}) \neq 0$ and $s_n(X) = 1$ otherwise. Let $\mathbf{s}(X)$ be the sequence of integers $\{s_1(X), s_2(X) , \cdots\}$. If $f : X \to Y$ is a map , then $\mathbf{s}(f)$ is defined to be the sequence of integers $\{s_1(X)s_1(Y), s_2(X)s_2(Y), \cdots\}$.

On the other hand , given a H-space X with primitively generated $H^*(X,\mathbb{Q})$ and a sequence of integers $\mathbf{s} = \{s_1, s_2, \cdots\}$, an $\mathbf{s}$-maximal map for X is a map $\varphi_X : X \to K(X)$ such that for each n there exist $\{x_1, \cdots, x_r\} \in H^n(X,\mathbb{Z})$ which projects to a basis of $PH^n(X,\mathbb{Q})$ and a basis $\{\iota_1, \cdots, \iota_r\}$ for $PH^n(K(X),\mathbb{Z})/torsion$ with $(\varphi_X)^*(\iota_i) = s_n x_i$. If X is an H-space and φ_X is an H-map, then the maximal map will be called primitive. An elementary but useful result is as follows:

Lemma 2.4. *$s_n t_n(X)$ can be divided by the integer $exp(\pi_n(fiber(\varphi_X)))$ for an $\mathbf{s}$-maximal map of X .*

Remark 2.5. Any $\mathbf{s}(X)$-maximal map of an H-space X is primitive.

Now we define the integer $\hat{t}$ in Theorem1.6 as follow:

Definition 2.6. Given an H-map $f : X \to Y$ between two H-spaces,

$$\hat{t} = t(X)(t(Y))^2 \prod_{n \leq \max(N(X), N(Y))} s_n(X)s_n(Y)$$

Let $f : X \to Y$ be a map between two rational H-spaces with primitively generated rational cohomology and $\mathbf{s}$ be a sequence of integers $\{s_1, s_2, \cdots\}$. Then an $\mathbf{s}$-maximal map for f is a homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\varphi_X}\big\downarrow & & {\scriptstyle\varphi_Y}\big\downarrow \\ K(X) & \xrightarrow{f_X} & K(Y) \end{array}$$

where φ_X, φ_Y are $\mathbf{s}$-maximal maps. If the map f is also an H-map, then an $\mathbf{s}$-maximal map for f is called primitive if $f_X, \varphi_X, \varphi_Y$ are H-maps.

Remark 2.7. Any $\mathbf{s}(f)$-maximal map for f is primitive.

In this section we will prove Theorem 1.6. Before that , however, we need several preliminary results .

Theorem 2.8. *Let X be an H-space with primitively generated $H^*(X, \mathbb{Q})$ and with only finitely nontrivial homotopy groups and W be any space. Then*

(i)For every map $f : W_p \to X_p$, there exist an integer n and a map $g : W \to X$ such that $(n, p) = 1$ and $g_{(p)} \sim \phi_n f$.

(ii)Given two maps $f_1, f_2 : W \to X$ so that $\varphi_X f_1 \sim \varphi_X f_2$ where $\varphi_X : X \to K(X)$ is a primitive $\mathbf{s}$-maximal map, then $\phi_{t'} f_2 \sim \phi_{t'} f_1$ where $t' = \prod_{n \leq N(X)} t_n(X) s_n$.

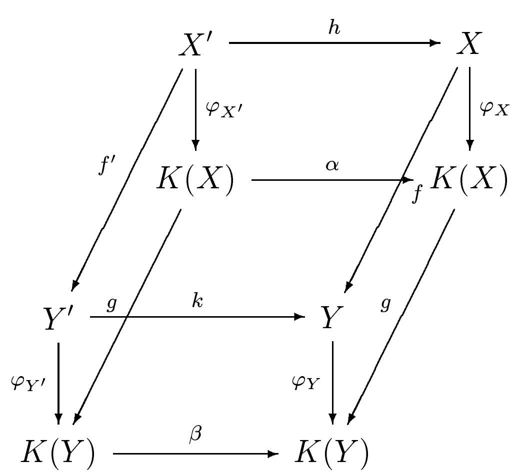
Proof. The statement (i) is essentially the same as that of Theorem 2.1 in [5].

To prove the part (ii),it suffices to prove that $\phi_{t'}k \sim *$ by Lemma 2.2 where $k = D_{f_1,f_2}$. The given conditions and Lemma 2.2 imply that $\varphi_X k \sim *$. The result follows from an induction by Postnikov tower of map φ_X , see the proof of Theorem 3.8 for details of a dual proof. □

Theorem 2.9. *Let $f : X \to Y$ be an H-map between two H-spaces and l be an integer with $l = p_1^{w_1} \cdots p_s^{w_s}$. Than given two spaces X', Y', a function $f' : X' \to Y'$ and homotopy equivalences $h_{p_i} : X'_{(p_i)} \to X_{(p_i)}$, $k_{p_i} : Y'_{(p_i)} \to Y_{(p_i)}$ satisfying $f_{(p_i)}h_{p_i} \sim k_{p_i}f'_{(p_i)}$, there exist l-equivalences $h : X' \to X$, $Y' \to Y$ so that $fh \sim kf'$.*

Proof. It is Theorem 2.2 in [5] □

Theorem 2.10. *If $f : X \to Y$ is any map between rational H-spaces. Suppose that an $\mathbf{s}(f)$-maximal map of f is given as in the following diagram.*



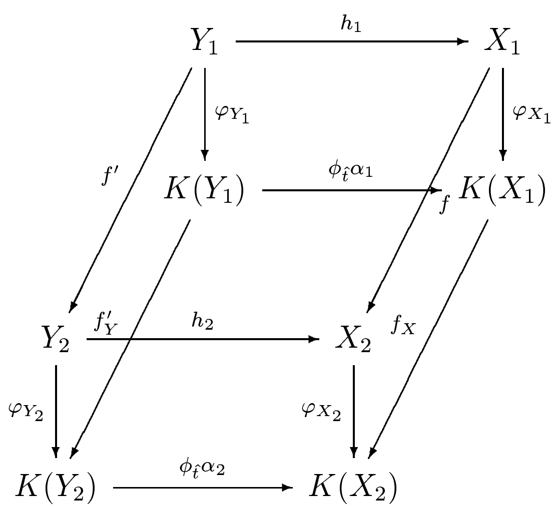
Then for every pair (α, β), where $\alpha : K(X) \to K(X)$ and $\beta : K(Y) \to K(Y)$ are $\hat{t}$-equivalences such that $\beta g \sim g\alpha$, there exist a map $f' : X' \to Y'$, $f' \in G(f)$ and $\hat{t}$-equivalences $h : X' \to X$ and $k : Y' \to Y$ such that the every diagram in the following cube commutes up to homotopy and $\varphi_{X'}, \varphi_{Y'}$ are $\mathbf{s}(f)$-maximal maps.

Remark 2.11. It is easy to know that, if α, β are homotopy equivalences, then f' is equivalent to f.

Proof. It is Theorem 3.4 in [5]. □

Theorem 2.12. *Let $f : X_1 \to X_2$ be an H-map between H-spaces such that $H^*(X_1, \mathbf{Q})$ and $H^*(X_2, \mathbf{Q})$ are primitively generated. Let $f' : Y_1 \to Y_2$ be a map between rational H-spaces. Choose $\mathbf{s}(f)$-maximal maps f_X, f'_Y for f, f' respectively so that the maximal map for f is primitive.*

Then given a pair of maps (α_1, α_2) as in the following diagram satisfying $\alpha_2 f'_Y = f_X \alpha_1$, there exist mappings $h_i : Y_i \to X_i$, $i = 1, 2$, so that all the diagrams in the following cube commute up to homotopy.

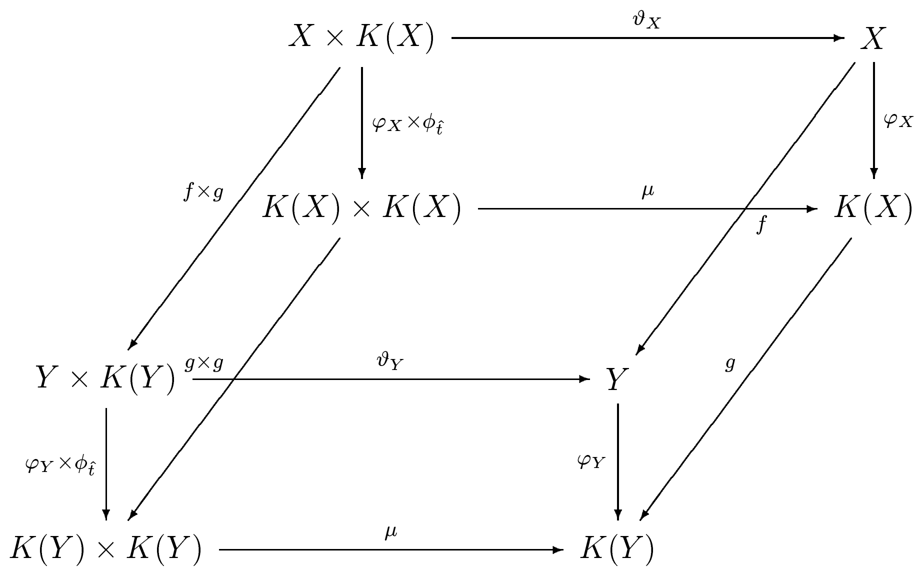


Proof. By Proposition 1.8 in [10] , for each pair (α_1, α_2) in the Theorem, there exist mappings $h'_i : Y_i \to X_i$ so that $\varphi_{X_i} h'_i \sim \varphi_{Y_i} \phi_{t(X_1)t(X_2)} \alpha_i$. It is easy to know that $\varphi_{X_2} h'_2 g \sim \varphi_{X_2} f h'_1$. Theorem 2.8 implies that

$$\phi_{(t(X_2))^2} h'_2 g \sim \phi_{(t(X_2))^2} f h'_1 \sim f \phi_{(t(X_2))^2} h'_1$$

since f is an H-map. The result follows by taking $h_i = \phi_{t'} h'_i$ where $t' = \hat{t}/t(X_1)t(X_2)$. □

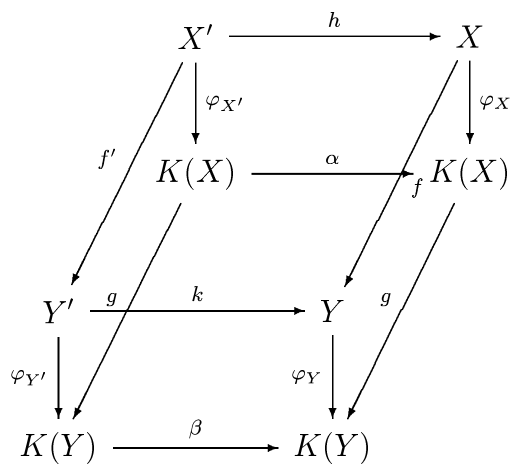
Theorem 2.13. *Let f be as in Theorem 1.6. Given a primitive* $\mathbf{s}(f)$*-maximal map g of f, there exists a commutative cube up to homotopy as follows*



where ϑ_X, ϑ_Y restrict to the identity on the first factor and μ are the standard multiplication on product of Eilenberg-MacLane spaces and left face is the given maximal map.

Proof. The proof is similar to that of Theorem 2.12 using Proposition 4.6 in [6]. Although we modified the notion of maximal map, we have also modified the integer $\hat{t}(X)$ to ensure that it remains valid. □

Theorem 2.14. *Let $f : X \to Y$ be as in Theorem 1.6 . Given $[f'] \in G(f)$, then there is a commutative cube up to homotopy*



where h, k are $\hat{t}$-equivalences , all the vertical maps are maximal maps, the back and front face diagrams are homotopy pullback diagrams and

$$\alpha = \prod_i \alpha_i : K(X) \to K(X), \beta = \prod_j \beta_j : K(Y) \to K(Y)$$

where

$$\alpha_i : K(\pi_i(X)/torsion, i) \to K(\pi_i(X)/torsion, i)$$

$$\beta_j : K(\pi_j(Y)/torsion, j) \to K(\pi_j(Y)/torsion, j)$$

and

$$K(X) = \prod_i K(\pi_i(X)/torsion, i), K(Y) = \prod_j K(\pi_j(Y)/torsion, j)$$

Proof. Take $\hat{t}$-equivalences $h : X' \to X, k : Y' \to Y$ as granted by Theorem 2.9 so that $fh \sim kf'$. We will try to define the other maps in the above diagram so that the Theorem is true. Without loss of generality, we can assume that there is only one l such that $\pi_i(X) \bigotimes \mathbb{Q} = \pi_i(Y) \bigotimes \mathbb{Q} = 0$ if $i \neq l$. Choose rational H-space structures on X', Y' such that $h_{(0)}, k_{(0)}, f'_{(0)}$ are H-maps. It is always possible to choose bases$\{x_1, \cdots, x_{r_1}\}$ for $PH^l(X, \mathbb{Z})/torsion$ and $\{x'_1, \cdots, x'_{r_2}\}$

for $PH^l(Y,\mathbb{Z})/torsion$ so that $PH^*(f,\mathbb{Z})/torsion$ is diagonal with respect to these bases and $\{x_1,\cdots,x_{r_1}\}$ and $\{x'_1,\cdots,x'_{r_2}\}$ are bases for $PH^*(X,\mathbb{Q})$ and $PH^*(Y,\mathbb{Q})$. Let φ_X,φ_Y be defined by $s_l(f)x_i = (\varphi_X)^*(\iota_i)$ and $s_l(f)x'_j = (\varphi_Y)^*(\iota'_j)$ where $\{\iota_1,\cdots,\iota_{r_1}\}$ is a basis for $H^l(K(X),\mathbb{Z})/torsion$ and $\{\iota'_1,\cdots,\iota'_{r_2}\}$ is a basis for $H^l(K(Y),\mathbb{Z})/torsion$. Then g is also of the diagonal form under the bases. Choose elements $\{y_1,\cdots,y_{r_1}\}\in H^l(X',\mathbb{Z})$ and $\{y'_1,\cdots,y'_{r_2}\}\in H^l(Y',\mathbb{Z})$ such that they projects to bases of $PH^*(X',\mathbb{Q})$ and $PH^*(Y',\mathbb{Q})$ respectively. Define the maps $\varphi_{X'},\varphi_{Y'}$ similarly. Since k,h are $\hat{t}$-equivalences and rational H-maps, we can write

$$(h)^*(x_i)=\sum\lambda_{ii'}y_{i'}+v_i,(k)^*(x'_j)=\sum\lambda'_{jj'}y'_{j'}+w_j$$

where $\det\lambda_{ii'},\det\lambda'_{jj'}$ are prime relative to $\hat{t}$ and v_i,w_j are torsions. Certainly we can find $\bar{v}_i$ and $\bar{w}_j$ so that

$$(h)^*(x_i)=\sum\lambda_{ii'}(y_{i'}+\bar{v}_{i'}),(k)^*(x'_j)=\sum\lambda'_{jj'}(y'_{j'}+\bar{w}_{j'})$$

Let $\varphi_{X'}$ be the map which sends each ι_i to $s_l(f)(y_i+\bar{v}_i)$ and $\varphi_{Y'}$ be the map which sends each ι'_j to $s_l(f)(y'_j+\bar{w}_j)$ and set β,α to be the maps which send ι_i and ι'_j to $\sum\lambda_{ii'}\iota_{i'}$ and $\sum\lambda'_{jj'}\iota'_{j'}$ respectively. Obviously what we have to verify is that $\varphi_{Y'}f'\sim g\varphi_{X'}$. It is easy to know from diagram chasing argument that $\beta\varphi_{Y'}f'\sim\beta g\varphi_{X'}$. Now $(\varphi_{Y'}f')^*(\iota_i)-(g\varphi_{X'})^*(\iota_i)=u_i$ which are torsion of order dividing $\hat{t}$ for all i. $u_i=0$ for all i iff $\sum\lambda_{ii'}u_{i'}=0$ for all i iff $\beta\varphi_{Y'}f'\sim\beta g\varphi_{X'}$ since $\det\lambda_{ii'}$ is prime relative to $\hat{t}$. Now it is clear that the front and back squares are homotopy pullback. □

Proof of Theorem 1.6. Without loss of generality, we can assume that there is an integer l such that $\pi_i(X)\bigotimes\mathbb{Q}=\pi_i(Y)\bigotimes\mathbb{Q}=0$ if $i\neq l$. By Theorem 2.10 and Theorem 2.14 it follows that there exists a surjection

$\xi : T \to G(f)$ where

$$T = \{(\alpha, \beta) | \beta g \sim g\alpha, \alpha \text{ and } \beta \text{ are } \hat{t}\text{-equivalences}\}.$$

In other words, there is a surjection $\xi' : T' \to G(f)$ where

$$T' = \{(A_1, A_2) | A_i \in M(\mathbb{Z}, r_i), (\det A_i, \hat{t}) = 1, i = 1, 2, A_2 C = C A_1\}$$

where $M(\mathbb{Z}, n)$ is the set of $n \times n$ matrices and A_1, A_2, C represents α, β, g respectively.

Given $(A_1, A_2), (B_1, B_2) \in T^{'}$ such that $\det A_i \equiv \det B_i \pmod{\hat{t}}$ for $i = 1, 2$, then an elementary calculation shows that $A_1 = B_1 G_1 + \hat{t} H_1$, $A_2 = B_2 G_2 + \hat{t} H_2$, $(G_1, G_2) \in T^{'}, \det G_1 \equiv \det G_2 \equiv 1 (mod \hat{t})$ and $H_2 C = C H_1$. Claim 2.15 implies that we can assume $\det G_1 = \det G_2 = 1$. It follows from Theorem 2.14 and Theorem 2.13 and Remark 2.11 that $\xi^{'}(A_1, A_2) = \xi^{'}(B_1, B_2)$. Thus $\xi^{'}$ factors through the map $\det : T^{'} \to (\mathbb{Z}_{\hat{t}}^{*})^{l(X)+l(Y)}$. The rest of the proof is the same as that of Hurvitz[5]. □

Claim 2.15. *Given* G_1, G_2 *with* $\det G_1 \equiv \det G_2 \equiv 1 (mod \hat{t})$ *and* $(G_1, G_2) \in T'$. *There exist* $H_1 \in GL(\mathbb{Z}, r_1)$, $H_2 \in GL(\mathbb{Z}, r_2)$ *such that* $(H_1, H_2) \in T'$ *and* $G_1 H_1 = Id + \hat{t} H_1'$, $H_2 G_2 = Id + \hat{t} H_2'$

Proof. Assume, without loss of generality, that the $m_1 \times m_2$ matrix C has entries zero unless those on diagonal. Then $(G_1, G_2) \in T'$ is equivalent to the equations

$$(G_1)_{ij} c_{jj} = c_{ii} (G_2)_{ij}, \text{ for } 0 \le i, j \le l_0$$

$$(G_2)_{ij} = 0 \text{ for } i \le l_0 \text{ and } j > l_0$$

$$(G_1)_{ij} = 0 \text{ for } j \le l_0 \text{ and } i > l_0$$

where c_{ii}, $0 \le i \le l_0$, are the nontrivial terms in matrix C. The proof of the existence of the matrices H_1, H_2 with the prescribed properties will be given by mathematical induction and it is obvious for $l = 1$. Assume it is also true for $l < s$, we will prove it for $l = s$. If $l_0 < m_1$, using elementary matrix we can find $H_1 \in GL(\mathbb{Z}, r_1)$ such that $(H_1, Id) \in T'$ and we have the following modulo $\hat{t}$ equation

$$(G_1H_1)_{ij} = \begin{cases} 0 & \text{if } i = m_1, l_0 < j < m_1 \\ 1 & \text{if } i = m_1, j = m_2 \\ (G_1)_{ij} & \text{otherwise} \end{cases}$$

Similarly we can find $H_2 \in GL(\mathbb{Z}, r_2)$ if $l_0 < m_2$ with similar properties. It follows that we can assume $l_0 = m_1 = m_2$. In this case $(G_1, G_2) \in T'$ is equivalent to the equations

$$(G_1)_{ij}c_{jj} = c_{ii}(G_2)_{ij}, \text{ for } i, j \leq l_0$$

The equations above imply that G_1 and G_2 are determined by one matrix. Thus the result follows from the corresponding result in the one matrix case. □

3. Genus of co-H-maps between co-H-spaces

As in the last section we fix some notations first. Throughout this section all spaces will be 1-connected finite CW complexes with rational co-H-space structures. Given space X, let $M(X)$ denote the bouquet of spheres whose reduced integral homology is isomorphic to $\tilde{H}_*(X, \mathbb{Z})/torsion$. For any space X, let $Q\pi_*(X)$ denote the quotient of $\pi_*(X)$ by the subgroup generated by all Whitehead products in $\pi_*(X)$. An $\mathbf{s}$-maximal map from $M(X)$ to X is one that satisfies a condition dual to that in the rational H-space case. A $\mathbf{s}$-maximal map of a co-H-space is called coprimitive if it is a co-H-map.

Let $s_n(X) = exp(torsion(\pi_k(X)))$ when $Q\pi_n(X) \bigotimes \mathbb{Q} \neq 0$ and $s_n(X) = 1$ otherwise. Let $\mathbf{s}(X)$ be the sequence of integers $\{s_1(X), s_2(X), \cdots\}$

. If $f : X \to Y$ is a map , then $\mathbf{s}(f)$ is defined to be the sequence of integers $\{s_1(X)s_1(Y), s_2(X)s_2(Y), \cdots\}$.

On the other hand , let $\sigma_n : Q\pi_n(X)/torsion \to H_n(X, \mathbb{Z})$ be induced by the Hurewicz homomorphism. Since X is a rational co-H-space, σ_n has finite kernel and finite cokernel for each n. Define

$$t_n(X) = exp(\text{coker}\sigma_{n+1})exp(\text{ker}\sigma_n)$$

and

$$t(X) = \prod_{n \leq N(X)} t_n(X)$$

where $N(X)$ is the least integer so that , for every $n > N(X), H_n(X) = 0$. As in the last section, we have the following

Lemma 3.1. *$s_n t_n(X)$ can be divided by the integer $exp(H_n(C(\varphi_X), \mathbb{Z}))$ where $C(f)$ is the homotopy cofiber of f and φ_X is an $\mathbf{s}$-maximal map of X.*

Remark 3.2. Any $\mathbf{s}(X)$-maximal map of a co-H-space is also coprimitive.

Now we define as before an integer $\hat{t}$ in Theorem 1.8 as follow

Definition 3.3. Given a co-H-map $f : X \to Y$ between two co-H-spaces,

$$\hat{t} = t(X)(t(Y))^2 \prod_{n \leq \max(N(X), N(Y))} s_n(X)s_n(Y)$$

Let X be a rational co-H-space, we say $\pi_*(X) \bigotimes Q$ is coprimitive generated as Lie algebra if there exists a Lie algebra basis $\{x_1, \cdots, x_n\} \in \pi_*(X) \bigotimes Q \cong \pi_*(X_{(0)})$ such that x_i is represented by a co-H-map $S^j \to X_{(0)}$.

Let $f : X \to Y$ be a map between two rational co-H-spaces with coprimitive generated rational homotopy groups. Then an $\mathbf{s}$-maximal map for f is a homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \varphi_X}\uparrow & & {\scriptstyle \varphi_Y}\uparrow \\ M(X) & \xrightarrow{f_X} & M(Y) \end{array}$$

where φ_X, φ_Y are $\mathbf{s}$-maximal maps. If the map f is also a co-H-map, then an $\mathbf{s}$-maximal map for f is called coprimitive if $f_X, \varphi_X, \varphi_Y$ are co-H-maps.

Remark 3.4. Any $\mathbf{s}(f)$-maximal map of a co-H-map f is coprimitive.

In this section we will prove Theorem 1.8. As before , we need several preliminary results which are dual to those in the previous section. Let (X, ν) be a co-H-space. We shall denote by $+$ the operation on $[X, W]$ induced by ν , by η_n the map

$$\eta_n = F \cdots (F \vee 1 \vee \cdots \vee 1) \circ (\psi \vee 1 \vee \cdots \vee 1) \cdots (\psi \vee 1)\psi$$

where F denotes the folding map.

In general the product ν is not assumed to have an inverse. Nevertheless we have the following elementary but important result

Lemma 3.5. *If (X, ν) is a co-H-space, then $[X, W]$ is an algebraic loop for any space W, i.e., for any pair $f, g \in [X, W]$ there exists a unique $D_{f,g} \in [X, W]$ such that $D_{f,g} + g = f$.*

Lemma 3.6. *Let $h : X_1 \to X_2$ be a co-H-map and $g : W_1 \to W_2$ be a map. For any pair $f_1, f_2 \in [X_2, W_1]$, we have $(D_{f_1,f_2})h = D_{f_1h,f_2h}$ and $gD_{f_1,f_2} = D_{gf_1,gf_2}$.*

Remark 3.7. The proof of the lemmas above are dual to that in [9]

The dual of Theorem 2.8 is

Theorem 3.8. *Let X be a 1-connected co-H-space and W be a space with a finite number of nontrivial homotopy groups. Then we have the following:*
(i)For every map $f : X_{(p)} \to W_{(p)}$ there exist an integer n, $(n, p) = 1$, and a map $\bar{f} : X \to W$ so that $\bar{f}_{(p)} \sim f\eta_n$.
(ii) If two maps $f_1, f_2 : X \to W$ satisfy $f_1\varphi_X \sim f_2\varphi_X$, then $f_2\eta_{t'} \sim f_1\eta_{t'}$ where $\varphi_X : M(X) \to X$ is a coprimitive **s**-*maximal map and $t' = \prod_{n \leq N(X)} t_n(X)s_n$.*

The part (i) is dual to Theorem 2.1(i) in [5], so we will only give a detailed proof of the part (ii) .

Proof. As mentioned before, the maximal map φ_X as in the Theorem exists. Let $k = D_{f_1,f_2}$. It suffices to prove that $k\eta_{t'} \sim *$ by Lemma 3.6. The given condition and Lemma 3.6 imply that $k\varphi_X \sim *$. Now consider the Moore-Postnikov decomposition of the map $\varphi_X : M(X) \to X$. A typical term in the decomposition fits into the following

$$\begin{array}{ccc} X^{(n)} & \xrightarrow{g^{(n)}} & X^{(n+1)} \\ \uparrow & & \downarrow \\ M(X) & \xrightarrow{\varphi_X} & X \end{array}$$

where $X^{(0)} = M(X)$ and map $X^{(n)} \to X^{(n+1)}$ is the mapping cone of a map $k^{(n)} : M(H_{n+1}(C(\varphi_X), \mathbb{Z}), n) \to X^{(n)}$ and $M(G, n)$ is the type $G - n$ Moore space :

$$\tilde{H}_i(M(G, n), \mathbb{Z}) = \begin{cases} G & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases}$$

By the condition on X , there is an integer s such that $X^{(s)} = X$. Let $t_n = \prod_{i \leq n} t_i(X)s_i$. We will prove that $k_n\eta_{t_n} \sim *$ where $k_n : X^{(n)} \to W$

is the map induced by $k : X \to W$ which will completes the proof since for $n = s$, $k = k_n$ and $t' = t_n$. We will proceed by induction. It is obvious for the case $n = 0$. If the statement is true for n , we want to prove it for $n + 1$. From the cofibration

$$M(H_{n+1}(C(\varphi_X), \mathbb{Z}), n) \to X^{(n)} \to X^{(n+1)}$$

the following exact sequence is exact:

$$[M(H_{n+1}(C(\varphi_X), \mathbb{Z}), n + 1), W] \to [X^{(n+1)}, W] \to [X^{(n)}, W]$$

Now $(\eta_{t_n})^*(g^{(n)})^*(k_{n+1}) = (\eta_{t_n})^* k_n = 0$. Since $g^{(n)}$ is a co-H-map, it follows by Theorem 4.1 that $k_{n+1}\eta_{t_n}$ belongs to image of the group $[M(H_{n+1}(C(\varphi_X), \mathbb{Z}), n + 1), W]$ which has exponent dividing $t_{n+1}(X)$. Therefore it follows that $k_{n+1}\eta_{t_{n+1}s_{n+1}} \sim *$ which completes the induction. □

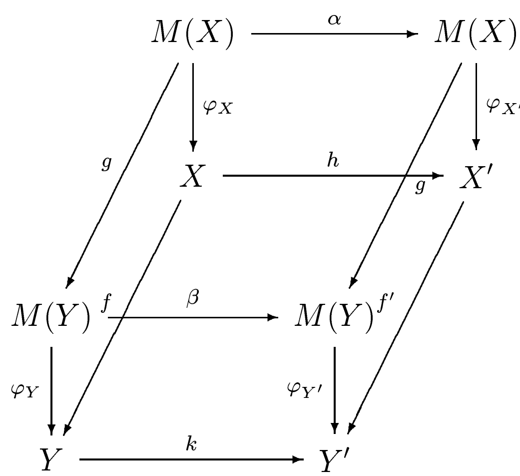
The result dual to Theorem 2.9 is

Theorem 3.9. *Let $f : X \to Y$ be a co-H-map between two co-H-spaces, l be an integer with $l = p_1^{w_1} \cdots p_s^{w_s}$. Given two spaces X', Y', a function $f' : X' \to Y'$ and homotopy equivalences $h_{p_i} : X_{p_i} \to X'_{p_i}$, $k_{p_i} : Y_{p_i} \to Y'_{p_i}$ satisfying $k_{p_i} f_{(p_i)} \sim f'_{(p_i)} h_{p_i}$, then there exist l-equivalences $h : X \to X'$, $Y \to Y'$ so that $kf \sim f'h$.*

Proof. The proof is exactly the dual of that of Theorem 2.2 in [5]. □

The following is what is dual to Theorem 2.10

Theorem 3.10. *If $f : X \to Y$ is any map between rational co-H-spaces , Suppose that an $\mathbf{s}(f)$-maximal map of f is given as in the following* *diagram.*



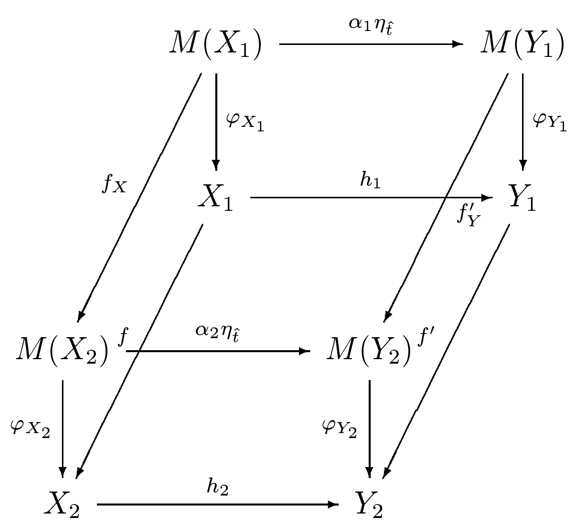
Then for every pair (α, β) where $\alpha : M(X) \to M(X)$, $\beta : M(Y) \to M(Y)$ are $\hat{t}$-equivalences such that $\beta g \sim g\alpha$, there exist a map $f' : X' \to Y', f' \in G(f)$ and $\hat{t}$-equivalences $h : X \to X'$ and $k : Y \to Y'$ so that the every diagram in the above cube commutes up to homotopy and $\varphi_{X'}, \varphi_{Y'}$ are $\mathbf{s}(f)$-maximal maps.

Remark 3.11. It is easy to know that, if α, β are homotopy equivalences, then f' is equivalent to f.

Proof. The proof is dual to that of Theorem 3.4 in [5]. □

That Dual to Theorem 2.12 is the following

Theorem 3.12. *Let $f : X_1 \to X_2$ be a co-H-map between co-H-spaces such that $\pi_*(X_1) \bigotimes \mathbb{Q}$ and $\pi_*(X_2) \bigotimes \mathbb{Q}$ are coprimitively generated. Let $f' : Y_1 \to Y_2$ be a map between rational co-H-spaces. Choose $\mathbf{s}(f)$-maximal maps f_X, f'_Y for f, f', respectively, so that the maximal map for f is coprimitive. Then given a pair of maps (α_1, α_2) satisfying $\alpha_2 f_X = f'_Y \alpha_1$ as in the following diagram, there exist mappings $h_i : X_i \to Y_i$, $i = 1, 2$, so that all the diagrams in the following cube* *commute up to homotopy.*

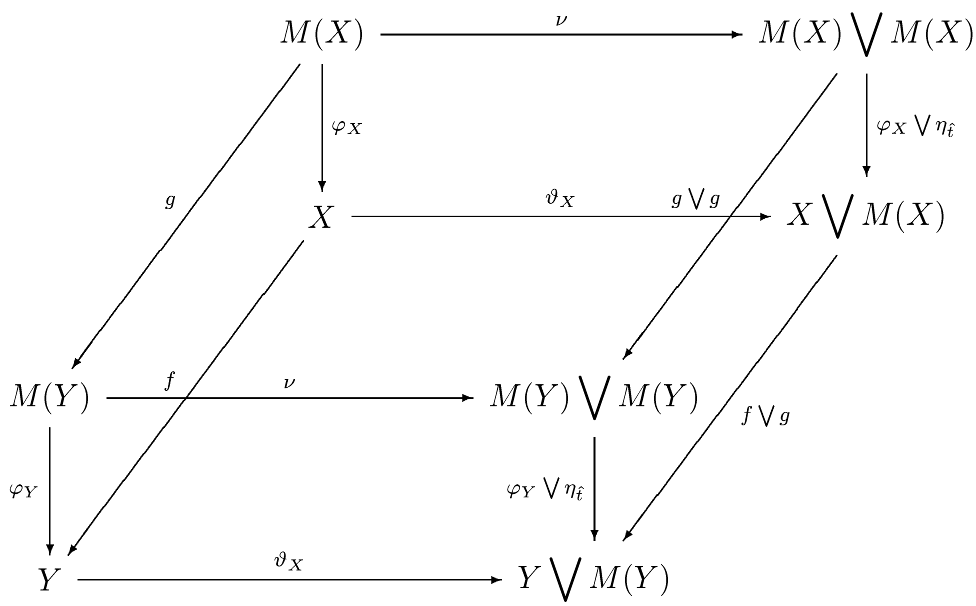


Proof. An exactly dual proof can be given for Theorem 2.12 , except that, instead of using Theorem 2.8 and Proposition 1.8 in [10] , we appeal to Theorem 3.8 and Theorem 3.13 below which is dual to Proposition 1.8 in [10]. □

Theorem 3.13. *Let X be a finite co-H-space and $\varphi_X : M(X) \to X$ be a* $\mathbf{s}$*-maximal map and $g : M(X) \to W$ be any map into a space W. Then there exists a map $h : X \to W$ such that $h\varphi_X \sim g\eta_{t(X)}$.*

Proof. Let $\vartheta : X \to X \bigvee M(X)$ be the co-action given in Proposition 5.6 [6] with respect to maximal map φ_X. Then h is the composite $X \stackrel{\vartheta}{\to} X \bigvee M(X) \stackrel{g\eta_{t(X)} \bigvee \mathrm{id}}{\longrightarrow} W \bigvee X \stackrel{\nabla}{\to} W$ where $\nabla|_W = \mathrm{id}$ and $\nabla|_X = *$. It is easy to verify that h is what we want. □

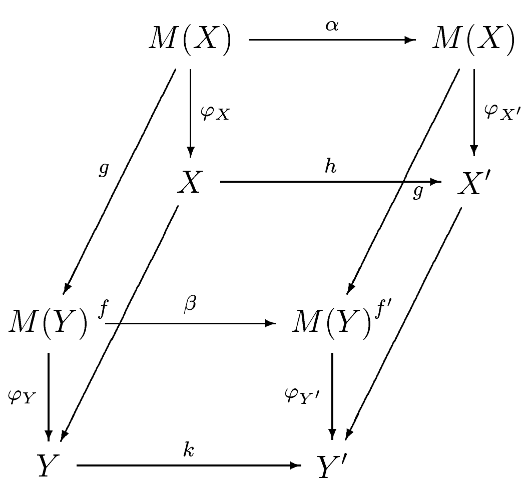
Theorem 3.14. *Let f be as in Theorem 1.8. Given a coprimitive* $\mathbf{s}(f)$*-maximal map g of f, there exists a commutative cube up to homotopy* *as follows*



where ϑ_X, ϑ_Y project to the identity on the first factor and ν are the standard multiplication on bouquet of spheres.

Proof. The proof is similar to that of Theorem 2.12 using Proposition 5.6 in[6]. □

Theorem 3.15. *Let $f : X \to Y$ be as in Theorem 1.8 . Given $[f'] \in G(f)$, then there is a commutative cube up to homotopy*



where h, k are $\hat{t}$-equivalences, the vertical maps are all $\mathbf{s}(f)$-maximal maps , the back and front face diagrams are homotopy pushout diagrams and

$$\alpha = \bigvee_i \alpha_i : M(X) \to M(X), \beta = \bigvee_j \beta_j : M(Y) \to M(Y)$$

where

$$\alpha_i : M(H_i(X, \mathbb{Z})/torsion, i) \to M(H_i(X, \mathbb{Z})/torsion, i)$$

$$\beta_j : M(H_j(Y, \mathbb{Z})/torsion, j) \to M(H_j(Y, \mathbb{Z})/torsion, j)$$

and

$$M(X) = \sum_i M(H_i(X, \mathbb{Z})/torsion, i), M(Y) = \sum_j M(H_j(Y, \mathbb{Z})/torsion, j)$$

Proof. Take $\hat{t}$-equivalences $h : X \to Y, k : Y \to Y'$ as granted by Theorem 3.9 such that $kf \sim f'h$. We will try to define the other maps in the above diagram so that the Theorem is true. As in the dual case , we can assume there is only one l such that $H_i(X, \mathbb{Q}) = H_i(Y, \mathbb{Q}) = 0$ if $i \neq l$. Choose rational co-H-space structures on X', Y' such that $h_{(0)}, k_{(0)}, f'$ are co-H-maps. As in the dual case we can choose elements $\{x_1, \cdots, x_{r_1}\} \in \pi_*(X)$, $\{x'_1, \cdots, x'_{r_2} \in \pi_*(Y)$, $\{y_1, \cdots, y_{r_1}\} \in \pi_*(X')$ and $\{y'_1, \cdots, y'_{r_2}\} \in \pi_*(Y')$ such that they project to bases of $\pi_*(X) \bigotimes \mathbb{Q}$, $\pi_*(Y) \bigotimes \mathbb{Q}$, $\pi_*(X') \bigotimes \mathbb{Q}$ and $\pi_*(Y') \bigotimes \mathbb{Q}$ respectively. As in the dual case $\pi_*(f)/torsion$ is of the diagonal form with respect to these bases and

$$(h)_*(x_i) = \sum \lambda_{ii'} y_{i'} + v_i, (k)_*(x'_j) = \sum \lambda'_{jj'} y'_{j'} + w_j$$

where $\det \lambda_{ii'}, \det \lambda'_{jj'}$ are prime relative to $\hat{t}$ and v_i, w_j are torsions. Certainly we can find $\bar{v}_i$ and $\bar{w}_j$ so that

$$(h)_*(x_i) = \sum \lambda_{ii'}(y_{i'} + \bar{v}_{i'}), (k)_*(x'_j) = \sum \lambda'_{jj'}(y'_{j'} + \bar{w}_{j'})$$

Let $\{\iota_1, \cdots, \iota_{r_1}\}$ be a base for $H_*(M(X), \mathbb{Z})/torsion$ and $\{\iota'_1, \cdots, \iota'_{r_2}\}$ be a basis for $H_*(M(Y), \mathbb{Z})/torsion$. Define φ_Y, φ_X by $s_l(f)x_i = (\varphi_X)_*(\iota_i)$, $s_l(f)x'_j = (\varphi_Y)_*(\iota'_j)$ respectively and define g so that its matrix with respect to $\{\iota_1, \cdots, \iota_{r_1}\}$, $\{\iota'_1, \cdots, \iota'_{r_2}\}$ is the same as that of $\pi_*(f)/torsion$. Let $\varphi_{X'}$ be the map which sends each ι_i to $s_l(f)(y_i+\bar{v}_i)$ and $\varphi_{Y'}$ be the map which sends each ι'_j to $s_l(f)(y'_j+\bar{w}_j)$ and set α, β to be the maps which send ι_i and ι'_j to $\sum \lambda_{ii'}\iota_{i'}$ and $\sum \lambda'_{jj'}\iota'_{j'}$, respectively. Obviously the only thing we have to verify is that $\varphi_{Y'}g \sim f'\varphi_{X'}$. It is easy to know from diagram chasing argument that $g\varphi_{Y'}\alpha \sim f'\varphi_{X'}\alpha$. Now $(g\varphi_{Y'})_*(\iota_i) - (\varphi_{X'}f')_*(\iota_i) = u_i$ which are torsions of order dividing $\hat{t}$ for all i. $u_i = 0$ for all i iff $\sum \lambda_{ii'}u_{i'} = 0$ for all i iff $g\varphi_{Y'}\alpha \sim f'\varphi_{X'}\alpha$ since $\det \lambda_{ii'}$ is prime relative to $\hat{t}$. $\square$

Proof of Theorem1.8. Without loss of generality, we can assume that there is an integer l such that $Q\pi_i(X) \bigotimes \mathbb{Q} = Q\pi_i(Y) \bigotimes \mathbb{Q} = 0$ if $i \neq l$. By Theorem 3.10 and Theorem 3.15 it follows that there exists a surjection $\xi' : T \to G(f)$ where

$$T = \{(\alpha, \beta) | \beta g \sim g\alpha, \alpha \text{ and } \beta \text{ are } \hat{t}\text{-equivalences}\}.$$

In other words, there is a surjection $\xi' : T' \to G(f)$ where

$$T' = \{(A_1, A_2) | A_1 \in M(\mathbb{Z}, r_i), (\det A_1, \hat{t}) = 1, i = 1, 2, A_2C = CA_1\}.$$

and $M(\mathbb{Z}, n)$ is the set of $n \times n$ matrices and A_1, A_2, C represent α, β, g respectively. Given $(A_1, A_2), (B_1, B_2) \in T^{'}$ such that $\det A_i \equiv \det B_i \pmod{\hat{t}}$ for $i = 1, 2$, then an elementary calculation shows that $A_1 = B_1G_1 + \hat{t}H_1$, $A_2 = B_2G_2 + \hat{t}H_2$, $(G_1, G_2) \in T^{'}, \det G_1 \equiv \det G_2 \equiv 1 (mod \hat{t})$ and $G_2C = CH_2$. Claim 2.15 implies that we can assume $\det G_1 = \det G_2 = 1$. It follows from Theorem 3.15 and Theorem 3.14 and Remark 3.11 that $\xi^{'}(A_1, A_2) = \xi^{'}(B_1, B_2)$. Thus $\xi^{'}$ factors through the map $\det : T^{'} \to (\mathbb{Z}^*_{\hat{t}})^{l(X)+l(Y)}$. The rest of the proof is dual to that of Hurvitz[5]. □

4. Some results about co-H-spaces

In this section we will prove a result about co-H-spaces which is needed in the last section and which may have independent interest. This result concerns about the relative homology decomposition of a map . Zabrodsky had proved that the Postnikov decomposition of an H-map consists of H-spaces and H-maps (c.f., Corollary 2.3.2 in [9]). The dual result is also true under our assumption that all co-H-spaces we are concerned are 1-connected.

Theorem 4.1. *Let $f : X \to Y$ be a co-H-map between co-H-spaces. Then the homology decomposition of f consists of co-H-spaces and co-H-maps.*

Remark 4.2. When the map f is the inclusion of the base point into the space X , the homology decomposition of f is the Moore decomposition of X. In that special case the above Theorem was already obtained by M.Arkowitz[2] and M.Golasinski and John R. Klein[4]

To prove the Theorem above we will follow Zabrodsky's approach to the dual result. First let us recall results about the obstruction of a map between co-H-spaces to be a co-H-map , see Arkowitz[1] for a comprehensive survey about co-H-space.

Let XbX be the space of paths in $X \times X$ beginning in $X \bigvee X$ and ending at the base point and let $i : XbX \to X \bigvee X$ assign to a path its initial point. Then we have a short exact sequence

$$0 \to [\Omega X, \Omega(XbX)] \stackrel{(\Omega i)_*}{\to} [\Omega X, \Omega(X \bigvee X)] \stackrel{(\Omega j)_*}{\to} [\Omega X, \Omega(X \times X)] \to 0$$

where i_1, i_2 are the inclusions to the first and second factors of $X \bigvee X$. The comultiplication $\nu : XX \bigvee X$ determines an element

$$\mu_X = -\Omega i_1 - \Omega i_2 + \Omega\nu \in [\Omega X, \Omega(X \bigvee X)]$$

such that $(\Omega j)_*(\mu_X) = 0$. Thus there exists a unique element $H(\nu) \in [\Omega X, \Omega(XbX)]$ such that $(\Omega i)_*(H(\nu)) = \mu_X$. We call $H(\nu)$ the dual Hopf construction (applied to ν). It is well known that $\beta : \Sigma A \to X$ is a co-H-map if and only if $H(\nu)\bar{\beta} = 0$ in $[A, \Omega(XbX)]$ where ΣA has the standard comultiplication on the suspension and $\bar{\beta}$ is the adjoint of the map β .

Lemma 4.3. *Let $f : (X, \nu_X) \to (Y, \nu_Y)$ be a co-H-map between co-H-spaces and $\beta : \Sigma A \to X$ be a map. Then*

$$H(\nu_Y)(\Omega f\bar{\beta}) = \Omega(fbf)H(\nu_X)\bar{\beta}$$

Proof. It follows from the equation $\mu_Y(\Omega f\bar{\beta}) = \Omega(fbf)\mu_X\bar{\beta}$ which can be verified directly. □

It is well known that XbX has the weak homotopy type of $\Sigma\Omega X \bigwedge \Omega X$. The following is a special case of a general result given by Golasinski and Klein.

Lemma 4.4. [4] *If $f : X \to Y$ is a co-H-map between co-H-spaces and f is n-connected with $n \geq 1$. Then fbf is $n+1$-connected.*

Remark 4.5. The authors learned of this result through the discussion list of Hopf archive. Thanks go to all those who have shared information about this with us.

Proposition 4.6. *Let $f : (X, \nu_X) \to (Y, \nu_Y)$ be a co-H-map. Suppose f is n-connected with $n \geq 2$. Given any map $\beta : \Sigma M(G, n-1) = M(G, n) \to X$ so that $f\beta \sim *$, then β is a co-H-map.*

Proof. By Lemma 4.3 , we have

$$* = H(\nu_Y)(*) = H(\nu_Y)(\Omega f \bar{\beta}) = \Omega(fbf)H(\nu_X)\bar{\beta}$$

On the other hand , $H(\nu_X)\bar{\beta} \in [M(G, n-1), \Omega(XbX)] = \pi_{n-1}(\Omega(XbX), G)$. Thus β is a co-H-map if $H(\nu_X)\bar{\beta} = 0$. Now $\Omega(fbf)H(\nu_X)\bar{\beta} = 0$. The universal coefficient formula for homotopy group with coefficient and Lemma 4.4 imply that $\Omega(fbf)$ induces a monomorphism which gives the desired equation $H(\nu_X)\bar{\beta} = 0$. □

Proof of Theorem4.1. Now the proof of Theorem 4.1 follows from an induction by the homology decomposition of the co-H-map f together with Proposition 4.6 and the fact that cofiber of a co-H-map is a co-H-space and the inclusion of the cofiber is a co-H-map. □

References

[1] M. Arkowitz, *Co-H-spaces.* (English) James, I. M. (ed.), Handbook of algebraic topology. Amsterdam: North-Holland. pp.1143-1173 (1995).

[2] M. Arkowitz,*Induced mappings of homology decompositions.* (English) Oprea, J. (ed.) et al., Homotopy and geometry. Banach Cent. Publ. 45, pp.225-233 (1998). Warsaw

[3] C.R.Curjel,*On H-space structure of finite complexes*, Comment. Math. Helv. **43** (1967), pp.1-17

[4] M. Golasinski , John R.Klein, *On maps into a co-H-space*, Hiroshima Math. J. **28** (1998), no. 2, pp.321–327

[5] S.Hurvitz,*The genus of a map*, Trans. Amer.Soc. **264** (1981), pp.1-28

[6] C.A.McGibbon, *On the localization genus of a space*, Progress in Math., vol.136, pp.285–306 Birkhauser Verlag ,1996

[7] G.Mislin, *The genus of an H-space*, Lecture Notes in Math., vol.249, Springer-Verlag,1971, pp.75-83

[8] C. Wilkerson, *Genus and cancellation*, (English) Topology **14**, (1975), pp.29-36.

[9] A.Zabrodsky, *Hopf spaces*, North-Holland Math. Studies , vol.**22**, North-Holland, Amsterdam,1976

[10] A.Zabrodsky, *p-equivalences and homotopy type*, Lecture Notes in Math., vol.418, Springer-Verlag, 1974, pp.161-171

Institute of Math.,Academia Sinica ,Beijing China and Department of Mathematics Education , Korea University , Seoul , Korea

E-mail address: pjz@math03.math.ac.cn

Department of Mathematics Education , Korea University , Seoul , Korea

E-mail address: woomh@kuccnx.korea.ac.kr