

EQUIVARIANT PHANTOM MAPS

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ABSTRACT. A successful generalization of phantom map theory to the equivariant case for all compact Lie groups is obtained in this paper. One of the key observations is the discovery of the fact that homotopy fiber of equivariant completion splits as product of equivariant Eilenberg-MacLane spaces which seems impossible at first sight by the example of Triantafyllou[19].

1. INTRODUCTION

A map $f : X \rightarrow Y$ from a CW complex X is called a phantom map if the restriction of f to the skeleton X_n , $f|X_n$, is trivial for each n . After the discovery of the first example of phantom map by Adams and Walker[1], theory of phantom map receives a lot of attention. Among the many studies on phantom maps the most successful study is those conducted by Zabrodsky, McGibbon and Roitberg, see [8] for the references. Beside the study on the phantom map itself, remarkable applications of theory of phantom maps was given by Harper and Roitberg[3],[15] who applied it to compute $SNT(X)$ and $Aut(X)$, by Roitberg[16] and Pan[12] where several conjectures of McGibbon were settled, by Pan and Woo[13] where a deep relation between Tsukiyama problem about self homotopy equivalence and a generalization of phantom map was found.

In the category of G -spaces with base point, a map $f : X \rightarrow Y$ from a G -CW complex X is called a phantom map if the restriction of f to the skeleton X_n , $f|X_n$, is trivial for each n where X_n is the

Date: Mar. 15, 2001.

1991 Mathematics Subject Classification. 55P91,55P60.

Key words and phrases. Phantom map, equivariant homotopy.

The author is partially supported by ZD9603 of Chinese Academy of Science and the NSFC projects 19701032, 10071087.

G - n -skeleton X . Generalization of theory of phantom maps to the equivariant case was first given by Oda and Shitanda[10, 11] which is successful only when the group is cyclic. The trouble one meets when following the standard approach to the theory of phantom map is the fact that the argument depends strongly on the following well known result in the nonequivariant homotopy theory.

Theorem 1.1. *Let X be a CW complex of finite type which is also a rational H -space. Then X is rationally equivalent to a product of Eilenberg-MacLane spaces*

The corresponding result in equivariant case was obtained for finite cyclic group by Triantafyllou[19] and counterexample for the above result in case the group is a product of two cyclic groups in the same paper as cited above.

It is thus easy to understand why Oda and Shitanda's equivariant generalization is successful only in case the group is cyclic.

A key observation of this paper is the following which makes it possible the equivariant generalization of an alternative approach to the theory of phantom maps.

Theorem 1.2. *Let Y be a G -CW complex of finite type and $c : Y \rightarrow \hat{Y}$ be the equivariant completion of Y . Then Y_ρ is homotopy equivalent to a product of equivariant Eilenberg-MacLane spaces where Y_ρ is the homotopy fiber of c .*

The organization of this paper is as follows.

In section 2 definition of equivariant phantom map is recalled and several simple but needed later results are stated without proof since their proof is parallel to that of their nonequivariant analogue.

In section 3 we will study equivariant localization and completion where the key Theorem1.2 among others will be proved.

In section4 The Theorem1.2 will be applied to compute the set of homotopy classes of phantom maps from a G -CW complex X to another G -space Y following a recent approach to the computation in the nonequivariant case given by Roitberg and Touhey[17]. The paper ends with computation of equivariant phantom maps between certain

pairs of spaces. In concluding the Introduction , we 'd like to give the following

Question 1.3. *Is the condition that Y is of finite type essential?*

Remark 1.4. 1. The condition that Y is of finite type is indeed essential to our argument in proving Theorem1.2.

2. All the arguments in this paper generalizes directly to the case of phantom pair as defined in [11].

In this paper, G will denote a compact Lie group. For simplicity we will assume throughout the paper that all G -spaces are based G -1-connected although almost all results in this paper remains true if we assume all G -spaces are nilpotent , the target is G -simple and all fixed point set of the domain have finite fundamental groups. $[X, Y]$ will be the set of based nonequivariant homotopy classes of maps while $[X, Y]_G$ will be he set of based equivariant homotopy classes of equivariant maps when both X and Y are G -spaces. $map_*^G(X, Y)$ will be the space of based equivariant maps between G -spaces X and Y .

2. PHANTOM MAPS:PRELIMINARIES

Let's begin with definition.

Definition 2.1. Let X be a G -CW complex, Y be a G -space . Then a map $f : X \rightarrow Y$ is called an G -phantom map if $f|X_n = 0$ for all $n \geq 0$. Denoted by

$$Ph^G(X, Y) = \{f : X \rightarrow Y | f \text{ is an } G\text{-phantom map}\}$$

An equivariant analogue of Gray's description of $Ph(X, Y)$ is true[11]

$$Ph^G(X, Y) = \lim_{\rightarrow n}^1 [\Sigma X_n, Y]_G$$

which follows from the following well known exact sequence

$$* \rightarrow \lim_{\rightarrow n}^1 [\Sigma X_n, Y]_G \rightarrow [X, Y]_G \rightarrow \lim_{\rightarrow n} [X_n, Y]_G \rightarrow *$$

By the above formula we get the following according to Proposition 4.3 in [8] .

Lemma 2.2. [11] *Let X and Y be G -nilpotent G -CW complexes of finite type. Then*

$$Ph^G(X, \hat{Y}) = *$$

Another result which we need is the following whose nonequivariant analogue is well known.

Proposition 2.3. *Let X be a G -CW complex and Y be a G -space of finite type. Then*

$$Ph^G(X, Y_0) = *$$

Actually this is an easy corollary of the following whose proof is exactly the same

Theorem 2.4. [18] *Let X be a G -CW complex and Y be a G -space of finite type. Let $\{X_\alpha\}$ be the set of all finite subcomplexes of X . Then*

$$[X, Y_0]_G \cong \lim_{\leftarrow \alpha} [X_\alpha, Y_0]_G$$

3. EQUIVARIANT LOCALIZATION AND COMPLETION

Let X be a G -nilpotent . Then the equivariant localization and completion are defined in [6],[7]. Denote by $l : X \rightarrow X_0$ the equivariant rationalization and $c : X \rightarrow \hat{X}$ the equivariant completion. Let Y_ρ be the homotopy fiber of c and X_τ be the homotopy fiber of l . Then it is easy to show that $X_\tau \rightarrow X \rightarrow X_0$ is a cofibration. Now we can state first of our results concerning localization and completion which is well known in nonequivariant case.

Theorem 3.1. *Let Y be G -CW complex of finite type. Then*

- $\pi_n(\hat{Y}^H) = \prod_p Ext(\mathbb{Z}_p^\infty, \pi_n(Y^H))$ for all closed subgroups H of G .
- For W a finite CW complex, $c_* : [W, Y]_G \rightarrow [W, \hat{Y}]_G$ is injective

Proof. The first part can be found in Theorem 14 of [7].

The second part can be proved by an argument similar to that of Theorem 2.5.3 of [4]. \square

An immediate consequence of the above Theorem is

Corollary 3.2. *Let Y be a G -CW complex of finite type. Then $j : Y_\rho \rightarrow Y$ is a G -phantom map*

Proposition 3.3. *Let X be G -CW complex and Y be G -space . Then the followings hold:*

- $[\Sigma^n X_{(0)}, \hat{Y}]_G = *$ for all $n \geq 0$
- $[\Sigma^n X_\tau, Y_\rho]_G = *$ for all $n \geq 0$

For a proof see that of Proposition 2.3 in nonequivariant case in [14].

Proposition 3.4. *Let X be G -CW complex and Y be G -space. Then the followings hold :*

- $\tau^* : [X, \hat{Y}]_G \simeq [X_\tau, \hat{Y}]_G$
- $\rho_* : [X_{(0)}, Y_\rho]_G \simeq [X_{(0)}, Y]_G$
- $c_* : [X_\tau, Y]_G \simeq [X_\tau, \hat{Y}]_G$
- $l^* : [X_{(0)}, Y_\rho]_G \simeq [X, Y_\rho]_G$

Proof. First part of the theorem follows from Proposition 1,6 and Theorem 6 of [7] as noted by Oda and Shitanda[11].

The second and the fourth parts follows from long homotopy exact sequence.

For the proof of third part, it follows from the following exact sequence that it suffices to prove the third part for each X_n

$$* \rightarrow \lim_{\rightarrow n}^1 [\Sigma(X_n)_\tau, Y]_G \rightarrow [X_\tau, Y]_G \rightarrow \lim_{\rightarrow n} [(X_n)_\tau, Y]_G \rightarrow *$$

The above sequence is exact since an easy argument shows that there is a cofibration for each n

$$\bigvee G/H \times (S^n)_\tau \rightarrow (X_n)_\tau \rightarrow (X_{n+1})_\tau$$

and X_τ is the direct limit of $(X_n)_\tau$.

An induction argument using above cofibration reduces the proof of third part to that of a nonequivariant analogue of it which is of course true.

□

Now we proceed to prove Theorem1.2. Let us record the theorem again

Theorem 3.5. *Let Y be a G -CW complex of finite type and $c : Y \rightarrow \hat{Y}$ be the equivariant completion of Y . Then Y_ρ is homotopy equivalent to a product of equivariant Eilenberg-MacLane spaces where Y_ρ is the homotopy fiber of c .*

Proof. To begin the proof, consider the following commutative diagram following Roitberg and Touhey[17]

$$\begin{array}{ccccccccc}
\Omega Y & \xrightarrow{\quad} & \Omega \hat{Y} & \xrightarrow{\quad} & Y_\rho & \xrightarrow{\quad} & Y & \xrightarrow{\quad} & \hat{Y} \\
\downarrow & & \downarrow & & \downarrow \parallel & & \downarrow r & & \downarrow \\
\Omega Y_0 & \longrightarrow & \Omega(\hat{Y})_0 & \xrightarrow{\quad} & Y_\rho & \xrightarrow{\quad} & Y_0 & \longrightarrow & (\hat{Y})_0
\end{array}$$

The right hand square is the pull-back by the G -Arithmetic Square Theorem as in [5] and horizontal sequences are easy to see to be fibration sequences.

Now Corollary 3.2 implies that $j : Y_\rho \rightarrow Y$ is a G -phantom and thus the composite $j' = l \circ j : Y_\rho \rightarrow Y_0$ is G -phantom. Then Proposition 2.3 shows that $j' \simeq *$ and thus $d' : \Omega(\hat{Y})_0 \rightarrow Y_\rho$ has a right G -homotopy inverse. It follows that Y_ρ is an equivariant H-space. As in nonequivariant case Y_ρ is rational. The proof completes if we prove that the equivariant k -invariants of Y_ρ are all trivial.

To prove it, note first that the above result is also true for $Y^{(n)}$ where $Y^{(n)}$ is the equivariant n -th Postnikov section of G -space Y [2]. Thus $(d')^{(n)} : \Omega(Y^{(n)})_0 \rightarrow (Y^{(n)})_\rho$ has a right G -homotopy inverse.

let's consider the following commutative diagram

$$\begin{array}{ccccc}
\Omega Y^{(n+2)} & \xrightarrow{\quad} & \Omega \hat{Y}^{(n+2)} & \xrightarrow{\quad} & (Y^{(n+2)})_\rho \\
\downarrow k_1 & & \downarrow k_2 & & \downarrow k_3 \\
K(\pi_1, n+2) & \xrightarrow{\quad} & K(\pi_2, n+2) & \xrightarrow{\quad} & K(\pi_3, n+2)
\end{array}$$

where $\pi_1 = \pi_{n+1}(\Omega Y^{(n+2)})$, $\pi_2 = \pi_{n+1}(\Omega \hat{Y}^{(n+2)})$ and $\pi_3 = \pi_{n+1}((Y^{(n+2)})_\rho)$. The fact that $\Omega c^{(n)}$ is completion and the above commutative diagram imply that $k_3 \circ d^{(n+2)}$ is trivial. Since $\pi_{n+1}((Y^{(n+2)})_\rho)$ is rational, it follows that $k_3 \circ d'^{(n+2)}$ is trivial. On the other hand $d'^{(n+2)}$ has a right

G -homotopy inverse . It follows that k_3 is trivial which completes the proof. □

4. PHANTOM MAPS: LOCALIZATION AND COMPLETION APPROACH

The most fundamental result in the theory of phantom map is the following

Theorem 4.1. *Let X be a G -CW complex of finite type , Y be 1-connected G -space of finite type. Then the followings are equivalent:*

- $f : X \rightarrow Y$ is a G -phantom map
- the composite $X \rightarrow Y \rightarrow \hat{Y}$ is trivial
- $f \simeq l \circ \tilde{f}$ where $\tilde{f} : X_{(0)} \rightarrow Y$

Proof. The proof follows easily from Proposition 2.2, Theorem 3.1 and 3.4 as in [11]. □

Until now our argument follows closely the standard approach to the theory of phantom map as developed by Zabrodsky[20] and this was what Oda and Shitanda would have been done. After that we will face the computation problem. The standard approach suggests to take $[X_{(0)}, Y]_G$ as an upper bound for $Ph^G(X, Y)$ and compute it. Then the standard approach runs into trouble since the equivariant analogue of Theorem 1.1 is not true. This is the reason why $Ph^G(X, Y)$ is computed only in case G is a finite cyclic group. Fortunately we have an alternative approach to the computation first given by Roitberg and Touhey [17] . In their approach they used the completion fibration. Now Theorem 1.2 enables us to follow their approach. Thus we have the following whose proof is parallel to that of Theorem 1.1 of [17] and will be omitted.

Theorem 4.2. *Let X be a G -CW complex of finite type , Y be 1-connected G -space of finite type. Then*

$$Ph^G(X, Y) \approx [X, Y_\rho]_G / [X, \Omega \hat{Y}]_G$$

and

$$[X, Y_\rho]_G = \prod_{n \geq 1} H_G^n(X; \pi_{n+1}(Y) \otimes \mathbb{R})$$

where \mathbb{R} is a vector space over \mathbb{Q} of uncountable dimension.

Furthermore if X is a G -co- H -space or Y is a G - H -space, the above set theoretic bijections are group isomorphisms. Moreover the group structure on $Ph^G(X, Y)$ is abelian, divisible and independent of the G -co- H -structure on X or the G - H -structure on Y .

Now it's time to discuss when all maps between two spaces are phantom.

As noted by Oda and Shitanda [10], The following Theorem is especially useful in applying nonequivariant results to equivariant case.

Theorem 4.3. [2] *Let X be a G -CW complex and Y be a G -space. If $[\Sigma^n X^K, Y^H] = *$ for all closed subgroups K, H of G . Then $[\Sigma^n X, Y]_G = *$ for all $n \geq 0$.*

As a corollary we have the following equivariant Miller-Zabrodsky Theorem as noted by Oda and Shitanda [10]

Theorem 4.4. *Let X be a G -CW complex with the following property:*

*There exists an integer n such that, for all closed subgroups H of G , $\pi_i(X^H) = *$ for all $i \geq n$ and $\pi_i(X^H)$ is locally finite for all $i < n$.*

Then $map_^G(X, \hat{Y})$ is weakly contractible and thus*

$$Ph^G(X, Y) = [X, Y]_G$$

for all finite dimensional G -CW complex Y .

Combined with Theorem 4.2, the above Theorem implies the following

Theorem 4.5. *Let X be a G -CW complex of finite type and Y be a finite G -CW complex. If there exists an integer n such that $\pi_i(X^H) = *$ for all $i \geq n$ and all closed subgroups H of G , then*

$$Ph^G(X, Y) = \prod_{n \geq 1} H_G^n(X; \pi_{n+1}(Y) \otimes \mathbb{R})$$

where \mathbb{R} is a vector space over \mathbb{Q} of uncountable dimension.

Remark 4.6. Equivariant Postnikov section of any G -CW complex satisfies the condition for X in the above theorem. In particular any Equivariant Eilenberg-MacLane space can be taken as X in the above theorem. This provides us with a large class of computable phantom maps.

Theorem 4.3 combined with results in [9] yields another class of phantom maps

Theorem 4.7. *Let K be a G -CW complex and Y be a finite G -CW complex. Let $X = \mathbf{Q}K$ where $\mathbf{Q}K = \lim^n \Omega^n \Sigma^n K$. Then $\text{map}_*^G(X, \hat{Y})$ is weakly contractible and thus*

$$\text{Ph}^G(X, Y) = [X, Y]_G$$

Remark 4.8. It follows from above theorem that the conclusion of Theorem 4.5 is also true.

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