EXOTIC NORMAL FUSION SUBSYSTEMS OF GENERAL LINEAR GROUPS

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Abstract. We classify the saturated fusion subsystems of index prime to p of the general linear group over $\mathbb{F}_q$ over a Sylow p-subgroup, where q is a prime power prime to an odd prime p. In this classification we get some of the exotic p-local finite groups discovered by C. Broto and J. Møller as saturated fusion subsystems of the general linear group.

1. Introduction

The concept of p-local finite group arose in the work of C. Broto, R. Levi and B. Oliver [3] as a formalization of the p-local structure of a finite group. A p-local finite group consists of a triple $(S, \mathcal{F}, \mathcal{L})$ where S is a p-group, $\mathcal{F}$ is a category defined in an axiomatic way which models the fusion category over S, a Sylow p-subgroup of a finite group, and $\mathcal{L}$ is an extension of $\mathcal{F}$ which contains extra information so that its p-completed nerve has many of the same properties as the p-completion of the classifying space of a finite group.

One source of examples of p-local finite groups is the ones constructed from finite groups: when G is a finite group and S is a Sylow p-subgroup in G, we can construct a triple $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ which is a p-local finite group. We can recover the p-primary information of G using the fact that in this case $|BG|_p^\wedge \simeq |\mathcal{L}_S^c(G)|_p^\wedge$.

The examples which cannot be constructed from a finite group are called exotic examples. There are known exotic examples of p-local finite groups for all primes p, and these have been constructed in two different ways:

(i) Examples by C. Broto, R. Levi and B. Oliver ([3], [4]), and A. Díaz, A. Viruel and the author ([6], [10]) are constructed in a combinatorial way: they start with the saturated fusion system of a finite group and they add morphisms to the automorphism group of some proper subgroups.
Also Solomon's example by R. Levi and B. Oliver ([7]): they show that the fusion on the Sylow 2-subgroup of $\mathrm{Spin}_7(q)$ (q an odd prime power) considered by Solomon, which is known not to occur as the fusion of any finite group, has a p-local finite group structure.

(ii) Homotopy fixed point sets of p-compact groups by C. Broto and J. Møller [5].

In this paper we use the study of saturated fusion subsystems by C. Broto, N. Castellana, J. Grodal, R. Levi and B. Oliver [2] to obtain some of the classifying spaces of exotic p-local finite groups from [3] and [5] as, up to homotopy equivalence, covering spaces of the classifying spaces of a non-exotic p-local finite groups (Remark 6.4).

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More exactly we classify the saturated fusion subsystems of index prime to p of the general linear group $\mathrm{GL}_n(q)$ for q a prime power prime to p.

Now we will describe the contents of the paper, giving simplified versions of the main results.

In Section 3 we review the classification of the saturated fusion subsystems of index prime to p from [2]. For any $(S, \mathcal{F}, \mathcal{L})$ a p-local finite group, this classification is given in terms of the $\mathcal{F}$-centric subgroups of S and $O_*^{p'}(\mathcal{F})$, the smallest subcategory with the same objects as $\mathcal{F}$ and which contains all the restrictions of all the automorphisms in $\mathcal{F}$ of p-power order. With this data there is a bijection between saturated fusion subsystems of index prime to p in $\mathcal{F}$ and subgroups of

$$\Gamma_{p'}(\mathcal{F}) \overset{\text{def}}{=} \mathrm{Out}_{\mathcal{F}}(S)/\,\mathrm{Out}^0_{\mathcal{F}}(S)\,,$$

where $\mathrm{Out}^0_{\mathcal{F}}(S)$ is defined as

$$\mathrm{Out}^0_{\mathcal{F}}(S) \overset{\text{def}}{=} \langle \alpha \in \mathrm{Out}_{\mathcal{F}}(S) \mid \alpha|_P \in \mathrm{Mor}_{O_*^{p'}(\mathcal{F})}(P, S),\ \text{some } \mathcal{F}\text{-centric } P \rangle\,.$$

If we want to compute $O_*^{p'}(\mathcal{F})$ in some particular cases we would like to restrict the family of subgroups of S involved in this computation to a smaller one. So in this section we give the following general result (Theorem 3.4):

Theorem A. *Let $\mathcal{F}$ be a saturated fusion system over S. Then for each morphism $\psi \in \mathrm{Hom}_{O_*^{p'}(\mathcal{F})}(P, P')$, there exists a sequence of subgroups of S*

$$P = P_0, P_1, \ldots, P_k = P' \quad \text{and} \quad Q_1, Q_2, \ldots, Q_k,$$

and morphisms $\psi_i \in \mathrm{Aut}_{O_*^{p'}(\mathcal{F})}(Q_i)$, such that

- *Q_i is fully $\mathcal{F}$-normalized, $\mathcal{F}$-centric and $\mathcal{F}$-radical for each i;*
- *$P_{i-1}, P_i \leq Q_i$ and $\psi_i(P_{i-1}) = P_i$ for each i; and*
- *$\psi(u) = \psi_k \circ \psi_{k-1} \circ \cdots \circ \psi_1(u)$, $\forall u \in P$.*

In Sections 4 and 5 we use J.L. Alperin and P. Fong results in [1] to describe the possible $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups in $\mathrm{GL}_n(q)$. This description enables us to classify the saturated fusion subsystems of index prime to p in $\mathrm{GL}_n(q)$, using the following result obtained here as Theorem 5.10:

Theorem B. *Let p be a prime and q a prime power prime to p. Let e be the multiplicative order of q modulo p, and $n \geq ep$. Consider $(S_{n,q}, \mathcal{F}_{n,q}, \mathcal{L}_{n,q})$ the p-local finite group induced by $\mathrm{GL}_n(q)$ over $S_{n,q}$, a Sylow p-subgroup. Then*

$$\Gamma_{p'}(\mathcal{F}_{n,q}) \cong \mathbb{Z}/e\,.$$

In Section 6 we identify the p-local finite groups corresponding to these saturated fusion subsystems, getting that these correspond to the p-local finite groups described by C. Broto and J. Møller as the finite Chevalley version of the generalized Grassmannians, denoted by $X(e, r, m)(q')$, for any positive integers e, r, m such that r divides e, e divides $(p-1)$ and q' a p-adic unit. More precisely the result that we get in Proposition 6.1 and Theorem 6.3 is:

Theorem C. *Fix* p, q, e *and* $(S_{n,q}, \mathcal{F}_{n,q}, \mathcal{L}_{n,q})$ *as in Theorem B, and, for each* r *a divisor of* e, *let* $(S_{n,q}, \mathcal{F}_{n,q,r}, \mathcal{L}_{n,q,r})$ *the unique* p*-local finite group such that* $(S_{n,q}, \mathcal{F}_{n,q,r})$ *is the saturated fusion subsystem of index prime to* p *in* $(S_{n,q}, \mathcal{F}_{n,q})$ *such that* $\operatorname{Out}_{\mathcal{F}_{n,q}}(S_{n,q})/\operatorname{Out}_{\mathcal{F}_{n,q,r}}(S_{n,q}) \cong \mathbb{Z}/r$. *Then:*

(a) *Up to homotopy equivalence, there is a fibration*

$$|\mathcal{L}_{n,q,r}| \to |\mathcal{L}_{n,q}| \to B(\mathbb{Z}/r)\,.$$

(b) $|\mathcal{L}_{n,q,r}| \simeq BX(e,r,[n/e])(q^e)$ *up to* p*-completion, where* $[n/e]$ *is the greatest integer less or equal than* n/e.

Using that $BX(e,r,m)(q')$ are known to be exotic p-local finite groups when $m \geq p$ and $r > 2$ [5, Proposition 11.5], this result provides us examples of extensions of p-local finite groups where one of the involved elements is exotic, and the other two correspond to finite groups. These examples answer a question proposed by B. Oliver in the Banff conference on *Homotopy theory and group actions* (November 2005).

Remark 1.1. These results give us a new construction of the generalized p-adic Grassmannians. For all integer $j \geq 0$ we have natural maps $|\mathcal{L}_{n,q^{p^j},r}| \to |\mathcal{L}_{n,q^{p^{j+1}},r}|$ that at the level of maximal tori induce the inclusion

$$(\mathbb{Z}/p^{l+j})^{[n/e]} \leq (\mathbb{Z}/p^{l+j+1})^{[n/e]}\,,$$

where $l = \nu_p(q^e - 1)$. The telescope construction of these maps gives us a homotopy equivalence

$$BX(e,r,[n/e]) \simeq \operatorname*{hocolim}_{j} |\mathcal{L}_{n,q^{p^j},r}|$$

up to p-completion.

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2. p-LOCAL FINITE GROUPS

In this section we review the concept of a p-local finite group introduced in [3] that is based on a previous unpublished work of L. Puig, where the axioms for fusion systems are already established. See [4] for a survey on this subject.

If P and Q are subgroups of a group G we consider $\operatorname{Hom}_G(P,Q)$ the group morphisms from P to Q induced by conjugation of elements in G, and $\operatorname{Inj}(P,Q)$ are the injective group morphisms from P to Q.

Definition 2.1. A fusion system $\mathcal{F}$ over a finite p-group S is a category whose objects are the subgroups of S, and whose morphisms sets $\operatorname{Hom}_{\mathcal{F}}(P,Q)$ satisfy the following two conditions:

(a) $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$ for all P and Q subgroups of S.
(b) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

We say that two subgroups P,$Q \leq S$ are *$\mathcal{F}$-conjugate* if there is an isomorphism between them in $\mathcal{F}$. As all the morphisms are injective by condition (b), we write by $\mathrm{Aut}_{\mathcal{F}}(P)$ the group $\mathrm{Hom}_{\mathcal{F}}(P,P)$. $\mathrm{Out}_{\mathcal{F}}(P)$ denotes the quotient group $\mathrm{Aut}_{\mathcal{F}}(P)/\,\mathrm{Aut}_P(P)$.

The fusion systems that we consider are saturated, so we need the following definitions:

Definition 2.2. Let $\mathcal{F}$ be a fusion system over a p-group S.

- A subgroup $P \leq S$ is *fully $\mathcal{F}$-centralized* if $|C_S(P)| \geq |C_S(P')|$ for all P' which is $\mathcal{F}$-conjugate to P.
- A subgroup $P \leq S$ is *fully $\mathcal{F}$-normalized* if $|N_S(P)| \geq |N_S(P')|$ for all P' which is $\mathcal{F}$-conjugate to P.
- $\mathcal{F}$ is a *saturated fusion system* if the following two conditions hold:
  (I) Every fully $\mathcal{F}$-normalized subgroup $P \leq S$ is fully $\mathcal{F}$-centralized and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.
  (II) If $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ are such that φP is fully $\mathcal{F}$-centralized, and if we set
  $$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi P)\},$$
  then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\overline{\varphi}|_P = \varphi$.

As expected, every finite group G gives rise to a saturated fusion system [3, Proposition 1.3], which provides valuable information about $BG^\wedge_p$. Some classical results for finite groups can be generalized to saturated fusion systems, as for example, Alperin's fusion theorem for saturated fusion systems [3, Theorem A.10]:

Definition 2.3. Let $\mathcal{F}$ be any fusion system over a p-group S. A subgroup $P \leq S$ is:

- *$\mathcal{F}$-centric* if P and all its $\mathcal{F}$-conjugates contain their S-centralizers.
- *$\mathcal{F}$-radical* if $\mathrm{Out}_{\mathcal{F}}(P)$ is p-reduced, that is, if $\mathrm{Out}_{\mathcal{F}}(P)$ has no nontrivial normal p-subgroups.

Theorem 2.4 ((Alperin's fusion theorem for saturated fusion systems)). *Let $\mathcal{F}$ be a saturated fusion system over S. Then for each morphism $\psi \in \mathrm{Aut}_{\mathcal{F}}(P,P')$, there exists a sequence of subgroups of S*
$$P = P_0, P_1, \ldots, P_k = P' \quad \text{and} \quad Q_1, Q_2, \ldots, Q_k,$$
and morphisms $\psi_i \in \mathrm{Aut}_{\mathcal{F}}(Q_i)$, such that

- *Q_i is fully $\mathcal{F}$-normalized, $\mathcal{F}$-centric and $\mathcal{F}$-radical for each i;*
- *$P_{i-1}, P_i \leq Q_i$ and $\psi_i(P_{i-1}) = P_i$ for each i; and*
- *$\psi(u) = \psi_k \circ \psi_{k-1} \circ \cdots \circ \psi_1(u)$, $\forall u \in P$.*

3. SATURATED FUSION SUBSYSTEMS OF INDEX PRIME TO p

The saturated fusion subsystems of index prime to p in any saturated fusion system $(S,\mathcal{F})$ are described in [2] in terms of:

Definition 3.1.
- If G is a finite group, $O^{p'}(G)$ is the smallest normal subgroup of index prime to p (equivalently the subgroup generated by elements of p-power order in G).
- A *restriction-closed category over* S is a category $\mathcal{F}$ such that $\mathrm{Ob}(\mathcal{F})$ is the set of subgroups of S, such that all morphisms in $\mathcal{F}$ are group monomorphisms between the subgroups, and with the following additional property: for each $P' \leq P \leq S$ and $Q' \leq Q \leq S$, and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ such that $\varphi(P') \leq Q'$ then $\varphi|_{P'} \in \mathrm{Hom}_{\mathcal{F}}(P', Q')$.
- If $\mathcal{F}$ is a saturated fusion system $O^{p'}_*(\mathcal{F}) \subset \mathcal{F}$ denotes the smallest restriction-closed subcategory of $\mathcal{F}$ whose morphism set contains $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ for all subgroups $P \leq S$.
- If $\mathcal{F}$ is a saturated fusion system over S we write:

$$\mathrm{Out}^0_{\mathcal{F}}(S) = \langle \alpha \in \mathrm{Out}_{\mathcal{F}}(S) \mid \alpha|_P \in \mathrm{Mor}_{O^{p'}_*(\mathcal{F})}(P, S), \text{ some } \mathcal{F}\text{-centric } P \leq S \rangle \,.$$

The result giving the classification of the saturated fusion subsystems of index prime to p is the following [2, Theorem 5.4 and Proposition 5.2]:

Theorem 3.2. *For any saturated fusion system $\mathcal{F}$ over a p-group S, there is a bijective correspondence between subgroups*

$$H \leq \Gamma_{p'}(\mathcal{F}) \overset{def}{=} \mathrm{Out}_{\mathcal{F}}(S)/\,\mathrm{Out}^0_{\mathcal{F}}(S)$$

and saturated fusion subsystems $\mathcal{F}_H$ of $\mathcal{F}$ over S of index prime to p in $\mathcal{F}$. The correspondence is given by associating to H the fusion system generated by $\widehat{\theta}^{-1}(\mathcal{B}(H))$, where $\mathcal{B}(H)$ is a category with one object and with morphism monoid the group H, and $\widehat{\theta}$ is the unique functor

$$\widehat{\theta}\colon \mathcal{F}^c \to \mathcal{B}(\Gamma_{p'}(\mathcal{F}))$$

with the following properties:

(a) $\widehat{\theta}(\alpha) = \alpha$ *(modulo* $\mathrm{Out}^0_{\mathcal{F}}(S)$*) for all* $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$.
(b) $\widehat{\theta}(\varphi) = 1$ *if* $\varphi \in \mathrm{Mor}(O^{p'}_*(\mathcal{F})^c)$.

The rest of the section is dedicated to reduce the family of subgroups involved in the calculation of $\mathrm{Out}^0_{\mathcal{F}}(S)$.

Consider ${}^R O^{p'}_*(\mathcal{F})$ the smallest restriction-closed category of $\mathcal{F}$ whose morphism set contains $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ for all fully $\mathcal{F}$-normalized, $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups $P \leq S$.

Lemma 3.3. *Fix $\mathcal{F}$ is a saturated fusion system over S. Then:*

(a) $\mathrm{Aut}_{\mathcal{F}}(S)$ *normalizes* ${}^R O^{p'}_*(\mathcal{F})$.
(b) $\mathcal{F} = \langle {}^R O^{p'}_*(\mathcal{F}), \mathrm{Aut}_{\mathcal{F}}(S) \rangle$.

Proof. (a) Consider $\psi \in \mathrm{Hom}_{{}^R O^{p'}_*(\mathcal{F})}(P, P')$ and $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$. We should check that $\alpha\psi\alpha^{-1} \in \mathrm{Hom}_{{}^R O^{p'}_*(\mathcal{F})}(\alpha(P), \alpha(P'))$.
ψ is the restriction of composition of $\phi_1, \cdots, \phi_k$, which are p-power order elements of automorphisms of $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups $R_1, \ldots, R_k$. But now $\alpha\psi\alpha^{-1}$ can be written as the composition of $\alpha\phi_j\alpha^{-1}$, which are p-power order elements of automorphisms of $\alpha(R_j)$, which are again fully $\mathcal{F}$-normalized, $\mathcal{F}$-centric and $\mathcal{F}$-radical because this two properties are kept under $\mathrm{Aut}_{\mathcal{F}}(S)$-conjugation.

(b) By Alperin's theorem for saturated fusion systems (Theorem 2.4) it is enough to check that $\mathrm{Aut}_{\mathcal{F}}(Q) \leq \langle {}^R O_*^{p'}(\mathcal{F}), \mathrm{Aut}_{\mathcal{F}}(S)\rangle$ for Q an $\mathcal{F}$-centric, $\mathcal{F}$-radical and fully $\mathcal{F}$-normalized subgroup of S. We will proceed by downward induction. $\mathrm{Aut}_{\mathcal{F}}(S) \leq \langle {}^R O_*^{p'}(\mathcal{F}), \mathrm{Aut}_{\mathcal{F}}(S)\rangle$, so the result holds for $Q = S$. Assume now that the $\mathrm{Aut}_{\mathcal{F}}(Q') \leq \langle {}^R O_*^{p'}(\mathcal{F}), \mathrm{Aut}_{\mathcal{F}}(S)\rangle$ for all Q' of bigger order than a fixed Q, and $\psi \in \mathrm{Aut}_{\mathcal{F}}(Q)$.

Consider the subgroup $K \overset{\text{def}}{=} \psi \,\mathrm{Aut}_S(Q)\psi^{-1}$, which is a p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$, so $K \leq \mathrm{Aut}_{{}^R O_*^{p'}(\mathcal{F})}(Q)$. Since Q is fully $\mathcal{F}$-normalized, $\mathrm{Aut}_S(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. As $\mathrm{Aut}_{{}^R O_*^{p'}(\mathcal{F})}(Q) \leq \mathrm{Aut}_{\mathcal{F}}(Q)$, we get that both $\mathrm{Aut}_S(Q)$ and K are Sylow p-subgroups in $\mathrm{Aut}_{{}^R O_*^{p'}(\mathcal{F})}(Q)$, so they are conjugated by an element $\phi \in \mathrm{Aut}_{{}^R O_*^{p'}(\mathcal{F})}(Q)$:

$$\phi K \phi^{-1} \leq \mathrm{Aut}_S(Q)\,. \tag{1}$$

So, as $\mathcal{F}$ is saturated, by condition (II) in Definition 2.2 $\phi\psi$ must extend to a morphism $\overline{\phi\psi}$ defined over $N_{\phi\psi}$ which by Equation (1) is equal to $N_S(Q)$, so it is always bigger than Q. So by induction hypothesis $\overline{\phi\psi}\colon N_S(Q) \to S$ is an element in $\langle {}^R O_*^{p'}(\mathcal{F}), \mathrm{Aut}_{\mathcal{F}}(S)\rangle$, so its restriction $\phi\psi$ is again in $\langle {}^R O_*^{p'}(\mathcal{F}), \mathrm{Aut}_{\mathcal{F}}(S)\rangle$. Recall now that $\phi \in \mathrm{Aut}_{{}^R O_*^{p'}(\mathcal{F})}(Q)$ so $\psi = \phi^{-1}(\phi\psi)$ is an element in $\langle {}^R O_*^{p'}(\mathcal{F}), \mathrm{Aut}_{\mathcal{F}}(S)\rangle$. □

Now we are ready to prove the main result of this section, which is analogous to the Alperin's Theorem (Theorem 2.4) for the category $O_*^{p'}(\mathcal{F})$:

Theorem 3.4. *Let $\mathcal{F}$ be a saturated fusion system over S. Then for each morphism $\psi \in \mathrm{Hom}_{O_*^{p'}(\mathcal{F})}(P, P')$, there exists a sequence of subgroups of S*

$$P = P_0, P_1, \ldots, P_k = P' \quad \textit{and} \quad Q_1, Q_2, \ldots, Q_k,$$

and morphisms $\psi_i \in \mathrm{Aut}_{O_*^{p'}(\mathcal{F})}(Q_i)$, such that

- *Q_i is fully $\mathcal{F}$-normalized, $\mathcal{F}$-centric and $\mathcal{F}$-radical for each i;*
- *$P_{i-1}, P_i \leq Q_i$ and $\psi_i(P_{i-1}) = P_i$ for each i; and*
- *$\psi(u) = \psi_k \circ \psi_{k-1} \circ \cdots \circ \psi_1(u)$, $\forall u \in P$.*

Proof. We have just to check that $O_*^{p'}(\mathcal{F}) \leq {}^R O_*^{p'}(\mathcal{F})$.

Consider $\alpha\colon P \to P'$ an element in $O_*^{p'}(\mathcal{F})$. Then α is the composition of restrictions of automorphisms of order a power of p in $\mathrm{Aut}_{\mathcal{F}}(Q_i)$, for some subgroups $Q_1, \ldots, Q_r$. So it is enough to check that these elements are in ${}^R O_*^{p'}(\mathcal{F})$.

Consider then $\alpha\colon Q \to Q$ an element of order a power of p, for Q any subgroup in S. We will check that it is a morphism in ${}^R O_*^{p'}(\mathcal{F})$ by downward induction. The result is true for $Q = S$ (both morphisms sets are empty sets). Assume now that for a fixed Q, all the elements of order a power of p in $\mathrm{Aut}_{\mathcal{F}}(Q')$ are in ${}^R O_*^{p'}(\mathcal{F})$ for all Q' of order bigger than the order of Q.

If Q is not fully $\mathcal{F}$-normalized, consider $g\colon Q \to Q'$ an isomorphism in $\mathrm{Hom}_{\mathcal{F}}(Q, Q')$ such that Q' is fully $\mathcal{F}$-normalized. Using Lemma 3.3 and [2, Lemma 3.4 (c)] we get that there exists $\beta \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $\varphi \in \mathrm{Hom}_{{}^R O_*^{p'}(\mathcal{F})}(\beta(Q), Q')$ such that $g = \varphi \circ \beta|_Q$.

If $g \circ \alpha \circ g^{-1} \in \mathrm{Hom}_{{}^R O_*^{p'}(\mathcal{F})}(Q', Q')$ then $\beta|_Q \circ \alpha \circ \beta^{-1}|_{\beta(Q)} = \varphi^{-1} \circ g \circ \alpha \circ g^{-1} \circ \varphi$ is also in $\mathrm{Hom}_{{}^R O_*^{p'}(\mathcal{F})}(Q', Q')$. Use now that α is $\mathrm{Aut}_{\mathcal{F}}(S)$-conjugated to $\beta|_Q \circ \alpha \circ \beta^{-1}|_{\beta(Q)}$ and $\mathrm{Aut}_{\mathcal{F}}(S)$

normalizes ${}^{R}O_{*}^{p'}(\mathcal{F})$ and we obtain that α is a morphism in ${}^{R}O_{*}^{p'}(\mathcal{F})$. So we can assume that Q is fully $\mathcal{F}$-normalized.

If Q is not $\mathcal{F}$-centric, then by condition (II) in [3, Definition 1.2] we get that α extends to a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(Q \cdot C_S(Q), S)$. The image of $\overline{\varphi}$ must be again in $Q \cdot C_S(Q)$, so $\overline{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(Q \cdot C_S(Q))$ and $\overline{\varphi}|_Q = \varphi$. If the order of $\overline{\varphi}$ is a power of p, we can apply the induction hypothesis and the result follows. If not, we can consider an integer r such that $\overline{\varphi}^r|_Q = \varphi$ and the order of $\overline{\varphi}^r|_Q$ is a power of p, and apply the induction hypothesis.

Assume now that Q is $\mathcal{F}$-centric and fully $\mathcal{F}$-normalized, and $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q)$ an element of p-power order. If Q is $\mathcal{F}$-radical, we have finished, so suppose that Q is not $\mathcal{F}$-radical. That means that $O_p(\mathrm{Out}_{\mathcal{F}}(Q)) \neq \langle 1 \rangle$, and so $K \overset{\text{def}}{=} O_p(\mathrm{Aut}_{\mathcal{F}}(Q))$ is strictly bigger than $\mathrm{Aut}_Q(Q)$. As Q is fully $\mathcal{F}$-normalized, we have that $\mathrm{Aut}_S(Q)$ is a Sylow p-subgroup for $\mathrm{Aut}_{\mathcal{F}}(Q)$, and so $K \leq \mathrm{Aut}_S(Q)$. Consider

$$N_S^K(Q) \overset{\text{def}}{=} \{x \in N_S(Q) \mid c_x \in K\},$$

which is strictly bigger than Q. If $x \in N_S^K(Q)$, $\alpha c_x \alpha^{-1}$ is an element in K (K is a normal subgroup in $\mathrm{Aut}_{\mathcal{F}}(Q)$), so, in this case,

$$N_S^K(Q) \leq N_\alpha \overset{\text{def}}{=} \{x \in N_S(Q) \mid \alpha c_x \alpha^{-1} \in \mathrm{Aut}_S(Q)\}$$

and by condition (II) in [3, Definition 1.2] α extends to a morphism $\overline{\alpha}\colon N_S^K(Q) \to S$. Moreover the image of $\overline{\alpha}$ is contained in $N_S^K(Q)$: the extension to elements of the normalizer in S of Q must give elements in the normalizer of the image, which is again Q; also $c_{\overline{\alpha}(x)} = \alpha \circ c_x \circ \alpha^{-1}$, so if c_x is in K, $c_{\overline{\alpha}(x)}$ is again in K. So we have an extension of α, which can be taken of p-power order (take $\overline{\alpha}^r$ as before if necessary) and which is in ${}^{R}O_{*}^{p'}(\mathcal{F})$ by induction hypothesis. As we are in a restriction closed category, α is also a morphism in ${}^{R}O_{*}^{p'}(\mathcal{F})$. □

4. A REPRESENTATION OF THE EXTRASPECIAL GROUP $p_+^{1+2\gamma}$ IN $\mathrm{GL}_n(q)$

In this section we introduce some subgroups needed to understand the fusion system of $\mathrm{GL}_n(q)$ over a prime p such that $p \nmid q$. We divide this study in two subsections, the first one concerning to the particular case $q \equiv 1 \bmod p$ and the second one, where we extend the results to the general case.

Notation 4.1. Consider the extraspecial group of order $p^{1+2\gamma}$ and exponent p, noted as $p_+^{1+2\gamma}$, as the group generated by $A_0, \ldots, A_{\gamma-1}$, $B_0, \ldots, B_{\gamma-1}$ and C with the relations $[A_i, A_j] = 1$, $[B_i, B_j] = 1$ and $[A_i, B_j] = C^{\delta_{ij}}$ for all $i, j \in \{0, 1, \ldots, \gamma - 1\}$.

4.1. **Case $q \equiv 1$ mod p.**

Fix first q a prime power such that $p \mid (q-1)$ and ζ a p-root of the unity in $\mathbb{F}_q$.

We will construct a representation of $p_+^{1+2\gamma}$ in $\mathrm{GL}_{p^\gamma}(q)$, so we assume that $n \geq p^\gamma$. In the case $n > p^\gamma$ we consider the inclusion of $p_+^{1+2\gamma}$ in $\mathrm{GL}_n(q)$, as the composition $p_+^{1+2\gamma} \leq \mathrm{GL}_{p^\gamma} \leq \mathrm{GL}_n(q)$.

A faithful representation can be constructed as follows: consider V a $\mathbb{F}_q$-vector space of dimension p^γ, with basis $\{v_0, \ldots, v_{p^\gamma - 1}\}$; write each integer k such that $0 \leq k \leq p^\gamma - 1$ in base p, getting γ unique numbers $a_i \in \{0, \ldots, p-1\}$:

$$k = a_0 + a_1 p + a_2 p^2 + \cdots + a_{\gamma-1} p^{\gamma-1}.$$

Then, using this basis consider the following morphisms:

- A_i is the linear transformation which sends
$$v_{a_0+\cdots+a_ip^i+\cdots} \mapsto \zeta^{a_i} v_{a_0+\cdots+a_ip^i+\cdots}\,,$$
- B_i is the permutation in the basis which sends
$$v_{a_0+\cdots+a_ip^i+\cdots} \mapsto v_{a_0+\cdots+(a_i+1)p^i+\cdots}\,,$$
where the coefficient (a_i+1) is reduced mod p.

Lemma 4.2. *The elements A_i and B_i defined above give a faithful representation of $p_+^{1+2\gamma}$ in $\mathrm{GL}_{p^\gamma}(q)$.*

Proof. A direct computation shows us that these elements carry out the relations in Notation 4.1, taking C as $\zeta\,\mathrm{Id}$. □

We will also be interested in the following transformation: for $i \in \{0,1,\ldots,n-1\}$ we consider $\sigma_i\colon V \to V$ as the linear map which in the basis $v_0,\ldots,v_{p^\gamma-1}$ acts as:

$$\sigma_i(v_{a_0+\cdots+a_ip^i+\cdots}) \overset{\text{def}}{=} v_{a_0+\cdots+(\xi a_i)p^i+\cdots}\,, \tag{2}$$

where ξ is a $(p-1)$-root of unity in $\mathbb{F}_p$ and again the coefficient (ξa_{i-1}) is reduced mod p.

We can check with a direct computation that:

$$\sigma_i A_j \sigma_i^{-1} = \begin{cases} A_j & \text{if } i \neq j \\ A_j^{\xi^{-1}} & \text{if } i = j \end{cases} \tag{3}$$

$$\sigma_i B_j \sigma_i^{-1} = \begin{cases} B_j & \text{if } i \neq j \\ B_j^{\xi} & \text{if } i = j \end{cases} \tag{4}$$

where $A_j^{\xi^{-1}}$ (respectively B_j^{ξ}) means the matrix A_j to the power ξ^{-1} (respectively to the power ξ).

4.2. **General case.** Now we consider e the order of q mod p. We assume that $e > 1$, and observe that e must divide $(p-1)$.

Consider $\mathbb{F}_{q^e}$ a Galois extension of $\mathbb{F}_q$, with Galois group $\mathbb{Z}/e\mathbb{Z}$. $\mathbb{F}_{q^e}$ can be regarded as the quotient $\mathbb{F}_q[x]/r(x)$, where $r(x)$ is an irreducible polynomial of degree e with coefficients in $\mathbb{F}_q$.

Now consider $\mathbb{F}_{q^e}$ as $\mathbb{F}_q$-vector space with basis $\{1,x,\ldots,x^{e-1}\}$. This gives an inclusion of $\mathrm{GL}_1(q^e) \subset \mathrm{GL}_e(q)$, and more generally of $\mathrm{GL}_n(q^e) \subset \mathrm{GL}_{en}(q)$ (convert every coefficient of a matrix in $\mathrm{GL}_n(q^e)$ to an $e \times e$-matrix with coefficients in $\mathbb{F}_q$).

Using this inclusion we get that we can apply the previous subsection to the study of $\mathrm{GL}_n(q^e)$, getting the corresponding elements A_j, B_j and σ_j in $\mathrm{GL}_{en}(q)$, and also the corresponding inclusion of $p_+^{1+2\gamma} \leq \mathrm{GL}_{ep^\gamma}(q)$.

Consider the Frobenius automorphism of $\mathbb{F}_{q^e}$, which is the $\mathbb{F}_q$-linear map which sends every $y \overset{\text{def}}{=} a_0 + \cdots + a_{e-1}x^{e-1}$ to y^q, and is a generator of the Galois group of $\mathbb{F}_{q^e}$ over $\mathbb{F}_q$. This can be thought as an $e \times e$ matrix with coefficients in $\mathbb{F}_q$.

Consider the following fixed isomorphism $\mathbb{F}_{q^e}^n \cong \mathbb{F}_q^{en}$ as $\mathbb{F}_q$-vector space: if $\{v_1, \ldots, v_n\}$ is a basis of $\mathbb{F}_{q^e}^n$ as $\mathbb{F}_{q^e}$-vector space, then

$$\{v_1, xv_1, \ldots, x^{e-1}v_1, v_2, xv_2, \ldots, x^{e-1}v_2, \ldots, v_n, xv_n, \ldots, x^{e-1}v_n\}$$

is a basis of $\mathbb{F}_q^{en}$ as $\mathbb{F}_q$-vector space.

Now define φ_n the $\mathbb{F}_q$-linear automorphism in $\mathbb{F}_{q^e}^n$ which sends a vector

$$\varphi_n(r_1(x)v_1 + \cdots + r_n(x)v_n) \overset{\text{def}}{=} r_1(x)^q v_1 + \cdots + r_n(x)^q v_n\,, \tag{5}$$

and regard it as an automorphism of $\mathbb{F}_q^{en}$ in the previous basis.

With these definitions, we can compute the following conjugations:

- $\varphi_n A_j \varphi_n^{-1} = A_j^q$ and
- $\varphi_n B_j \varphi_n^{-1} = B_j$.

Note that, as the order of the Frobenius automorphism is q^e, the inverse φ_n^{-1} is given by:

$$\varphi_n^{-1}(r_1(x)v_1 + \cdots + r_n(x)v_n) = r_1(x)^{q^{e-1}} v_1 + \cdots + r_n(x)^{q^{e-1}} v_n\,,$$

So φ_n^{-1} sends $r_i(x)v_i$ to $r_i(x)^{q^{e-1}}v_i$. If we apply A_j, each $r_i(x)^{q^{e-1}}v_i$ is multiplied by a coefficient a_{ij} which only depends on the expression of i in base p and j. In we apply now φ_n to $a_{ij}r_i(x)^{q^{e-1}}v_i$ we get $a_{ij}^q v_i$. Now as A_j are diagonal matrices, we get A_j^q.

B_j permutes the basis v_i, so it commutes with the action of φ_n.

5. The fusion systems of $\mathrm{GL}_n(q)$

Notation 5.1. In all this section fix p and odd prime and q a prime power prime to p. Let e be the multiplicative order of q modulo p, and $l = \nu_p(q^e - 1)$. Consider also V an n-dimensional $\mathbb{F}_q$-vector space, in such a way that we write $\mathrm{GL}(V)$ instead of $\mathrm{GL}_n(q)$ if we have to deal with subgroups.

We are interested in the $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups of the general linear groups. Fix S a Sylow p-subgroup of a finite group G, and P a subgroup of S.

Note that when G is a finite group and S a Sylow p-subgroup, then a subgroup $P \leq S$ is p-radical in G when $N_G(P)/P$ does not contain any non trivial normal p-subgroup. Also a subgroup $P \leq S$ is p-centric in G if $C_G(P) = Z(P) \times C'_G(P)$, where $C'_G(P)$ has order prime to p.

For P, being p-radical in G and being $\mathcal{F}_S(G)$-radical ($\mathcal{F}_S(G)$ is the saturated fusion system given by G over S) are independent definitions. If we require P to be p-centric in G, then we have that when P is $\mathcal{F}_S(G)$-radical, P is also p-radical in G. So the list of subgroups of G that we are interested in is contained in the list of p-radical subgroups of G.

The possible p-radical subgroups of $\mathrm{GL}_n(q)$ over the prime p are described in [1, Section 4] and depend on some parameters α, γ, m, and $\mathfrak{c} = (c_1, \ldots, c_t)$. These are constructed as follows:

Fixed $\alpha \geq 0$ and $\gamma \geq 0$ consider $(\mathbb{Z}/p^{l+\alpha})p_+^{1+2\gamma}$ the central product of the cyclic group of order $p^{l+\alpha}$ and an extraspecial group of order $p^{1+2\gamma}$ and exponent p over the center of the extraspecial group. Consider $R_{\alpha,\gamma}$ the image of $(\mathbb{Z}/p^{l+\alpha})p^{1+2\gamma}$ by the composition

$$(\mathbb{Z}/p^{l+\alpha})p_+^{1+2\gamma} \leq \mathrm{GL}_{p^\gamma}(q^{ep^\alpha}) \leq \mathrm{GL}_{ep^{\alpha+\gamma}}(q)\,, \tag{6}$$

where the first inclusion works as follows:

- The subgroup $\mathbb{Z}/p^{l+\alpha}$ corresponds to the matrices of the form $\lambda\,\mathrm{Id}$, where λ is a $p^{l+\alpha}$-root of unity in $\mathbb{F}_{q^{ep^\alpha}}$.
- The inclusion $p_+^{1+2\gamma} \leq \mathrm{GL}_{p^\gamma}(q^{ep^\alpha})$ is as in Section 4.

Fix now $m \geq 1$ and $R_{m,\alpha,\gamma}$ the image of $R_{\alpha,\gamma}$ in $\mathrm{GL}_{mep^{\alpha+\gamma}}(q)$ which sends g to the m-fold diagonal map.

Finally consider $\mathfrak{c} = (c_1, \ldots, c_t)$ a sequence of positive integers and define $F_\mathfrak{c} \overset{\mathrm{def}}{=} (\mathbb{Z}/p)^{c_1} \wr (\mathbb{Z}/p)^{c_2} \wr \cdots \wr (\mathbb{Z}/p)^{c_t}$. Let $d \overset{\mathrm{def}}{=} mep^{\alpha+\gamma+c_1+\cdots+c_t}$ and denote by $R_{m,\alpha,\gamma,\mathfrak{c}}$ the image of $R_{m,\alpha,\gamma} \wr F_\mathfrak{c}$ in $\mathrm{GL}_d(q)$. Call a subgroup of this type as a *basic subgroup of* $\mathrm{GL}_d(q)$.

Theorem 5.2 ([1])**.** *Fix V an n-dimensional $\mathbb{F}_q$-vector space. Let $G = \mathrm{GL}(V)$, and R be a p-radical subgroup of G. Then there exist decompositions*

$$\begin{aligned} W &= V_0 \oplus V_1 \oplus \cdots \oplus V_s, \\ R &= R_0 \times R_1 \times \cdots \times R_s, \end{aligned}$$

such that R_0 is the trivial subgroup of $\mathrm{GL}(V_0)$, and R_i are basic subgroups of $\mathrm{GL}(V_i)$ for $i \geq 1$.

Now we are able to give the Sylow p-subgroup in terms of e, (the order of q modulo p), l (the p-adic valuation of $q^e - 1$) and the coefficients $a_0, \ldots, a_k$, all of them $0 \leq a_i \leq (p-1)$ such that:

$$[n/e] = a_0 + a_1 p + \cdots + a_k p^k \,.$$

Lemma 5.3. *$S_{n,q}$, a Sylow p-subgroup of $\mathrm{GL}_n(q)$, is given by the following construction:*

$$\begin{aligned} S_{n,q} \overset{def}{=} \; & (\mathbb{Z}/p^l \wr \mathbb{Z}/p \wr \overset{(k)}{\cdots} \wr \mathbb{Z}/p)^{a_k} \times (\mathbb{Z}/p^l \wr \mathbb{Z}/p \wr \overset{(k-1)}{\cdots} \wr \mathbb{Z}/p)^{a_{k-1}} \times \cdots \\ & \cdots \times (\mathbb{Z}/p^l \wr \mathbb{Z}/p)^{a_1} \times (\mathbb{Z}/p^l)^{a_0} \,. \end{aligned}$$

where e, l and $a_1, \ldots, a_k$ are defined above.

Moreover, if $(S_{n,q}, \mathcal{F}_{n,q})$ is the saturated fusion system induced by $\mathrm{GL}_n(q)$ over $S_{n,q}$, then $\mathrm{Out}_{\mathcal{F}_{n,q}}(S_{n,q})$ is isomorphic to

$$\big((\mathbb{Z}/e \times \mathbb{Z}/(p-1)^k) \wr \Sigma_{a_k}\big) \times \big((\mathbb{Z}/e \times \mathbb{Z}/(p-1)^{k-1}) \wr \Sigma_{a_{k-1}}\big) \times \cdots \times \big((\mathbb{Z}/e) \wr \Sigma_{a_0}\big) \,.$$

Proof. This combination of direct products and wreath products is included in $\mathrm{GL}_{e[n/e]}(q) \leq \mathrm{GL}_n(q)$. Checking the orders of $S_{n,q}$ and the Sylow p-subgroup of $\mathrm{GL}_n(q)$ we have finished.

Finally the outer automorphisms group is computed in [1, Section 4]. □

For fixed p and q we can compute e, and the Sylow p-subgroup and the fusion does not change between $\mathrm{GL}_n(q)$ and $\mathrm{GL}_{e[n/e]}(q)$. Sometimes we will consider that we are working in rank multiples of e and we will write in $\mathrm{GL}_{em}(q)$ instead of $\mathrm{GL}_n(q)$.

Notation 5.4. Consider $S_{n,q}$ a Sylow p-subgroup of $\mathrm{GL}_n(q)$ as computed in Lemma 5.3, and $(S_{n,q}, \mathcal{F}_{n,q}, \mathcal{L}_{n,q})$ the corresponding p-local finite group.

Let us begin now with the first non-trivial case:

Lemma 5.5. *Let p be an odd prime and q a prime power such that $p|(q-1)$. Consider $(S_{p,q}, \mathcal{F}_{p,q})$ as in Notation 5.4. Then $\mathrm{Out}^0_{\mathcal{F}_{p,q}}(S) = \mathrm{Out}_{\mathcal{F}_{p,q}}(S)$.*

Proof. To simplify the notation, consider in this proof $(S,\mathcal{F}) \overset{\mathrm{def}}{=} (S_{p,q},\mathcal{F}_{p,q})$. In this case, by Lemma 5.3 we have $\mathrm{Out}_{\mathcal{F}}(S) \cong \mathbb{Z}/(p-1)$. We can see that σ defined in Equation (2) is a generator, so we must check that $\sigma \in \mathrm{Out}^0_{\mathcal{F}}(S)$.

Consider $R_{0,1}$ as in Equation (6). We get that $R_{0,1}$ is isomorphic to a central extension of p^{1+2}_+, generated by A and B as defined in Equations (3) and (4), by $\mathbb{Z}/p^l$, the p-primary part in the center of $\mathrm{GL}_p(q)$.

A direct computation tells us that $R_{0,1}$ is $\mathcal{F}$-centric.

Equation (2) gives us the action of A and B:

$$\sigma A \sigma^{-1} = A^{\zeta^{-1}} \text{ and } \sigma B \sigma^{-1} = B^{\zeta}\,,$$

where ζ is a $(p-1)$-root in $\mathbb{F}_p$. So σ restricts to an automorphism of $R_{0,1}$ which, as element in $\mathrm{Out}_{\mathcal{F}}(R_{0,1})$, must be considered as the matrix $\left(\begin{smallmatrix}\zeta^{-1} & 0\\ 0 & \zeta\end{smallmatrix}\right)$.

Use now that in this case, [1, Section 4] tells us that $\mathrm{Out}_{\mathcal{F}}(R_{0,1}) \cong \mathrm{SL}_2(p)$, and now apply the fact that $\mathrm{SL}_2(p)$ is generated by its elements of order p. $\square$

Proposition 5.6. *Consider $(S_{n,q},\mathcal{F}_{n,q})$ as in Notation 5.4, with $n \geq ep$. There is a subgroup H in $\mathrm{Out}_{\mathcal{F}_{n,q}}(S_{n,q})$ such that $\mathrm{Out}_{\mathcal{F}_{n,q}}(S_{n,q})/H \cong \mathbb{Z}/e$ and $\mathrm{Out}^0_{\mathcal{F}_{n,q}}(S) \leq H$.*

Proof. Let us consider the form of $\mathrm{Out}_{\mathcal{F}_{n,q}}(S_{n,q})$ given in Lemma 5.3.

Observe that all the elements in $\mathrm{Out}_{\mathcal{F}_{n,q}}(S_{n,q})$ restrict to automorphisms of the maximal torus $\mathbb{T}^{[n/e]}_{p^l}$, and that different elements in $\mathrm{Out}_{\mathcal{F}_{n,q}}(S_{n,q})$ give different restrictions. This implies that if $\mathcal{F}_0$ is the minimal saturated fusion subsystem of $(S_{n,q},\mathcal{F}_{n,q})$ of index prime to p over S, then

$$\begin{aligned} \mathrm{Out}_{\mathcal{F}_{n,q}}(S_{n,q})/\,\mathrm{Out}^0_{\mathcal{F}_{n,q}}(S_{n,q}) &\cong \mathrm{Out}_{\mathcal{F}_{n,q}}(S_{n,q})/\,\mathrm{Out}_{\mathcal{F}_0}(S) \cong \\ &\cong \mathrm{Out}_{\mathcal{F}_{n,q}}(\mathbb{T}^{[n/e]}_{p^l})/\,\mathrm{Out}_{\mathcal{F}_0}(\mathbb{T}^{[n/e]}_{p^l})\,. \end{aligned} \tag{7}$$

We have that $\mathrm{Out}_{\mathcal{F}_{n,q}}(\mathbb{T}^{[n/e]}_{p^l}) \cong \mathbb{Z}/e \wr \Sigma_{[n/e]}$. Now consider $O_{p'}(\mathrm{Out}_{\mathcal{F}_{n,q}}(\mathbb{T}^{[n/e]}_{p^l}))$, the subgroup generated by all the elements of order a power of p. As $p \leq [n/e]$, there are elements of order p in $\Sigma_{[n/e]}$, and as $O_{p'}(\mathrm{Out}_{\mathcal{F}_{n,q}}(\mathbb{T}^{[n/e]}_{p^l}))$ is a normal subgroup, it musts contain all the alternating group $A_{[n/e]} \leq \Sigma_{[n/e]}$ and its normal closure in $(\mathbb{Z}/e)^{[n/e]} \rtimes A_{[n/e]}$, which is $(\mathbb{Z}/e)^{[n/e]-1} \rtimes A_{[n/e]}$ ($(\mathbb{Z}/e)^{[n/e]-1} \leq (\mathbb{Z}/e)^{[n/e]}$ is the kernel of the map $(a_1,\ldots,a_{[n/e]}) \mapsto \sum a_i$).

To finish the proof, take the restriction of σ, which by Lemma 5.5 is in $\mathrm{Out}^0_{\mathcal{F}_{n,q}}(S)$, and which gives an odd permutation of $\Sigma_{[n/e]}$, getting that

$$H_{\mathbb{T}} \overset{\mathrm{def}}{=} (\mathbb{Z}/e)^{[n/e]-1} \rtimes \Sigma_{[n/e]} \leq \mathrm{Out}_{\mathcal{F}_0}(\mathbb{T}^{[n/e]}_{p^l})\,.$$

But, by Equation (7), $\mathrm{Out}^0_{\mathcal{F}_{n,q}}(S)$ must be contained in H defined as the kernel of the morphism:

$$\mathrm{Out}_{\mathcal{F}_{n,q}}(S) \to \mathrm{Out}_{\mathcal{F}_{n,q}}(\mathbb{T}^{[n/e]}_{p^l})/H_{\mathbb{T}} \cong \mathbb{Z}/e\,,$$

and the result follows. $\square$

From the previous proposition we get that $\mathrm{Out}_{\mathcal{F}_{n,q}}(S)/\,\mathrm{Out}^0_{\mathcal{F}_{n,q}}(S)$ has at most e elements. The rest of the section is dedicated to prove that there is an element of order e in $\mathrm{Out}_{\mathcal{F}_{n,q}}(S)/\,\mathrm{Out}^0_{\mathcal{F}_{n,q}}(S)$.

As we are interested just in the $\mathcal{F}_{n,q}$-centric, $\mathcal{F}_{n,q}$-radical subgroups, we can remove some subgroups in the list:

Lemma 5.7. *If $m > 1$ then $R_{m,\alpha,\gamma,\mathfrak{c}}$ is not p-centric in S.*

Proof. Assume $m > 1$. We are considering the inclusion of $R_{\alpha,\gamma}$ in $\mathrm{GL}_{mep^{\alpha+\gamma}}(q)$ diagonally:

$$g \mapsto \begin{pmatrix} g & 0 & \cdots & 0 \\ 0 & g & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g \end{pmatrix}.$$

Fix g a non-trivial element in the center or $R_{\alpha,\gamma}$. We have that

$$h \overset{\text{def}}{=} \begin{pmatrix} g & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

is an element which centralizes $R_{m,\alpha,\gamma}$ which is not in $R_{m,\alpha,\gamma}$.

Recall now that in a semidirect product $G \rtimes H$, if we have an element g in the center of G invariant under the action of H, then $(g, 1)$ is in the centre of $G \rtimes H$. Apply this argument to the element $(h, h, \ldots, h) \in (R_{m,\alpha,\gamma})^{p^{c_1+\cdots+c_r}}$, which is invariant by the action of Σ_n. □

So, from now on, we just consider the case $m = 1$.

Lemma 5.8. *Consider λ a p^l-root of unity in $\mathbb{F}_{q^e}$. The maximal torus $\mathbb{T}^m_{p^l}$ can be seen as the image of $(\mathbb{Z}/p^l)^m$ under the composition*

$$(a_1, \ldots, a_m) \mapsto \mathit{diag}(\lambda^{a_1}, \ldots, \lambda^{a_m}) \in \mathrm{GL}_m(q^e) \leq \mathrm{GL}_{em}(q)\,.$$

Fix R an $\mathcal{F}_{em,q}$-centric, $\mathcal{F}_{em,q}$-radical subgroup in $S_{em,q}$ (Notation 5.4) and consider χ the image in $\mathrm{GL}_{em}(q)$ of the subset (not a subgroup) of elements in $(\mathbb{Z}/p^l)^m$ such that the product $a_1 \cdots a_m = 1$. If $\psi \in O^{p'}(\mathrm{Aut}_{\mathcal{F}_{em,q}}(R))$, then $\psi(\chi \cap R) = \chi \cap R$.

Proof. To simplify the notation in this proof consider $(S, \mathcal{F}) \overset{\text{def}}{=} (S_{em,q}, \mathcal{F}_{em,q})$ Consider R an $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroup of $\mathrm{GL}(V)$, with V an n-dimensional $\mathbb{F}_p$-vector space. By the description in Theorem 5.2 we get that there is a decomposition which can be given, after reordering and grouping by isomorphism type, as:

$$V = V_0 \oplus V_{1,1} \oplus \cdots \oplus V_{1,m_1} \oplus V_{2,1} \oplus \cdots \oplus V_{k,m_k}$$

such that R can be written as:

$$R = R_0 \times (R_1)^{m_1} \times \cdots \times (R_k)^{m_k}$$

with R_i basic subgroups in $\mathrm{GL}(V_{i,j})$, and such that $m_1 d_1 + \cdots + m_k d_k = m$, where $d_i = \dim(V_{i,1})$. For each $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, m_i\}$, consider $H_{i,j}$ the image of $\mathrm{GL}(V_{i,j})$ in $\mathrm{GL}(V)$. Consider $\overline{S}_{i,j} = S \cap H_{i,j}$, and $\overline{\mathcal{F}}_{i,j}$ the saturated fusion system $\mathcal{F}_{\overline{S}_{i,j}}(\mathrm{GL}(V_{i,j}))$. Observe that $\overline{S}_{i,j} \cong \overline{S}_{i,j'}$ and $\overline{\mathcal{F}}_{i,j} \cong \overline{\mathcal{F}}_{i,j'}$ for $j, j' \in \{1, \ldots, m_i\}$.

With all this notation, we have the isomorphism:

$$\mathrm{Aut}_{\mathcal{F}}(R)/R \cong \prod_{i=1}^{k} (\mathrm{Aut}_{\overline{\mathcal{F}}_{i,1}}(R_i)/R_i) \wr \Sigma_{m_i}$$

The action of R on itself by conjugation is included in the action of S over R, and this preserves χ, as these are permutations.

Consider now an element in $\mathrm{Aut}_{\mathcal{F}}(R)$ such that its class belongs to $O^{p'}(\mathrm{Out}_{\mathcal{F}}(R))$. We have to deal with two possibilities:

- As R_i are $\overline{\mathcal{F}}_{i,j}$-centric basic subgroups, using Lemma 5.7 and [1, Section 4] all the elements of order a power of p in $\mathrm{Out}_{\overline{\mathcal{F}}_{i,1}}(R_i)$ are permutations, so they preserve χ.
- If there exists i such that $m_i \geq p$ then the elements of the form $\tau\sigma(\tau^{-1})$ are also in $O^{p'}(\mathrm{Out}_{\mathcal{F}}(R))$, for all σ in $A_{m_i} \leq \Sigma_{m_i}$ and all $\tau \in \mathrm{Aut}_{\mathcal{F}'_{i,1}}(R_i)$ (A_{m_i} the alternating subgroup in Σ_i). To simplify notation consider the elements in $\chi \cap S_{i,j}$ as $(a_1, \ldots, a_{d_i}) \in (\mathbb{Z}/p^l)^{d_i}$. But τ sends $(a_1, \ldots, a_{d_i})$ to $(a_{\tau(1)}q_1, \ldots, a_{\tau(d_i)}q_{d_i})$, and so $\sigma(\tau^{-1})$ permutes the components and multiply each one by q_i^{-1} respectively. In any case, the product of the elements is
$$a_1 \cdots a_{d_i} q_1 \cdots q_{d_i} q_1^{-1} \cdots q_{d_i}^{-1} = a_1 \cdots a_{d_i}.$$
  So the product $a_1 \cdots a_{d_i}$ doesn't change.

This implies that the elements of $\psi(\chi \cap R) \leq \chi$, but as ψ is an automorphism of R the result follows. □

To get an element of order e in $\mathrm{Aut}^0_{\mathcal{F}_{n,q}}(S_{n,q})$ we use the automorphism φ_1 defined in Equation (5) as follows: as we consider the case $n \geq e$, consider ϕ the $\mathbb{F}_q$-linear map defined as

$$\phi \overset{\mathrm{def}}{=} \varphi_1 \oplus \mathrm{Id}_{[n/e]-1} \; . \tag{8}$$

Observe that ϕ is an element of order e.

Proposition 5.9. *Consider $(S_{n,q}, \mathcal{F}_{n,q})$ as in Notation 5.4, with $n \geq e$. For every $\mathcal{F}_{n,q}$-centric subgroup P, and j such that $1 \leq j \leq e-1$, the restriction of ϕ^j, where ϕ is defined in Equation (8), to P is not in $\mathrm{Hom}_{O^{p'}_*(\mathcal{F}_{n,q})}(P, S_{n,q})$.*

Proof. Consider the restriction of ϕ^j an $\mathcal{F}_{n,q}$-centric subgroup P. As P is $\mathcal{F}_{n,q}$-centric, it contains the center of $S_{n,q}$, which can be seen as the image of the matrices $\lambda\,\mathrm{Id}$ in the inclusion $\mathrm{GL}_m(q^e) \leq \mathrm{GL}_{em}(q)$. Assume now that $\phi^j|_P$ is in $\mathrm{Hom}_{O^{p'}_*(\mathcal{F}_{n,q})}(P, S_{n,q})$.

By Theorem 3.4, ϕ^j can be written as a composition of restriction of automorphisms $\phi_i \in O^{p'}(\mathrm{Aut}_{\mathcal{F}_{n,q}}(R_i))$ ($i \in \{1, \ldots, k\}$), for R_i $\mathcal{F}_{n,q}$-centric, $\mathcal{F}_{n,q}$-radical subgroups. Consider now $\rho\,\mathrm{Id}$, ρ a primitive p-root of unity in $\mathbb{F}_{q^e}$, which is an element in $\chi \cap P$ (χ as defined in Lemma 5.8). This implies $\phi^j(\rho\,\mathrm{Id}) = \phi_k \circ \phi_{k-1} \circ \cdots \circ \phi_1(\rho\,\mathrm{Id}) \in \chi$.

A direct computation show us that $\phi(\rho\,\mathrm{Id}) = \rho^{q^{-1}}\,\mathrm{Id}_1 \oplus \rho\,\mathrm{Id}_{[n/e]-1}$, and then $\phi^j(\rho\,\mathrm{Id}) = \rho^{q^{-j}}\,\mathrm{Id}_1 \oplus \rho\,\mathrm{Id}_{[n/e]-1}$. So this element is not in χ, as $q^{-j} \neq 1 \bmod p$.

So we get a contradiction which comes from assuming that

$$\phi^j|_P \in \mathrm{Hom}_{O^{p'}_*(\mathcal{F}_{n,q})}(P, S_{n,q})\,.$$

□

Theorem 5.10. *Let p be an odd prime and q prime power such that $p \nmid q$. Consider e the multiplicative order of q modulo p. Fix $S_{n,q}$ a Sylow p-subgroup of $\mathrm{GL}_n(q)$ and $(S_{n,q}, \mathcal{F}_{n,q})$ the induced saturated fusion system. Then,*

$$\mathrm{Out}_{\mathcal{F}}(S_{n,q})/\,\mathrm{Out}^0_{\mathcal{F}}(S_{n,q}) \cong \begin{cases} (\mathbb{Z}/e) \wr \Sigma_{[n/e]} & \text{if } e \le n < ep, \\ \mathbb{Z}/e & \text{if } ep \le n. \end{cases}$$

Proof. We have to consider $n \ge e$ to have a nontrivial Sylow p-subgroup in $\mathrm{GL}_n(q)$.

For $n < ep$ we have that $S_{n,q} \cong (\mathbb{Z}/p^l)^{[n/e]}$, which is abelian. So the only $\mathcal{F}$-centric subgroup is the total and $\mathrm{Out}^0_{\mathcal{F}_{n,q}}(S_{n,q})$ is trivial.

When $n \ge ep$, use Proposition 5.6 to see that the quotient is at most $\mathbb{Z}/e$ and Proposition 5.9 implies the result. □

6. Exotic fusion subsystems in the general linear group

Fix as before p a prime, q a prime power prime to p. Fix e the multiplicative order of q modulo p. Consider $(S_{n,q}, \mathcal{F}_{n,q}, \mathcal{L}_{n,q})$ the saturated fusion system of $\mathrm{GL}_n(q)$ at the prime p, with $n \ge e$.

Proposition 6.1. (a) *For each r dividing e there is a p-local finite group $(S_{n,q}, \mathcal{F}_{n,q,r}, \mathcal{L}_{n,q,r})$, such that $\mathcal{F}_{n,q,r}$ is of index prime to p in $\mathcal{F}_{n,q}$ over $S_{n,q}$ and $\mathrm{Out}_{\mathcal{F}_{n,q,r}}(S_{n,q})$ is a subgroup of index r in $\mathrm{Out}_{\mathcal{F}_{n,q}}(S_{n,q})$.*

(b) *If $n \ge ep$, there is just one $(S_{n,q}, \mathcal{F}_{n,q,r}, \mathcal{L}_{n,q,r})$ satisfying* (a).

(c) *Up to homotopy equivalence, there is a fibration*

$$|\mathcal{L}_{n,q,r}| \longrightarrow |\mathcal{L}_{n,q}| \longrightarrow B(\mathbb{Z}/r)\,.$$

Proof. For $e \le n < ep$ consider the saturated fusion subsystem of index prime to p corresponding to the kernel of the group epimorphism from

$$\mathrm{Out}_{\mathcal{F}}(S_{n,q})/\,\mathrm{Out}^0_{\mathcal{F}}(S_{n,q}) \cong (\mathbb{Z}/e) \wr \Sigma_{[n/e]}$$

to $\mathbb{Z}/r$ defined as

$$\begin{array}{ccc} (\mathbb{Z}/e)^{[n/e]} \rtimes \Sigma_{[n/e]} & \longrightarrow & \mathbb{Z}/r \\ (b_1, \dots, b_{[n/e]}; \sigma) & \mapsto & \dfrac{e}{r}(b_1 + \cdots + b_{[n/e]})\,. \end{array}$$

getting (a) and (c) for this case, applying [2, Theorem 5.5].

For $n \ge ep$ we have $\mathrm{Out}_{\mathcal{F}}(S_{n,q})/\,\mathrm{Out}^0_{\mathcal{F}}(S_{n,q}) \cong (\mathbb{Z}/e)$, so (a), (b) and (c) follows directly again from [2, Theorem 5.5]. □

We proceed now to identify these saturated fusion subsystems with the ones studied in [5, Section 11].

Let p be an odd prime and $r \ge 1$, $e \ge 1$ natural numbers such that $r|e|(p-1)$. Consider $G(e, r, m)$ the subgroup of $\mathrm{GL}_m(\mathbb{Z}^\wedge_p)$ as the subgroup generated by:

$$A(e, r, m) = \{\mathrm{diag}(a_1, \dots, a_m) \mid a_i^e = 1 \text{ and } (a_1 \cdots a_m)^{e/r} = 1\}$$

and the matrices corresponding to permutations in the coordinates. As a group, we have that $G(e, r, m) = A(e, r, m) \rtimes \Sigma_n$.

Consider now $m \geq 1$ and let $X(e,r,m)$ be the p-compact group which realizes the pseudoreflection group $G(e,r,m) \leq \mathrm{GL}_m(\mathbb{Z}_p^\wedge)$ (see [8] and [9] for further information). If $m \geq 2$ these are called the Generalized p-adic Grassmannians and we are interested in their finite Chevalley version. That is the space $BX(e,r,m)(q)$ defined as the pullback diagram:

$$\begin{tikzcd}
BX(e,r,m)(q) \arrow[r] \arrow[d] & BX(e,r,m) \arrow[d, "\Delta"] \\
BX(e,r,m) \arrow[r, "1\times\varphi^q"] & BX(e,r,m) \times BX(e,r,m)
\end{tikzcd}$$

where Δ is the diagonal map and φ^q is the unstable Adams operation of exponent q, where q is a p-adic unit.

So if we consider q a power of a prime not divisible by p, it is a p-adic unit and the previous construction makes sense. Consider e the multiplicative order of q modulo p. In this case we have the following equivalences up to p-completion [5, Remark 11.1]:

- $BX(1,1,n)(q) \simeq \mathrm{BGL}_n(q)$.
- $BX(1,1,n)(q) \simeq BX(1,1,e[n/e])(q)$.
- If q' is a another prime power such that $p|(q'-1)$ and $\nu_p(q'-1) = \nu_p(q^e-1)$ then $BX(1,1,n)(q) \simeq BX(e,1,[n/e])(q')$. Observe that we can take $q' = q^e$.

Where $[n/e]$ is the greater integer less or equal than n/e.

Before the main statement of the section we need to compute some centralizers.

Lemma 6.2. *Consider $(S_{n,q}, \mathcal{F}_{n,q,r}, \mathcal{L}_{n,q,r})$ as in Proposition 6.1, with $n \geq e$, and V a nontrivial elementary abelian p-subgroup contained in $S_{n,q}$. Then*

(a) *V is $\mathcal{F}_{n,q,r}$-conjugate to a subgroup of the maximal torus $\mathbb{T}_{p^l}^{[n/e]}$.*

(b) *For V contained in $\mathbb{T}_{p^l}^{[n/e]}$, consider the centralizer p-local finite group*

$$(C_{S_{n,q}}(V), C_{\mathcal{F}_{n,q,r}}(V), C_{\mathcal{L}_{n,q,r}}(V))$$

defined in [3, Proposition 2.5]. *Then there are integers m_0, m_1, ..., m_s such that*

$$|C_{\mathcal{L}_{n,q,r}}(V)| \simeq |\mathcal{L}_{em_0,q,r}| \times \mathrm{BGL}_{m_1}(q^e) \times \cdots \times \mathrm{BGL}_{m_s}(q^e)$$

up to p-completion. Moreover $em_0 < n$ and $[n/e] = m_0 + m_1 + \cdots + m_s$.

Proof. Consider V a nontrivial elementary abelian p-group contained in S. Consider the saturated fusion system $(S_{n,q}, \mathcal{F}_{S_{n,q}}(\mathrm{GL}_{[n/e]}(q^e)))$ which is contained in $(S_{n,q}, \mathcal{F}_{n,q,r})$ for all $r|e$. Then we can use the fact that V is $\mathcal{F}_{S_{n,q}}(\mathrm{GL}_{[n/e]}(q^e))$-conjugate to a toral subgroup, getting (a).

To compute the centralizer fusion system, consider the point-wise stabilizer of V in

$$\mathrm{Aut}_{\mathcal{F}_{n,q,r}}(\mathbb{T}_{p^l}^{[n/e]}) = G(e,r,[n/e]),$$

which is isomorphic to $G(e,r,m_0) \times \Sigma_{m_1} \times \cdots \times \Sigma_{m_s}$. Note that as V is a nontrivial group, $m_0 < [n/e]$.

We can see that $C_{\mathcal{F}_{n,q,r}}(V)$ is a saturated fusion system over $C_{S_{n,q}}(V)$, which can be written as $C_{S_{n,q}}(V) \cong S_{em_0,q} \times S_{m_1,q^e} \times \cdots \times S_{m_s,q^e}$, where we are using that Lemma 5.3 identifies S_{m,q^e} and $S_{em,q}$. Also $C_{\mathcal{F}_{n,q,r}}(V)$ contains $\mathcal{F}_{em_0,q,r} \times \mathcal{F}_{S_{m_1,q^e}}(\mathrm{GL}_{m_1}(q^e)) \times \cdots \times \mathcal{F}_{S_{m_s,q^e}}(\mathrm{GL}_{m_s}(q^e))$, as all the morphisms in this last saturated fusion system centralize V and are contained in $\mathcal{F}_{n,q,r}$.

Considering now that $\mathcal{F}_{n,q,r}$ is a fusion subsystem of $\mathcal{F}_{n,q}$, the morphisms in the centralizer of V in $\mathcal{F}_{n,q,r}$ are again morphisms in the centralizer of V in $\mathcal{F}_{n,q}$.

The centralizer p-local finite group of V in $\mathcal{F}_{n,q}$ is computed in [5, Proposition 11.2], and its classifying space is mod p equivalent to $\mathrm{BGL}_{em_0}(q) \times \mathrm{BGL}_{m_1}(q^e) \times \cdots \times \mathrm{BGL}_{m_s}(q^e)$, and also has the same Sylow p-subgroup as $C_{\mathcal{F}_{n,q,r}}(V)$.

With all this data we get that $C_{\mathcal{F}_{n,q,r}}(V)$ is a saturated fusion subsystem of index prime to p in $C_{\mathcal{F}_{n,q}}(V)$ such that in the maximal torus $\mathrm{Aut}_{C_{\mathcal{F}_{n,q,r}}(V)}(\mathbb{T}_{p^l}^{[n/e]})$ is an index r subgroup of $\mathrm{Aut}_{C_{\mathcal{F}_{n,q}}(V)}(\mathbb{T}_{p^l}^{[n/e]})$, and so

$$C_{\mathcal{F}_{n,q,r}}(V) \simeq \mathcal{F}_{em_0,q,r} \times \mathcal{F}_{S_{m_1,q^e}}(\mathrm{GL}_{m_1}(q^e)) \times \cdots \times \mathcal{F}_{S_{m_s,q^e}}(\mathrm{GL}_{m_s}(q^e)) .$$

So (b) follows considering the centric linking systems associated to these saturated fusion systems. □

Theorem 6.3. *For each r dividing e and $n \geq e$ consider $(S_n, \mathcal{F}_{n,q,r}, \mathcal{L}_{n,q,r})$, the p-local finite group from Proposition 6.1. Then*

$$|\mathcal{L}_{n,q,r}| \simeq BX(e, r, [n/e])(q^e)$$

up to p-completion.

Proof. We proceed by induction on n, beginning by the smallest considered case, that is $n = e$.

For $\mathrm{GL}_e(q)$ we have that the p-local finite group $(S_{e,q}, \mathcal{F}_{e,q,r}, \mathcal{L}_{e,q,r})$ is characterized by $S_{e,q} = \mathbb{Z}/p^l$, and $\mathrm{Aut}_{\mathcal{F}_{e,q,r}}(S_{e,q}) = \mathbb{Z}/h$, where $h \overset{\text{def}}{=} e/r$. So it corresponds to the fusion system of $\mathbb{Z}/p^l \rtimes \mathbb{Z}/h$, where $h \overset{\text{def}}{=} e/r$. The Sylow p-subgroup of $BX(e, h, 1)(q)$ is also isomorphic to $\mathbb{Z}/p^l$, with outer automorphism group $G(e, h, 1) = \mathbb{Z}/h$, so both are the same p-local finite group.

Now assume that the result is true for any $n' \leq n$.

To proceed with the induction argument we use the centralizer decomposition from [3, Theorem 2.6].

Fix $V \leq S$ an elementary abelian fully $\mathcal{F}_{n,q,r}$-centralized subgroup. If necessary we can conjugate it to get V contained in the torus, and still being fully $\mathcal{F}_{n,q,r}$-centralized.

Consider $(C_{S_{n,q}}(V), C_{\mathcal{F}_{n,q,r}}(V), C_{\mathcal{L}_{n,q,r}}(V))$ the centralizer p-local finite group, whose classifying space is, by Lemma 6.2, homotopy equivalent up to p-completion to

$$|\mathcal{L}_{em_0,q,r}| \times \mathrm{BGL}_{m_1}(q^e) \times \cdots \times \mathrm{BGL}_{m_s}(q^e) ,$$

with $em_0 < n$. By induction hypothesis we have, up to p-completion, $|\mathcal{L}_{em_0,q,r}| \simeq BX(e, r, m_0)(q^e)$ and so

$$\begin{aligned} |\mathcal{L}_{em_0,q,r}| \times \mathrm{BGL}_{m_1}(q^e) \times \cdots \times \mathrm{BGL}_{m_s}(q^e) \simeq \\ BX(e, r, m_0)(q^e) \times \mathrm{BGL}_{m_1}(q^e) \times \cdots \times \mathrm{BGL}_{m_s}(q^e) . \end{aligned} \tag{9}$$

Now consider the centralizer decomposition from [3, Theorem 2.6], obtaining

$$\underset{V \in (\mathcal{F}^e)^{op}}{\underrightarrow{\mathrm{hocolim}}} |C_{\mathcal{L}_{n,q,r}}(V)| \simeq |\mathcal{L}_{n,q,r}|$$

while using Equation (9) and [5, Proposition 11.3] we also get

$$\underset{V\in(\mathcal{F}^e)^{op}}{\underrightarrow{\text{hocolim}}}\,|C_{\mathcal{L}_{n,q,r}}(V)| \simeq BX(e,r,[n/e])(q^e)$$

getting the result. □

We finally get one of the results which motivated this work.

Remark 6.4. Consider p a prime, q a prime power such that e, the order of q modulo p, is bigger than 2. Consider r a divisor of e such that $r > 2$. Then, by Proposition 6.1 we have the fibration (up to homotopy equivalence):

$$|\mathcal{L}_{n,q,r}| \longrightarrow |\mathcal{L}_{n,q}| \longrightarrow B(\mathbb{Z}/r)\,.$$

By Theorem 6.3, we have

$$|\mathcal{L}_{n,q,r}| \simeq BX(e,r,[n/e])(q^e)$$

up to p-completion, and by [5, Proposition 11.5], when $r > 2$ and $n \geq ep$, these are classifying spaces of exotic p-local finite groups. So we get examples of extensions of p-local finite groups where two of them correspond to finite groups and the third one is an exotic p-local finite group.

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