HAVING THE H-SPACE STRUCTURE IS NOT A GENERIC PROPERTY

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ABSTRACT. In this note, we answer in negative a question posed by McGibbon[7] about the generic property of H-space structure. In fact we verify the conjecture of Roitberg [12]. Incidentally, the same example also answers in negative the open problem 10 in McGibbon[7]

1. INTRODUCTION

Let X be a connected CW complex, the L-S category of X, $cat(X)$, of X is the least integer $k \geq 0$ such that X can be covered by $k+1$ open subsets which are contractible in X. Of course the condition that X is a CW complex is unnecessary for the above definition to have a meaning . However it is in this context that a rich theory of category exists. Recent works in rational homotopy theory gave rise to a theory of rational L-S category. This makes it possible to calculate the rational L-S category and thus attack the rational Ganea Conjecture[1]. On the other hand works by N.Iwase, H.Scheerer, and D.Stanley provided the method to determine the L-S category itself in some case which lead to the construction of counterexamples to the Ganea Conjecture, see,

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e.g., [14]. Besides the application to the Ganea Conjecture , another interesting application of these ideas was given by Roitberg[12]. To explain this we have to state one problem posed by McGibbon[7]:

Question 1.1. *If X and Y have the same Mislin genus ,i.e. $X_{(p)} \simeq Y_{(p)}$ for all p where $X_{(p)}$ is the p-localization of X, does it follow that $cat(X) = cat(Y)$?*

In his paper[12], applying some results about category by Iwase[4] and results about phantom maps , Roitberg was able to answer the above question negatively. His main result can be stated as follows

Theorem 1.2. *Let $\phi : \Sigma K(\mathbb{Z}, 5) \to S^4$ be an essential, special, phantom map and X be the mapping cone of ϕ .Then $cat(X) = 2$.*

Remark 1.3. It is well known that $cat(X) = 1$ iff X is a co-H-space . It follows from the above theorem that X is not a co-H-space . On the other hand it easy to know that $S^4 \vee \Sigma^2 K(Z, 5)$ has the same Mislin genus with X and is a co-H-space .

From this point of view , the theorem above answers in negative the following

Question 1.4. *If X and Y have the same Mislin genus and X is co-H-space , does it follow that Y is also a co-H-space?*

Question1.1 has an obvious Eckmann-Hilton dual . However it may not be a good question at present time since the dual L-S category is not well developed. A manageable problem is the obvious dual of Question1.4 which has been posed by McGibbon in [7]. A more precise conjecture was given by Roitberg in his paper[12]. The purpose of this paper is to establish Roitberg's conjecture and thus also answers in negative McGibbon's problem.

Theorem 1.5. *Let* $\psi : K(\mathbb{Z}, 2) \to \Omega S^6$ *be an essential, special, phantom map and* Z *be the homotopy fiber of* ψ*. Then* Z *and* $K(\mathbb{Z}, 2) \times \Omega^2 S^6$ *have the same Mislin genus and* Z *is not an H-space.*

Since the dual L-S category is not well developed and Iwase's paper[4] is unavailable to the author, method from the well developed theory of H-space will be used in stead. Another feature is the application of the newly developed Gray index of phantom map [5],[9]. Actually , by duality , our method also gives an alternative proof of the main results in [12].

Incidentally , the example constructed above combined with Theorem 3.4 in [3] also provides a negative answer to the open problem 10 in [7]: Is X an H-space if each of its Postnikov approximations $X^{(n)}$ is ? Actually we have the following

Theorem 1.6. *Let* $\psi : K(\mathbb{Z}, 2) \to \Omega S^6$ *be an essential, special, phantom map and* Z *be the homotopy fiber of* ψ*. Then* $Z^{(n)}$ *is an H-space for each n but* Z *is not.*

Proof. It follows immediately from Theorem 3.4 in [3] that $Z^{(n)}$ and $(K(\mathbb{Z}, 2) \times \Omega^2 S^6)^{(n)}$ have the same homotopy type and thus $Z^{(n)}$ is an H-space for each n. □

In this paper all spaces involved are assumed to be 1-connected CW complexes with finite type.

The author would like to thank Prof. Roitberg for his interest in this work and for pointing out a fatal error in the earlier version of this paper . It is to correct that error that we find the application of Gray index to this work.

2. Background about H-spaces and Phantom maps

First we will recall some backgrouds about H-spaces , see [17] for details. An H-space is a space X with a map $\mu : X \times X \to X$ such that $\mu|_{X\vee X} = F$ where $F : X \bigvee X \to X$ is the natural folding map. An H-map between H-spaces is a map of spaces $f : X \to Y$ such that the following diagram commutes up to homotopy.

$$\begin{tikzcd}
X \times X \arrow[r, "f\times f"] \arrow[d, "\mu_X"'] & Y \times Y \arrow[d, "\mu_Y"'] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

In this case we say that f is a $\mu_X - \mu_Y$ H-map. Two elementary but important results are the followings

Proposition 2.1. *Let (X, μ_X) be an H-space. Then, for any space M, $[M, X]$ is an algebraic loop, i.e., for any $f, g \in [X, Y]$ there exists a unique $D_{f,g} \in [M, X]$ such that*

$$\mu_*(D_{f,g}, g) = f$$

Proposition 2.2. *If $f : (X, \mu_X) \to (Y, \mu_Y)$ is an H-map , then the homotopy fiber of f is an H-space.*

Thus it is important to know when is a map an H-map or what is the obstruction for a map to be an H-map.

Definition 2.3. Let (X, μ) and (Y, μ') be H-spaces and $f : X \to Y$ be a map of spaces. H-derivation of f is the map

$$HD(f) \in [X \wedge X, Y]$$

which is defined by

$$HD(f)\Lambda = D_{f\mu, \mu'(f\times f)}$$

where $\Lambda : X \times X \to X \wedge X$ is the natural quotient map.

Remark 2.4. The definition of H-derivation depends on the H-space structures on both X and Y.

Remark 2.5. Let (X, μ) and (Y, μ') be H-spaces and $f : X \to Y$ be a map of spaces. It is well known that $f : (X, \mu_X) \to (Y, \mu_Y)$ is an H-map iff $HD(f) = *$.

An easy but crucial corollary of this last remark is the following

Corollary 2.6. *Let (X, μ) and (X', μ') be H-spaces and $f : X \to X'$ be a map of spaces. Assume further that $\pi_i X' = 0$ for $i \geq 2d$ and X is $(d-1)$-connected . Then f is a $\mu - \mu'$ H-map.*

Following is one of the fundamental properties of H-derivation

Proposition 2.7. *Let (X_i, μ_i) be H-spaces, $i = 0, 1, 2$ and $f : X_0 \to X_1$, $g : X_1 \to X_2$ be maps of spaces.*
(a)If $f : X_0 \to X_1$ is a $\mu_0 - \mu_1$ H-map, then $HD(gf) = HD(g)(f \wedge f)$
(b)If $g : X_1 \to X_2$ is a $\mu_1 - \mu_2$ H-map, then $HD(gf) = gHD(f)$

Another ingredient for the main result is the phantom map . Recall that a map f from a CW complex X is called an phantom map if its restriction to the n-th skeleton is inessential for any integer n. Let $Ph(X, Y)$ denote the set of homotopy classes of phantom maps from X to Y. The following result which follows from the Sullivan conjecture provides us many examples of phantom maps.

Theorem 2.8. [8] *Let $Y = \Omega^i K$ and $X = \Sigma^j Z$ such that $i, j \geq 0$,K is a 1-connected finite CW complex. Then every map from X to Y is a phantom map if Z is as follows:*

- *Z is the classifying space of a 1-connected compact Lie group*
- *Z is an infinite loop space with torsion fundamental group*
- *Z has only finitely number of nontrivial homotopy groups*

and in this case we have

$$Ph(X,Y) = [X,Y] = [X_{(0)},Y] = \prod_{n>0} H^n(X, \pi_{n+1}(Y) \bigotimes R)$$

Let $Ph(X,Y)$ denote the set of homotopy classes of phantom maps from X to Y. The p-localization l_p induces a natural map

$$l_p^* : Ph(X,Y) \to Ph(X, Y_{(p)})$$

It follows that there is a natural map

$$l : Ph(X,Y) \to \prod_p Ph(X, Y_{(p)})$$

It is well known that l is an epimorphism [15] and $Ker(l)$ is nontrivial iff $Ph(X,Y)$ is nontrivial[7], see also [3],[11] . The phantom map in $Ker(l)$ is called special , following Roitberg[12], see also,[6] where it is called the clone of constant map.

On the other hand , Gray, Le Minh Ha , McGibbon and Strom [2], [5],[9] introduced the notion of Gray index which is defined as follows:

Definition 2.9. Let $f : X \to Y$ be a phantom map. Then f can be factorized as the composition $X \to X/X_k \xrightarrow{\bar{f}} Y$ for each k . The Gray index of f , denoted by $G(f)$, is the largest integer k such that the $\bar{f}$ can be chosen to be a phantom map. $G(f) = \infty$ if no such k exists.

Remark 2.10. Let $f : X \to Y$ be a phantom map. Then f can be lifted to the k-th connected covering for each k and $G(f) + 1$ is the largest integer k such that the kth lifting can be chosen to be a phantom map.

A useful fact we need is

Proposition 2.11. *Let $f : X \to Y$ be a phantom map. Then*
(i)$G(f) \geq n$ if X is n-connected or Y is $n+1$-connected.
(ii) $G(f) \in \{k | H^n(X, \pi_{n+1}(Y) \bigotimes Q) \neq 0\}$ if $H^n(X, \pi_{n+1}(Y) \bigotimes Q) = 0$ for n sufficiently large.

For the proof of the Proposition above, see [5] and [9].

An immediate corollary of the above Proposition which is crucial to our purpose is

Corollary 2.12. *Let* $f : K(Z, 2m) \wedge K(Z, 2m) \to \Omega^1 S^{4m+2}$ *be any essential map. Then* $G(f) = 4m$ *where* $m \geq 1$*.*

Now we are ready to prove the main result.

3. Proof of Theorem1.5

First is a preliminary lemma needed later.

Lemma 3.1. *Let* X *be an H-space which is* $(e-1)$*-connected and* $\pi_i(X) = 0$ *for* $i \geq 2e$ *and* $Y = \Omega^j K$ *where* K *is a* $(d+j-1)$*-connected finite CW-complex with* $j \geq 1$ *and* $d > e \geq 2$*. Let* $\psi : X \to Y$ *be any essential map. Then* Z*(=homotopy fiber of* ψ*) is not an H-space if* $HD(\psi) \circ (i \wedge i)$ *is essential where* $i : Z \to X$ *is the homotopy fiber of* ψ *.*

Proof. If Z is an H-space , then $* = \psi \circ i : Z \to Y$ is an H-map and $HD(*) = 0$. Since i is an H-map by Corollary2.6 , it follows by Proposition2.7 that

$$* = HD(*) = HD(\psi \circ i) = HD(\psi) \circ (i \wedge i)$$

which is in contradiction to the condition. □

Remark 3.2. To apply the above lemma it suffices to discuss when ψ is not an H-map and when $i \wedge i$ induces an injective.

Theorem 3.3. *Let* $X = K(\mathbb{Z}, 2m)$ *and* $Y = \Omega^1 S^{4m+2}$ *with* $m \geq 1$*. Then there is no essential H-map from* X *to* Y*.*

Proof. By Proposition2.8, the rationalization $r : X \to X_{(0)}$ which is an H-map induces an isomorphism of groups

$$r^* : [X_{(0)}, Y] \to [X, Y]$$

It follows from Proposition2.7 that it suffices to prove that there is no essential H-map from $X_{(0)}$ to Y.

On the other hand the map $h : S^{4m+2} \to K(Z, 4m+2)$ which represents a generator of $H^{4m+2}(S^{4m+2}; \mathbb{Z}) \cong \mathbb{Z}$ induces an isomorphism of groups

$$(\Omega^3 h)_* : [X_{(0)}, Y] \to [X_{(0)}, K(\mathbb{Z}, 4m+1)]$$

Again the Proposition2.7 implies that it suffices to prove that there is no essential H-map from $X_{(0)}$ to $K(\mathbb{Z}, 4m+1)$ which is well known to be equivalent to the injectivity of the following homomorphism

$$\theta^* : H^{4m+2}(\Sigma X_{(0)}; \mathbb{Z}) \to H^{4m+2}(\Sigma X_{(0)} \wedge X_{(0)}; \mathbb{Z})$$

where θ is defined as follows:

Let $X * X = X \times I \times X / \{(x, 0, y) \sim (x, 0, y'), (x, 1, y) \sim (x', 1, y)\}$ be the join . There is a well defined map

$$k : X * X \to \Sigma X \wedge X$$

by k[x,t,y]=(x,y,t). It is well known that $X * X$ is homotopy equivalent to $\Sigma X \wedge X$. If X is an H-space with multiplication μ , then θ is the composite map

$$\Sigma X \wedge X \simeq X * X \overset{k}{\to} \Sigma(X \times X) \to \Sigma X$$

where the last map is the map $-\Sigma\pi_1 + \Sigma\mu - \Sigma\pi_2$ and π_1, π_2 are the projection of $X \times X$ to the first and second factors respectively .

Consider the following commutative diagram where the horizontal maps which are isomorphisms come from the universal coefficient Theorem

$$\begin{tikzcd}
H^{4m+2}(\Sigma X_{(0)};\mathbb{Z}) \arrow[r] \arrow[d, "\theta^*"'] & \mathrm{Ext}(H_{4m+1}(\Sigma X_{(0)};\mathbb{Z}),\mathbb{Z}) \arrow[d, "\mathrm{Ext}(\theta_*,\mathbb{Z})"'] \\
H^{4m+2}(\Sigma X_{(0)}\wedge X_{(0)};\mathbb{Z}) \arrow[r] & \mathrm{Ext}(H_{4m+1}(\Sigma X_{(0)}\wedge X_{(0)};\mathbb{Z}),\mathbb{Z})
\end{tikzcd}$$

It follows that it suffices to prove that the map

$$\theta_* : H_{4m+1}(\Sigma X_{(0)}\wedge X_{(0)};\mathbb{Z}) \to H_{4m+1}(\Sigma X_{(0)};\mathbb{Z})$$

or equivalently the map

$$\theta_* : H_{4m}(X\wedge X;\mathbb{Q}) \to H_{4m}(X;\mathbb{Q})$$

is injective . On the other hand, it is well known that the map θ_* is dual to the reduced coproduct which is an isomorphism in this case and thus completes the proof. □

The Theorem1.5 is actually a corollary of the following more general Theorem

Theorem 3.4. *Let $X = K(\mathbb{Z}, 2m)$ and $Y = \Omega S^{4m+2}$ with $m \geq 1$. Let $\psi : X \to Y$ be any essential, special, phantom map. Then Z(=homotopy fiber of ψ) is not an H-space and has the same Mislin genus with $X \times \Omega Y$.*

Proof. That Z and $X \times \Omega Y$ have the same Mislin genus follows from the condition that $\psi : X \to Y$ is a special phantom map.

On the other hand Lemma3.1 and Theorem3.3 apply here. Thus the Theorem above follows from the following Proposition. □

Proposition 3.5. *Let $X = K(\mathbb{Z}, 2m)$ and $Y = \Omega S^{4m+2}$ with $m \geq 1$. Let $\psi : X \to Y$ be any essential, special, phantom map which exists by*

Proposition2.8 and the remark after it. Then $(i \wedge i)^* : [X \wedge X, Y] \to [Z \wedge Z, Y]$ *is injective where* $i : Z \to X$ *is the homotopy fiber of* ψ *.*

Proof. Let $f : X \wedge X \to Y$ be any essential map . If $f \circ (i \wedge i) \sim *$ we will prove that this leads to a contradiction which concludes the proof. Since f is a phantom map , $f \circ (i \wedge i) \sim *$ is also a phantom map.Thus we have the following commutative diagram up to homotopy.

$$\begin{tikzcd}
Z \wedge Z \arrow[r, "i \wedge i"] \arrow[d] & X \wedge X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "id"'] \\
Z_{(0)} \wedge Z_{(0)} \arrow[r, "i_{(0)} \wedge i_{(0)}"] & X_{(0)} \wedge X_{(0)} \arrow[r, "\tilde{f}"] & Y
\end{tikzcd}$$

If $f \circ (i \wedge i) \sim *$, then $\tilde{f} \circ (i_{(0)} \wedge i_{(0)})$ is the composite $Z_{(0)} \wedge Z_{(0)} \to \Sigma Z_\tau \wedge Z_\tau \xrightarrow{h} Y$ where Z_τ is the homotopy fiber of the rationalization $X \to X_{(0)}$. On the other hand we claim that

Claim 3.6. *Any map* $h : \Sigma Z_\tau \wedge Z_\tau \to Y$ *factors through a map* $\Sigma Z_\tau \wedge Z_\tau \to \Sigma F_\tau \wedge F_\tau$ *where* $F = \Omega^2 S^{4m+2}$.

Assuming this , note that $i_{(0)} \wedge i_{(0)}$ admits a right inverse, we have that f is the composite $X \wedge X \to X_{(0)} \wedge X_{(0)} \to \Sigma F_\tau \wedge F_\tau \to Y$.

It is easy to know that ΣF_τ is $4m-1$-connected and thus $\Sigma F_\tau \wedge F_\tau$ is $8m-2$-connected. It follows that f is the composite $X \wedge X \to X_{(0)} \wedge X_{(0)} \to Y < 8m-2 > \to Y$. By Remark2.10, $G(f) \geq 8m-3$ which contradicts Corollary2.12. □

Remark 3.7. Roitberg has shown us how the use of Gray index can be avoided.

It remains to prove the Claim3.6 which follows from the following

Lemma 3.8. *There is a map* $\Sigma g \wedge g : \Sigma Z_\tau \wedge Z_\tau \to \Sigma F_\tau \wedge F_\tau$ *such that the following map is a weak homotopy equivalence*

$$(\Sigma g \wedge g)^* : map_*(\Sigma F_\tau \wedge F_\tau, Y) \to map_*(\Sigma Z_\tau \wedge Z_\tau, Y)$$

Proof. Roitberg and P. Touhey proved in [13] that , if X, Y have the same Mislin genus, then $X_\tau \simeq Y_\tau$. So we have a map g which is a composite

$$Z_\tau \simeq (X \times F)_\tau \xrightarrow{\pi_\tau} F_\tau$$

where $\pi : X \times F \to F$ is the projection.

To prove $\Sigma g \wedge g$ induces a homotopy equivalence it suffices to prove that $\pi_\tau : (X \times F)_\tau \to (F)_\tau$ induces a homotopy equivalence

$$(\Sigma\pi_\tau \wedge \pi_\tau)^* : map_*(\Sigma F_\tau \wedge F_\tau, Y) \to map_*(\Sigma(X \times F)_\tau \wedge (X \times F)_\tau, Y)$$

which follows directly from the fact that $map_*(X_\tau, Y)$ is weakly contractible and the fact

$$map_*(\Sigma X_\tau, Y) \simeq map_*(\Sigma X, \hat{Y})$$

which can be found in [16] , for a stronger result , see Pan and Woo [10]. $\square$

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