

On the relation of Voevodsky's algebraic cobordism to Quillen's K -theory

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Abstract

Quillen's algebraic K -theory is reconstructed via Voevodsky's algebraic cobordism. More precisely, for a ground field k the algebraic cobordism \mathbf{P}^1 -spectrum MGL of Voevodsky is considered as a commutative \mathbf{P}^1 -ring spectrum. Setting $\mathrm{MGL}^i = \bigoplus_{2q-p=i} \mathrm{MGL}^{p,q}$ we regard the bigraded theory $\mathrm{MGL}^{p,q}$ as just a graded theory. There is a unique ring morphism $\phi: \mathrm{MGL}^0(k) \rightarrow \mathbb{Z}$ which sends the class $[X]_{\mathrm{MGL}}$ of a smooth projective k -variety X to the Euler characteristic $\chi(X, \mathcal{O}_X)$ of the structure sheaf \mathcal{O}_X . Our main result states that there is a canonical grade preserving isomorphism of ring cohomology theories on the category $\mathrm{Sm}\mathcal{O}p/k$

$$\varphi: \mathrm{MGL}^*(X, U) \otimes_{\mathrm{MGL}^0(k)} \mathbb{Z} \rightarrow K_{-*}^{TT}(X, U) = K'_{-*}(X - U),$$

in the sense of [PS1], where K_*^{TT} is the Thomason-Trobaugh K -theory and K'_* is Quillen's K' -theory. In particular, the left hand side is a ring cohomology theory. Moreover both theories are oriented in the sense of [PS1] and φ respects the orientations. The result is an algebraic version of a theorem due to Conner and Floyd. That theorem reconstructs complex K -theory via complex cobordism [CF].

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1 A motivic version of a theorem by Conner and Floyd

Our main result relates Voevodsky’s algebraic cobordism theory $\mathrm{MGL}^{*,*}$ to Quillen’s K' -theory. We refer to [PPR1, Appendix] for the basic terminology, notation, constructions, definitions, results. Let S be a Noetherian separated finite-dimensional scheme S . One may think of S being the spectrum of a field or the integers. A *motivic space over S* is a functor

$$A: \mathcal{S}m\mathcal{O}p/S \rightarrow \mathbf{sSet}$$

(see [PPR1, Appendix]). The category of motivic spaces over S is denoted $\mathbf{M}(S)$. This definition of a motivic space is different from the one considered by Morel and Voevodsky in [MV] – they consider only those simplicial presheaves which are sheaves in the Nisnevich topology on $\mathcal{S}m/S$. With our definition the Thomason-Trobaugh K -theory functor obtained by using big vector bundles is a motivic space on the nose. It is not a simplicial Nisnevich sheaf. This is why we prefer to work with the above notion of “space”.

We write $\mathbf{H}_\bullet^{\mathrm{cm}}(S)$ for the pointed motivic homotopy category and $\mathrm{SH}^{\mathrm{cm}}(S)$ for the stable motivic homotopy category over S as constructed in [PPR1, A.3.9, A.5.6]. By [PPR1, A.3.11 resp. A.5.6] there are canonical equivalences to $\mathbf{H}_\bullet(S)$ of [MV] resp. $\mathrm{SH}(S)$ of [V1]. Both $\mathbf{H}_\bullet^{\mathrm{cm}}(S)$ and $\mathrm{SH}^{\mathrm{cm}}(S)$ are equipped with closed symmetric monoidal structures such that the \mathbf{P}^1 -suspension spectrum functor is a strict symmetric monoidal functor

$$\Sigma_{\mathbf{P}^1}^\infty: \mathbf{H}_\bullet^{\mathrm{cm}}(S) \rightarrow \mathrm{SH}^{\mathrm{cm}}(S).$$

Here \mathbf{P}^1 is considered as a motivic space pointed by $\infty \in \mathbf{P}^1$. The symmetric monoidal structure $(\wedge, \mathbb{I}_S = \Sigma_{\mathbf{P}^1}^\infty S_+)$ on the homotopy category $\mathrm{SH}^{\mathrm{cm}}(S)$ is constructed on the model category level by employing the category $\mathbf{MSS}(S)$ of symmetric \mathbf{P}^1 -spectra. This symmetric monoidal category satisfies the properties required by Theorem 5.6 of Voevodsky congress talk [V1]. From now on we will usually omit the superscript $(-)^{\mathrm{cm}}$.

Every \mathbf{P}^1 -spectrum E represents a cohomology theory on the category of pointed motivic spaces. Namely, for a pointed motivic space (A, a) set $E^{p,q}(A, a) = \mathrm{Hom}_{\mathbf{H}_\bullet^{\mathrm{cm}}(S)}(\Sigma_{\mathbf{P}^1}^\infty(A, a), \Sigma^{p,q}(E))$ and $E^{*,*}(A, a) = \bigoplus_{p,q} E^{p,q}(A, a)$. This definition extends to motivic spaces via the functor $A \mapsto A_+$ which adds a disjoint basepoint. That is, for a non-pointed motivic space A set $E^{p,q}(A) = E^{p,q}(A_+, +)$ and $E^{*,*}(A) = \bigoplus_{p,q} E^{p,q}(A)$.

Every $X \in \mathcal{S}m/S$ defines a representable motivic space constant in the simplicial direction taking an S -smooth scheme U to $\mathrm{Hom}_{\mathcal{S}m/S}(U, X)$. It is

not possible in general to choose a basepoint for representable motivic spaces. So we regard S -smooth varieties as motivic spaces (non-pointed) and set

$$E^{p,q}(X) = E^{p,q}(X_+, +).$$

Given a \mathbf{P}^1 -spectrum E we will reduce the double grading on the cohomology theory $E^{*,*}$ to a grading. Namely, set $E^m = \bigoplus_{m=p-2q} E^{p,q}$ and $E^* = \bigoplus_m E^m$. We often write $E^*(k)$ for $E^*(\mathrm{Spec}(k))$ below.

A \mathbf{P}^1 -ring spectrum is a monoid (E, μ, e) in $(\mathrm{SH}(S), \wedge, \mathbb{1}_S)$. A commutative \mathbf{P}^1 -ring spectrum is a commutative monoid (E, μ, e) in $(\mathrm{SH}(S), \wedge, 1)$.

The cohomology theory E^* defined by a \mathbf{P}^1 -ring spectrum is a ring cohomology theory. The cohomology theory E^* defined by a commutative \mathbf{P}^1 -ring spectrum is a ring cohomology theory, however it is not necessarily graded commutative. The cohomology theory E^* defined by an oriented commutative \mathbf{P}^1 -ring spectrum (to be defined below) is a graded commutative ring cohomology theory.

Occasionally a \mathbf{P}^1 -ring spectrum (E, μ, e) might have a model (E', μ', e') which is a symmetric \mathbf{P}^1 -ring spectrum, that is, a symmetric \mathbf{P}^1 -spectrum E' equipped with a strict multiplication $\mu': E' \wedge E' \rightarrow E'$ which is strictly associative and strictly unital for the unit $e': \Sigma_{\mathbf{P}^1}^\infty(S_+) \rightarrow E'$. This is the case for the algebraic cobordism \mathbf{P}^1 -ring spectrum MGL, as described below. Such a model for the algebraic K -theory \mathbf{P}^1 -ring spectrum BGL is currently not known to us.

For the rest of the paper let k be a field and $S = \mathrm{Spec}(k)$. Usually S will be replaced by k in the notation. We work in this text with the algebraic cobordism \mathbf{P}^1 -spectrum MGL and the algebraic K -theory \mathbf{P}^1 -spectrum BGL as described in [PPR1, Defn. 1.2.4] and [PPR2, Sect. 2.1] respectively. The spectrum MGL is a commutative ring \mathbf{P}^1 -spectrum by that construction. The spectrum BGL is equipped with a structure of a commutative \mathbf{P}^1 -ring spectrum as explained in [PPR1, Thm. 2.1.1]. Let K_*^{TT} be Thomason-Trobaugh K -theory functor [TT]. There is a canonical isomorphism

$$Ad: K_{-*}^{TT} \rightarrow \mathrm{BGL}^{*,0}$$

of ring cohomology theories on the category SmOp/S in the sense of [PS1]. An invertible Bott element $\beta \in \mathrm{BGL}^{2,1}(\mathrm{Spec}(k))$ is constructed in [PPR1, Section 1.3]. For every pointed motivic space A the morphism

$$\mathrm{BGL}^{*,0}(A) \otimes \mathrm{BGL}^0(\mathrm{Spec}(k)) \rightarrow \mathrm{BGL}^{*,*}(A) \quad (1)$$

given by $a \otimes b \mapsto a \cup b$ is a ring isomorphism by [PPR1, Sect. 1.3]. Furthermore $\mathrm{BGL}^0(\mathrm{Spec}(k)) = \mathbb{Z}[\beta, \beta^{-1}]$ is the ring of Laurent polynomials on the Bott

element β . To say the same in a different way,

$$\mathrm{BGL}^{*,0}(A)[\beta, \beta^{-1}] \cong \mathrm{BGL}^{*,*}(A). \quad (2)$$

The special case $A = X/(X \setminus Z)$ where X is a smooth k -variety and $Z \subset X$ is a closed subset implies the following result [PPR1, Cor. 1.3.6].

Corollary 1.0.1. *Let X be a smooth k -scheme, Z a closed subset of X and $U = X \setminus Z$ its open complement. Then there are isomorphisms*

$$K_{-*,Z}^{TT}(X)[\beta, \beta^{-1}] \cong \mathrm{BGL}^{*,*}(X/U) = \mathrm{BGL}^*(X/U) \quad (3)$$

$$K_{-*,Z}^{TT}(X) \cong \mathrm{BGL}^{*,*}(X/U)/(\beta + 1)\mathrm{BGL}^{*,*}(X/U) \quad (4)$$

of ring cohomology theories on SmOp/k in the sense of [PS1].

We refer to [PPR2] for a construction of the commutative \mathbf{P}^1 -ring spectrum MGL. For the purposes of the present preprint we will need to know only two properties of that \mathbf{P}^1 -spectrum, which we refer to as Quillen universality and BGL-cellularity (see Subsection 2.1 below).

1.1 Oriented commutative ring spectra

Following Adams and Morel we define an orientation of a commutative \mathbf{P}^1 -ring spectrum. However we prefer to use Thom classes instead of Chern classes. Consider the pointed motivic space $\mathbf{P}^\infty = \mathrm{colim}_{n \geq 0} \mathbf{P}^n$ having base point $g_1: S = \mathbf{P}^0 \hookrightarrow \mathbf{P}^\infty$.

The tautological “vector bundle” $\mathcal{J}(1) = \mathcal{O}_{\mathbf{P}^\infty}(-1)$ is also known as the Hopf bundle. It has zero section $z: \mathbf{P}^\infty \hookrightarrow \mathcal{J}(1)$. The fiber over the point $g_1 \in \mathbf{P}^\infty$ is \mathbb{A}^1 . For a vector bundle V over a smooth S -scheme X , with zero section $z: X \hookrightarrow V$, let the *Thom space* $\mathrm{Th}(V)$ of V be the Nisnevich sheaf associated to the presheaf $Y \mapsto V(Y)/(V \setminus z(X))(Y)$ on the Nisnevich site $\mathcal{S}m/S$. In particular $\mathrm{Th}(V)$ is a pointed motivic space in the sense of [PPR1, Defn. A.1.1]. It coincides with Voevodsky’s Thom space [V1, p. 422], since $\mathrm{Th}(V)$ is already a Nisnevich sheaf. The Thom space of the Hopf bundle is then defined as the colimit $\mathrm{Th}(\mathcal{J}(1)) = \mathrm{colim}_{n \geq 0} \mathrm{Th}(\mathcal{O}_{\mathbf{P}^n}(-1))$. Abbreviate $T = \mathrm{Th}(\mathbf{A}_S^1)$.

Let E be a commutative \mathbf{P}^1 -ring spectrum. The unit gives rise to an element $1 \in E^{0,0}(\mathrm{Spec}(k)_+)$. Applying the \mathbf{P}^1 -suspension isomorphism to that element we get an element $\Sigma_{\mathbf{P}^1}(1) \in E^{2,1}(\mathbf{P}^1, \infty)$. The canonical covering of \mathbf{P}^1 defines motivic weak equivalences

$$\mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1/\mathbf{A}^1 \xleftarrow{\sim} \mathbf{A}^1/\mathbf{A}^1 \setminus \{0\} = T$$

of pointed motivic spaces inducing isomorphisms $E(\mathbf{P}^1, \infty) \leftarrow E(\mathbf{A}^1/\mathbf{A}^1 \setminus \{0\}) \rightarrow E(T)$. Let $\Sigma_T(1)$ be the image of $\Sigma_{\mathbf{P}^1}(1)$ in $E^{2,1}(T)$.

Definition 1.1.1. Let E be a commutative ring \mathbf{P}^1 -spectrum. A *Thom orientation* of E is an element $th \in E^{2,1}(\mathrm{Th}(\mathcal{J}(1)))$ such that its restriction to the Thom space of the fibre over the distinguished point coincides with the element $\Sigma_T(1) \in E^{2,1}(T)$. A *Chern orientation* of E is an element $c \in E^{2,1}(\mathbf{P}^\infty)$ such that $c|_{\mathbf{P}^1} = -\Sigma_{\mathbf{P}^1}(1)$. An *orientation* of E is either a Thom orientation or a Chern orientation. One says that a Thom orientation th of E coincides with a Chern orientation c of E provided that $c = z^*(th)$ or equivalently the element th coincides with the one $th(\mathcal{O}(-1))$ given by (6) below.

Remark 1.1.2. The element th should be regarded as the Thom class of the tautological line bundle $\mathcal{J}(1) = \mathcal{O}(-1)$ over \mathbf{P}^∞ . The element c should be regarded as the Chern class of the tautological line bundle $\mathcal{J}(1) = \mathcal{O}(-1)$ over \mathbf{P}^∞ .

Example 1.1.3. The following orientations given right below are relevant for our work. Here MGL denotes the \mathbf{P}^1 -ring spectrum representing algebraic cobordism obtained in [PPR2, Defn 2.1.1] and BGL denotes the \mathbf{P}^1 -ring spectrum representing algebraic K -theory constructed in [PPR1, Theorem 2.2.1].

- Let $u_1 : \Sigma_{\mathbf{P}^1}^\infty(\mathrm{Th}(\mathcal{J}(1)))(-1) \rightarrow \mathrm{MGL}$ be the canonical map of \mathbf{P}^1 -spectra. Set $th^{\mathrm{MGL}} = u_1 \in \mathrm{MGL}^{2,1}(\mathrm{Th}(\mathcal{J}(1)))$. Since $th^{\mathrm{MGL}}|_{\mathrm{Th}(\mathbf{1})} = \Sigma_{\mathbf{P}^1}(1)$ in $\mathrm{MGL}^{2,1}(\mathrm{Th}(\mathbf{1}))$, the class th^{MGL} is an orientation of MGL .
- Set $c = (-\beta) \cup ([\mathcal{O}] - [\mathcal{O}(1)]) \in \mathrm{BGL}^{2,1}(\mathbf{P}^\infty)$. The relation (11) from [PPR1] shows that the class c is an orientation of BGL .

2 Oriented cohomology theories

Let (E, c) be an oriented commutative \mathbf{P}^1 -ring spectrum. In this section we compute the E -cohomology of infinite Grassmannians. The results are the expected ones – see Theorem 2.0.6.

The oriented \mathbf{P}^1 -ring spectrum (E, c) defines an oriented cohomology theory on $\mathcal{S}m\mathcal{O}p$ in the sense of [PS1, Defn. 3.1] as follows. The restriction of the functor $E^{*,*}$ to the category $\mathcal{S}m/S$ is a ring cohomology theory. By [PS1, Th. 3.35] it remains to construct a Chern structure on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ in the sense of [PS1, Defn.3.2]. Let $\mathrm{H}_\bullet(k)$ be the homotopy category of pointed motivic spaces over k . The functor isomorphism $\mathrm{Hom}_{\mathrm{H}_\bullet(k)}(-, \mathbf{P}^\infty) \rightarrow \mathrm{Pic}(-)$ on the category $\mathcal{S}m/S$ provided by [MV, Thm. 4.3.8] sends the class of the identity map $\mathbf{P}^\infty \rightarrow \mathbf{P}^\infty$ to the class of the tautological line bundle $\mathcal{O}(-1)$ over \mathbf{P}^∞ . For a line bundle L over $X \in \mathcal{S}m/S$ let $[L]$ be the class

of L in the group $\text{Pic}(X)$. Let $f_L: X \rightarrow \mathbf{P}^\infty$ be the morphism in $\mathbf{H}_\bullet(k)$ corresponding to the class $[L]$ under the functor isomorphism above. For a line bundle L over $X \in \mathcal{S}m/S$ set $c(L) = f_L^*(c) \in E^{2,1}(X)$. Clearly, $c(\mathcal{O}(-1)) = c$. The assignment $L/X \mapsto c(L)$ is a Chern structure on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ since $c|_{\mathbf{P}^1} = -\Sigma_{\mathbf{P}^1}(1) \in E^{2,1}(\mathbf{P}^1, \infty)$. With that Chern structure $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ is an oriented ring cohomology theory in the sense of [PS1]. In particular, (BGL, c^K) defines an oriented ring cohomology theory on $\mathcal{S}m\mathcal{O}p$.

Given this Chern structure, one obtains a theory of Thom classes $V/X \mapsto th(V) \in E^{2\text{rank}(V), \text{rank}(V)}(\text{Th}_X(V))$ on the cohomology theory $E^{*,*}|_{\mathcal{S}m\mathcal{O}p/S}$ in the sense of [PS1, Defn. 3.32] as follows. There is a unique theory of Chern classes $V \mapsto c_i(V) \in E^{2i,i}(X)$ such that for every line bundle L on X one has $c_1(L) = c(L)$. For a rank r vector bundle V over X consider the vector bundle $W := \mathbf{1} \oplus V$ and the associated projective vector bundle $\mathbf{P}(W)$ of lines in W . Set

$$\bar{th}(V) = c_r(p^*(V) \otimes \mathcal{O}_{\mathbf{P}(W)}(1)) \in E^{2r,r}(\mathbf{P}(W)). \quad (5)$$

It follows from [PS1, Cor. 3.18] that the support extension map

$$E^{2r,r}(\mathbf{P}(W)/(\mathbf{P}(W) \setminus \mathbf{P}(\mathbf{1}))) \rightarrow E^{2r,r}(\mathbf{P}(W))$$

is injective and $\bar{th}(E) \in E^{2r,r}(\mathbf{P}(W)/(\mathbf{P}(W) \setminus \mathbf{P}(\mathbf{1})))$. Set

$$th(E) = j^*(\bar{th}(E)) \in E^{2r,r}(\text{Th}_X(V)), \quad (6)$$

where $j: \text{Th}_X(V) \rightarrow \mathbf{P}(W)/(\mathbf{P}(W) \setminus \mathbf{P}(\mathbf{1}))$ is the canonical motivic weak equivalence of pointed motivic spaces induced by the open embedding $V \hookrightarrow \mathbf{P}(W)$. The assignment V/X to $th(V)$ is a theory of Thom classes on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ (see the proof of [PS1, Thm. 3.35]). So the Thom classes are natural, multiplicative and satisfy the following Thom isomorphism property.

Theorem 2.0.4. *For a rank r vector bundle $p: V \rightarrow X$ on $X \in \mathcal{S}m/S$ with zero section $z: X \hookrightarrow V$, the map*

$$-\cup th(V): E^{*,*}(X) \rightarrow E^{*+2r, *+r}(V/(V \setminus z(X)))$$

is an isomorphism of two-sided $E^{,*}(X)$ -modules, where $-\cup th(V)$ is written for the composition map $(-\cup th(V)) \circ p^*$.*

Proof. See [PS1, Defn. 3.32.(4)]. □

Analogous to [V1, p. 422] one obtains for vector bundles $V \rightarrow X$ and $W \rightarrow Y$ in $\mathcal{S}m/S$ a canonical map of pointed motivic spaces $\text{Th}(V) \wedge \text{Th}(W) \rightarrow \text{Th}(V \times_S W)$ which is a motivic weak equivalence as defined

in [PPR1, Defn. 3.1.6]. In fact, the canonical map becomes an isomorphism after Nisnevich (even Zariski) sheafification. Taking $Y = S$ and $W = \mathbf{1}$ the trivial line bundle yields a motivic weak equivalence $\mathrm{Th}(V) \wedge T \rightarrow \mathrm{Th}(V \oplus \mathbf{1})$. The motivic weak equivalences

$$T = \mathbf{A}^1/\mathbf{A}^1 \setminus \{0\} \xrightarrow{\sim} \mathbf{P}^1/\mathbf{A}^1 \xleftarrow{\sim} \mathbf{P}^1$$

introduced above imply that one may switch between T and \mathbf{P}^1 as desired.

Corollary 2.0.5. *For $W = V \oplus \mathbf{1}$ consider the motivic weak equivalences*

$$\epsilon: \mathrm{Th}(V) \wedge \mathbf{P}^1 \rightarrow \mathrm{Th}(V) \wedge \mathbf{P}^1/\mathbf{A}^1 \leftarrow \mathrm{Th}(V) \wedge T \rightarrow \mathrm{Th}(W)$$

of pointed motivic spaces. The diagram

$$\begin{array}{ccc} E^{*+2r,*+r}(\mathrm{Th}(V)) & \xrightarrow{\Sigma_{\mathbf{P}^1}} & E^{*+2r+2,*+r+1}(\mathrm{Th}(V) \wedge \mathbf{P}^1) \\ \uparrow \mathrm{id} & & \uparrow \epsilon^* \\ E^{*+2r,*+r}(\mathrm{Th}(V)) & \xrightarrow{\Sigma_T} & E^{*+2r+2,*+r+1}(\mathrm{Th}(W)) \\ \uparrow -\cup th(V) & & \uparrow -\cup th(W) \\ E^{*,*}(X) & \xrightarrow{\mathrm{id}} & E^{*,*}(X) \end{array}$$

commutes.

Let $\mathcal{J}(n) \rightarrow \mathrm{Gr}(n)$ be the tautological vector bundle of n -dimensional linear subspaces.

Theorem 2.0.6. *Let $c_i = c_i(\mathcal{J}(n)) \in E^{2i,i}(\mathrm{Gr}(n))$ be the i -th Chern class of the tautological bundle $\mathcal{J}(n)$. Then*

$$E^{*,*}(\mathrm{Gr}(n)) = E^{*,*}(k)[[c_1, c_2, \dots, c_n]]$$

is the formal power series on the c_i 's. The inclusion $i: \mathrm{Gr}(n) \hookrightarrow \mathrm{Gr}(n+1)$ satisfies $i^*(c_m) = c_m$ for $m < n+1$ and $i^*(c_{n+1}) = 0$.

2.1 A general result

The main result of this Section is Theorem 2.1.4. The complex cobordism spectrum, equipped with its natural orientation, is a universal oriented ring cohomology theory by Quillen's universality theorem [Qu1]. A motivic version of this universality theorem is proved in [PPR2] (see [Ve] for the original statement). We consider MGL with the commutative monoid structure described in [PPR2, Defn 2.1.1] and with the orientation th^{MGL} described in 1.1.3.

Definition 2.1.1 (Universality Property). Let (U, u) be an oriented commutative ring \mathbf{P}^1 -spectrum over a field k . We say that (U, u) is *Quillen universal* if for every commutative ring \mathbf{P}^1 -spectrum E over k the assignment $\varphi \mapsto \varphi(u) \in U^{2,1}(\mathrm{Th}(\mathcal{T}(1)))$ identifies the set of monoid homomorphisms

$$\varphi: U \rightarrow E \tag{7}$$

in the motivic stable homotopy category $\mathrm{SH}^{\mathrm{cm}}(S)$ with the set of orientations of E .

Remark 2.1.2. The Universality Theorem ([Ve] or [PPR2]) implies that the \mathbf{P}^1 -spectrum MGL , equipped with its canonical orientation, is Quillen universal.

If E is a commutative \mathbf{P}^1 -ring spectrum over k and A is a pointed motivic space over k , $E^*(A)$ is an $E^0(k)$ -module in a natural way. A monoid homomorphism $\phi: E_1 \rightarrow E_2$ induces an $E^0(k)$ -module homomorphism $E_1^*(A) \rightarrow E_2^*(A)$. In particular, if (U, u) is a Quillen universal oriented commutative ring \mathbf{P}^1 -spectrum over k and (E, th) is an oriented commutative ring \mathbf{P}^1 -spectrum over k , the monoid homomorphism

$$\varphi: U \rightarrow E \tag{8}$$

in $\mathrm{SH}(k)$ induces the homomorphism

$$\bar{\varphi}_A: U^*(A) \otimes_{U^0(k)} E^0(k) \rightarrow E^*(A) \tag{9}$$

and in particular the homomorphism

$$\bar{\varphi}_A^0: U^0(A) \otimes_{U^0(k)} E^0(k) \rightarrow E^0(A). \tag{10}$$

Both are natural in A .

Since this moment choose $(\mathrm{BGL}, \mathrm{th}^K)$ for (E, th) (see Example 1.1.3). Set $\bar{U}^*(X, Y) = U^*(X, Y) \otimes_{U^0(k)} \mathrm{BGL}^0(k)$, $\bar{U}^0(X, Y) = U^0(X, Y) \otimes_{U^0(k)} \mathrm{BGL}^0(k)$.

Definition 2.1.3 (Weakly BGL-Cellular). Let (U, u) be a Quillen universal \mathbf{P}^1 -ring spectrum, and let $\bar{\phi}_A^0: U^0(A) \otimes_{U^0(k)} \mathrm{BGL}^0(k) \rightarrow \mathrm{BGL}^0(A)$ be the homomorphism induced by the orientation th^K on BGL (see Example 1.1.3). Then (U, u) is called *weakly BGL-cellular* if there exists an integer N such that the map $\bar{\varphi}_{U_n}^0$ is an isomorphism for $n \geq N$.

A pointed motivic space A is called *small* if the covariant functor

$$\mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma_{\mathbf{P}^1}^\infty A, -)$$

on $\mathrm{SH}(S)$ commutes with arbitrary coproducts.

Theorem 2.1.4. *Let (U, u) be a Quillen universal oriented commutative \mathbf{P}^1 -ring spectrum over a field k . Suppose (U, u) is weakly BGL-cellular. Then the homomorphism $\bar{\varphi}_A$ is an isomorphism for all small pointed motivic spaces A .*

Proof. The proof consists of several steps. Let Our first aim is to prove that the homomorphisms $\bar{\varphi}_A^0$ are isomorphisms, where A is a small pointed motivic space. First we construct a section of the natural transformation

$$\varphi^{0,0}: U^{0,0} \rightarrow \text{BGL}^{0,0}$$

of functors on the category of small pointed motivic spaces. To do this we begin with recalling that for every oriented commutative \mathbf{P}^1 -ring spectrum (E, th) the ring cohomology theory $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ is an oriented cohomology theory on the category $\mathcal{S}m\mathcal{O}p$ (see Section 2). Let $\mathbb{F}_{E,th}$ be the induced commutative formal group law over the ring $E^0(k)$. Let Ω be the complex cobordism ring and let $l_{E,th}: \Omega \rightarrow E^0(k)$ be the unique ring homomorphism sending the universal formal group \mathbb{F}_Ω to $\mathbb{F}_{E,th}$. Set

$$[\mathbf{P}^n]_E = l_{E,th}([\mathbb{C}\mathbf{P}^n]), \quad (11)$$

where $[\mathbb{C}\mathbf{P}^n]$ is the class of the complex projective space $\mathbb{C}\mathbf{P}^n$ in Ω . Although the class $[\mathbf{P}^n]_E$ depends on the orientation class th , we use the notation $[\mathbf{P}^n]_E$ instead. If (E', th') is another oriented commutative \mathbf{P}^1 -ring spectrum and $\psi: E \rightarrow E'$ is a monoid homomorphism in the category $\text{SH}^{\text{cm}}(S)$ which preserves orientation classes, then it sends the formal group law $\mathbb{F}_{E,th}$ to $\mathbb{F}_{E',th'}$. In particular $\psi([\mathbf{P}^n]_E) = [\mathbf{P}^n]_{E'}$. Applying this observation to the monoid homomorphism φ one obtains

$$\varphi([\mathbf{P}^1]_U) = [\mathbf{P}^1]_{\text{BGL}}.$$

To compute $[\mathbf{P}^1]_{\text{BGL}}$ recall that the coefficient at XY in the formal group law \mathbb{F}_Ω coincides with the class $-\mathbb{C}\mathbf{P}^1$ in Ω . The formal group law \mathbb{F}_{BGL} coincides with $X + Y + \beta^{-1}XY$, since $c^{\text{BGL}}(L) = ([\mathbf{1}] - [\mathbf{L}^\vee])(-\beta)$. Thus the equality

$$[\mathbf{P}^1]_{\text{BGL}} = -\beta^{-1}$$

holds. *We are ready to construct a section.* Set $\text{Gr} = \text{colim}_{n \geq 0} \text{Gr}(n)$ and consider the map

$$s: \Sigma_{\mathbf{P}^1}^\infty(\mathbb{Z} \times \text{Gr}) \rightarrow U \quad (12)$$

in the stable homotopy category category $\text{SH}(S)$ given by the element

$$c_1^U(\infty - \tau_\infty^\vee) \cup [\mathbf{P}^1]_U \in U^{0,0}(\mathbb{Z} \times \text{Gr}).$$

Claim 2.1.5. One has $\varphi(c_1^U(\infty - \tau_\infty^\vee) \cup [\mathbf{P}^1]_U) = \tau_\infty - \infty \in \mathrm{BGL}^{0,0}(\mathbb{Z} \times \mathrm{Gr})$.

In fact,

$$\begin{aligned} \varphi(c_1^U(\infty - \tau_\infty^\vee) \cup [\mathbf{P}^1]_U) &= c_1^{\mathrm{BGL}}(\infty - \tau_\infty^\vee) \cup [\mathbf{P}^1]_{\mathrm{BGL}} \\ &= (\infty - \tau_\infty) \cup \beta \cup (-\beta^{-1}) \\ &= \tau_\infty - \infty \end{aligned}$$

Claim 2.1.5 shows that the composition

$$\varphi \circ s: \Sigma_{\mathbf{P}^1}^\infty(\mathbb{Z} \times \mathrm{Gr}) \rightarrow \mathrm{BGL}$$

coincides with the adjoint of the motivic weak equivalence $i: \mathbb{Z} \times \mathrm{Gr} \rightarrow \mathcal{K} = \mathcal{K}_0$ from [PPR1, Lemma 1.2.2]. Thus for every pointed motivic space A the map

$$s_A: \mathrm{BGL}^{0,0}(A) = [A, \mathcal{K}_0] = [A, \mathbb{Z} \times \mathrm{Gr}] \rightarrow [\Sigma_{\mathbf{P}^1}^\infty(A), U] = U^{0,0}(A)$$

is a *section* of the map $\varphi_A^{0,0}: U^{0,0}(A) \rightarrow \mathrm{BGL}^{0,0}(A)$. Moreover, the section s_A is natural in A .

Let $\overline{U}^*(A) = U(A) \otimes_{U^0(k)} \mathrm{BGL}^0(k)$ and recall that $\bar{\varphi}_A: \overline{U}^*(A) \rightarrow \mathrm{BGL}^*(A)$ is the induced $\mathrm{BGL}^0(k)$ -module homomorphism. To extend the section s to a section $\bar{s}^0: \mathrm{BGL}^0 \rightarrow \overline{U}^0$ of the natural transformation $\bar{\varphi}^0: \overline{U}^0 \rightarrow \mathrm{BGL}^0$ of functors on pointed motivic spaces. note that

$$\mathrm{BGL}^0 = \mathrm{BGL}^{0,0}[\beta, \beta^{-1}]$$

for the Bott element $\beta \in \mathrm{BGL}^{2,1}(k)$ (see (2)). Thus for every pointed motivic space A , every element $\alpha \in \mathrm{BGL}^0(A)$ can be presented in a unique way in the form $a \cup \beta^i$ with $a \in \mathrm{BGL}^{0,0}(A)$. Define

$$\bar{s}_A^0: \mathrm{BGL}^0 \rightarrow \overline{U}^0 \tag{13}$$

by $\bar{s}_A^0(a \cup \beta^i) = s_A(a) \otimes \beta^i \in \overline{U}^0(A)$, where $a \in \mathrm{BGL}^{0,0}(A)$. It is immediate that \bar{s}_A^0 is natural in A . The computation

$$\bar{\varphi}_A^0(\bar{s}_A^0(a \cup \beta^i)) = \bar{\varphi}_A^0(s_A(a) \otimes \beta^i) = \varphi(s_A(a)) \cup \beta^i = a \cup \beta^i$$

proves the following

Claim 2.1.6. The map \bar{s}_A^0 is a section of $\bar{\varphi}_A^0$.

If for a pointed motivic space A the map $\bar{\varphi}_A^0$ is an isomorphism, then \bar{s}_A^0 is an isomorphism inverse to $\bar{\varphi}_A^0$. In particular, one has $\bar{s}_A^0 \circ \bar{\varphi}_A^0 = \text{id}$.

The homomorphism $\bar{\varphi}_A^0$ is an isomorphism for the pointed motivic spaces U_n with $n \geq N$, since U is weakly BGL-cellular. The class $[u_n] \in U^{2n,n}(U_n, *)$ of the canonical morphism $u_n: \Sigma_{\mathbf{P}^1}^\infty U_n(-n) \rightarrow U$ then satisfies the following relation:

$$(\bar{s}_{U_n}^0 \circ \varphi_{U_n}^0)([u_n]) = [u_n] \otimes 1 \in \bar{U}^0(U_n). \quad (14)$$

Now we are ready to check that $\bar{\varphi}_A^0$ is an isomorphism for all small pointed motivic spaces. Recall that for a small pointed motivic space A there is a canonical isomorphism of the form

$$U^{2i,i}(A) = \text{colim}_n [\Sigma^{2n,n}(A), U_{i+n}]_{H_\bullet(S)} \quad (15)$$

where $\Sigma^{2n,n} = \Sigma_{\mathbf{P}^1}^n$. This isomorphism implies that for every element $a \in U^{2i,i}(A)$ there exists an integer $n \geq 0$ such that $\Sigma^{2n,n}(a) = f^*([u_n])$ for an appropriate map $f: \Sigma^{2n,n}(A) \rightarrow U_{i+n}$ in the homotopy category $H_\bullet(S)$. Here $\Sigma^{2n,n}(a)$ is the n -fold $\Sigma_{\mathbf{P}^1}$ -suspension of a .

The surjectivity of $\bar{\varphi}_A^0$ is clear, since \bar{s}_A^0 is its section. It remains to check the injectivity of $\bar{\varphi}_A^0$. Take a homogeneous element $\alpha \in \bar{U}^{2i,i}(A) \subseteq \bar{U}^0(A)$ such that $\bar{\varphi}_A^0(\alpha) = 0$. It has the form $\alpha = a \otimes \beta^m$ for a homogeneous element $a \in U^0(A)$. Since the element β is invertible in $\text{BGL}^{*,*}(k)$, one concludes $\varphi_A^0(a) = 0$.

Choose an integer $n \geq 0$ such that $\Sigma^{2n,n}(a) = f^*([u_n])$. The map φ of \mathbf{P}^1 -spectra respects the suspension isomorphisms. Thus $\varphi_{\Sigma^{2n,n}A}(\Sigma^{2n,n}(a)) = \Sigma^{2n,n}(\varphi_A(a)) = 0$ and $(\bar{s}_{\Sigma^{2n,n}A}^0 \circ \varphi_{\Sigma^{2n,n}A})(\Sigma^{2n,n}(a)) = 0$ too. The chain of relations in $\bar{U}^0(\Sigma^{2n,n}A)$ given by

$$\begin{aligned} 0 &= (\bar{s}_{\Sigma^{2n,n}A}^0 \circ \varphi_{\Sigma^{2n,n}A})(\Sigma^{2n,n}(a)) = (\bar{s}_{\Sigma^{2n,n}A}^0 \circ \varphi_{\Sigma^{2n,n}A})(f^*([u_n])) \\ &= f^*((\bar{s}_{U_{n+i}}^0 \circ \varphi_{U_{n+i}})([u_n])) = f^*([u_n] \otimes 1) = f^*([u_n]) \otimes 1 \\ &= \Sigma^{2n,n}(a) \otimes 1 \end{aligned}$$

implies that $\Sigma^{2n,n}(a \otimes 1) = \Sigma^{2n,n}(a) \otimes 1 = 0$. Because the n -fold suspension map

$$\Sigma^{2n,n}: \bar{U}^0(A) \rightarrow \bar{U}^0(\Sigma^{2n,n}A)$$

is an isomorphism, $a \otimes 1 = 0$ in $\bar{U}^0(A) = \bar{U}^0(A)$. This proves the injectivity and hence the bijectivity of $\bar{\varphi}_A^0$ for cofibrations of all small motivic spaces.

To prove that $\bar{\varphi}_A$ is an isomorphism for all small motivic spaces we will use the fact that $\bar{\varphi}_A$ respects the \mathbf{P}^1 -suspension isomorphisms. For every

integer $i \in \mathbb{Z}$ choose an integer $n \geq 0$ with $n \geq i$. Then for a pointed motivic space A one may form the suspension $\mathbb{G}_m^{\wedge n} \wedge S_s^{n-i} \wedge A = S^{n,n} \wedge S^{n-i,0} \wedge A$ in the category of pointed motivic spaces, which supplies the commutative diagram

$$\begin{array}{ccccc} \mathrm{BGL}^i(A) & \xrightarrow[\cong]{\Sigma^{2n,n}} & \mathrm{BGL}^i(S^{2n,n} \wedge A) & \xleftarrow[\cong]{\Sigma^{i,0}} & \mathrm{BGL}^0(S^{n,n} \wedge S^{n-i,0} \wedge A) \\ \bar{\varphi}_A^i \uparrow & & \bar{\varphi}_{S^{2n,n} \wedge A}^i \uparrow & & \cong \uparrow \bar{\varphi}_{S^{n,n} \wedge S^{n-i,0} \wedge A}^0 \\ \mathrm{U}^i(A) & \xrightarrow[\cong]{\Sigma^{2n,n}} & \mathrm{U}^i(S^{2n,n} \wedge A) & \xleftarrow[\cong]{\Sigma^{i,0}} & \mathrm{U}^0(S^{n,n} \wedge S^{n-i,0} \wedge A) \end{array}$$

with the suspension isomorphisms $\Sigma^{2n,n} = \Sigma_{\mathbf{P}^1}^n$ and $\Sigma^{i,0}$. The map $\bar{\varphi}_B^0$ is an isomorphism for B a small pointed motivic space, hence so is $\bar{\varphi}_A^i$. We proved that the map $\bar{\varphi}_A$ is an isomorphism for A being small. \square

2.2 The BGL-cellularity of algebraic cobordism

Theorem 2.2.1. *The oriented commutative ring \mathbf{P}^1 -spectrum $(\mathrm{MGL}, th^{\mathrm{MGL}})$ from Example 1.1.3 is weakly BGL-cellular.*

Proof. We will prove that the homomorphism $\bar{\varphi}_A^0$ is an isomorphism for A being one of the pointed motivic spaces $\mathrm{Spec}(k)_+$, \mathbf{P}_+^∞ , $\mathrm{Gr}(n)_+$ and $\mathrm{Th}(\mathcal{T}_n) = \mathrm{MGL}_n, *$.

The map $\bar{\varphi}_k^0$ is an isomorphism, since it is the identity map. The case $n = 1$ of Theorem 2.0.6 implies that $\overline{\mathrm{MGL}}^*(\mathbf{P}^\infty) = \overline{\mathrm{MGL}}^*(k)[[c^{\mathrm{MGL}}]]$, whence

$$\overline{\mathrm{MGL}}^0(\mathbf{P}^\infty) = \overline{\mathrm{MGL}}^0(k)[[c^{\mathrm{MGL}}]]$$

(the formal power series on the first Chern class c^{MGL} of the tautological line bundle $\mathcal{O}(-1)$). The same holds for BGL. Namely

$$\mathrm{BGL}^0(\mathbf{P}^\infty) = \mathrm{BGL}^0(k)[[c^{\mathrm{BGL}}]].$$

By its definition the morphism φ takes the orientation class th^{MGL} to the orientation class th^K and so it preserves the first Chern class. Whence the map $\bar{\varphi}_{\mathbf{P}^\infty}^0$ coincides with the map of formal power series induced by the isomorphism $\bar{\varphi}_k^0$ of coefficients rings. Hence $\bar{\varphi}_{\mathbf{P}^\infty}^0$ is an isomorphism as well.

Consider now $A = \mathrm{Gr}(n)_+$. By Theorem 2.0.6 its MGL-cohomology ring is the ring of formal power series on the Chern classes of the tautological bundle \mathcal{T}_n over the coefficient ring $\mathrm{MGL}^{*,*}(k)$. The same holds for the BGL-cohomology ring. As observed above, the map φ preserves the first Chern class, thus it preserves all Chern classes. Thus $\bar{\varphi}_{\mathrm{Gr}(n)}^0$ is an isomorphism as well.

Now consider $A = \text{Th}(\mathcal{J}_n)$. The morphism φ respects Thom classes (see (5) and (6)). The vertical arrows in the commutative diagram

$$\begin{array}{ccc} \overline{\text{MGL}}^0((\text{Th}(\mathcal{J}_n), *)) & \xrightarrow{\bar{\varphi}_{\text{Th}(\mathcal{J}_n, *)}^0} & \text{BGL}^0(\text{Th}(\mathcal{J}_n)) \\ \uparrow \text{thom}^{\text{MGL}} & & \uparrow \text{thom}^{\text{BGL}} \\ \overline{\text{MGL}}^0(\text{Gr}(n)) & \xrightarrow{\bar{\varphi}_{\text{Gr}(n)}^0} & \text{BGL}^0(\text{Gr}(n)) \end{array}$$

are isomorphisms induced by the the Thom isomorphism 2.0.4. The map $\bar{\varphi}_{\text{Gr}(n)}^0$ is an isomorphism by the preceding case, whence $\bar{\varphi}_{\text{Th}(\mathcal{J}_n, *)}^0$ is an isomorphism too. □

2.3 The main result

Let k be a field and $S = \text{Spec}(k)$. By Theorem [PPR2, Theorem 2.2.1] and Example 1.1.3 there exists a unique monoid homomorphism

$$\varphi: \text{MGL} \rightarrow \text{BGL} \tag{16}$$

in $\text{SH}(S)$ such that $\varphi(th^{\text{MGL}}) = th^K$. It induces a natural transformation

$$\bar{\varphi}_A: \overline{\text{MGL}}^*(A) := \text{MGL}^*(A) \otimes_{\text{MGL}^0(k)} \text{BGL}^0(k) \rightarrow \text{BGL}^*(A). \tag{17}$$

Theorem 2.3.1. *The homomorphism*

$$\bar{\varphi}_A: \text{MGL}^*(A) \otimes_{\text{MGL}^0(k)} \text{BGL}^0(k) \rightarrow \text{BGL}^*(A)$$

is an isomorphism for all small pointed motivic spaces.

Proof. In fact, $(\text{MGL}, th^{\text{MGL}})$ is Quillen universal by [Ve] and [PPR2], and weakly BGL-cellular by Theorem 2.2.1. Theorem 2.1.4 completes the proof. □

Remark 2.3.2. There is an unpublished result due to Morel and Hopkins, which states that there is a canonical isomorphism of the form

$$\text{MGL}^{*,*}(X) \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \rightarrow \text{BGL}^{*,*}(X)$$

where \mathbb{L} denotes the Lazard ring carrying the universal formal group law. If the canonical homomorphism $\mathbb{L} \rightarrow \text{MGL}^0(k)$ is an isomorphism, Theorem 2.3.1 implies their result.

Let X be a smooth k -scheme and $Z \subseteq X$ a closed subset, with open complement $U \subseteq X$. Consider the small pointed motivic space X/U and take the quotients of both sides of the isomorphism (17) modulo the principal ideal generated by the element $1 \otimes (\beta + 1)$. Corollary 1.0.1 then implies that the natural transformation

$$\bar{\varphi}_{X/U}: \mathrm{MGL}^*(X/U) \otimes_{\mathrm{MGL}^0(k)} \mathbb{Z} \rightarrow K_{-*}^{TT}(X) \quad (18)$$

is an isomorphism, where $K_{*,Z}^{TT}(X)$ are the Thomason-Trobaugh K -groups with supports. This family of isomorphisms shows that the functor

$$(X, X \setminus Z) \mapsto \mathrm{MGL}^*(X/(X \setminus Z)) \otimes_{\mathrm{MGL}^0(k)} \mathbb{Z}$$

is a ring cohomology theory in the sense of [PS1]. This implies the first part of our main result.

Theorem 2.3.3 (Main Theorem). *Let $X \in \mathrm{Sm}_k$ and $Z \subseteq X$ be a closed subset.*

- *The family of isomorphisms*

$$\bar{\varphi}_{X/(X-Z)}: \mathrm{MGL}^*(X/U) \otimes_{\mathrm{MGL}^0(k)} \mathbb{Z} \rightarrow K_{-*}^{TT}(X) \quad (19)$$

form an isomorphism $\bar{\varphi}$ of ring cohomology theories on SmOp/k .

- *The natural isomorphism $\bar{\varphi}$ respects orientations provided that MGL^* and K_{-*}^{TT} are considered as oriented cohomology theories in the sense of [PS1] with orientations given by the Thom class $th^{\mathrm{MGL}} \otimes 1$ from 1.1.3 and the Chern structure $L/X \mapsto [\mathcal{O}] - [L^{-1}]$. In particular, the composition*

$$\mathrm{MGL}^0(k) \xrightarrow{a \mapsto a \otimes 1} \mathrm{MGL}^0(k) \otimes \mathbb{Z} \xrightarrow{b \otimes c \mapsto \varphi(b) \cdot c} \mathbb{Z}$$

sends the class $[X] \in \mathrm{MGL}^0(X)$ of a smooth projective k -variety X to the Euler characteristic $\chi(X, \mathcal{O}_X)$ of the structure sheaf \mathcal{O}_X .

Proof. The first part is already proven. To prove the second part, consider the orientations th^{MGL} and th^K from 1.1.3. Note that by the very definition of φ it sends th^{MGL} to th^K . Thus it respects the Chern structures on MGL^* and BGL^* described in Section 2.

The quotient map $\mathrm{BGL}^* \rightarrow K_{-*}^{TT}$ takes the Bott element β to (-1) . Thus it takes the Chern structure on BGL^* to the Chern structure on K_{-*}^{TT} given by $L/X \mapsto [\mathcal{O}] - [L^{-1}] \in K_0(X)$. This shows that $\bar{\varphi}: \mathrm{MGL}^*(-) \otimes_{\mathrm{MGL}^0(k)} \mathbb{Z}^* \rightarrow K_{-*}^{TT}$ respects the orientations described in the Theorem 2.3.3.

Let $f \mapsto f_{\text{MGL}}$ resp. $f \mapsto f_K$ be the integrations on MGL^* resp. K_{-*}^{TT} given by these Chern structures via Theorem [PS3, Thm. 4.1.4]. By Theorem [PS2, Thm. 1.1.10] the composition $\text{MGL}^* \rightarrow \text{BGL}^* \rightarrow K_{-*}^{TT}$ respects the integrations on MGL^* and K_{-*}^{TT} since it preserves the Chern structures. In particular, given a smooth projective S -scheme $f: X \rightarrow \text{Spec}(k)$, the diagram

$$\begin{array}{ccc} \text{MGL}^0(X) \otimes_{\text{MGL}^0(k)} \mathbb{Z}^0(X) & \xrightarrow{\bar{\varphi}} & K_0^{TT}(X) \\ f_{\text{MGL}} \downarrow & & \downarrow f_K \\ \text{MGL}^0(X) \otimes_{\text{MGL}^0(k)} \mathbb{Z}^0(k) & \xrightarrow{\bar{\varphi}} & K_0^{TT}(k) \end{array}$$

commutes where f_{MGL} and f_K are the push-forward maps for MGL^* and K_{-*}^{TT} respectively. The integration $f \mapsto f_K$ on K_{-*}^{TT} respecting the Chern structure $L \mapsto [\mathcal{O}] - [L^{-1}]$ coincides with the one given by the higher direct images by Theorem [PS2, Thm. 1.1.11]. The last one sends the class $[V] \in K_0(X)$ of a vector bundle V over a smooth projective variety X to the Euler characteristic $\chi(X, \mathcal{V})$ of the sheaf \mathcal{V} of sections of V .

Recall that for an oriented cohomology theory A with a Chern structure $L \mapsto c(L)$ and for a smooth projective variety $f: X \rightarrow \text{Spec}(k)$ its class $[X]_A \in A^{\text{even}}(\text{Spec}(k))$ is defined as $f_A(1)$, where $f_A: A(X) \rightarrow A(\text{Spec}(k))$ is the push-forward morphism respecting the Chern structure (see [PS3, Thm. 4.1.4]). The notation f_A is misleading, since it depends on the Chern structure. Taking the element $1 \in \text{MGL}^{0,0}(X)$ and using the commutativity of the very last diagram we see that

$$\bar{\varphi}([X]_{\text{MGL}} \otimes 1) = \chi(X, \mathcal{O}_X).$$

Whence the Theorem. □

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