RATIONAL HOMOTOPY THEORY AND NONNEGATIVE CURVATURE

JIANZHONG PAN

ABSTRACT. In this note , we answer positively a question by Belegradek and Kapovitch[2] about the relation between rational homotopy theory and a problem in Riemannian geometry which asks that total spaces of which vector bundles over compact nonnegative curved manifolds admit (complete) metrics with nonnegative curvature.

1. Introduction

Given a Riemannian manifold M with metric

$$<>: TM \times TM \to TM$$

an affine connection is a bilinear map

$$\nabla : Vec(M) \times Vec(M) \to Vec(M)$$

which satisfies the following

- $\nabla_{fV} W = f \nabla_V W$
- $\nabla_V (fW) = (Vf)W + f \nabla_V W$

where $f \in C^\infty(M), V, W \in Vec(M)$

An affine connection is called Levi-Civita connection if it satisfies also the following

- $X < V, W >=< \nabla_X V, W > + < V, \nabla_X W >$
- $\nabla_V W - \nabla_W V - [V, W] = 0$

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where $[V, W]f = (XY - YX)f$ is the Lie bracket.

A fundamental result in Riemannian geometry asserts that

Theorem 1.1. *For each Riemannian metric, there exists a unique Levi-Civita connection.*

Given a Riemannian manifold M with Levi-Civita connection , there is defined a curvature operator

$$R : Vec(M) \times Vec(M) \times Vec(M) \to Vec(M)$$

defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

From it one arrives at an important geometric invariant which is called *Sectional curvature* defined by

$$K(\sigma) = \frac{< R(v, w)w, v >}{< v \wedge w, v \wedge w >}$$

where $\sigma \subset T_p M$ is a tangent plane at $p \in M$ and $v, w \in \sigma$ span it. It is well known that $K(\sigma)$ does not depend on the choice of spanning vectors.

A well known question in Riemannian geometry is

Question 1.2. *Does the restriction on curvature imply the restriction on topology and vice versa?*

In particular, how does the positive(nonnegative) curvature restrict the topology of the underlining manifold?

A Riemannian manifold is called positively (or nonnegatively) curved if, for any σ, $K(\sigma) > 0$ (or $K(\sigma) \geq 0$).

For compact manifold, we have the following classical

Theorem 1.3. *Let M be a compact Riemannian manifold with positive curvature. Then*

$$\pi_1(M) = \begin{cases} \text{finite group} & \text{if } \dim M \text{ is odd} \\ 0 & \text{if } \dim M \text{ is even and } M \text{ is orientable} \\ Z_2 & \text{if } \dim M \text{ is even and } M \text{ is nonorientable} \end{cases}$$

The main concern of this note is on noncompact manifold. In this case there is the following

Theorem 1.4. *Let M be a complete noncompact Riemannian manifold with nonnegative curvature. Then M is diffeomorphic to the total space of the normal bundle of a compact totally geodesic submanifold which is called the soul.*

Another central question in Riemannian geometry is to what extent the converse is true, or in other words

Question 1.5. *Total spaces of which vector bundles over compact nonnegatively curved manifolds admit (complete) metrics with nonnegative curvature?*

Previously, obstructions to the existence of nonnegatively curved metrics on vector bundles were only known for a flat soul [7]. No obstructions are known when the soul is simply-connected. In [2] an approach to the reduction of the problem to the vector bundle over simply connected manifold was initiated. The start point is another result of Cheeger and Gromoll [4] that a finite cover of any closed nonnegatively curved manifold is diffeomorphic to a product of a torus and a simply-connected closed nonnegatively curved manifold. It turns out that a similar statement holds for open complete nonnegatively curved manifolds which is the basis of their analysis.

Lemma 1.6. [2] *Let (N, g) be a complete nonnegatively curved manifold. Then there exists a finite cover N' of N diffeomorphic to a product $M \times T^k$ where M is a complete open simply connected nonnegatively curved manifold. Moreover, if S' is a soul of N', then this diffeomorphism can be chosen in such a way that it takes S' onto $C \times T^k$ where C is a soul of M.*

By using this and characteristic classes technique, they proved that, in various case, the total spaces of rank k vector bundles over $C \times T$ admit no nonnegatively curved metric if they do not become the pullback of a bundle over C in a finite cover. The following is such an example

Corollary 1.7. [1] *Let B be a closed nonnegatively curved manifold. If $\pi_1(B)$ contains a free abelian subgroup of rank four (two, respectively),* *then for each $k \geq 2$ (for $k = 2$, respectively) there exists a finite cover of B over which there exist infinitely many rank k vector bundles whose total spaces admit no nonnegatively curved metrics.*

Belegradek and Kapovitch [2] are thus lead to the following

Definition 1.8. Given a closed smooth simply connected manifold C, a torus T, and a positive integer k, we say that a triple (C, T, k) is *splitting rigid* if any rank k vector bundle over $C \times T$ with nonnegatively curved total space splits, after passing to a finite cover, as the product of a rank k bundle over C and a rank zero bundle over T.

Let $\mathcal{H}$ be the class of simply-connected CW-complexes whose rational cohomology algebra is finite dimensional, as a rational vector space, and has no nonzero derivations of negative degree(a homomorphism $f : A \to A$ between graded algebras over rational is said to be derivations of negative degree k if $f(uv) = f(u)v + (-1)^{kp} u f(v)$ where $u \in A, v \in A^p$), see [2] for the reason to choose such a class $\mathcal{H}$. For example, $\mathcal{H}$ contains any compact simply-connected Kähler manifold [6].

A natural question is

Question 1.9. [2] *Let $C \in \mathcal{H}$ be a closed smooth manifold. Is (C, T, k) splitting rigid for any T and k?*

The main result in this note is a positive answer to this question

Theorem 1.10. *Let $C \in \mathcal{H}$ be a closed smooth manifold. Then (C, T, k) is splitting rigid for any T and k.*

In this paper, all cohomology groups have rational coefficients, all manifolds and vector bundles are smooth; all topological spaces are homotopy equivalent to connected CW-complexes. $[X, Y]$ will be the based homotopy classes of based maps between them. $map(X, Y)$ is the space of maps from X to Y and $map(X, Y)_f$ is the connected component of $map(X, Y)$ which contains the map $f : X \to Y$.

2. A splitting criterion

Given a finite cell complex C, define $Char(k, C)$ to be the subspace of $H^*(C)$ which is the direct sum of $\oplus_{i=1}^{[(k-1)/2]} H^{4i}(C)$ and the subspace equal to $H^k(C)$ if k is even, and to $H^{4[k/2]}(C)$ if k is odd. Note that any rational characteristic class of a rank k vector bundle over C lies in the subalgebra of $H^*(C)$ generated by $Char(k, C)$.

Belegradek and Kapovitch transform the problem of a triple (C, T, k) being splitting rigid into a homotopy problem as follows

Proposition 2.1. [2] *Let C be a closed simply-connected manifold, T be a torus, k be a positive integer. If any self-homotopy equivalence of $C \times T$ maps $Char(k, C)$ to itself, then the triple (C, T, k) is splitting rigid.*

We are thus led to compute the group of homotopy classes of self homotopy equivalences $Aut(C \times T)$ of $C \times T$.

Before we can do this , let's recall a work by Booth and Heath [3]

Given spaces with base point (X, x_0) and (Y, y_0), there is a natural map

$$\varphi : map(X \times Y, X \times Y) \to map(X, X) \times map(Y, Y)$$

defined by $\varphi(f) = (g, h)$ where $g(x) = \pi_X \circ f(x, y_0)$, $h(y) = \pi_Y \circ f(x_0, y)$ and $\pi_X : X \times X, \pi_Y : X \times Y \to Y$ are projections to the factors X and Y respectively.

Definition 2.2. Let X and Y be two spaces . We say X and Y have the induced equivalence property(IEP) if whenever f is a homotopy equivalence , then g, h defined above are homotopy equivalences.

Remark 2.3. Let X and Y be such that for each $i > 0$, at least one of $\pi_i(X)$ and $\pi_i(Y)$ is zero. Then they satisfy the IEP by Whitehead theorem.

With the above notion, we can quote the following

Theorem 2.4. *Let X and Y be two spaces having IEP. Suppose further that $[X, map(Y, Y)_{id}] = 0$, then there is a short exact sequence of groups and homomorphisms*

$$1 \to [Y, map(X, X)_{id}] \xrightarrow{\theta} Aut(X \times Y) \to Aut(X) \times Aut(Y) \to 1$$

which splits by a homomorphism $\sigma : Aut(X) \times Aut(Y) \to Aut(X \times Y)$ given by $\sigma(g, h) = g \times h$

Let $X = C$ and $Y = T$ where C, T be as in Theorem1.10. Then X and Y have IEP by the remark following the definition2.2. On the other hand, $[C, map(T,T)_{id}] = 0$ since it is well known that $map(T,T)_{id} = T$ and C is 1-connected and thus first cohomology of C is trivial.

Now given $f \in Aut(C \times T)$, to prove that the induced homomorphism in cohomology maps $Char(k, C)$ to itself , it suffices to assume that $f \in Im([T, map(C,C)_{id}])$ by the exact sequence above. Recall that the map $\theta : [T, map(C,C)_{id}] \to Aut(C \times T)$ is given by $\theta(f)(x, y) = (f(y)(x), y)$. The above argument gives the following

Corollary 2.5. *Let C be a closed simply-connected manifold, T be a torus, k be a positive integer. Then the triple (C, T, k) is splitting rigid if, for any map $f : T \to map(C,C)_{id}$, the adjoint $\tilde{f} : T \times C \to C$ induces a homomorphism in cohomology given by $\tilde{f}^*(u) = 1 \otimes u$ for any $u \in H^*(C)$.*

3. The proof of Theorem1.10

Proof of Theorem1.10. Let $T = (S^1)^s$. By Corollary 2.5, to prove Theorem1.10 , it suffices to prove that , for map $f : T \times C \to C$ such that $f(y_0, -)$ homotopic to id, it induces a homomorphism in cohomology given by $f^*(u) = 1 \otimes u$. Given such f, for any $u \in H^*(C)$,

$$f^*(u) = 1 \otimes u + \sum_k \sum_{i_1 \cdots i_k} \lambda_{i_1 \cdots i_k}(u) \otimes \iota_{i_1} \cdots \iota_{i_k}$$

where the first sum is taken over k from 1 to s and the second sum is taken over all $(i_1 \cdots i_k)'s$ such that $0 < i_1 < \cdots < i_k < s+1$. Thus we get a sequence of maps $\lambda_{i_1 \cdots i_k} : H^n(C) \to H^{n-k}(C)$ where $0 < k < s+1$ and $0 < i_1 < \cdots < i_k < s+1$.

To prove that $f^*(u) = 1 \otimes u$, it suffices to prove that $\lambda_{i_1 \cdots i_k} = 0$ where $0 < k < s+1$ and $0 < i_1 < \cdots < i_k < s+1$ while for the proof of later we need to study the behaviour of these maps with respect to the cup product of cohomology.

If $u, v \in H^*(C)$, then

$$f^*(uv) = 1 \otimes uv + \sum_k \sum_{i_1 \cdots i_k} \lambda_{i_1 \cdots i_k}(uv) \otimes \iota_{i_1} \cdots \iota_{i_k}$$

On the other hand $f^*(uv) = f^*(u)f^*(v)$. Using the formula for $f^*(uv), f^*(u), f^*(v)$ and comparing the terms associated with $\iota_{i_1} \cdots \iota_{i_k}$, we find the following equations

$$\lambda_{i_1\cdots i_k}(uv) = \lambda_{i_1\cdots i_k}(u)v+(-1)^{k|v|}u\lambda_{i_1\cdots i_k}(v)\pm\sum \lambda_{j_1\cdots j_p}(u)\lambda_{l_1\cdots l_q}(v)\otimes\iota_{i_1}\cdots\iota_{i_k}$$

where $p+q=k$ with $p>0, q>0$ and the sum is taken over all partitions of $i_1, \cdots, i_k$ into $j_1 < \cdots < j_p$ and $l_1 < \cdots < j_q$.

Let $k=1$. Then then above formula implies that λ_{i_1} is a derivation of degree -1 which is trivial by the condition of the Theorem1.10. The above formula in case $k=2$ implies that $\lambda_{i_1 i_2}$ is a derivation of degree -2 modulo products of derivations of degree -1 which are trivial. It follows that $\lambda_{i_1 i_2}$ is a derivation of degree -2 and thus is trivial by the condition of the Theorem1.10. Inductively we can prove that all $\lambda_{i_1\cdots i_k}$ are trivial which completes the proof of the Theorem1.10.

□

References

[1] I. Belegradek and V. Kapovitch, *Topological obstructions to nonnegative curvature*, Math.DG/0001125.

[2] I. Belegradek and V. Kapovitch, *Obstructions to nonnegative curvature and rational homotopy theory*, Math.DG/0007007.

[3] P.I. Booth and P.R. Heath, *On the groups $E(X \times Y)$ and $E_B^B(X \times_B Y)$*. (English) Groups of self-equivalences and related topics, Proc. Conf., Montreal/Can. 1988, Lect. Notes Math. 1425, 17-31 (1990).

[4] J. Cheeger and D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. **96** (1972), 413–443.

[5] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geom **6** (1971/72), 119–128.

[6] W. Meier, *Some topological properties of Kähler manifolds and homogeneous spaces*, Math. Z. **183** (1983), no. 4, 473–481.

[7] M. Özaydin and G. Walschap, *Vector bundles with no soul*, Proc. Amer. Math. Soc. **120** (1994), no. 2, 565–567.

Institute of Math.,Academia Sinica ,Beijing 100080, China

E-mail address: pjz@math03.math.ac.cn