A simplicial model for the Hopf map

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Abstract

We give an explicit simplicial model for the Hopf map $S^3 \to S^2$. For this purpose, we construct a model of S^3 as a principal twisted cartesian product $K \times_\eta S^2$, where K is a simplicial model for S^1 acting by left multiplication on itself, S^2 is given the simplest simplicial model and the twisting map is $\eta : (S^2)_n \to (K)_{n-1}$. We construct a Kan complex for the simplicial model K of S^1. The simplicial model for the Hopf map is then the projection $K \times_\eta S^2 \to S^2$.

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1 Introduction

The motivation for finding a simplicial model for the Hopf map arose when trying to find a simple test to decide whether the stabilisation of a certain interesting model category M is different from that of the category of chain complexes of abelian groups. As detailed in [2, chapter 6], consider the following situation. Let M be a symmetric monoidal model category whose stabilisation exists and suppose there is a monoidal Quillen adjunction $F : \mathsf{sS} \leftrightarrow \mathsf{M} : G$ between the category of simplicial sets and M. In the stable category of chain complexes, the Hopf map vanishes. Therefore, if we have a good simplicial model for the Hopf map that allows us to show that the

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multiply suspended images under the functor F of this simplicial model never vanish, then the stabilisation of M is different from that of chain complexes.

The main result of this paper is that a very good simplicial model of the Hopf map is the projection $p : K \times_\eta S^2 \to S^2$ of a principal twisted cartesian product of a simplicial model K of S^1 with the simplest simplicial model for S^2. The proof of this result shows that we are able to model simplicially any S^1-bundle of base S^2.

This paper is structured in the following manner. Section 2 recalls the notions and results related to principal twisted cartesian products. In section 3 we construct a Kan model for S^1, which is required to carry enough structure. Section 4 gives explicit computations of the Kan model as well as of the twisting map. Finally, we prove the main result in section 5.

We are indebted to Kathryn Hess and Andrew Tonks for the idea of using principal twisted cartesian products. The idea of using the Hopf map as a test came out from a discussion with Stefan Schwede. This work has been carried out with the financial support of the Swiss National Science Foundation.

2 Principal twisted cartesian products

This section is devoted to explaining the tools for building a simplicial model for S^3 with enough structure to capture the Hopf map.

Definition 2.1. *Let F and B be two simplicial sets. Let H be a simplicial group acting on the left on F. Let $\zeta : B \to H$ be a map of graded sets of degree -1 such that $\zeta_n : B_n \to H_{n-1}$ satisfies the following identities:*

$$\begin{aligned}
\partial_0 \zeta(b) &= (\zeta(\partial_0 b))^{-1} \zeta(\partial_1 b) \\
\partial_i \zeta(b) &= \zeta(\partial_{i+1} b) \quad \text{for } i > 0 \\
s_i \zeta(b) &= \zeta(s_{i+1} b) \quad \text{for } i \geq 0 \\
\zeta(s_0 b) &= id_n \quad \text{for } b \in B_n.
\end{aligned}$$

The map ζ is the twisting map. A twisted cartesian product of fibre F, base B and group H is a simplicial set denoted $F \times_\zeta B$ satisfying

$$(F \times_\zeta B)_n = F_n \times B_n$$

with faces and degeneracies as follows :

1. $\partial_i(f, b) = (\partial_i f, \partial_i b)$ *for* $i > 0$

2. $\partial_0(f,b)) = (\zeta(b)\partial_0 f, \partial_0 b)$

3. $s_i(f,b) = (s_i f, s_i b)$ *for* $i \geq 0$.

Furthermore, if $F = H$ *acting on itself by left multiplication, then* $F \times_\zeta B$ *is a principal twisted cartesian product (PTCP).*

We will also use the terminology "twisted cartesian product" for the projection $p : F \times_\zeta B \to B$.

The following proposition is a classical result whose proof can be found in [1, proposition 18.4].

Proposition 2.2. *Let* $p : F \times_\zeta B \to B$ *be a twisted cartesian product with group* H. *If the fiber* F *is a Kan complex, then :*

1. *the projection* p *is a Kan fibration, and*
2. *if* $F = H$, p *is a principal fibration.*

Remark 2.3. *Let* $S : \mathsf{TOP} \to \mathsf{sS}$ *be the singular functor from the category* TOP *of topological spaces to the category* sS *of simplicial sets. A map* f *is a (Serre) fibration if and only if* $S(f)$ *is a Kan fibration. Thus, a principal fibration (or fibre bundle) in* TOP *passes via the functor* S *to a principal fibration in* sS. *Since the Hopf map* $S^3 \to S^2$ *is a fibration in* TOP, *the corresponding simplicial model has to be a Kan fibration. As a consequence, if we want to model* S^3 *as a PTCP* $K \times_\eta S^2 \to S^2$, *this has to be a Kan fibration, which it is when* K *is a Kan complex, by proposition 2.2. We construct such a PTCP in the following sections.*

3 The simplicial model for S^1

In short, to build a Kan model of S^1 we let $\mathbb{Z}(2)$ denote a chain complex concentrated in degree two, and we apply a functor Γ to obtain a simplicial abelian group $\Gamma\mathbb{Z}(2)$. By applying the loop group functor G, the model of S^1 is given by $G\Gamma\mathbb{Z}(2)$. The latter is always a Kan complex, since every simplicial group is a Kan complex. More precisely we give the following definitions.

Definition 3.1. *Let* sAb *be the category of simplicial abelian groups and let* CC *be the category of chain complexes of abelian groups. We define the functor* $\Gamma : \mathsf{CC} \to \mathsf{sAb}$ *as follows. For any* $(X, \partial) \in \mathsf{CC}$, *the simplicial abelian group* $\Gamma(X)$ *is given by :*

1.

$$\Gamma_n(X) = X_n \bigoplus_{r=0}^{n-1} \sum_{k=n-r} \sigma_{j_k} \dots \sigma_{j_1} X_r \tag{1}$$

where $\sigma_{j_k} \dots \sigma_{j_1} X_r$ *is the abelian group whose elements are the symbols* $\sigma_{j_k} \dots \sigma_{j_1} x$ *with* $x \in X_r$*. The sum* $\sum_{k=n-r}$ *is taken over all sequences of indices* $\{j_i\}$ *such that* $0 \le j_1 < j_2 < \cdots < j_k < n$*.*

The addition of symbols is defined by

$$\sigma_{j_k} \dots \sigma_{j_1} x + \sigma_{j_k} \dots \sigma_{j_1} y = \sigma_{j_k} \dots \sigma_{j_1} (x + y).$$

Degeneracies and faces are given by :

2. $s_i : \Gamma_n(X) \to \Gamma_{n+1}(X)$ *is defined by*

   (a) $s_i x = \sigma_i x$ *for* $x \in X_n$

   (b) if $k = n - r$ *and* $x \in X_r$ *then*

$$s_i \sigma_{j_k} \dots \sigma_{j_1} x = \sigma_{h_{k+1}} \dots \sigma_{h_1} x$$

   when $s_i s_{j_k} \dots s_{j_1} = s_{h_{k+1}} \dots s_{h_1}$ *and where* $s_{h_{k+1}} \dots s_{h_1}$ *is written in the canonical form*[1]*, i.e.* $h_{k+1} > h_k > ... > h_1$*.*

3. $\partial_i : \Gamma_n(X) \to \Gamma_{n-1}(X)$ *is defined by*

   (a) $\partial_n x = \partial(x)$ *and* $\partial_i x = 0$ *if* $i < n$ *and* $x \in X_n$*.*

   (b) if $k = n - r$ *and* $x \in X_r$ *then*

$$\partial_i \sigma_{j_k} \dots \sigma_{j_1} x = \begin{cases} \sigma_{h_{k-1}} \dots \sigma_{h_1} x \\ \sigma_{h_k} \dots \sigma_{h_1} \partial(x) \\ 0 \end{cases}$$

   if respectively

$$\partial_i s_{j_k} \dots s_{j_1} = \begin{cases} s_{h_{k-1}} \dots s_{h_1} \\ s_{h_k} \dots s_{h_1} \partial_r \\ s_{h_k} \dots s_{h_1} \partial_j \quad j < r \end{cases}$$

   where the right hand side is written in the canonical form.

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[1] Every composition of degeneracies and/or faces can be written in the canonical form with the aid of the simplicial identities.

We now define the functor G.

Definition 3.2. *Let* sGr *the category of simplicial groups and let* K *be a simplicial set. We define the functor* $G : \mathsf{sS} \to \mathsf{sGr}$ *as follows. The group* $G_n(K) = G(K)_n$ *is the free group generated by the elements of* K_{n+1} *modulo the relations* $s_0 x = id_n$ *for all* $x \in K_n$*.*

If $x \in K_{n+1}$*, let* $\zeta(x)$ *be the class of* x *in* $G_n(K)$*. Faces and degeneracies of* $G(K)$ *are defined on generators by the relations :*

$$\begin{aligned} \zeta(\partial_0 x)\partial_0\zeta(x) &= \zeta(\partial_1 x) && (2)\\ \partial_i\zeta(x) &= \zeta(\partial_{i+1}x) \qquad \textit{if } i > 0 && (3)\\ s_i\zeta(x) &= \zeta(s_{i+1}x) \qquad \textit{if } i \geq 0. && (4) \end{aligned}$$

By extension we have homomorphisms $\partial_i : G_n(K) \to G_{n-1}(K)$ and $s_i : G_n(K) \to G_{n+1}(K)$. Clearly, $G(K)$ is a simplicial group.

Remark 3.3. *The morphism* ζ *of definition 3.2 is clearly a twisting map. Hence, for every simplicial abelian group* K *we have a twisted cartesian product* $G(K) \times_\zeta K$*, which is acyclic. The reader may refer to [1, pp. 118–123] for details.*

By [1, Remarks 23.7], $\Gamma\mathbb{Z}(2)$ is a $K(\mathbb{Z}, 2)$, hence a simplicial model for BS^1. $G\Gamma\mathbb{Z}(2)$ is then a model for ΩBS^1, hence for S^1.

4 Some computations

This section is devoted to clarifying the previous construction by giving explicit computations of $G\Gamma\mathbb{Z}(2)$ and the map η. For this we will choose a simplicial model for S^2 consisting in one non degenerate simplex in degree two and only degeneracies above.

To compute $\Gamma_n\mathbb{Z}(2)$ we use formula (1). Since $\mathbb{Z}(2)$ is concentrated in degree two, we obtain

$$\Gamma_n\mathbb{Z}(2) = \bigoplus_{0 \leq j_1 < \cdots < j_{n-2} < n} \sigma_{j_{n-2}} \cdots \sigma_{j_1}\mathbb{Z}. \tag{5}$$

As an example, in degree three, the faces and degeneracies are given for all $z \in \mathbb{Z}$ by

$$\begin{array}{llll} \partial_0(\sigma_0 z) = z & \partial_1(\sigma_0 z) = z & \partial_2(\sigma_0 z) = 0 & \partial_3(\sigma_0 z) = 0\\ \partial_0(\sigma_1 z) = 0 & \partial_1(\sigma_1 z) = z & \partial_2(\sigma_1 z) = z & \partial_3(\sigma_1 z) = 0\\ \partial_0(\sigma_2 z) = 0 & \partial_1(\sigma_2 z) = 0 & \partial_2(\sigma_2 z) = z & \partial_3(\sigma_2 z) = z \end{array}$$

$$\begin{array}{llll}
s_0(\sigma_0 z) = \sigma_1\sigma_0 z & s_1(\sigma_0 z) = \sigma_1\sigma_0 z & s_2(\sigma_0 z) = \sigma_2\sigma_0 z & s_3(\sigma_0 z) = \sigma_3\sigma_0 z \\
s_0(\sigma_1 z) = \sigma_2\sigma_0 z & s_1(\sigma_1 z) = \sigma_2\sigma_1 z & s_2(\sigma_1 z) = \sigma_2\sigma_1 z & s_3(\sigma_1 z) = \sigma_3\sigma_1 z \\
s_0(\sigma_2 z) = \sigma_3\sigma_0 z & s_1(\sigma_2 z) = \sigma_3\sigma_1 z & s_2(\sigma_2 z) = \sigma_3\sigma_2 z & s_3(\sigma_2 z) = \sigma_3\sigma_2 z.
\end{array}$$

For $G\Gamma\mathbb{Z}(2)$, we have

$$\begin{array}{rcl}
(G\Gamma\mathbb{Z}(2))_0 & = & \{e\}, \qquad (G\Gamma\mathbb{Z}(2))_1 = \mathcal{F}\{\mathbb{Z}\backslash\{0\}\} \\
(G\Gamma\mathbb{Z}(2))_2 & = & \mathcal{F}\{\sigma_2\mathbb{Z} \oplus \sigma_1\mathbb{Z}\} \\
(G\Gamma\mathbb{Z}(2))_3 & = & \mathcal{F}\{\sigma_2\sigma_1\mathbb{Z} \oplus \sigma_3\sigma_2\mathbb{Z} \oplus \sigma_3\sigma_1\mathbb{Z}\} \\
(G\Gamma\mathbb{Z}(2))_4 & = & \mathcal{F}\{\sigma_4\sigma_2\sigma_1\mathbb{Z} \oplus \sigma_4\sigma_3\sigma_2\mathbb{Z} \oplus \sigma_4\sigma_3\sigma_1\mathbb{Z} \oplus \sigma_3\sigma_2\sigma_1\mathbb{Z}\} \\
 & \vdots & \\
(G\Gamma\mathbb{Z}(2))_n & = & \mathcal{F}\left\{ \bigoplus_{0<j_1<\cdots<j_{n-1}<n+1} \sigma_{j_{n-1}}\ldots\sigma_{j_1}\mathbb{Z} \right\}.
\end{array} \tag{6}$$

where $\mathcal{F}\{\}$ stands for the free group generated by elements inside $\{\}$. Notice that $s_0(\sigma_{j_{n-1}}\ldots\sigma_{j_1}z)$ can always be expressed in a form ending by $\sigma_0 z$. Hence each term containing $\sigma_0\mathbb{Z}$ is trivial and gives the first strict inequality in $0 < j_i < \cdots < j_{n-1} < n+1$. Faces and degeneracies are given by the formulae (2)–(4).

Let $\overline{x}$ be the class of $x \in \Gamma_{n+1}\mathbb{Z}(2)$ in $(G\Gamma\mathbb{Z}(2))_n$. For S^2 we consider the simplicial model consisting in one generator y in degree two and only degeneracies above. The twisting morphism $\eta : S^2 \to G\Gamma\mathbb{Z}(2)$ is defined by the relations :

$$\begin{array}{rcl}
\eta_0(*) & = & e, \qquad \eta_1(*) = e \\
\eta_2(y) & = & \overline{1} \\
\eta_3(s_1 y) & = & \overline{\sigma_1 1} \\
\eta_3(s_2 y) & = & \overline{\sigma_2 1} \\
\eta_3(s_0 y) & = & e \\
\eta_4(s_2 s_1 y) & = & \overline{\sigma_2\sigma_1 1} \\
\eta_4(s_3 s_2 y) & = & \overline{\sigma_3\sigma_2 1} \\
\eta_4(s_3 s_1 y) & = & \overline{\sigma_3\sigma_1 1} \\
\eta_4(s_i s_0 y) & = & e \qquad \text{for } 0 \le i < 4 \\
 & \vdots &
\end{array}$$

where $\bar{1}$ is a generator of $\mathcal{F}\{\mathbb{Z}\backslash\{0\}\}$. In general, for $n \geq 2$

$$\eta_n(s_{j_{n-2}} \dots s_{j_1} y) = \begin{cases} e & \text{if } s_{j_1} = s_0 \\ \overline{\sigma_{j_{n-2}} \dots \sigma_{j_1} 1} & \text{otherwise} \end{cases}$$

where $s_{j_{n-2}} \dots s_{j_1} y$ is written in the canonical form.

The map η is then determined by its value on the generator y of the model of S^2, as is clear from the formula (4).

5 The simplicial model for the Hopf map

We now have all the tools to build our simplicial model for S^3. Denote by $\mathbb{Z}$ the set of integers and by $\mathbb{Z}(2)$ the chain complex of abelian groups consisting in one copy of $\mathbb{Z}$ in degree two and 0 elsewhere. We apply the functor Γ to get a simplicial abelian group $\Gamma\mathbb{Z}(2)$. Therefore, by remark 3.3,

$$p : G\Gamma\mathbb{Z}(2) \times_\eta S^2 \to S^2$$

is a principal twisted cartesian product whose fiber is $G\Gamma\mathbb{Z}(2)$ acting on itself by left multiplication. The map $\eta : S^2 \to G\Gamma\mathbb{Z}(2)$ is explained in the previous section.

Theorem 5.1. *Let S^2 be endowed with the above simplicial model. A simplicial model for the Hopf map $S^3 \to S^2$ is then given by the principal twisted cartesian product*

$$p : G\Gamma\mathbb{Z}(2) \times_\eta S^2 \to S^2.$$

Proof. The fibration $p : G\Gamma\mathbb{Z}(2) \times_\eta S^2 \to S^2$ is a model for an element of the set of S^1-bundles of base S^2, which contains the Hopf map. Now, S^1-bundles of base S^2 are classified by $\mathbb{Z}$, and the Hopf map corresponds to the class $1 \in \mathbb{Z}$. All we have to show is that our model $G\Gamma\mathbb{Z}(2) \times_\eta S^2 \to S^2$ corresponds indeed to the class $1 \in \mathbb{Z}$. Consider the diagramm

$$\begin{tikzcd}
G\Gamma\mathbb{Z}(2) \arrow[d] & G\Gamma\mathbb{Z}(2) \arrow[d] & \\
G\Gamma\mathbb{Z}(2) \times_\eta S^2 \arrow[r] \arrow[d] & G\Gamma\mathbb{Z}(2) \times_\zeta \Gamma\mathbb{Z}(2) \arrow[d] & \\
S^2 \arrow[r, "\alpha"] & \Gamma\mathbb{Z}(2) \arrow[r, "\beta"] & G\Gamma\mathbb{Z}(2)
\end{tikzcd}$$

where the two columns are fibrations and the composition $\beta\alpha$ is the twisting map η. Recall from last section that the bottom composition η sends the generator y of S^2 to the class of the generator $1 \in \mathbb{Z}$. Note that $G\Gamma\mathbb{Z}(2) \times_\zeta \Gamma\mathbb{Z}(2)$ is acyclic and that the first vertical fibration is classified by the map $\beta\alpha = \eta$. By choosing η to send a generator of S^2 to the generator $1 \in \mathbb{Z}$ we guaranty that our fibration $G\Gamma\mathbb{Z}(2) \times_\eta S^2 \to S^2$ lies in the same class as the Hopf map does and hence is a model of the later. $\square$

Remark 5.2. *In the previous proof, if we choose to send* $y \in S^2$ *to* $m1 \in \mathbb{Z}$ *via the map* α*, our fibration can model any* S^1*-bundle of base* S^2 *by letting* m *vary over* $\mathbb{Z}$*.*

References

[1] J. P. May, *Simplicial objects in algebraic topology*, Mathematical Studies 11, Van Nostrand, 1967.

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