A universality theorem for Voevodsky’s algebraic cobordism spectrum

I. Panin† K. Pimenov† O. Röndigs‡

June 06, 2007§

Abstract

An algebraic version of a theorem due to Quillen is proved. More precisely, for a ground field $k$ we consider the motivic stable homotopy category $\text{SH}(k)$ of $\mathbb{P}^1$-spectra, equipped with the symmetric monoidal structure described in [PPRI]. The algebraic cobordism $\mathbb{P}^1$-spectrum $\text{MGL}$ is considered as a commutative monoid equipped with a canonical orientation $\text{th}^{\text{MGL}} \in \text{MGL}^2(\mathbb{Q}(\{-1\}))$. For a commutative monoid $E$ in the category $\text{SH}(k)$ it is proved that assignment $\varphi \mapsto \varphi(\text{th}^{\text{MGL}})$ identifies the set of monoid homomorphisms $\varphi: \text{MGL} \to E$ in the motivic stable homotopy category $\text{SH}(k)$ with the set of all orientations of $E$. The result was stated originally in a slightly different form by G. Vezzosi in [Ve].

1 Oriented commutative ring spectra

We refer to [PPRI] Appendix for the basic terminology, notation, constructions, definitions, results. For the convenience of the reader we recall the basic definitions. Let $S$ be a Noetherian scheme of finite Krull dimension. One may think of $S$ being the spectrum of a field or the integers. Let $\text{Sm}/S$ be the category of smooth quasi-projective $S$-schemes, and let $\textbf{sSet}$ be the category of simplicial sets. A motivic space over $S$ is a functor

$$A: \text{SmOp}/S \to \textbf{sSet}$$

(see [PPRI A.1.1]). The category of motivic spaces over $S$ is denoted $\text{M}(S)$. This definition of a motivic space is different from the one considered by Morel and Voevodsky in [MV] – they consider only those simplicial presheaves which are sheaves in the Nisnevich topology on $\text{Sm}/S$. With our definition the Thomason-Trobaugh $K$-theory functor obtained by using big vector bundles is a motivic space on the nose. It is not a simplicial Nisnevich sheaf. This is why we prefer to work with the above notion of “space”.

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*Universität Bielefeld, SFB 701, Bielefeld, Germany
†Steklov Institute of Mathematics at St. Petersburg, Russia
‡Institut für Mathematik, Universität Osnabrück, Osnabrück, Germany
§The authors thank the SFB-701 at the Universität Bielefeld, the RTN-Network HPRN-CT-2002-00287, the RFFI-grant 03-01-00633a, and INTAS-03-51-3251 for their support.
We write $H^m_\bullet(S)$ for the pointed motivic homotopy category and $\text{SH}^{\text{cm}}(S)$ for the stable motivic homotopy category over $S$ as constructed in \cite[A.3.9, A.5.6]{PPR1}. By \cite[A.3.11 resp. A.5.6]{PPR1} there are canonical equivalences to $H_\bullet(S)$ of \cite{MV} resp. $\text{SH}(S)$ of \cite{MV}. Both $H^m_\bullet(S)$ and $\text{SH}^{\text{cm}}(S)$ are equipped with closed symmetric monoidal structures such that the $\mathbb{P}^1$-suspension spectrum functor is a strict symmetric monoidal functor

$$\Sigma^\infty_\mathbb{P}^1 : H^m_\bullet(S) \to \text{SH}^{\text{cm}}(S).$$

Here $\mathbb{P}^1$ is considered as a motivic space pointed by $\infty \in \mathbb{P}^1$. The symmetric monoidal structure $(\wedge, \sqcup_S = \Sigma^\infty_\mathbb{P}^1 S_+)$ on the homotopy category $\text{SH}^{\text{cm}}(S)$ is constructed on the model category level by employing symmetric $\mathbb{P}^1$-spectra. It satisfies the properties required by Theorem 5.6 of Voevodsky congress talk \cite{MV}. From now on we will usually omit the superscript $(\_)^{\text{cm}}$.

Every $\mathbb{P}^1$-spectrum $E = (E_0, E_1, \ldots)$ represents a cohomology theory on the category of pointed motivic spaces. Namely, for a pointed motivic space $(A, a)$ set

$$E^{p,q}(A, a) = \text{Hom}_{\text{SH}(\mathcal{S})}(\Sigma^\infty_\mathbb{P}^1 (A, a), \Sigma^{p,q}(E))$$

and $E^{*,*}(A, a) = \bigoplus_{p,q} E^{p,q}(A, a)$. This definition extends to motivic spaces via the functor $A \mapsto A_+$ which adds a disjoint basepoint. That is, for a non-pointed motivic space $A$ set $E^{p,q}(A) = E^{p,q}(A_+, +)$ and $E^{*,*}(A) = \bigoplus_{p,q} E^{p,q}(A)$. Recall that there is a canonical element in $E^{2n,n}(E_n)$, denoted as $\Sigma^\infty_\mathbb{P}^1 E_n(-n) \to E$. It is represented by the canonical map $(*, \ldots, *, E_n, E_n \wedge \mathbb{P}^1, \ldots) \to (E_0, E_1, \ldots, E_n, \ldots)$ of $\mathbb{P}^1$-spectra.

Every $X \in \mathcal{S}m/S$ defines a representable motivic space constant in the simplicial direction taking an $S$-smooth scheme $U$ to $\text{Hom}_{\mathcal{S}m/S}(U, X)$. It is not possible in general to choose a basepoint for representable motivic spaces. So we regard $S$-smooth varieties as motivic spaces (non-pointed) and set

$$E^{p,q}(X) = E^{p,q}(X_+, +).$$

Given a $\mathbb{P}^1$-spectrum $E$ we will reduce the double grading on the cohomology theory $E^{*,*}$ to a grading. Namely, set $E^m = \bigoplus_{m=p-2q} E^{p,q}$ and $E^* = \bigoplus_m E^m$. We often write $E^*(k)$ for $E^*(\text{Spec}(k))$ below.

To complete this section, note that for us a $\mathbb{P}^1$-ring spectrum is a monoid $(E, \mu, e)$ in $(\text{SH(S)}, \wedge, \sqcup_S)$. A commutative $\mathbb{P}^1$-ring spectrum is a commutative monoid $(E, \mu, e)$ in $(\text{SH(S)}, \wedge, 1)$. The cohomology theory $E^*$ defined by a $\mathbb{P}^1$-ring spectrum is a ring cohomology theory. The cohomology theory $E^*$ defined by a commutative $\mathbb{P}^1$-ring spectrum is a ring cohomology theory, however it is not necessary graded commutative. The cohomology theory $E^*$ defined by an oriented commutative $\mathbb{P}^1$-ring spectrum is a graded commutative ring cohomology theory.

### 1.1 Oriented commutative ring spectra

Following Adams and Morel we define an orientation of a commutative $\mathbb{P}^1$-ring spectrum. However we prefer to use Thom classes instead of Chern classes. Consider the pointed motivic space $\mathbb{P}^\infty = \text{colim}_{n \geq 0} \mathbb{P}^n$ having base point $g_1 : S = \mathbb{P}^0 \hookrightarrow \mathbb{P}^\infty$. 


The tautological "vector bundle" $\mathcal{I}(1) = \mathcal{O}_{\mathbb{P}^\infty}(-1)$ is also known as the Hopf bundle. It has zero section $z: \mathbb{P}^\infty \rightarrow \mathcal{I}(1)$. The fiber over the point $g_1 \in \mathbb{P}^\infty$ is $\mathbb{A}^1$. For a vector bundle $V$ over a smooth $S$-scheme $X$, with zero section $z: X \rightarrow V$, its Thom space $Th(V)$ is the Nisnevich sheaf associated to the presheaf $Y \mapsto V(Y)/(V \smallsetminus z(X))(Y)$ on the Nisnevich site $\mathcal{S}m/S$. In particular, $Th(V)$ is a pointed motivic space in the sense of [PPRI Defn. A.1.1]. It coincides with Voevodsky's Thom space [V], p. 422, since $Th(V)$ already is a Nisnevich sheaf. The Thom space of the Hopf bundle is then defined as the colimit $Th(\mathcal{I}(1)) = \text{colim}_{n \geq 0} Th(\mathcal{O}_{\mathbb{P}^n}(-1))$. Abbreviate $T = Th(\mathbb{A}^1_S)$.

Let $E$ be a commutative $\mathbb{P}^1$-ring spectrum. The unit gives rise to an element $1 \in E^{0,0}(\text{Spec}(k))$. Applying the $\mathbb{P}^1$-suspension isomorphism to that element we get an element $\Sigma_{\mathbb{P}^1}(1) \in E^{2,1}(\mathbb{P}^1, \infty)$. The canonical covering of $\mathbb{P}^1$ defines motivic weak equivalences

$$\mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1/\mathbb{A}^1 \leftarrow \mathbb{A}^1/\mathbb{A}^1 \smallsetminus \{0\} = T$$

of pointed motivic spaces inducing isomorphisms $E(\mathbb{P}^1, \infty) \leftarrow E(\mathbb{A}^1/\mathbb{A}^1 \smallsetminus \{0\}) \rightarrow E(T)$. Let $\Sigma_T(1)$ be the image of $\Sigma_{\mathbb{P}^1}(1)$ in $E^{2,1}(T)$.

**Definition 1.1.1.** Let $E$ be a commutative ring $\mathbb{P}^1$-spectrum. A Thom orientation of $E$ is an element $th \in E^{2,1}(Th(\mathcal{I}(1)))$ such that its restriction to the Thom space of the fibre over the distinguished point coincides with the element $\Sigma_T(1) \in E^{2,1}(T)$. A Chern orientation of $E$ is an element $c \in E^{2,1}(\mathbb{P}^\infty)$ such that $c|_{\mathbb{P}^1} = -\Sigma_{\mathbb{P}^1}(1)$. An orientation of $E$ is either a Thom orientation or a Chern orientation. One says that a Thom orientation $th$ of $E$ coincides with a Chern orientation $c$ of $E$ provided that $c = z^*(th)$ or equivalently the element $th$ coincides with the one $th(\mathcal{O}(-1))$ given by (2) below.

**Remark 1.1.2.** The element $th$ should be regarded as the Thom class of the tautological line bundle $\mathcal{I}(1) = \mathcal{O}(-1)$ over $\mathbb{P}^\infty$. The element $c$ should be regarded as the Chern class of the tautological line bundle $\mathcal{I}(1) = \mathcal{O}(-1)$ over $\mathbb{P}^\infty$.

**Example 1.1.3.** The following orientations given right below are relevant for our work. Here $MGL$ denotes the $\mathbb{P}^1$-ring spectrum representing algebraic cobordism obtained below in Definition 2.1.1 and $BGL$ denotes the $\mathbb{P}^1$-ring spectrum representing algebraic $K$-theory constructed in [PPRI Theorem 2.2.1].

- Let $u_1 : \Sigma_{\mathcal{I}(1)}(\text{Th}(\mathcal{I}(1)))(-1) \rightarrow MGL$ be the canonical map of $\mathbb{P}^1$-spectra. Set $th^{MGL} = u_1 \in MGL^{2,1}(\text{Th}(\mathcal{I}(1)))$. Since $th^{MGL}|_{\text{Th}(1)} = \Sigma_{\mathbb{P}^1}(1)$ in $MGL^{2,1}(\text{Th}(1))$, the class $th^{MGL}$ is an orientation of $MGL$.
- Set $c = (-\beta) \cup ([\mathcal{O}] - [\mathcal{O}(1)]) \in BGL^{2,1}(\mathbb{P}^\infty)$. The relation (11) from [PPRI] shows that the class $c$ is an orientation of $BGL$.

## 2 Oriented ring spectra and infinite Grassmannians

Let $(E, c)$ be an oriented commutative $\mathbb{P}^1$-ring spectrum. In this section we compute the $E$-cohomology of infinite Grassmannians and their products. The results are the expected ones – see Theorems 2.0.6 and 2.0.7.
The oriented $\mathbb{P}^1$-ring spectrum $(E, c)$ defines an oriented cohomology theory on $Sm\mathcal{O}p$ in the sense of [PSI, Defn. 3.1] as follows. The restriction of the functor $E^{*, *}$ to the category $Sm/S$ is a ring cohomology theory. By [PSI, Th. 3.35] it remains to construct a Chern structure on $E^{*, *}|_{Sm\mathcal{O}p}$ in the sense of [PSI, Defn. 3.2]. Let $H_*(k)$ be the homotopy category of pointed motivic spaces over $k$. The functor isomorphism $\text{Hom}_{H_*(k)}(-, \mathbb{P}^\infty) \to \text{Pic}(-)$ on the category $Sm/S$ provided by [MV] Thm. 4.3.8 sends the class of the identity map $\mathbb{P}^\infty \to \mathbb{P}^\infty$ to the class of the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{P}^\infty$. For a line bundle $L$ over $X \in Sm/S$ let $[L]$ be the class of $L$ in the group $\text{Pic}(X)$. Let $f_L: X \to \mathbb{P}^\infty$ be a morphism in $\text{H}(k)$ corresponding to the class $[L]$ under the functor isomorphism above. For a line bundle $L$ over $X \in Sm/S$ set $c(L) = f_L^*(c) \in E^{2, 1}(X)$. Clearly, $c(\mathcal{O}(-1)) = c$. The assignment $L/X \mapsto c(L)$ is a Chern structure on $E^{*, *}|_{Sm\mathcal{O}p}$ since $c|_{\mathbb{P}^1} = -\Sigma_{\mathbb{P}^1}(1) \in E^{2, 1}(\mathbb{P}^1, \infty)$. With that Chern structure $E^{*, *}|_{Sm\mathcal{O}p}$ is an oriented ring cohomology theory in the sense of [PSI]. In particular, $(BGL, c^K)$ defines an oriented ring cohomology theory on $Sm\mathcal{O}p$.

Given this Chern structure, one obtains a theory of Thom classes $V/X \mapsto th(V) \in E^{2\text{rank}(V), \text{rank}(V)}(\text{Th}_X(V))$ on the cohomology theory $E^{*, *}|_{Sm\mathcal{O}p/S}$ in the sense of [PSI, Defn. 3.32] as follows. There is a unique theory of Chern classes $V \mapsto c_i(V) \in E^{2i, i}(X)$ such that for every line bundle $L$ on $X$ one has $c_1(L) = c(L)$. For a rank $r$ vector bundle $V$ over $X$ consider the vector bundle $W := 1 \oplus V$ and the associated projective vector bundle $\mathbb{P}(W)$ of lines in $W$. Set

$$\overline{th}(V) = c_r(p^*(V) \otimes \mathcal{O}_{\mathbb{P}(W)}(1)) \in E^{2r, r}(\mathbb{P}(W)).$$

It follows from [PSI, Cor. 3.18] that the support extension map

$$E^{2r, r}(\mathbb{P}(W)/(\mathbb{P}(W) \setminus \mathbb{P}(1))) \to E^{2r, r}(\mathbb{P}(W))$$

is injective and $\overline{th}(E) \in E^{2r, r}(\mathbb{P}(W)/(\mathbb{P}(W) \setminus \mathbb{P}(1)))$. Set

$$th(E) = j^*(\overline{th}(E)) \in E^{2r, r}(\text{Th}_X(V)),$$

where $j: \text{Th}_X(V) \to \mathbb{P}(W)/(\mathbb{P}(W) \setminus \mathbb{P}(1))$ is the canonical motivic weak equivalence of pointed motivic spaces induced by the open embedding $V \hookrightarrow \mathbb{P}(W)$. The assignment $V/X$ to $th(V)$ is a theory of Thom classes on $E^{*, *}|_{Sm\mathcal{O}p}$ (see the proof of [PSI, Thm. 3.35]). Hence the Thom classes are natural, multiplicative and satisfy the following Thom isomorphism property.

**Theorem 2.0.4.** For a rank $r$ vector bundle $p: V \to X$ on $X \in Sm/S$ with zero section $z: X \hookrightarrow V$, the map

$$- \cup th(V): E^{*, *}(X) \to E^{*+2r, *+r}(V/(V \setminus z(X)))$$

is an isomorphism of two-sided $E^{*, *}(X)$-modules, where $- \cup th(V)$ is written for the composition map $(- \cup th(V)) \circ p^*$.

**Proof.** See [PSI, Defn. 3.32.(4)].
Analogous to [V1] p. 422 one obtains for vector bundles $V \to X$ and $W \to Y$ in $\mathcal{S}m/S$ a canonical map of pointed motivic spaces $Th(V) \wedge Th(W) \to Th(V \times_S W)$ which is a motivic weak equivalence as defined in [P2K1, Defn. 3.1.6]. In fact, the canonical map becomes an isomorphism after Nisnevich (even Zariski) sheafification. Taking $Y = S$ and $W = 1$ the trivial line bundle yields a motivic weak equivalence $Th(V) \wedge T \to Th(V \oplus 1)$. The canonical covering of $\mathbb{P}^1$ defines motivic weak equivalences

$$T = A^1/A^1 \setminus \{0\} \sim \mathbb{P}^1/A^1 \sim \mathbb{P}^1$$

and the arrow $T = A^1/A^1 \setminus \{0\} \to \mathbb{P}^1/A^1 \setminus \{0\}$ is an isomorphism. Hence one may switch between $T$ and $\mathbb{P}^1$ as desired.

**Corollary 2.0.5.** For $W = V \oplus 1$ consider the motivic weak equivalences

$$\epsilon: Th(V) \wedge \mathbb{P}^1 \to Th(V) \wedge \mathbb{P}^1/A^1 \leftarrow Th(V) \wedge T \to Th(W)$$

of pointed motivic spaces over $S$. The diagram

\[
\begin{array}{ccc}
E^*+2r^*+r^*(Th(V)) & \xrightarrow{\Sigma^r_{\mathbb{P}^1}} & E^*+2r^*+2s^*+r^*(Th(V) \wedge \mathbb{P}^1) \\
\text{id} & & \epsilon_* \\
E^*+2r^*+r^*(Th(V)) & \xrightarrow{\Sigma^r_{Th(V)}} & E^*+2r^*+2s^*+r^*(Th(W)) \\
-\cup th(V) & & -\cup th(W) \\
E^*,(X) & \xrightarrow{\text{id}} & E^*,(X)
\end{array}
\]

commutes.

Let $Gr(n, n+m)$ be the Grassmann scheme of $n$-dimensional linear subspaces of $A_S^{n+m}$. The closed embedding $A^{n+m} = A^{n+m} \times \{0\} \hookrightarrow A^{n+m+1}$ defines a closed embedding

$$Gr(n, n+m) \hookrightarrow Gr(n, n+m+1).$$

The tautological vector bundle is denoted $\mathcal{I}(n, n+m) \to Gr(n, n+m)$. The closed embedding $\mathcal{I}$ is covered by a map of vector bundles $\mathcal{I}(n, n+m) \hookrightarrow \mathcal{I}(n, n+m+1)$. Let $Gr(n) = \text{colim}_{m \geq 0} Gr(n, n+m)$, $\mathcal{I} = \text{colim}_{m \geq 0} \mathcal{I}(n, n+m)$ and $Th(\mathcal{I}(n)) = \text{colim}_{m \geq 0} Th(\mathcal{I}(n, n+m))$. These colimits are taken in the category of motivic spaces over $S$.

**Theorem 2.0.6.** Let $E$ be an oriented $\mathbb{P}^1$-ring spectrum. Then

$$E^*,(Gr(n)) = E^*,(k)[[c_1, c_2, \ldots, c_n]]$$

is the formal power series ring, where $c_i := c_i(\mathcal{I}(n)) \in E^{2i,i}(Gr(n))$ denotes the $i$-th Chern class of the tautological bundle $\mathcal{I}(n)$. The inclusion $\text{inc}_n: Gr(n) \hookrightarrow Gr(n+1)$ satisfies $\text{inc}_n^*(c_m) = c_m$ for $m < n+1$ and $\text{inc}_n^*(c_{n+1}) = 0$. 

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Proof. The case $n = 1$ is well-known (see for instance [PS1 Thm. 3.9]). For a finite dimensional vector space $W$ and a positive integer $m$ let $F(m, W)$ be the flag variety of flags $W_1 \subset W_2 \subset \cdots \subset W_m$ of linear subspaces of $W$ such that the dimension of $W_i$ is $i$. Let $\mathcal{T}^i(m, W)$ be the tautological rank $i$ vector bundle on $F(m, W)$.

Let $V = \mathbb{A}^\infty$ be an infinite dimensional vector bundle over $S$ and set $e = (1, 0, \ldots)$. Then $V_n$ denotes the $n$-fold product of $V$, and $e^n_i \in V_n$ the vector $(0, \ldots, 0, e, 0, \ldots, 0)$ having $e$ precisely at the $i$th position. Let $F(m) = \operatorname{colim}_W F(m, W)$ and let $\mathcal{T}^i(m) = \operatorname{colim}_W \mathcal{T}^i(m, W)$, where $W$ runs over all finite-dimensional vector subspaces of $V_n$. Thus we have a flag $\mathcal{T}^i(m) \subset \mathcal{T}^j(m) \subset \cdots \subset \mathcal{T}^m(m)$ of vector bundles over $F(m)$. Set $L^i(m) = \mathcal{T}^i(m)/\mathcal{T}^{i-1}(m)$. It is a line bundle over $F(m)$.

Consider the morphism $p_m : F(m) \to F(m-1)$ which takes a flag $W_1 \subset W_2 \subset \cdots \subset W_m$ to the flag $W_1 \subset W_2 \subset \cdots \subset W_{m-1}$. It is a projective vector bundle over $F(m-1)$ such that the line bundle $L^i(m)$ is its tautological line bundle. Thus there exists a tower of projective vector bundles $F(m) \to F(m-1) \to \cdots \to F(1) = \mathbb{P}(V_n)$. The projective bundle theorem implies that

$$E^* \cdot \cdot \cdot (F(n)) = E^*(k)[[t_1, t_2, \ldots, t_n]]$$

(the formal power series in $n$ variables), where $t_i = c(L^i(n))$ is the first Chern class of the line bundle $L^i(n)$ over $F(n)$.

Consider the morphism $q : F(n) \to \operatorname{Gr}(n)$, which takes a flag $W_1 \subset W_2 \subset \cdots \subset W_n$ to the space $V_n$. It can be decomposed as a tower of projective vector bundles. In particular, the pull-back map $q^* : E^*(\operatorname{Gr}(n)) \to E^*(F(n))$ is a monomorphism. It takes the class $c_i$ to the symmetric polynomial $\sigma_i = t_{1}t_{2}\cdots t_{i} + \cdots + t_{n-i+1}\cdots t_{n-1}t_{n}$. So the image of $q^*$ contains $E^*(k)[[\sigma_1, \sigma_2, \ldots, \sigma_n]]$. It remains to check that the image of $q^*$ is contained in $E^*(k)[[\sigma_1, \sigma_2, \ldots, \sigma_n]]$. To do that consider another variety.

Namely, let $V^0$ be the $n$-dimensional subspace of $V_n$ generated by the vectors $e^n_i$’s. Let $l^n_i$ be the line generated by the vector $e^n_i$. Let $V^0_i$ be a subspace of $V^0$ generated by all $e^n_j$s with $j \leq i$. So one has a flag $V^0_1 \subset V^0_2 \subset \cdots \subset V^0_n$. We denote this flag $F^0$. For each vector subspace $W$ in $V_n$ containing $V^0$ consider three algebraic subgroups of the general linear group $\mathbb{G}_L W$. Namely, set

$$P_W = \operatorname{Stab}(V^0), \quad B_W = \operatorname{Stab}(F^0), \quad T_W = \operatorname{Stab}(l^n_1, l^n_2, \ldots, l^n_n).$$

The group $T_W$ stabilizes each line $l^n_i$. Clearly, $T_W \subset B_W \subset P_W$ and $\operatorname{Gr}(n, W) = \mathbb{G}_L W/P_W$, $F(n, W) = \mathbb{G}_L W/B_W$ Set $M(n, W) = \mathbb{G}_L W/T_W$. One has a tower of obvious morphisms

$$M(n, W) \xrightarrow{r_W} F(n, W) \xrightarrow{g_W} \operatorname{Gr}(n, W).$$

Set $M(n) = \operatorname{colim}_W M(n, W)$, where $W$ runs over all finite dimensional subspace $W$ of $V_n$ containing $V^0$. Now one has a tower of morphisms

$$M(n) \xrightarrow{r} F(n) \xrightarrow{g} \operatorname{Gr}(n).$$

The morphisms $r_W$ can be decomposed in a tower of affine bundles. Hence it induces an isomorphism on any cohomology theory. The same then holds for the morphism $r$ and

$$E^*(M(n)) = E^*(k)[[t_1, t_2, \ldots, t_n]].$$

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Permuting vectors $e_i^n$'s yields an inclusion $\Sigma_n \subset GL(V^0)$ of the symmetric group $\Sigma_n$ in $GL(V^0)$. The action of $\Sigma_n$ by the conjugation on $GL(V)$ normalizes the subgroups $T_V$ and $P_V$. Thus $\Sigma_n$ acts as on $M(n)$ so on $\text{Gr}(n)$ and the morphism $q \circ r : M(n) \to \text{Gr}(n)$ respects this action. Note that the action of $\Sigma_n$ on $\text{Gr}(n)$ is trivial and the action of $\Sigma_n$ on $E^*,*(M(n))$ permutes the variable $t_1, t_2, \ldots, t_n$. Thus the image of $(q \circ r)^*$ is contained in $E^*,*(k)[[\sigma_1, \sigma_2, \ldots, \sigma_n]]$. Whence the same holds for the image of $q^*$. The Theorem is proven. \qed

The projection from the product $\text{Gr}(m) \times \text{Gr}(n)$, to the $j$-th factor is called $p_j$. For every integer $i \geq 0$ set $c'_i = p_1^*(c_i(\mathcal{F}(m)))$ and $c''_i = p_2^*(c_i(\mathcal{F}(n)))$

**Theorem 2.0.7.** Suppose $E$ is an oriented commutative $\mathbf{P}^1$-ring spectrum. There is an isomorphism

$$E^*,*((\text{Gr}(m) \times \text{Gr}(n))) = E^*,*(k)[[c'_1, c'_2, \ldots, c'_m, c''_1, c''_2, \ldots, c''_n]]$$

is the formal power series on the $c'_i$'s and $c''_i$'s. The inclusion $i_{m,n} : G(m) \times \text{Gr}(n) \hookrightarrow G(m+1) \times G(n+1)$ satisfies $i_{m,n}^*(c'_r) = c'_r$ for $r < m+1$, $i_{m,n}^*(c'_{m+1}) = 0$, and $i_{m,n}^*(c''_r) = c''_r$ for $r < n+1$, $i_{m,n}^*(c''_{n+1}) = 0$.

**Proof.** Follows as in the proof of Theorem 2.0.6 \qed

### 2.1 The symmetric ring spectrum representing algebraic cobordism

To give a construction of the symmetric $\mathbf{P}^1$-ring spectrum $\text{MGL}$, recall the external product of Thom spaces described in [V1, p. 422]. For vector bundles $V \to X$ and $W \to Y$ in $\text{Sm}/S$ one obtains a canonical map of pointed motivic spaces $\text{Th}(V) \wedge \text{Th}(W) \to \text{Th}(V \times_S W)$ which is a motivic weak equivalence as defined in [PPR1, Defn. 3.1.6]. In fact, the canonical map becomes an isomorphism after Nisnevich (even Zariski) sheafification.

The algebraic cobordism spectrum appears naturally as a $T$-spectrum, not as a $\mathbf{P}^1$-spectrum. Hence we describe it as a symmetric $T$-ring spectrum and obtain a symmetric $\mathbf{P}^1$-ring spectrum (and in particular a $\mathbf{P}^1$-ring spectrum) by switching the suspension coordinate (see [PPR1, A.6.9]). For $m, n \geq 0$ let $\mathcal{F}(n, mn) \to \text{Gr}(n, mn)$ denote the tautological vector bundle over the Grassmannian scheme of $n$-dimensional linear subspaces of $A^n_S = A^n_S \times_S \cdots \times_S A^n_S$. Permuting the copies of $A^n_S$ induces a $\Sigma_n$-action on $\mathcal{F}(n, mn)$ and $\text{Gr}(n, mn)$ such that the bundle projection is equivariant. The closed embedding $A^n_S = A^n_S \times \{0\} \hookrightarrow A^{n+1}_S$ defines a closed $\Sigma_n$-equivariant embedding $\text{Gr}(n, mn) \hookrightarrow \text{Gr}(n, (m+1)n)$. In particular, $\text{Gr}(n, mn)$ is pointed by $g_n : S = \text{Gr}(n, n) \hookrightarrow \text{Gr}(n, mn)$. The fiber of $\text{Gr}(n, mn)$ over $g_n$ is $A^n_S$. Let $\text{Gr}(n)$ be the colimit of the sequence

$$\text{Gr}(n, n) \hookrightarrow \text{Gr}(n, 2n) \hookrightarrow \cdots \hookrightarrow \text{Gr}(n, mn) \hookrightarrow \cdots$$
in the category of pointed motivic spaces over \( S \). The pullback diagram

\[
\begin{array}{ccc}
\mathcal{T}(n, mn) & \longrightarrow & \mathcal{T}(n, (m + 1)n) \\
\downarrow & & \downarrow \\
\text{Gr}(n, mn) & \longrightarrow & \text{Gr}(n, (m + 1)n)
\end{array}
\]

induces a \( \Sigma_n \)-equivariant inclusion of Thom spaces

\[\text{Th}(\mathcal{T}(n, mn)) \hookrightarrow \text{Th}(\mathcal{T}(n, (m + 1)n)).\]

Let \( \mathbb{MGL}_n \) denote the colimit of the resulting sequence

\[\mathbb{MGL}_n = \underset{m \geq n}{\text{colim}} \text{Th}(\mathcal{T}(n, mn))\]  \hspace{1cm} (4)

with the induced \( \Sigma_n \)-action. There is a closed embedding

\[
\text{Gr}(n, mn) \times \text{Gr}(p, mp) \hookrightarrow \text{Gr}(n + p, m(n + p))
\]  \hspace{1cm} (5)

which sends the linear subspaces \( V \hookrightarrow A^{mn} \) and \( W \hookrightarrow A^{mp} \) to the product subspace

\[V \times W \hookrightarrow A^{mn} \times A^{mp} = A^{m(n + p)}\]. In particular \((g_n, g_p)\) maps to \(g_{n+p}\). The inclusion (5) is covered by a map of tautological vector bundles and thus gives a canonical map of Thom spaces

\[
\text{Th}(\mathcal{T}(n, mn)) \wedge \text{Th}(\mathcal{T}(p, mp)) \to \text{Th}(\mathcal{T}(n + p, m(n + p)))
\]  \hspace{1cm} (6)

which is compatible with the colimit (4). Furthermore, the map (6) is \( \Sigma_n \times \Sigma_p \)-equivariant, where the product acts on the target via the standard inclusion \( \Sigma_n \times \Sigma_p \subseteq \Sigma_{n+p} \). After taking colimits, the result is a \( \Sigma_n \times \Sigma_p \)-equivariant map

\[
\mu_{n,p} : \mathbb{MGL}_n \wedge \mathbb{MGL}_p \to \mathbb{MGL}_{n+p}
\]  \hspace{1cm} (7)

of pointed motivic spaces (see [V1] p. 422]). The inclusion of the fiber \( A^p \) over \( g_p \) in \( \mathcal{T}(p) \) induces an inclusion \( \text{Th}(A^p) \subseteq \text{Th}(\mathcal{T}(p)) = \mathbb{MGL}_p \). Precomposing it with the canonical \( \Sigma_p \)-equivariant map of pointed motivic spaces

\[
\text{Th}(A^1) \wedge \text{Th}(A^1) \wedge \cdots \wedge \text{Th}(A^1) \to \text{Th}(A^p)
\]

defines a family of maps \( \epsilon_p : (\Sigma_p^\infty S_+)_p = T^{\wedge p} \to \mathbb{MGL}_p \). Inserting it in the inclusion (7) yields \( \Sigma_n \times \Sigma_p \)-equivariant structure maps

\[
\mathbb{MGL}_n \wedge \text{Th}(A^1) \wedge \text{Th}(A^1) \wedge \cdots \wedge \text{Th}(A^1) \to \mathbb{MGL}_{n+p}
\]  \hspace{1cm} (8)

of the symmetric \( T \)-spectrum \( \mathbb{MGL} \). The family of \( \Sigma_n \times \Sigma_p \)-equivariant maps (7) form a commutative, associative and unital multiplication on the symmetric \( T \)-spectrum \( \mathbb{MGL} \) (see [L] Sect. 4.3]). Regarded as a \( T \)-spectrum it coincides with Voevodsky’s spectrum \( \mathbb{MGL} \) described in [V1] 6.3].
Let $T$ be the Nisnevich sheaf associated to the presheaf $X \mapsto \mathbb{P}^1(X)/(\mathbb{P}^1 - \{0\})(X)$ on the Nisnevich site $Sm/S$. The canonical covering of $\mathbb{P}^1$ supplies an isomorphism
\[ T = \text{Th}(A^1_\Delta) \xrightarrow{\cong} T \]
of pointed motivic spaces. This isomorphism induces an isomorphism $\text{MSS}_T(S) \cong \text{MSS}_T(S)$ of the categories of symmetric $T$-spectra and symmetric $T$-spectra. In particular, $\text{MGL}$ may be regarded as a symmetric $T$-spectrum by just changing the structure maps up to an isomorphism. Note that the isomorphism of categories respects both the symmetric monoidal structure and the model structure. The canonical projection $p: \mathbb{P}^1 \to T$ is a motivic weak equivalence, because $A^1$ is contractible. It induces a Quillen equivalence
\[ \text{MSS}(S) = \text{MSS}_{\mathbb{P}^1}(S) \xrightarrow{p_!} \text{MSS}_T(S) \]
when equipped with model structures as described in [J] (see [PPRI A.6.9]). The right adjoint $p^*$ is very simple: it sends a symmetric $T$-spectrum $E$ to the symmetric $\mathbb{P}^1$-spectrum having terms $(p^*(E))_n = E_n$ and structure maps
\[ E_n \land \mathbb{P}^1 \xrightarrow{E_n \land p} E \land T \xrightarrow{\text{structure map}} E_{n+1}. \]
In particular $\text{MGL} := p^*\text{MGL}$ is a symmetric $\mathbb{P}^1$-spectrum by just changing the structure maps. Since $p^*$ is a lax symmetric monoidal functor, $\text{MGL}$ is a commutative monoid in a canonical way. Finally, the identity is a left Quillen equivalence from the model category $\text{MSS}^{\text{sm}}(S)$ used in [PPRI] to Jardine’s model structure by the proof of [PPRI A.6.4]. Let $\gamma: \text{Ho}(\text{MSS}^{\text{sm}}(S)) \to \text{SH}(S)$ denote the equivalence obtained by regarding a symmetric $\mathbb{P}^1$-spectrum just as a $\mathbb{P}^1$-spectrum.

**Definition 2.1.1.** Let $(\text{MGL}, \mu_{\text{MGL}}, e_{\text{MGL}})$ denote the commutative $\mathbb{P}^1$-ring spectrum which is the image $\gamma(\text{MGL})$ of the commutative symmetric $\mathbb{P}^1$-ring spectrum $\text{MGL}$ in the motivic stable homotopy category $\text{SH}(S)$.

**2.2 A universality theorem for the algebraic cobordism spectrum**

The complex cobordism spectrum, equipped with its natural orientation, is a universal oriented ring cohomology theory by Quillen’s universality theorem [Q]. In this section we prove a motivic version of Quillen’s universality theorem. The statement is contained already in [V2]. Recall that the $\mathbb{P}^1$-ring spectrum $\text{MGL}$ carries a canonical orientation $\text{th}^{\text{MGL}}$ as defined in [L1.3] It is the canonical map $\text{th}^{\text{MGL}}: \Sigma^\infty \text{Th}(\mathcal{O}(-1))(-1) \to \text{MGL}$ of $\mathbb{P}^1$-spectra.

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Theorem 2.2.1 (Universality Theorem). Let $E$ be a commutative $\mathbb{P}^1$-ring spectrum and let $S = \text{Spec}(k)$ for a field $k$. The assignment $\varphi \mapsto \varphi(th^{\text{MGL}}) \in E^{2,1}(\text{Th}(\mathcal{F}((1))))$ identifies the set of monoid homomorphisms
\[
\varphi : \text{MGL} \to E
\]
in the motivic stable homotopy category $\text{SH}^m(S)$ with the set of orientations of $E$. The inverse bijection sends an orientation $th \in E^{2,1}(\text{Th}(\mathcal{F}((1))))$ to the unique morphism
\[
\varphi \in E^{0,0}(\text{MGL}) = \text{Hom}_{\text{SH}(S)}(\text{MGL}, E)
\]
such that $u_i^*(\varphi) = \text{th}(\mathcal{F}(i)) \in E^{2i,i}(\text{Th}(\mathcal{F}(i)))$, where $\text{th}(\mathcal{F}(i))$ is given by (2) and $u_i : \Sigma_{\mathbb{P}^i}(\text{Th}(\mathcal{F}(i)))(-i) \to \text{MGL}$ is the canonical map of $\mathbb{P}^1$-spectra.

Proof. Let $\varphi : \text{MGL} \to E$ be a homomorphism of monoids in $\text{SH}(S)$. The class $\text{th} := \varphi(th^{\text{MGL}})$ is an orientation of $E$, because
\[
\varphi(\text{th})|_{\text{Th}(1)} = \varphi(\text{th}|_{\text{Th}(1)}) = \varphi(\Sigma_{\mathbb{P}^i}(1)) = \Sigma_{\mathbb{P}^i}(\varphi(1)) = \Sigma_{\mathbb{P}^i}(1).
\]
Now suppose $\text{th}^E \in E^{2i,i}(\text{Th}(\mathcal{F}(0))))$ is an orientation of $E$. We will construct a monoid homomorphism $\varphi : \text{MGL} \to E$ in $\text{SH}(S)$ such that $u_i^*(\varphi) = \text{th}(\mathcal{F}(i))$ and prove its uniqueness. To do so, we compute $E^{*,*}(\text{MGL})$. By [PPRT Cor. 2.1.4], this group fits into the short exact sequence
\[
0 \to \lim_{\leftarrow} E^{*+2i-1,*+i}(\text{Th}(\mathcal{F}(i))) \to E^{*,*}(\text{MGL}) \to \lim_{\leftarrow} E^{*+2i,*+i}(\text{Th}(\mathcal{F}(i))) \to 0
\]
where the connecting maps in the tower are given by the top line of the commutative diagram
\[
\begin{array}{ccccc}
E^{*+2i-1,*+i}(\text{Th}(i)) & \xrightarrow{\nabla_{\mathbb{P}^i}^{-1}} & E^{*+2i+1,*+i+1}(\text{Th}(i) \wedge \mathbb{P}^1) & \xleftarrow{\epsilon^*\text{inc}^*(\text{th}(\mathcal{F}(i) \oplus 1))} & E^{*+2i+1,*+i+1}(\text{Th}(i+1)) \\
\downarrow \text{th}(\mathcal{F}(i)) & & \downarrow \text{id} & & \downarrow \text{th}(\mathcal{F}(i+1)) \\
E^{*,*}(\text{Gr}(i)) & \leftarrow & E^{*,*}(\text{Gr}(i)) & \leftarrow & E^{*,*}(\text{Gr}(i+1))
\end{array}
\]
Here $\epsilon : \text{Th}(V) \wedge \mathbb{P}^1 \to \text{Th}(V \oplus 1)$ is the canonical map. The pull-backs $\text{inc}^*_i$ are all surjective by Theorem [2.0.4]. So we proved the following

Claim 2.2.2. The canonical map
\[
E^{*,*}(\text{MGL}) \to \lim_{\leftarrow} E^{*+2i,*+i}(\text{Th}(\mathcal{F}(i))) = E^{*,*}(k)[[c_1, c_2, c_3, \ldots]]
\]
is an isomorphism of two-sided $E^{*,*}(k)$-modules.
The family of elements \( th(\mathcal{F}(i)) \) is an element in the \( \lim \)-group, thus there is a unique element \( \varphi \in E^{0,0}(MGL) \) with \( u^*_i(\varphi) = th(\mathcal{F}(i)) \). We claim that \( \varphi \) is a monoid homomorphism. To check that it respects the multiplicative structure, consider the diagram

\[
\begin{array}{ccc}
\Sigma^\infty_{\mathbb{P}_1}(Th(\mathcal{F}(i)))(-i) \wedge \Sigma^\infty_{\mathbb{P}_1}(Th(\mathcal{F}(j)))(-j) & \xrightarrow{\Sigma^\infty_{\mathbb{P}_1}[\mu_{i,j}]} & \Sigma^\infty_{\mathbb{P}_1}(Th(\mathcal{F}(i+j)))(-i-j) \\
\downarrow_{u_i \wedge u_j} & & \downarrow_{u_{i+j}} \\
MGL \wedge MGL & \xrightarrow{\mu_{MGL}} & MGL \\
\downarrow_{\varphi \wedge \varphi} & & \downarrow_{\varphi} \\
E \wedge E & \xrightarrow{\mu_E} & E.
\end{array}
\]

Its enveloping square commutes in \( SH(S) \) by the chain of relations

\[
\varphi \circ u_{i+j} \circ \Sigma^\infty_{\mathbb{P}_1}(\mu_{i,j})(-i-j) = \mu^*_i(th(\mathcal{F}(i+j))) = th(\mathcal{F}(i) \times \mathcal{F}(j)) = th(\mathcal{F}(i)) \times th(\mathcal{F}(j)) = \mu_E(th(\mathcal{F}(i)) \wedge th(\mathcal{F}(j))) = \mu_E \circ ((\varphi \circ u_i) \wedge (\varphi \circ u_j)).
\]

To obtain the equality \( \mu_E \circ (\varphi \wedge \varphi) = \varphi \circ \mu_{MGL} \) in \( SH(k) \) consider the short exact sequence

\[
0 \to \lim_{i=1} E^{+i,1,*+2i}(\mathcal{F}(i)) \wedge th(\mathcal{F}(i)) \to E^{*,*}(MGL \wedge MGL) \\
\text{and these are surjective.}
\]

Note that since \( \mathcal{F}(i) \wedge \mathcal{F}(i) \cong \mathcal{F}(i) \times \mathcal{F}(i) \), there is a Thom isomorphism \( E^{+i-1,1,*+2i}(\mathcal{F}(i) \times \mathcal{F}(i)) \cong E^{*-1,*}(\text{Gr}(i) \times \text{Gr}(i)) \) by Theorem 2.0.4. The \( \lim \)-group is trivial because the connecting maps coincide with the pull-back maps

\[
E^{*-1,*}(\text{Gr}(i+1) \times \text{Gr}(i+1)) \to E^{*-1,*}(\text{Gr}(i) \times \text{Gr}(i))
\]

and these are surjective by Theorem 2.0.7. This implies the following

**Claim 2.2.3.** The canonical map

\[
E^{*,*}(MGL \wedge MGL) \to \lim_{\mathbb{P}_1} E^{+2i,*,*+i}(\mathcal{F}(i)) \wedge th(\mathcal{F}(i)) = E^{*,*}(k[[c'_1, c'_2, c'_3, \ldots]])
\]

is an isomorphism of two-sided \( E^{*,*}(k) \)-modules. Here \( c'_i \) is the \( i \)-th Chern class coming from the first factor of \( \text{Gr} \times \text{Gr} \) and \( c''_i \) is the \( i \)-th Chern class coming from the second factor.

Now the equality

\[
\varphi \circ u_{i+j} \circ \Sigma^\infty_{\mathbb{P}_1}(\mu_{i,j})(-2i) = \mu_E \circ ((\varphi \circ u_i) \wedge (\varphi \circ u_i))
\]

shows that \( \mu_E \circ (\varphi \wedge \varphi) = \varphi \circ \mu_{MGL} \) in \( SH(k) \).
To prove the Theorem it remains to check that the two assignments described in the Theorem are inverse to each other. An orientation \( th \in E^{2,1}(\text{Th} (\otimes (-1))) \) induces a morphism \( \varphi \) such that for each \( i \) one has \( \varphi \circ u_i = th (\mathcal{J}_i) \). And the new orientation \( th' := \varphi(th^{MGL}) \) coincides with the original one, due to the chain of relations

\[
\text{th}' = \varphi(th^{MGL}) = \varphi(u_1) = \varphi \circ u_1 = th(\mathcal{J}_1) = th(\otimes (-1)) = \text{th}.
\]

On the other hand a monoid homomorphism \( \varphi \) defines an orientation \( \text{th} := \varphi(th^{MGL}) \) of \( E \). The monoid homomorphism \( \varphi' \) we obtain then satisfies \( u_i^* (\varphi') = th(\mathcal{J}_i) \) for every \( i \geq 0 \). To check that \( \varphi' = \varphi \), recall that MGL is oriented, so we may use Claim 2.2.2 with \( E = \text{MGL} \) to deduce an isomorphism

\[
\text{MGL}^{*,*}(\text{MGL}) \to \varprojlim \text{MGL}^{*,*+2i,i}(\text{Th}(\mathcal{J}(i))).
\]

This isomorphism shows that the identity \( \varphi' = \varphi \) will follow from the identities \( u_i^* (\varphi') = u_i^*(\varphi) \) for every \( i \geq 0 \). Since \( u_i^*(\varphi) = th(\mathcal{J}_i) \) it remains to check the relation \( u_i^*(\varphi) = th(\mathcal{J}_i) \). It follows from the

**Claim 2.2.4.** There is an equality \( u_i = th^{MGL}(\mathcal{J}_i) \in \text{MGL}^{2i,i}(\text{Th}(\mathcal{J}(i))) \).

In fact, \( u_i^* (\varphi) = \varphi \circ u_i = \varphi(u_i) = \varphi(th^{MGL}(\mathcal{J}(i))) = th(\mathcal{J}(i)) \). The last equality in this chain of relations holds, because \( \varphi \) is a monoid homomorphism sending \( th^{MGL} \) to \( th \). It remains to prove the Claim. We will do this in the case \( i = 2 \). The general case can be proved similarly. The commutative diagram

\[
\begin{array}{ccc}
\Sigma_{\mathbf{p}, i} \text{Th} (\mathcal{J}(1)) (-1) \land \Sigma_{\mathbf{p}, i} \text{Th} (\mathcal{J}(1)) (-1) & \xrightarrow{\Sigma_{\mathbf{p}, i} (\mu_{1,1})(-2)} & \Sigma_{\mathbf{p}, i} \text{Th} (\mathcal{J}(2)) (-2) \\
\downarrow \quad u_1 \land u_1 & & \downarrow u_2 \\
\text{MGL} \land \text{MGL} & \xrightarrow{\mu_{MGL}} & \text{MGL}
\end{array}
\]

in \( \text{SH}(k) \) implies that

\[
\mu_{1,1}^*(u_2) = u_1 \times u_1 \in \text{MGL}^{4,2}(\text{Th} (\mathcal{J}(1)) \land \text{Th} (\mathcal{J}(1))) = \text{MGL}^{4,2}(\text{Th} (\mathcal{J}(1) \times \mathcal{J}(1))).
\]

The equalities

\[
\mu_{1,1}^*(th^{MGL}(\mathcal{J}(2))) = th^{MGL}(\mu_{1,1}^*(\mathcal{J}(2))) = th^{MGL}(\mathcal{J}(1) \times \mathcal{J}(1))
\]

\[
= th^{MGL}(\mathcal{J}(1)) \times th^{MGL}(\mathcal{J}(1))
\]

imply that it remains to prove the injectivity of the map \( \mu_{1,1}^* \). Consider the commutative diagram

\[
\begin{array}{ccc}
\text{MGL}^{*,*}(\text{Th}(\mathcal{J}(1) \times \mathcal{J}(1))) & \xleftarrow{\mu_{1,1}^*} & \text{MGL}^{*,*}(\text{Th}(\mathcal{J}(2))) \\
\text{Thom} \cong & & \cong \text{Thom} \\
\text{MGL}^{*,*}(\text{Gr}(1) \times \text{Gr}(1)) & \xleftarrow{\mu_{1,1}^*} & \text{MGL}^{*,*}(\text{Gr}(2))
\end{array}
\]

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where the vertical arrows are the Thom isomorphisms from Theorem 2.0.4 and \(\nu_{1,1} : \text{Gr}(1) \times \text{Gr}(1) \hookrightarrow \text{Gr}(2)\) is the embedding described by equation (2). For an oriented commutative ring \(\mathbb{P}^1\)-spectrum \((E, \theta)\) one has \(E^\ast(k)(\text{Gr}(2)) = E^\ast(k)[[c_1, c_2]]\) (the formal power series on \(c_1, c_2\)) by Theorem 2.0.6. From the other hand

\[
E^\ast(k)(\text{Gr}(1) \times \text{Gr}(1)) = E^\ast(k)[[t_1, t_2]]
\]

(the formal power series on \(t_1, t_2\)) by Theorem 2.0.7 and the map \(\nu_{1,1}^\ast\) takes \(c_1\) to \(t_1\) and \(c_2\) to \(t_2\). Whence \(\nu_{1,1}^\ast\) is injective. The proofs of the Claim and of the Theorem are completed.

\[
\square
\]

References


