THE GLOBAL STRUCTURE OF ODD-PRIMARY DICKSON ALGEBRAS AS ALGEBRAS OVER THE STEENROD ALGEBRA

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Dedicated to the memory of Franklin P. Peterson.

ABSTRACT. We prove a conjecture made by Frank Peterson on the global structure of the Dickson algebras arising as odd primary general linear group invariants. The Dickson algebra W_n of invariants in a rank n polynomial algebra over $\mathbb{F}_p$ is an unstable algebra over the mod p Steenrod algebra. We prove that W_n is a free unstable algebra on a certain cyclic module, modulo just one additional relation. The result is both similar to and different from the corresponding result we previously obtained with Frank Peterson at the prime 2. We also extend our characterization to the algebras of invariants under the special linear groups.

April 3, 2002

1. INTRODUCTION

With Frank Peterson we proved a global structure theorem for the algebras of mod 2 Dickson invariants, as algebras over the Steenrod algebra [PPW]. Before his untimely death in the year 2000, Frank conjectured to us a corresponding result at odd primes. He told us he had proved it for the first two Dickson algebras, and had largely completed a proof for the third. Here we prove that Frank's conjecture is correct for all the Dickson algebras, stated in an alternative but more tractable form, as we shall explain (Theorem 2.2). We also remark on other aspects of the representation provided by this global characterization. Finally, we extend the theorem to the subalgebras of invariants under the special linear groups (Theorem 3.3).

Let p be an odd prime, and let $\mathcal{A}$ denote the quotient of the Steenrod algebra by the two-sided ideal generated by the Bockstein; $\mathcal{A}$ is generated by the reduced powers $P^1, P^p, \ldots, P^{p^k}, \ldots$, where P^{p^k} has

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1991 *Mathematics Subject Classification.* Primary 55S05; Secondary 13A50, 16W30, 16W22, 16W50, 55S10.

Key words and phrases. Steenrod algebra, Dickson algebras, invariants, general linear group, special linear group.

degree $2\,(p-1)\,p^k$, subject to the Adem relations [SE]. The notation $\mathcal{A}_i$ shall indicate the subalgebra of $\mathcal{A}$ generated by $P^1, P^p, \dots, P^{p^i}$.

We begin by recalling the algebra of Dickson invariants (we refer to the corrected version of Wilkerson [W] for information about this algebra). Begin by considering, for $n \geq 1$, a graded polynomial algebra on n variables of degree two over the field $\mathbb{F}_p$. This algebra has a unique unstable $\mathcal{A}$-algebra structure [SE]. The general linear group acts on this algebra by acting on the homogeneous component in degree two. The invariants under this action form the Dickson algebra W_n, which is a polynomial subalgebra on n generators $c_{n-1}, \dots, c_0$, where the degree of c_i is $2(p^n - p^i)$. The algebra W_n inherits the unstable $\mathcal{A}$-algebra structure determined by

$$\begin{aligned} P^{p^{i-1}} c_i &= c_{i-1}, \\ P^{p^{n-1}} c_i &= -c_i c_{n-1}, \\ P^{p^k} c_i &= 0, \quad \text{if } k \neq n-1 \text{ or } i-1. \end{aligned}$$

We shall also consider another subalgebra, consisting of the invariants under the action of the special linear subgroup. We shall denote this subalgebra by V_n. Clearly V_n contains W_n. According to Wilkerson, V_n is isomorphic to the polynomial algebra on the Dickson generators $c_{n-1}, \dots, c_1$, together with a generator a satisfying $a^{p-1} = c_0$. The action of the Steenrod algebra on a is given by

$$\begin{aligned} P^{p^{n-1}} a &= a c_{n-1}, \\ P^{p^k} a &= 0, \quad \text{if } k \neq n-1. \end{aligned}$$

2. Global structure of Dickson algebras

We define an unstable cyclic $\mathcal{A}$-module $\mathcal{M}_n$, and from it an unstable $\mathcal{A}$-algebra $\mathcal{G}_n$, both with explicit generators and relations. We then prove that this yields the Dickson algebra W_n.

Definition 2.1. The module $\mathcal{M}_n$ has one generator u, where $|u| = 2(p^n - p^{n-1})$, and relations

(a) $$P^{p^k} u = 0, \quad \text{for } 0 \leq k \leq n-3,$$

(b) $$P^{p^{n-2}} P^{p^{n-2}} u = 0, \quad \text{and}$$

(c) $$P^{p^{n-2}} P^{p^{n-1}} u = 2 P^{p^{n-1}} P^{p^{n-2}} u.$$

The algebra $\mathcal{G}_n$ is the free unstable $\mathcal{A}$-algebra [SE] on the module $\mathcal{M}_n$, subject to the single additional relation

(d) $$P^{p^{n-1}} u = -u^2.$$

Theorem 2.2. *The algebra* $\mathcal{G}_n$ *is isomorphic as an* $\mathcal{A}$*-algebra to* W_n*, the Dickson algebra on* n *generators.*

Before proving the theorem, we make several remarks.

Remark 2.3. We describe how the theorem is equivalent to what Frank Peterson conjectured, thereby revealing how it directly generalizes the mod 2 result we proved together in [PPW]. Relations (a) above are completely analogous to those in the mod 2 result (via $P^{p^k} \leftrightarrow Sq^{2^k}$), while (b) is a relation whose mod 2 analog is trivial, and (d) is inherent mod 2 from instability. Now in the presence of (a) and (d), the left side of relation (c) transforms, via the calculation

$$P^{p^{n-2}} P^{p^{n-1}} u = P^{p^{n-2}} \left(-u^2\right) = -2u \cdot P^{p^{n-2}} u,$$

into the equivalent relation

$$\text{(c}'\text{)} \qquad -u \cdot P^{p^{n-2}} u = P^{p^{n-1}} P^{p^{n-2}} u,$$

which is perfectly analogous to the single additional relation imposed in the mod 2 result. Thus we see that relations (a),(b),(c′),(d) directly generalize the mod 2 result to all primes, and it was in this form that Frank Peterson presented us his conjecture. Once we realized that at odd primes, the algebra relation (c′) could be replaced by the equivalent relation (c), which is purely a module relation in $\mathcal{M}_n$, not openly involving the algebra structure of $\mathcal{G}$, Frank's conjecture seemed much more tractable to prove.

Remark 2.4. At the prime 2, we provided a basis for the analogous module underlying the construction, in terms of the Kudo-Araki-May algebra $\mathcal{K}$ [PPW], and were able to show that the module injects into the Dickson algebra [PPW, Proof of Thm. 2.11]. Our basis was used in proving the global structure theorem as well. This approach seems harder for odd primes. Our proof at odd primes of the global structure theorem above does not rely on a basis for $\mathcal{M}_n$. Nor have we proven that $\mathcal{M}_n$ injects into $\mathcal{G}_n$.

Remark 2.5. The relations in $\mathcal{G}_n$ are minimal, i.e., none are redundant. This can be verified from the Adem relations by careful upward induction in the indecomposable algebra quotient on the degrees of the relations.

Remark 2.6. Our method of proof is simple to describe. Verify that the defining relations of $\mathcal{G}_n$ hold also in W_n, yielding an $\mathcal{A}$-algebra map $\mathcal{G}_n \to W_n$, which is clearly seen to hit the algebra generators. Since W_n is a free commutative algebra, this will be an isomorphism provided the indecomposable quotients correspond. Most of the work is in this last step, i.e., showing that $\mathcal{G}_n$ has no more algebra generators than W_n.

We prepare for the detailed proof with two lemmas and a corollary about the Steenrod algebra.

Lemma 2.7. *Let* $a \leq b-2$. *Then*

$$P^{p^a} P^{p^b} \equiv P^{p^{b-1}} P^{(p-2)p^{b-1}+p^a} P^{p^{b-1}} \quad mod\ \mathcal{A}\overline{\mathcal{A}_{b-2}}\ .$$

Proof. The result follows from the following three formulas, the first line of each of which arises from an Adem relation [SE]:

$$\begin{aligned} P^{p^a} P^{p^b} &\equiv -\binom{(p-1)p^b-1}{p^a} P^{p^a+p^b} \\ &\equiv P^{p^a+p^b} \text{ mod } \mathcal{A}\overline{\mathcal{A}_{a-1}}\ , \end{aligned}$$

$$\begin{aligned} P^{p^{b-1}} P^{p^b-p^{b-1}+p^a} &\equiv -\binom{(p-1)(p^b-p^{b-1}+p^a)-1}{p^{b-1}} P^{p^a+p^b} \\ &\equiv -P^{p^a+p^b} \text{ mod } \mathcal{A}\overline{\mathcal{A}_{b-2}}\ , \end{aligned}$$

$$\begin{aligned} P^{(p-2)p^{b-1}+p^a} P^{p^{b-1}} &\equiv \binom{(p-1)p^{b-1}-1}{(p-2)p^{b-1}+p^a} P^{(p-1)p^{b-1}+p^a} \\ &\equiv -P^{(p-1)p^{b-1}+p^a} \text{ mod } \mathcal{A}\overline{\mathcal{A}_{b-2}}\ . \end{aligned}$$

□

Lemma 2.8. *Let* $r \leq m < l$. *Then*

$$P^{p^{n-l}+\cdots+p^{n-m-1}} P^{p^{n-m}+\cdots+p^{n-r}} \equiv P^{p^{n-l}+\cdots+p^{n-r}} \quad mod\ \mathcal{A}\overline{\mathcal{A}_{n-m-2}}\ .$$

Proof. We note that $p^{n-l-1}+\cdots+p^{n-m-2} < p^{n-m-1}$. An Adem relation gives us congruences modulo $\mathcal{A}\overline{\mathcal{A}_{n-m-2}}$:

$$\begin{aligned} & P^{p^{n-l}+\cdots+p^{n-m-1}} P^{p^{n-m}+\cdots+p^{n-r}} \\ \equiv\ & (-1)^{l-m} \binom{p^{n-r+1}-p^{n-m}-1}{p^{n-m-1}+\cdots+p^{n-l}} P^{p^{n-l}+\cdots+p^{n-r}} \\ \equiv\ & (-1)^{l-m} \binom{(p-1)p^{n-r}+\cdots+(p-2)p^{n-m}+\cdots+(p-1)}{p^{n-m-1}+\cdots+p^{n-l}} P^{p^{n-l}+\cdots+p^{n-r}} \\ \equiv\ & P^{p^{n-l}+\cdots+p^{n-r}}\ . \end{aligned}$$

□

Corollary 2.9. *We have, for* $r \leq l$, *that*

$$P^{p^{n-l}} \cdots P^{p^{n-r}} \equiv P^{p^{n-l}+\cdots+p^{n-r}} \quad mod\ \mathcal{A}\overline{\mathcal{A}_{n-r-2}}.$$

Proof of Theorem 2.2. It is easy to check, from the formulas determining the $\mathcal{A}$-action on W_n, that the action on the fundamental class $c_{n-1} \in W_n$ obeys the defining $\mathcal{A}$-action relations (a),(b),(c),(d) on the fundamental class $u \in \mathcal{G}_n$. Hence there is a unique $\mathcal{A}$-algebra map $\psi : \mathcal{G}_n \to W_n$ carrying u to c_{n-1}. Moreover, the $\mathcal{A}$-action on W_n also shows that ψ carries $P^{p^k} P^{p^{k+1}} \cdots P^{p^{n-2}} u$ to c_k for $0 \leq k \leq n-2$, and thus ψ is an algebra epimorphism. Since W_n is a free commutative algebra, our proof will be complete if we show that all other $P^I u$ are algebra decomposables in $\mathcal{G}_n$, for multi-indices I consisting of arbitrary powers of p. The remainder of the proof is devoted to confirming these decomposabilities.

We proceed by induction on the length of I. From the defining relations, clearly every such $P^I u$ is decomposable unless the right entry of I is p^{n-2}. It thus remains to show, by induction on $l \geq 2$, that each element of the form

$$P^{p^a} P^{p^{n-l}} P^{p^{n-l+1}} \cdots P^{p^{n-2}} u$$

is decomposable for $a \neq n-l-1$, where $2 \leq l \leq n$. From instability we need only consider $a \leq n-1$. In light of relation (a), the Corollary above shows that it will be equivalent to demonstrate that each such

$$P^{p^a} P^{p^{n-l}+\cdots+p^{n-2}} u$$

is decomposable. These are all inadmissible monomials on u since $a \leq n-1$, except for $a = n-1$ when $l = 2$.

In the remainder of the proof, we will use relation (a) and Corollary 2.9 liberally in calculations, and often without mention. We first use Adem relations to calculate

$$\begin{aligned} & P^{p^a} P^{p^{n-l}+\cdots+p^{n-2}} u \\ & = -\binom{p^{n-1} - p^{n-l} - 1}{p^a} P^{p^a+p^{n-l}+\cdots+p^{n-2}} u + P^{p^a - p^{a-1} + p^{n-l}+\cdots+p^{n-2}} P^{p^{a-1}} u \\ & = \begin{cases} P^{p^{n-l}+\cdots+p^{n-3}+p^{n-1}} P^{p^{n-2}} u \ \text{ for } a = n-1, & (1) \\ 2P^{p^{n-l}+p^{n-l}+\cdots+p^{n-2}} u \ \text{ for } a = n-l, & (2) \\ P^{p^a+p^{n-l}+\cdots+p^{n-2}} u \ \text{ otherwise.} & (3) \end{cases} \end{aligned}$$

We now verify the decomposability claimed for $a \neq n-l-1$, by considering four cases, each depending on where a lies in relation to $n-l$ and $n-1$.

Case 1. Suppose $a \leq n-l-2$.

Recall we are inducting upwards on length l. For the base instance $l = 2$ we have $a \leq n-4$, and Lemma 2.7 yields

$$P^{p^a}P^{p^{n-2}} = P^{p^{n-3}}P^{(p-2)p^{n-3}+p^a}P^{p^{n-3}} \bmod \mathcal{A}\overline{\mathcal{A}_{n-4}},$$

hence $P^{p^a}P^{p^{n-2}}u = 0$ from relation (a).

Now we may assume inductively that for all $a' \leq n-l-1$ we have

$$P^{p^{a'}}P^{p^{n-l+1}+\cdots+p^{n-2}}u = 0.$$

Recalling that $a \leq n-l-2$, we use relation (a) liberally and have, for $l \geq 3$,

$$\begin{aligned} P^{p^a}P^{p^{n-l}+\cdots+p^{n-2}}u &= P^{p^a}P^{p^{n-l}}P^{p^{n-l+1}+\cdots+p^{n-2}}u \text{ (Lemma 2.8)} \\ &\equiv P^{p^{n-l-1}}P^{(p-2)p^{n-l-1}+p^a}P^{p^{n-l-1}}P^{p^{n-l+1}+\cdots+p^{n-2}}u \\ &\bmod \mathcal{A}\overline{\mathcal{A}_{n-l-2}} \cdot P^{p^{n-l+1}+\cdots+p^{n-2}}u \text{ (Lemma 2.7)} \\ &= 0 \text{ (by induction on } l\text{)}. \end{aligned}$$

Case 2. Suppose $a = n-1$.

We prepare with the Adem relation (for $l \geq 3$)

$$\begin{aligned} P^{p^{n-l}+\cdots+p^{n-3}}P^{p^{n-1}} &\equiv (-1)^{l-2}\binom{(p-1)p^{n-1}-1}{p^{n-3}+\cdots+p^{n-l}}P^{p^{n-l}+\cdots+p^{n-3}+p^{n-1}} \\ &\equiv P^{p^{n-l}+\cdots+p^{n-3}+p^{n-1}} \bmod \mathcal{A}\overline{\mathcal{A}_{n-4}}\ . \end{aligned}$$

Combining this with (1) at the beginning of the proof produces, for $l \geq 2$,

$$P^{p^{n-1}}P^{p^{n-l}+\cdots+p^{n-2}}u \equiv P^{p^{n-l}+\cdots+p^{n-3}}P^{p^{n-1}}P^{p^{n-2}}u \bmod \mathcal{A}\overline{\mathcal{A}_{n-4}}P^{p^{n-2}}u.$$

But $P^{p^{n-1}}P^{p^{n-2}}u$ is decomposable from relations (c) and (d), and by Lemma 2.7 the indeterminacy is zero.

Case 3. Suppose $a = n-l$.

If $l = 2$, we have $P^{p^{n-2}}P^{p^{n-2}}u = 0$ from relation (b). For $3 \leq l \leq n$ we again prepare, from Adem relations, with

$$\begin{aligned} P^{p^{n-2}}P^{p^{n-l}+p^{n-l}+\cdots+p^{n-3}} &\equiv -\binom{p^{n-2}+p^{n-l+1}-2p^{n-l}-1}{p^{n-2}}P^{p^{n-l}+p^{n-l}+\cdots+p^{n-2}} \\ &\equiv -P^{p^{n-l}+p^{n-l}+\cdots+p^{n-2}} \bmod \mathcal{A}\overline{\mathcal{A}_{n-3}}. \end{aligned}$$

Since $p^{n-l}+p^{n-l}+\cdots+p^{n-3} < p^{n-2}$, this tells us that $P^{p^{n-l}+p^{n-l}+\cdots+p^{n-2}}u = 0$. Thus from (2) above, we obtain

$$P^{p^{n-l}}P^{p^{n-l}+\cdots+p^{n-2}}u = 0.$$

Case 4. Suppose $n-l+1 \leq a \leq n-2$.

Write $a = n-m$, so $2 \leq m < l$. We must consider the expression $P^{p^{n-m}}P^{p^{n-l}+\cdots+p^{n-2}}u$. Our philosophy is to move the two occurrences of p^{n-m} as far to the right as possible. We begin with

$$P^{p^{n-m}}P^{p^{n-l}+\cdots+p^{n-2}}u = P^{p^{n-m}}P^{p^{n-l}+\cdots+p^{n-m}}P^{p^{n-m+1}+\cdots+p^{n-2}}u,$$

from Lemma 2.8 and (a).

Now we appeal to Adem relations

$$\begin{aligned} P^{p^{n-l}+\cdots+p^{n-m}}P^{p^{n-m}} &\equiv (-1)^{l-m+1}\binom{(p-1)p^{n-m}-1}{p^{n-m}+\cdots+p^{n-l}}P^{p^{n-l}+\cdots+2p^{n-m}} \\ &\equiv 2P^{p^{n-l}+\cdots+2p^{n-m}} \bmod \mathcal{A}\overline{\mathcal{A}_{n-m-1}} \end{aligned}$$

and

$$\begin{aligned} P^{p^{n-m}}P^{p^{n-l}+\cdots+p^{n-m}} &\equiv -\binom{p^{n-m+1}-p^{n-l}-1}{p^{n-m}}P^{p^{n-l}+\cdots+2p^{n-m}} \\ &\equiv P^{p^{n-l}+\cdots+2p^{n-m}} \bmod \mathcal{A}\overline{\mathcal{A}_{n-m-1}} \text{ (since } m<l) \end{aligned}$$

to obtain the $\mathcal{A}$-relation

$$P^{p^{n-m}}P^{p^{n-l}+\cdots+p^{n-m}} \equiv \frac{1}{2}P^{p^{n-l}+\cdots+p^{n-m}}P^{p^{n-m}} \bmod \mathcal{A}\overline{\mathcal{A}_{n-m-1}}.$$

Combining this with our equation above on u yields congruence modulo $\mathcal{A}\overline{\mathcal{A}_{n-m-1}}\cdot P^{p^{n-m+1}+\cdots+p^{n-2}}u$:

$$\begin{aligned} & P^{p^{n-m}}P^{p^{n-l}+\cdots+p^{n-2}}u \\ \equiv\ & \frac{1}{2}P^{p^{n-l}+\cdots+p^{n-m}}P^{p^{n-m}}P^{p^{n-m+1}+\cdots+p^{n-2}}u \\ =\ & \frac{1}{2}P^{p^{n-l}+\cdots+p^{n-m}}P^{p^{n-m}+p^{n-m+1}+\cdots+p^{n-2}}u, \end{aligned}$$

the latter from Lemma 2.8, and the indeterminacy vanishes by induction on l. Now we continue this calculation, again using Lemma 2.8 to obtain

$$\begin{aligned} & \frac{1}{2}P^{p^{n-l}+\cdots+p^{n-m}}P^{p^{n-m}+p^{n-m+1}+\cdots+p^{n-2}}u \\ \equiv\ & \frac{1}{2}P^{p^{n-l}+\cdots+p^{n-m-1}}P^{p^{n-m}}P^{p^{n-m}+\cdots+p^{n-2}}u \\ & \bmod \mathcal{A}\overline{\mathcal{A}_{n-m-2}}\cdot P^{p^{n-m}+\cdots+p^{n-2}}u \\ =\ & 0 \text{ (both by induction on } l\text{, since } m<l). \end{aligned}$$

□

3. GLOBAL STRUCTURE OF THE SPECIAL LINEAR GROUP INVARIANTS

We define an $\mathcal{A}$-module $\mathcal{N}_n$ and an $\mathcal{A}$-algebra $\mathcal{H}_n$ as follows.

Definition 3.1. The module $\mathcal{N}_n$ has two generators u and t, where $|u| = 2(p^n - p^{n-1})$ and $|t| = 2(p^{n-1} + \cdots + 1)$, with relations

$$\begin{gathered} P^{p^k} u = 0, \quad \text{for } 0 \leq k \leq n-3, \\ P^{p^{n-2}} P^{p^{n-2}} u = 0, \\ P^{p^{n-2}} P^{p^{n-1}} u = 2P^{p^{n-1}} P^{p^{n-2}} u, \quad \text{and} \\ P^{p^k} t = 0, \quad \text{for } 0 \leq k \leq n-2. \end{gathered}$$

Definition 3.2. The algebra $\mathcal{H}_n$ is the free $\mathcal{A}$-algebra on the module $\mathcal{N}_n$, subject to the relations

$$\begin{aligned} P^{1+p+\cdots+p^{n-2}} u &= t^{p-1}, \\ P^{p^{n-1}} t &= tu, \quad \text{and} \\ P^{p^{n-1}} u &= -u^2. \end{aligned}$$

Theorem 3.3. *The algebra $\mathcal{H}_n$ is isomorphic as an $\mathcal{A}$-algebra to V_n, the algebra of invariants under the special linear group.*

Remark 3.4. The relations in $\mathcal{H}_n$ are minimal, i.e., none are redundant. This can be verified from the Adem relations by careful upward induction in the indecomposable algebra quotient on the degrees of the relations, and using what was already shown in the previous section.

Proof. We proceed as in the proof of the preceding theorem. Since the $\mathcal{A}$-action on t yields only algebra decomposables, the previous proof shows that the indecomposable algebra quotient of $\mathcal{H}_n$ is spanned by t, u and

$$P^{p^i} P^{p^{i+1}} \cdots P^{p^{n-2}} u, \quad \text{for } 1 \leq i \leq n-2,$$

since this time $i = 0$ produces a decomposable from

$$P^{p^0} \cdots P^{p^{n-2}} u = P^{1+\cdots+p^{n-2}} u = t^{p-1}.$$

If c_{n-1} and a are the classes in V_n described in the introduction, then the action of the Steenrod algebra on V_n clearly satisfies the relations on u and t, respectively, that define $\mathcal{H}_n$. Hence there is a unique $\mathcal{A}$-algebra map $\phi : \mathcal{H}_n \to V_n$ that takes u to c_{n-1} and t to a, and hits the algebra generators $a, c_{n-1}, \ldots, c_1$ of V_n. As in the previous proof, we thus see that the algebra generators correspond, and since V_n is a free commutative algebra, ϕ is an isomorphism. $\square$

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