Projective Bundle Ideals
Constructions of $\mathfrak{m}$-primary irreducible Ideals in Polynomial Algebras

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SUMMARY : *The study of $\mathfrak{m}$-primary irreducible ideals in a commutative graded connected Noetherean algebra over a field is in principal equivalent to the study of the corresponding quotient algebras. Such algebras are Poincaré duality algebras. The prototype of such an algebra (apart from the cosmetic difference of being graded commutative rather than commutative graded) is the cohomology with field coefficients of a closed oriented manifold. Topological constructions on closed manifolds often lead to algebraic constructions on Poincaré duality algebras. It is the purpose of this note to examine several of these and develop some of their basic properties.*

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THE STUDY of $\mathfrak{m}$-primary irreducible ideals in commutative graded[1] connected Noetherean algebras over a field is in principal equivalent to the study of the corresponding quotient algebras. Such algebras are **Poincaré duality algebras** (see e.g. [8] Section I.1), by which we mean the following: There is an integer d called the **formal dimension** of the algebra such that the homogeneous component of degree d is 1-dimensional, all homogeneous components of degree strictly greater than d are zero, and the pairing between elements of degree i and $d - i$, for $0 \le i \le d$ given by multiplication into the homogeneous component of degree d is nonsingular.

The prototype of such an algebra (apart from the cosmetic difference of being graded commutative rather than commutative graded) is the cohomology with field coefficients of a closed oriented manifold. There are several topological constructions on closed manifolds that lead to algebraic constructions on Poincaré duality algebras. To name but two, there is the projective bundle theorem and the notion of dualizing a line bundle. It is the purpose of this note to examine these and other constructions in a purely algebraic context, and develop some of their basic properties with an eye towards enhancing our store of examples (which are the basis for theorems in the end) of irreducible $\mathfrak{m}$-primary ideals in polynomial algebras $\mathbb{F}[z_1, \ldots, z_n]$.

If a Poincaré duality algebra is generated by its homogeneous component of degree one then it is called **standardly graded**. For standardly graded Poincaré duality algebras of the same formal dimension there is a operation called the connected sum (see § 6) that turns the set of isomorphism classes of such algebras into a semigroup. In [12] we determined all surface algebras (i.e., standardly graded Poincaré duality algebras of formal dimension two) by amongst other things computing the Grothendieck group of standardly graded surface algebras under the operation of connected sum. This group is finitely generated and its structure mirrors faithfully the topological classification of closed surfaces. Namely it is generated by the $\mathbb{F}_2$-cohomology $\mathbb{T}$ of the torus $S^1 \times S^1$ and $\mathbb{P}$ the projective plain $\mathbb{RP}(2)$ subject to the one relation $3[\mathbb{T}] = [\mathbb{T}] + [\mathbb{P}]$ in the Grothendieck group (where [ ] denotes the isomorphism class).

By contrast, for Poincaré duality algebras of formal dimension strictly greater than two the Grothendieck group fails to be finitely generated. This manuscript grew in part out of the search for generators for the Grothendieck of threefolds. We show that unfortunately there are Poincaré duality algebras $P(n)$ which are quotients of $\mathbb{F}[z_1, \ldots, z_n]/I(n)$ where the ideal $I(n)$ requires roughly $n!$ generators and the algebras $P(n)$ are indecomposable in the Grothendieck group.

One of our basic tools is the theory of *inverse systems* due to F.S. Macaulay [5] Part IV and we assume familiarity with it as presented for example in [8] §VI.1. We also make use of basic facts concerning irreducible ideals as found in [8] Parts I and II.

[1] We adhere to the conventions of J. C. Moore as far as graded objects go. This means only *homogeneous* elements are considered unless explicitly stated otherwise. The homogeneous component of degree d of a graded object is denoted by a subscript $_d$ attached to the object.

§1. Projective Bundle Ideals

If M is a closed smooth manifold and $\xi \downarrow M$ is a smooth $k+1$-dimensional vector bundle over M. Then one may form the corresponding projective bundle with fibres the projective space of dimension k. The projective bundle theorem (see e.g. [14] page 62) provides a relation between the cohomology of the manifold M and the total space $\mathbb{P}(\xi \downarrow M)$ of the corresponding projective bundle. It serves as the basis of the following definition.

Definition: *Let $I \subset \mathbb{F}[V, X]$ be a $\mathfrak{m}$-primary ideal and $J = I \cap \mathbb{F}[V]$. We call I a* **projective bundle ideal** *with* **base ideal** *J if $\mathbb{F}[V, X]/I$ is a free $\mathbb{F}[V]/J$-module with respect to the module structure defined by the canonical inclusion $\mathbb{F}[V]/J \hookrightarrow \mathbb{F}[V, X]/I$.*

Suppose that $I \subset \mathbb{F}[V, X]$ is an $\mathfrak{m}$-primary ideal and $J = I \cap \mathbb{F}[V]$. Then there is a commutative diagram

$$\begin{array}{ccccccccc} \mathbb{F} & \longrightarrow & \mathbb{F}[V] & \longrightarrow & \mathbb{F}[V, X] & \longrightarrow & \mathbb{F}[X] & \longrightarrow & \mathbb{F} \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{F} & \longrightarrow & \mathbb{F}[V]/J & \longrightarrow & \mathbb{F}[V, X] & \longrightarrow & \boxed{?} & \longrightarrow & \mathbb{F} \end{array}$$

where the rows are coexact and the vertical maps the natural projections. Therefore the *unknown* cokernel $\boxed{?}$ must be of the form $\mathbb{F}[X]/(X^{k+1})$ for some integer k. The integer $k+1$ is called the **bundle dimension**. We call $\mathbb{F}[V, X]/I$ the **bundle algebra**, $\mathbb{F}[V]/J$ the **base algebra**, and $\mathbb{F}[X]/(X^{k+1})$ the **fibre algebra**. This is because the coexact sequence

$$(\text{⌘}) \qquad \mathbb{F} \longrightarrow \mathbb{F}[V]/J \longrightarrow \mathbb{F}[V, X] \longrightarrow \mathbb{F}[X]/(X^{k+1}) \longrightarrow \mathbb{F}$$

is an analog of the the coexact sequence of cohomology algebras

$$\mathbb{F} \longrightarrow H^*(B; \mathbb{F}) \longrightarrow H^*(\mathbb{P}(\xi \downarrow B); \mathbb{F}) \longrightarrow H^*(\mathbb{CP}(k); \mathbb{F}) \longrightarrow \mathbb{F}$$

associated to a complex vector bundle $\xi \downarrow B$ of dimension $k+1$ over the base space B, where $\mathbb{P}(\xi \downarrow B)$ is the associated projective space bundle (see e.g. [14] loc.cit.).

Lemma 1.1: *Suppose that $I \subset \mathbb{F}[V, X]$ is a projective bundle ideal of bundle dimension $k+1$ and $J = I \cap \mathbb{F}[V]$. Then $\mathbb{F}[V, X]/I$ is a free $\mathbb{F}[V]/J$-module with basis $1, X, \ldots, X^k$.*

Proof: By hypothesis $\mathbb{F}[V, X]/I$ is a free $\mathbb{F}[V]/J$-module and the coexact sequence (⌘) shows that it is generated by $1, X, \ldots, X^k$. From the graded Nakayama Lemma (see e.g. [9] Proposition 5.1.3) it therefore follows that $1, X, \ldots, X^\ell$ is a basis where ℓ is the smallest integer such that $X^{\ell+1}$ can be written as an $\mathbb{F}[V]/J$-linear combination of $1, X, \ldots, X^\ell$. By definition ℓ is just k. □

This lemma means that for a projective bundle ideal $I \subset \mathbb{F}[V, X]$ of bundle dimension $k+1$ one may write

$$-X^{k+1} = \alpha_1 X^k + \cdots + \alpha_k X + \alpha_{k+1}$$

where $\alpha_1, \ldots, \alpha_{k+1} \in \mathbb{F}[V]/J$ and $\deg(\alpha_j) = j$ for $j = 1, \ldots, k+1$. We call this the **bundle relation**.If we choose $h_1, \ldots, h_{k+1} \in \mathbb{F}[V]$ lifting $\alpha_1, \ldots, \alpha_{k+1} \in F[V]/J$ respectively, then the form

$$h(X) = X^{k+1} + h_1 X^k + \cdots + h_k X + h_{k+1}$$

belongs to I. This is because

$$-X^{k+1} = \alpha_1 X^k + \cdots + \alpha_k X + \alpha_{k+1} \in \mathbb{F}[V]/J.$$

We call $h(X)$ a **homoginizing form** or **polynomial** for I.

LEMMA 1.2: *Suppose that $I \subset \mathbb{F}[V, X]$ is a projective bundle ideal of bundle dimension $k+1$, with bundle ideal $J = I \cap \mathbb{F}[V]$, and bundle relation*

$$-X^{k+1} = \alpha_1 X^k + \cdots + \alpha_k X + \alpha_{k+1}.$$

Then the kernel of the natural map

$$\big(\mathbb{F}[V]/J\big)[X] \xrightarrow{\varphi} \mathbb{F}[V, X]/I$$

is the principal ideal generated by

$$\alpha(X) = X^{k+1} + \alpha_1 X^k + \cdots + \alpha_k X + \alpha_{k+1}.$$

PROOF: We begin by assembling some facts.

(1) $\alpha(X) \in \ker(\varphi)$.

(2) No nonzero element $f(X) = f_0 + f_1 X + \cdots + f_m X^m$ of lower degree m in X belongs to $\ker(\varphi)$ since $1, X, \ldots, X^k$ are $\mathbb{F}[V]/J$-linearly independent.

(3) Any nonzero element of degree $k+1$ in X belonging to the kernel of φ is a scalar multiple of $\alpha(X)$.

Hence we may procede inductively on the degree of an element in X belonging to $\ker(\varphi)$ to show it belongs to $(\alpha(X))$. Choose an element $0 \neq f(X) = f_0 + f_1 X + \cdots + f_m X^m$ belonging to $\ker(\varphi)$. Then $m \geq k+1$. Hence $\alpha(X) \cdot X^{m-(k+1)}$ is a polynomial of degree m in X and moreover $f(X) - \alpha(X) \cdot X^{m-(k+1)}$ belongs to the kernel of φ and has degree in X strictly less than m, so by the induction hypothesis is also in $(\alpha(X))$ and the result follows. □

LEMMA 1.3: *Suppose that $I \subset \mathbb{F}[V, X]$ is a projective bundle ideal of bundle dimension $k+1$, $J = I \cap \mathbb{F}[V]$ is the base ideal, and*

$$-X^{k+1} = \alpha_1 X^k + \cdots + \alpha_k X + \alpha_{k+1} \in \mathbb{F}[V]/J$$

is the bundle relation. Let

$$h(X) = X^{k+1} + h_1 X^k + \cdots + h_k X + h_{k+1} \in \mathbb{F}[V]$$

be a homoginizing form. Then $I = (J, h(X)) \subset \mathbb{F}[V, X]$.

PROOF: Pass down from $\mathbb{F}[V, X]$ to $\big(\mathbb{F}[V]/J\big)[X]$ and note that the kernel of the natural map $\varphi : \big(\mathbb{F}[V]/J\big)[X] \longrightarrow \mathbb{F}[V, X]/I$ is the ideal I/J of $\big(\mathbb{F}[V]/J\big)[X]$. The result then follows from Lemma 1.2. □

The topological model for the following elementary example is a 2-plane bundle $\xi \downarrow \mathbb{RP}(n-1)$ with total Stiefel–Whitney class $1 + z \in H^*(\mathbb{RP}(n-1); \mathbb{F}_2) = \mathbb{F}_2[z]/(z^n)$. Although this example is quite simple, it exhibits some complex phenomena.

EXAMPLE 1: As base algebra we choose $\mathbb{F}_2[z]/(z^n)$, so the base ideal is $J = (z^n)$, and as homoginizing form $X^2 + zX \in \mathbb{F}[z, X]$. Then $I = (z^n, X^2 + zX)$. The corresponding quotient algebra $\mathbb{F}_2[z, X]/(z^n, X^2 + zX)$ turns out to be isomorphic to $\mathbb{F}_2[x]/(x^n) \# \mathbb{F}_2[y]/(y^n)$ where # denotes the connected sum operation of Poincaré duality algebras introduced in Section I.5 of [8]. The precise definition is at the beginning of § 6 of this manuscript.[2]

One way to see this is to consider the isomorphism φ of $\mathbb{F}_2[z, X]$ with $\mathbb{F}_2[x, y]$ induced by sending z to $x + y$ and X to y. Then

$$\varphi(X(X + z)) = xy$$
$$\varphi(z^n) = (x + y)^n \equiv x^n + y^n \bmod (xy),$$

[2] The # operation is another topologically motivated construction on irreducible ideals and is extensively studied in [12].

so φ maps $(z^n, X^2 + zX)$ isomorphically onto the ideal $(x^n + y^n, xy)$. An alternative version of this proof is to be found in § 2 Example 1. As is shown in § 6 this is the only projective bundle ideal with ground field $\mathbb{F}_2$ which produces a Poincaré duality algebra that is a connected sum.

REMARK: In the case where the ground field is a finite field $\mathbb{F}_q$ with q elements and $I \subseteq \mathbb{F}_q[V, X]$ is a projective bundle ideal with base ideal $J = I \cap \mathbb{F}_q[V]$ then Lemma 1.3 shows that I is closed under the action of the Steenrod algebra if and only if J is, and, one can choose a homoginizing form $h(X)$ such that its image $\alpha(X)$ in $\left(\mathbb{F}[V]/J\right)[X]$ is a Thom class.[3] The latter certainly is the case if one can choose $h(X)$ to be a product of linear forms in $\mathbb{F}_q[V, X]$.

LEMMA 1.4: *Suppose that $I \subseteq \mathbb{F}[X, y, z]$ is a projective bundle ideal. Then I is regular.*

PROOF: Let $J = I \cap \mathbb{F}[y, z]$ be the bundle ideal. By a result of W. Vasconcelos [15] J is generated by a regular sequence, say f', f''. So by Lemma 1.3 I is generated by f', f', h which must then be a regular sequence since I is $\mathfrak{m}$-primary. □

§2. Projective Bundle Ideals with an irreducible Base Ideal

We are particularly interested in projective bundle ideals which are irreducible. The next pair of results show that this is equivalent to assuming that the base ideal is irreducible. For related results see [13].

LEMMA 2.1: *If $I \subseteq \mathbb{F}[V, X]$ is a projective bundle ideal with a base ideal $J = I \cap F[V]$ that is irreducible in $\mathbb{F}[V]$, then $I \subseteq \mathbb{F}[V, X]$ is irreducible, and $\mathbb{F}[V, X]/I$ is a Poincaré duality algebra of formal dimension $d + k$, where d is the formal dimension of the base algebra $\mathbb{F}[V]/J$ and $k + 1$ is the bundle dimension.*

PROOF: By [8] Lemma I.1.3 and Proposition I.1.5 an $\mathfrak{m}$-primary ideal in a polynomial algebra is irreducible if and only if the corresponding quotient algebra is a Poincaré duality algebra. Thus $\mathbb{F}[V]/J$ is a Poincaré duality algebra and we need to show that $\mathbb{F}[V, X]/I$ is also. To this end note that the fundamental coexact sequence, § 1 (⌘) , shows that the Poincaré polynomial of $\mathbb{F}[V, X]/I$ is of degree $d + k$ and that the homogeneous component in this degree of $\mathbb{F}[V, X]/I$ is 1-dimensional. Choose a fundamnetal class $u \in \mathbb{F}[V]/J$. It has degree d. We will show that uX^k serves as a fundamental class for $\mathbb{F}[V, X]/I$.

Let $f \in \mathbb{F}[V, X]$ be a nonzero element and write $f = f_0 + f_1X + \cdots + f_kX^k$ where $f_0, \ldots, f_k \in \mathbb{F}[V]/J$. Let ℓ be the largest integer such that $f_\ell \neq 0$, so in point of fact $f = f_0 + f_1X + \cdots + f_\ell X^\ell$. By homogeniety $\deg(f_\ell) < \deg(f_{\ell-1}) < \cdots < \deg(f_0) = \deg(f)$. Let $f_\ell^\vee \in \mathbb{F}[V]/J$ be a Poincaré dual for f_ℓ. Then $\deg(f_\ell^\vee) = d - \deg(f_\ell)$ so for $i = 1, \ldots, \ell$ we have

$$\deg(f_\ell^\vee f_{\ell-i}) = \deg(f_\ell^\vee) + \deg(f_{\ell-i}) = d - \deg(f_\ell) + \deg(f_{\ell-i}) > d,$$

and hence $f_\ell^\vee f_{\ell-i} = 0 \in \mathbb{F}[V]/J$, $i = 1, \ldots, \ell$, for degree reasons. Therefore

$$f_\ell^\vee X^{k-\ell} \cdot f = f_\ell^\vee f_0 X^{k-\ell} + f_\ell^\vee f_1 X^{k-\ell+1} + \cdots + f_\ell^\vee f_\ell X^k = uX^k$$

showing that uX^k serves as a fundamnetal class for $\mathbb{F}[V, X]/I$ and completing the proof. □

LEMMA 2.2: *If $I \subseteq \mathbb{F}[V, X]$ is an irreducible projective bundle ideal with a base ideal $J = I \cap F[V]$, then $J \subseteq \mathbb{F}[V]$ is irreducible.*

[3] One says that an element $a \in A$ an unstable algebra A over the Steenrod algebra $\mathcal{P}^*$ is a **Thom class** if and only if the principal ideal it generates is closed under the action of the Steenrod algebra.

PROOF: Let $k+1$ be the bundle dimension of I. Consider the fundamental coexact sequence (⌘) of § 1. Since $\mathbb{F}[V, X]/I$ is a free $\mathbb{F}[V]/J$-module this sequence splits as a sequence of $\mathbb{F}[V]/J$-modules. So, if

$$s : \mathbb{F}[X]/(X^{k+1}) \longrightarrow \mathbb{F}[V, X]/I$$

is a splitting as $\mathbb{F}$-vector spaces of the quotient map $\mathbb{F}[V, X]/I \longrightarrow \mathbb{F}[X]/(X^{k+1})$, then the map

$$\mu : \big(\mathbb{F}[V]/J\big) \otimes \big(\mathbb{F}[X]/(X^{k+1})\big) \longrightarrow \mathbb{F}[V, X]$$

defined by $\mu(a, f) = a \cdot f$ is an isomorphism of $\mathbb{F}[V]/J$-modules. So for the Poincaré polynomials of the terms of the fundamental coexact sequence one has the product formula

$$P(\mathbb{F}[V, X]/I, t) = P(\mathbb{F}[V]/J, t) \cdot P(\mathbb{F}[X]/(X^{k+1}), t).$$

From this it follows that $P(\mathbb{F}[V]/J, t)$ has degree d, where $k + d = \text{f-dim}(\mathbb{F}[V, X]/I$, and moreover $\mathbb{F}[V]/J$ is one dimensional over $\mathbb{F}$ in homogeneous degree d. Choose a nonzero element $u \in \mathbb{F}[V]/J$ of degree d. To show $J \subset \mathbb{F}[V]$ is irreducible we invoke [8] Lemma I.1.3 and show instead that $\mathbb{F}[V]/J$ is a Poincaré duality algebra with fundamental class u.

To this end suppose that $0 \neq a \in \mathbb{F}[V]/J$. Then in $\mathbb{F}[V, X]/I$ the element a has a Poincaré dual, say

$$a^\vee = a_0^\vee + a_1^\vee X + \cdots + a_k^\vee X^k.$$

By homogenity

$$\deg(a_0^\vee) > \deg(a_1^\vee) > \cdots > \deg(a_k^\vee) = d,$$

so $a_0^\vee = \cdots = a_{k-1}^\vee = 0$ since $\mathbb{F}[V]/J$ is identically zero in homogeneous degrees exceeding d. Hence $uX^k = aa^\vee = aa_k^\vee X^k$ and therefore $u = aa_k^\vee$ since $1, X, \ldots, X^k$ is a basis for $\mathbb{F}[V, X]$ as an $\mathbb{F}[V]/J$-module by the graded Nakayama Lemma ([9] Proposition 5.1.3). Hence $a_k^\vee$ serves as a Poincaré dual to a in $\mathbb{F}[V]/J$ with respect to u. So u is a fundamental class for $\mathbb{F}[V]/J$ making it a Poincaré duality algebra. □

PROPOSITION 2.3: *If $I \subset \mathbb{F}[V, X]$ is a projective bundle ideal with base ideal $J = I \cap F[V]$, then I is irreducible in $\mathbb{F}[V, X]$ if and only if J is irreducible in $\mathbb{F}[V]$.*

PROOF: Combine Lemmas 2.1 and 2.2. □

Choose a basis $z_1, \ldots, z_n$ for V and let $\mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}]$ denote the algebra of inverse polynomials (see [8] Section VI.1). If $I \subset \mathbb{F}[V, X]$ is a projective bundle ideal with an irreducible base ideal $J = I \cap \mathbb{F}[V]$, then I is also irreducible, so both I, respectively J, have Macaulay inverses[4] (loc. cit. Section VI.2) $\theta_I \in \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}, X^{-1}]$, respectively $\theta_J \in \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}]$. In the theorem that follows we show that θ_I arises from θ_J by means of a homoginization process. We first require a lemma.

LEMMA 2.4: *Let $I \subset \mathbb{F}[V, X]$ be a projective bundle ideal of bundle dimension $k+1$ with irreducible base ideal $J = I \cap \mathbb{F}[V]$. Set $d = \text{f-dim}(\mathbb{F}[V]/J)$ and let $h(X) \in \mathbb{F}[V, X]$ be a homoginizing form for I. Then $X^{k+d+1} \in I$ and there exists a form of degree d in X*

$$\overline{h}(X) = X^d + \overline{h}_1 X^{d-1} + \cdots + \overline{h_d},$$

whose coefficients belong to $\mathbb{F}[V]$ and are well defined modulo the bundle ideal J, such that

$$\overline{h}(X)h(X) = X^{k+1+d} \in \big(\mathbb{F}[V]/J\big)[X].$$

[4] One needs to choose a basis $z_1, \ldots, z_n$ for V to define the $\mathbb{F}[z_1, \ldots, z_n]$-module structure on $F[z_1^{-1}, \ldots, z_n^{-1}]$ needed to apply Macaulay's theory. We always assume this has been done in a context involving Macaulay inverses.

PROOF: It follows directly from the hypotheses that the Poincaré series for $\mathbb{F}[V, X]/I$ is a polynomial of degree $d + k$, and hence $X^{k+d+1} \in I$ for degree reasons. By Lemma 1.3 $I = (J, h(X))$ so passing down to $(\mathbb{F}[V]/J)[X]$ one sees that X^{k+d+1} being zero in $\mathbb{F}[V, X]/I$ implies that in $(\mathbb{F}[V]/J)[X]$ it belongs to the principal ideal generated by $h(X)$ and the result follows. □

A form $\overline{h}(X) \in \mathbb{F}[V, X]$ with the properties of Lemma 2.4 is called a **dual homonigizing form** or **polynomial** for the projective bundle ideal $I \subset \mathbb{F}[V, X]$.

THEOREM 2.5: *Let $I \subset \mathbb{F}[V, X]$ be a projective bundle ideal of bundle dimension $k + 1$ with irreducible base ideal $J = I \cap \mathbb{F}[V]$. If $\overline{h}(X) = X^d + \overline{h}_1 X^{d-1} + \cdots + \overline{h_d} \in (\mathbb{F}[V]/J)[X]$ is a dual homoginizing form for I (so $d =$ f-dim$(\mathbb{F}[V]/J)$) and $\theta_J \in \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}]$ is a Macaulay inverse for J, then*

$$\theta_I = \overline{h}(X) \cap (\theta_J \cdot X^{-(d+k)}) = \theta_I X^{-k} + (\overline{h}_1 \cap \theta_J) X^{-(k+1)} + \cdots + (\overline{h}_d \cap \theta_J) X^{-(k+d)}$$

is a Macaulay inverse for I.

PROOF: We have the inclusion of ideals $(J, X^{k+d+1}) \subseteq I = (J, h(X))$. The ideal (J, X^{k+d+1}) is irreducible in $\mathbb{F}[V, X]$ since

$$\mathbb{F}[V, X]/(J, X^{k+d+1}) \cong (\mathbb{F}[V]/J) \otimes \left(\mathbb{F}[X]/(X^{k+d+1}\right)$$

is a tensor product of Poincaré duality algebras (see e.g. [8] Lemma I.1.3) and hence a Poincaré duality algebra in its' own right. Moreover this tensor product splitting shows that the Macaulay inverse for the ideal (J, X^{k+d+1}) is $\theta_J \cdot X^{-(k+d)} \in \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}] \otimes \mathbb{F}[X^{-1}] = \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}, X^{-1}]$. We next apply the $K \subset L$ paradigm (see [8] Theorem II.5.1) to the pair of ideals $(J, X^{k+d+1}) \subseteq I$. To do so we require a transition element for I over (J, X^{k+d+1}). To this end note that by standard properties of the $(- : -)$ construction one has

$$((J, X^{k+d+1}) : I) = ((J, X^{k+d+1}) : (J, h(X))) = ((J, X^{k+d+1}) : (h(X))) = (\overline{h}(X)) + (J, X^{k+d+1})$$

by the definition of $\overline{h}(X)$. So $\overline{h}(x)$ serves as a transition element for I over (J, X^{k+d+1}) and applying [8] Theorem II.5.1 yields the desired conclusion. □

EXAMPLE 1: Consider the situation $J = (z^n) \subset \mathbb{F}_2[z]$, corresponding to the base algebra being $\mathbb{F}_2[z]/(z^n)$, which has formal dimension $n - 1$, and the homoginizing quadratic form $h(X) = X^2 + zX \in \mathbb{F}_2[z, X]$ (see § 1 Example 1), which means the bundle dimension is 2. The corresponding dual homoginizing form (see Lemma 2.4) is then

$$\overline{h}(X) = X^{n-1} + zX^{n-2} + \cdots + z^{n-1},$$

as simple multiplication verifies. The inverse form $z^{-n} \in \mathbb{F}[z^{-1}]$ is a Macaulay inverse for the ideal $I = (z^n) \subset \mathbb{F}_2[z]$ defining the base algebra. As a consequence of Theorem 2.5 the bundle ideal $(z^n, X^2 + zX)$ is therefore defined by the inverse form

$$\theta_I = \overline{h}(X) \cap z^{-n} X^{-(n-1)} = z^{-n} + z^{-(n-1)} X^{-1} + \cdots + z^{-1} X^{-(n-1)}.$$

If we replace z by x and X by y then this inverse form becomes

$$\theta = x^{-n} + x^{-(n-1)} y^{-1} + \cdots + x^{-1} y^{-(n-1)}$$

and is in the same GL$(2, \mathbb{F}_2)$-orbit as the inverse form $x^{-n} + y^{-n} \in \mathbb{F}_2[x^{-1}, y^{-1}]$ since the transvection $x \rightsquigarrow x,\ y \rightsquigarrow x + y$ sends the inverse form $x^{-n} + y^{-n}$ to $x^{-n} + x^{-(n-1)} y^{-1} + \cdots + x^{-1} y^{-(n-1)}$. The inverse form $x^{-n} + y^{-n}$ is the Macaulay inverse for the ideal $(xy, x^n + y^n)$. To verify this one first writes down the catalecticant matrix (see [8] Section VI.2) $cat_\theta(1, n-1)$ from which one sees that $xy,\ x^n + y^n \in I(\theta)$. Then one notes that the induced map

$$\mathbb{F}_2[x, y]/(xy, x^n + y^n) \longrightarrow \mathbb{F}_2[x, y]/I(\theta)$$

has degree one since both these Poincaré duality algebras have x^n as a fundamental class. Finally one applies [8] Corollary I.2.4 to conclude this map is an isomorphism, so the inclusion $(xy,\ x^n + y^n) \subseteq I(\theta)$ must be an equality. This justifies the remark made in § 2 Example 1 concerning the structure of quotient algebra $\mathbb{F}_2[z, X]/J$.

§3. Bundling an $\mathfrak{m}$-primary Irreducible Ideal

We next show that a converse of Theorem 2.5 holds, allowing the construction of a family of $\mathfrak{m}$-primary irreducible ideals in $\mathbb{F}[V, X]$ from a single $\mathfrak{m}$-primary ideal in J in $\mathbb{F}[V]$. Suppose we are given an $\mathfrak{m}$-primary irreducible ideal $J \subset \mathbb{F}[V]$ and a monic polynomial in X of strictly positive degree. Guided by Lemma 1.3 we consider the ideal $I = (J,\ h(X)) \subset \mathbb{F}[V, X]$. Let $k+1$ be the degree of X in $h(X)$ so

$$h(X) = X^{k+1} + h_1X^k + \cdots + h_kX + h_{k+1},$$

where the elements $h_1, \ldots, h_{k+1} \in \mathbb{F}[V]$ have strictly positive degrees, so belong to the augmentation ideal of $\mathbb{F}[V]$. If d is the degree of the Poincaré polynomial of $\mathbb{F}[V]/J$, then $1, X, \ldots, X^k$ generate $\mathbb{F}[V, X]/I$ as an $\mathbb{F}[V]/J$-module, so one sees that the Poincaré series of $\mathbb{F}[V, X]/I$ is in fact a polynomial of degree $k+d$. Therefore we have proven the following lemma.

LEMMA 3.1: *If $J \subset \mathbb{F}[V]$ is an $\mathfrak{m}$-primary irreducible ideal and $h(X) \in \mathbb{F}[V, X]$ is a monic polynomial in X of strictly positive degree $k+1$, then $I = (J,\ h(X)) \subset \mathbb{F}[V, X]$ is an $\mathfrak{m}$-primary ideal, the Poincaré polynomial of $\mathbb{F}[V, X]/I$ has degree $k+d$, where d is the degree of the Poincaré polynomial of $\mathbb{F}[V]/J$, and $X^{k+d+1} \in I$.* □

Continuing in this vein one has:

LEMMA 3.2: *Let A be a commutative graded connected algebra over a field and $A[X]$ the standard polynomial algebra over A (so X has degree 1). If $\alpha(X) \in A[X]$ is a monic polynomial of degree m in X then $A[X]/(\alpha(X))$ is a free A-module with basis $1, X, \ldots, X^{m-1}$.*

PROOF: One notes that homogeniety requires that the coefficients of $\alpha(X)$ apart from the coefficient of X^m belong to the augmentation ideal of A. Hence $1 \otimes (X^i \cdot \alpha(X)) \equiv 1 \otimes X^{m+i}$ for $i = 0, 1, \ldots,$ in $\mathbb{F} \otimes_A A[X]$, so

$$1, X, \ldots, X^{m-1}, \alpha(X), X \cdot \alpha(X), \ldots,$$

project to an $\mathbb{F}$-vector space basis for the module of A-indecomposables $\mathbb{F} \otimes_A A[X]$. Since $A[X]$ is a free A-module they therefore are an A-basis for $A[X]$ by the graded Nakayama Lemma (see e.g. [9] Proposition 5.1.3). Since $\alpha(X), X \cdot \alpha(X), \ldots,$ span the ideal $(\alpha(X))$ as an A-module it follows that the inclusion $(\alpha(X)) \hookrightarrow A$ splits as a map of A-modules. Hence $A[X]/(\alpha(X))$ is also free as an A-module and $1, X, \ldots, X^{m-1}$ project to a basis for it. □

Lemma 3.2 says in particular that, for any ideal $J \subset \mathbb{F}[V]$ and any monic polynomial $h(X) \in \mathbb{F}[V, X]$ in X that, the quotient algebra $\mathbb{F}[V, X]/(J,\ h(X))$ is free as an $\mathbb{F}[V]/J$-module: Simply set $A = \mathbb{F}[V]/J$ in the lemma.

For an $\mathfrak{m}$-primary irreducible ideal $J \subset \mathbb{F}[V]$ denote by d_J, or d if J is clear from context, the degree of the Poincaré polynomial of $\mathbb{F}[V]/J$, i.e., $d = \text{f-dim}(\mathbb{F}[V]/J$ is the formal dimension of Poincaré duality quotient algebra $\mathbb{F}[V]/J$. Let $h(X)$ have strictly positive degree $k+1$, so $h(X) = X^{k+1} + h_1X^k + \cdots h_kX + h_{k+1}$, where $h_i \in \mathbb{F}[V]$ has degree i for $i = 1, \ldots, k+1$. The maximal ideal of $\mathbb{F}[V]/J$ is nilpotent, so the inhomogeneous element $1 + h_1 + \cdots + h_k + h_{k+1}$ has a formal inverse in the degraded algebra[5] $\text{Tot}(\mathbb{F}[V]/J)$. Say $1 + \overline{h}_1 + \cdots + \overline{h}_d$ is such an inverse, where $\overline{h}_j \in \mathbb{F}[V]/J$ has degree j for $j = 1, \ldots, d$, so

$$(⌘) \qquad (1 + h_1 + \cdots h_{k+1})(1 + \overline{h}_1 + \cdots + \overline{h}_d) = 1.$$

Introduce the polynomial

$$\overline{h}(X) = X^d + \overline{h}_1X^{d-1} + \cdots + \overline{h}_d.$$

Then (⌘) shows that

$$(\maltese) \qquad h(X) \cdot \overline{h}(X) = X^{k+d+1} \in \big(\mathbb{F}[V]/J\big)[X].$$

This leads to the following result.

Lemma 3.3: *With the notations preceding one has* $(J, X^{k+d+1}) \subseteq (J, h(X))$ *and*

$$\big((J, X^{k+d+1}) : (J, h(X))\big) = \big(\overline{h}(X)\big) + (J, X^{k+d+1}).$$

Proof: The first statement follows from Lemma 3.1. Pass down to the quotient algebra $B = \mathbb{F}[V, X]/(J, X^{k+d+1})$, then the second statement becomes equivalent to showing that $\text{Ann}_B(h(X)) = (\overline{h}(X))$. The equation ($\maltese$) shows that $(\overline{h}(X)) \subseteq \text{Ann}_B(h(X))$ so it remains to prove the reverse inclusion.

Suppose that $0 \neq f(X) = f_0X^m + \cdots + f_m \in \text{Ann}_B(h(X))$, where $f_0, \ldots, f_m \in A = \mathbb{F}[V]/(J)$. Since $f(X)$ is of degree m in X we have $f_0 \neq 0$. But

$$0 = f(X) \cdot h(X) = f_0X^{k+m+1} + \text{ terms of lower degree in } X,$$

so $k + m + 1 \geq k + d + 1$ since B is a free A-module with basis $1, X, \ldots, X^{k+d}$. If $m = d$, then $f(X)$ and $\overline{h}(X)$ both have degree d, so $f(X) - f_0\overline{h}(X) \in \text{Ann}_B(h(X))$ and this element has degree at most $d-1$ in X. By what was just shown this means $f(X) - f_0\overline{h}(X)$ is identically zero so $f(X) = f_0\overline{h}(X)$ and $f(X)$ belongs to the principal ideal generated by $\overline{h}(X)$. Hence we may proceed inductively and assume that for all $\overline{m} < m$, if $\overline{f}(X) \in B$ has degree $\overline{m}$ in X and annihilates $h(X)$ then $\overline{f}(X) \in ((\overline{h}(X))$. If $f(X) \in \text{Ann}_B(h(X))$ has degree m in X, then writting $f(X)$ as above one sees that $f(X) - f_0\overline{h}(X)$ has degree at most $m-1$ in X and it too annihilates $h(X)$. By the inductive assumption $f(X) - f_0\overline{h}(X)$ belongs to $(\overline{h}(X))$ and hence so does $f(X)$ completing the inductive step. □

Theorem 3.4: *Let* $J \subset \mathbb{F}[V]$ *be an* $\mathfrak{m}$*-primary ideal and* $h(X) \in \mathbb{F}[V, X]$ *a monic polynomial in* X *of strictly positive degree* $k+1$*. Then* $I = (J, h(X)) \subset \mathbb{F}[V, X]$ *is a projective bundle ideal of bundle dimension* $k+1$ *with base ideal* J*. If* J *is irreducible in* $\mathbb{F}[V]$ *then* $I = (J, h(X))$ *is irreducible in* $\mathbb{F}[V, X]$*.*

Proof: I is $\mathfrak{m}$-primary by Lemma 3.1 and $\mathbb{F}[V, X]/I$ is free as an $\mathbb{F}[V]/J$-module by Lemma 3.2. Since $J = I \cap \mathbb{F}[V]$ it follows from Lemma 1.3 that I is a projective bundle ideal with base ideal J and bundle dimension $k+1$. If J is irreducible in $\mathbb{F}[V]$, then (J, X^{k+d+1}) is irreducible in $\mathbb{F}[V, X]$, where d is the formal dimension of $\mathbb{F}[V]/J$. To see this one notes that

$$\mathbb{F}[V, X]/(J, X^{k+d+1}) \cong \big(\mathbb{F}[V]/J\big) \otimes \big(\mathbb{F}[X]/(X^{k+d+1})\big)$$

[5] $\text{Tot}(H)$ for any graded algebra H is the direct product of its homogeneous components.

is a Poincaré duality algebra by [8] Proposition I.1.5 and then applies [8] Lemma I.1.3 to conclude that $(J, X^{k+d+1}) \subset \mathbb{F}[V, X]$ is an irreducible ideal. By Lemma 3.1 $(J, X^{k+d+1}) \subseteq (J, h(X))$ and by Lemma 3.3

$$\big((J, X^{k+d+1}) : (J, h(X))\big) = \big(\overline{h}(X)\big) + (J, X^{k+d+1})$$

so I is irreducible by [8] Theorem I.2.1. □

Remark: In the situation of Lemma 3.3 one has shown in the quotient algebra

$$B = \mathbb{F}[V, X]/(J, X^{k+d+1}) = \Big(\mathbb{F}[V]/J\Big) \otimes \Big(\mathbb{F}[X]/(X^{k+d+1})\Big)$$

that

$$(0 \underset{B}{:} h(X)) = (\overline{h}(X))$$

or what is the same thing that

$$\mathrm{Ann}_B(h(X)) = (\overline{h}(X)).$$

The algebra B is Noetherean and $(0) \subset B$ is an irreducible ideal since B is a Poincaré duality algebra. Therefore by Emmy Noether's Involution Theorem (see [8] Section I.1.2) one also has

$$(0 \underset{B}{:} \overline{h}(X)) = (h(X))$$

or, again, what is the same thing that

$$\mathrm{Ann}_B(\overline{h}(X)) = (h(X)).$$

Thus in the algebra B the images of $h(X)$ and $\overline{h}(X)$ are mutual annihilators of each other. So by [8] Corollary I.2.3 one has the following conclusions.

(i) $B/(\overline{h}(X))$ is a Poincaré duality algebra of formal dimension

$$\text{f-dim}(B) - \deg(h(X)) = d + k + d - (k + 1) = 2d - 1.$$

(ii) $B/(h(X))$ is a Poincaré duality algebra of formal dimension

$$\text{f-dim}(B) - \deg(\overline{h}(X)) = d + k + d - d = k + d,$$

(as of course it should be since $B/(h(X)) \cong \mathbb{F}[V, X]/(J, h(X)))$.

If $X^{k+d+1} \in (J, \overline{h}(X)) \subset \mathbb{F}[V, X]$, then by Theorem 3.4 the relation of Lemma 3.3 would also hold with the roles of $h(X)$ and $\overline{h}(X)$ switched, viz.,

$$\big((J, X^{k+d+1}) : (J, \overline{h}(X))\big) = \big(h(X)\big) + (J, X^{k+d+1}).$$

This would give us another projective bundle ideal $(J, \overline{h}(X)) \subset \mathbb{F}[V, X]$. The condition $X^{k+d+1} \in (J, \overline{h}(X))$ holds if for example

$$k + d + 1 > \text{f-dim}(\mathbb{F}[V, X]/(J, \overline{h}(X)) = 2d - 1$$

i.e., if $k + 1 \geq d$.

It is also possible to reformulate the preceding discussion in terms of the Macaulay inverse θ_J of the ideal J. Here is how this works.

Corollary 3.5: *Let $J \subset \mathbb{F}[V]$ be an $\mathfrak{m}$-primary irreducible ideal with Macaulay inverse $\theta_J \in \mathbb{F}[V]$ of degree $-d$, so $d = \text{f-dim}(\mathbb{F}[V]/J)$. If $h(X) \in \mathbb{F}[V, X]$ is a monic polynomial in X of strictly positive degree $k+1$ choose a polynomial $\overline{h}(X)$ of degree d such that $h(X) \cdot \overline{h}(X) = X^{k+d+1} \in \Big(\mathbb{F}[V]/J\Big)[X]$ (see the discussion preceding Lemma 3.3). Then*

$$\theta = \overline{h}(X) \cap (\theta_J \cdot X^{-(k+d)}) = \theta_J \cdot X^{-k} + \overline{h}_1 \cap (\theta_J \cdot X^{-(k+1)}) + \cdots + \overline{h}_d \cap (\theta_J \cdot X^{-(k+d)}) \in \mathbb{F}[V^{-1}, X^{-1}]$$

defines an irreducible $\mathfrak{m}$*-primary ideal* $I(\theta) \subset \mathbb{F}[V, X]$ *which is a projective bundle ideal with bundle dimension* $k+1$ *and base ideal* J*. The formal dimension of the corresponding Poincaré duality quotient* $\mathbb{F}[V, X]/I(\theta)$ *is* $d + k$*.*

Proof: This follows from Theorems 3.4 and 2.5. □

§4. Pulling off Sections and Inverse Symmetric Forms

Note that in the equation

$$\theta_I = \overline{h} \cap (\theta_J \cdot X^{-(k+d)})$$

occuring in Corollary 3.5 one has

$$\deg(\overline{h}(X)) = d, \qquad \deg(\theta_J \cdot X^{-(k+d)}) = -2d - k.$$

Since $\deg(\overline{h}(X)) = d = -\deg(\theta_J)$ the formula

$$\theta_m = \overline{h}(X) \cap (\theta_J \cdot X^{-m})$$

defines an inverse form of degree $-m$ for any integer $m \in \mathbb{N}$. One may consider these inverse forms (if nonzero) as defining additional $\mathfrak{m}$-primary irreducible ideals in $\mathbb{F}[V, X]$ which may not be projective bundle ideals at all.

Topologically, for $m \le k + 1$, this would correspond to pulling sections off of a bundle. For example the bundle $\xi \downarrow \mathbb{RP}(n-1)$ considered in connection with Example 1 of § 1 could have been taken to be a line bundle, rather than a two plane bundle, since in the algebraic context of this example the only feature of ξ we used was the total Stiefel–Whitney class which was $1 + z \in H^*(\mathbb{RP}(n-1); \mathbb{F}_2) = \mathbb{F}_2[z]/(z^n)$. In such a topological context the geometric dimension of the bundle imposes a restriction on how many sections one may pull off, see e.g. [7] where this topological restriction was of central importance. Algebraically however there is only the restriction imposed by the requirement that $\theta_m = \overline{h}(X) \cap (\theta_J \cdot X^{-m})$ be an actual nonzero inverse form, so $m \in \mathbb{N}$. We exploit this next.

We start with an inverse form $\theta_J \in \mathbb{F}[V^{-1}]$ of degree $-d$ and a (dual homoginizing) form $\varphi(X) = X^d + \varphi_1 X^{d-1} + \cdots + \varphi_d \in \mathbb{F}[V, X]$ of degree d and consider the homoginizations of θ_J given by

$$\theta = \varphi(X) \cap (\theta_J \cdot X^{-m}) = \theta_J \cdot X^{-m} + \varphi_1 \cap (\theta_J \cdot X^{-(m+1)}) + \cdots + \varphi_d \cap (\theta_J \cdot X^{-(m+d)}) \in \mathbb{F}[V^{-1}, X^{-1}],$$

for $m \in \mathbb{N}$. These define $\mathfrak{m}$-primary irreducible ideals $I(\theta_m) \subset \mathbb{F}[V, X]$. We organize this section around an extensive family of ideals arising in this way, and use them to bring to the fore a number of the less obvious[6] properties of $\mathfrak{m}$-primary irreducible ideals.

Notation: *We denote by* $e_1, \ldots, e_n \in \mathbb{F}[z_1, \ldots, z_n]$ *the elementary symmetric polynomials in the variables* $z_1, \ldots, z_n$*, and by* $\sigma_1, \ldots, \sigma_n \in \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}]$ *their analogs as inverse polynomials. In other words* σ_i *is the* Σ_n*-orbit sum* $\mathfrak{S}(z_1^{-1} z_2^{-2} \cdots z_i^{-i})$ *of the inverse monomial* $z_1^{-1} z_2^{-2} \cdots z_i^{-i}$*. By convention* $e_0 = 1 = \sigma_0$*.*

With these notations one has

$$\sigma_i = e_{n-i} \cap \sigma_n \in \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}] \quad \text{for } i = 0, \ldots, n.$$

[6] It is very likely that many of these properties were known to F. S. Macaulay, but due to the enormous change in terminology that has taken place since he wrote [5] this is very diffucult to confirm by reference to it.

Introduce the form $\varphi(X) = e_n + e_{n-1}X + \cdots + e_1 X^{n-1} + X^n$ and define the inverse polynomial θ_n by

$$\begin{aligned}\theta_n &= \varphi(X) \cap (\sigma_n \cdot X^{-n}) = (e_n + e_{n-1}X + \cdots + e_1X^{n-1} + X^n) \cap (\sigma_n \cdot X^{-n}) \\ &= X^{-n} + \sigma_1 \cdot X^{-(n-1)} + \cdots + \sigma_{n-1}X^{-1} + \sigma_n = \prod_{i=1}^{n}(X^{-1} + z_i^{-1}) \in \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}, X^{-1}].\end{aligned}$$

Note that θ_n has degree $-n$ so the corresponding Poincaré duality quotient algebra has formal dimension n.

Notation: *The Poincaré duality quotient* $\mathbb{F}[z_1, \ldots, z_n, X]/I(\theta_n)$ *will be denoted by* $P(n)$.

If $I(\sigma_n) \subset \mathbb{F}[z_1, \ldots, z_n]$ denotes the $\mathfrak{m}$-primary irreducible ideal that is the Macaulay dual to σ_n then all squares of elements in $Q(n) = \mathbb{F}[z_1, \ldots, z_n]/I(\sigma_n)$ are zero since no monomial divisible by the square of one of the variables $z_1, \ldots, z_n$ occurs in the support of σ_n. The algebra $Q(n)$ is therefore a quotient of the exterior algebra $E(z_1, \ldots, z_n)$. Since both these algebras are Poincaré duality algebras of formal dimension n the quotient map $E(z_1, \ldots, z_n) \longrightarrow Q(n)$ must be an isomorphism (see e.g., [8] Lemma I.3.1). Hence $Q(n) = E(z_1, \ldots, z_n)$ is an exterior algebra on the generators $z_1, \ldots, z_n$. Observe that $f \cap \theta_n = 0$ for any $f \in I(\sigma_n)$ so there is a natural map

$$\mathbb{F}[z_1, \ldots, z_n]/I(\sigma_n) \longrightarrow \mathbb{F}[z_1, \ldots, z_n, X]/I(\theta_n)$$

which may be extended to a map

$$\big(\mathbb{F}[z_1, \ldots, z_n]/I(\sigma_n)\big)[X] \longrightarrow \mathbb{F}[z_1, \ldots, z_n, X]/I(\theta_n)$$

in the obvious way. Hence we have shown the following.

Lemma 4.1: *With the preceding notations one has* $z_i^2 = 0 \in P(n)$ *for* $i = 1, \ldots, n$. □

Before we delve deeper into the structure of the ideals $I(\theta_n)$ we indicate how these ideals arise from projective bundle ideals by stripping off sections as described above. To simplify this discussion we *assume that the ground field has characteristic* 2 so

$$\varphi(X)^2 = X^{2n} \in \big(\mathbb{F}[z_1, \ldots, z_n]/I(\sigma_n)\big)[X]$$

by Lemma 4.1. Therefore from the $\mathfrak{m}$-primary ideal $I(\sigma_n) \subset \mathbb{F}[z_1, \ldots, z_n]$ and the homoginizing polynomial $\varphi(X)$ we obtain a projective bundle ideal $(I(\sigma_n), \varphi(X)) \subset \mathbb{F}[z_1, \ldots, z_n, X]$ with base ideal $I(\sigma_n)$ and bundle dimension n. A Macaulay dual for this ideal is

$$\varphi(X) \cap (\sigma_n \cdot X^{2n-1})$$

since $\varphi(X)$ serves as its own dual homoginizing form.[7] The Macaulay inverse θ_n of the the ideal $I(\theta_n)$ may be thought of as arising from the projective bundle ideal $(I(\sigma_n), \varphi(X))$ by stripping off n-sections.

Lemma 4.2: *The Poincaré duality algebra* $P(n)$ *has rank* $n + 1$ *(in the sense of [8] Section I.1), i.e.,* $P(n)_1$ *has dimension* $n + 1$ *as an* $\mathbb{F}$*-vector space.*

Proof: The value of θ_n on e_n is 1, which shows for $i = 1, \ldots, n$ that $z_1 \cdots \widehat{z_i} \cdots z_n$ serves as a Poincaré dual for $z_i \in P(n)$ and is zero on z_j for $j \neq i$, so these elements are linearly independent. Likewise θ_n evaluates to 1 on X^n so not only is X nonzero but so are all its powers up to the n-th. Hence X cannot be a linear combination of the elements $z_1, \ldots, z_n$ whose squares are zero. □

[7] For another context in which this type of self duality was exploited see [10].

To analyze in more detail the multiplication of $P(n)$ we employ the natural map

$$E(z_1, \ldots, z_n)[X] = \big(\mathbb{F}[z_1, \ldots, z_n]/I(\sigma_n)\big)[X] \longrightarrow \mathbb{F}[z_1, \ldots, z_n, X]/I(\theta_n) = P(n)$$

introduced above. We note that as $z_1, \ldots, z_n, X$ generate $P(n)$ as an algebra this map is an epimorphism. So every element of $P(n)$ may be written as a sum of monomials of the form $X^t z_S$ where $S = \{i_1, \ldots, i_s\} \subseteq \{1, \ldots, n\}$, $z_S = z_{i_1} \cdots z_{i_s}$, and $s = |S|$ the number of elements in S. Since $P(n)$ has formal dimension n we need only consider such monomials for $t + |S| \le n$. For two such monomials $X^{t_1} z_{S_1}$ and $X^{t_2} z_{S_2}$ of complimentary degree (i.e., for which $t_1 + |S_1| + t_2 + |S_2| = n$) their product is given by

$$X^{t_1} z_{S_1} \cdot X^{t_2} z_{S_2} = \begin{cases} 0 & \text{if } S_1 \cap S_2 \neq \varnothing \\ 1 & \text{otherwise.} \end{cases}$$

Fix an integer j with $2j \le n$. We introduce a matrix $\mathbf{M}(j, n-j)$ that encodes the products of elements of degree j with elements of degree $n-j$. The rows of the matrix are to be indexed by

$$X^j, X^{j-1}z_1, \ldots, X^{j-1}z_n, X^{j-2}z_1z_2 + X^{j-1}(z_1 + z_2), \ldots$$

with the general term being

$$X^{j-|S|}z_S + \sum_{\varnothing \subsetneqq T \subsetneqq S} X^{j-|T|}z_T, \quad |S| \le j,$$

so the terms get larger (in the sense they have larger support among the monomials we are using) with increasing row number. The columns are indexed by

$$X^{n-j}, X^{n-j-1}z_1 + X^{n-j}, X^{n-j-1}z_2 + X^{n-j}, \ldots, X^{n-j-|T|}z_T + X^{n-j}, \ldots, \quad |T| \le j.$$

The entries of the matrix are the value of θ_n on the product of the forms indexing the rows and columns. This defines a square matrix $\mathbf{M}(j, n-j)$ of size

$$m = 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{j}.$$

which is an analog of the catalecticant matrix (see e.g., [8] §VI.2) The first row of $\mathbf{M}(j, n-j)$ is $(1, 0, \ldots, 0)$ since $X^jX^{n-j} = 1$ and $X^j(X^{n-j-|T|}z_T + X^{n-j}) = 1 + 1 = 0$. The first column of the matrix consists entirely of 1s since in the first row of that column one has $X^jX^{n-j} = 1$, and in the remaining rows $\big(\sum_{\varnothing \subsetneqq T \subseteq S} X^{j-|T|}z_T\big)X^{n-j}$, which is a sum of $2^{|S|} - 1$ entries which are all 1s.

Next one considers the product

$$(⌘) \qquad \Big(\sum_{\varnothing \subsetneqq T \subseteq S} X^{j-|T|}z_T\Big)\big(X^{n-j-|U|}z_U + X^{n-j}\big)$$

for subsets $S, U \subset \{1, \ldots, n\}$. One notes that

$$\big(X^{j-|T|}z_T\big)\big(X^{n-j-|U|}z_U + X_{n-j}\big) = \begin{cases} 1 & \text{if } T \cap U \neq \varnothing \\ 0 & \text{otherwise,} \end{cases}$$

so the product (⌘) being the sum of all these terms is the number of $\varnothing \subsetneqq T \subseteq S$ for which $T \cap U \neq \varnothing$. If $U \supseteq T$ this sum is $2^{|S|} - 1 \neq 0 \in \mathbb{F}$ whereas if $U \cap S = \varnothing$ the sum is 0. So it remains to consider the situation $\varnothing \neq U \cap S \neq S$ where one finds that the nonempty sets T with $T \cap U = \varnothing$ are those contained in $S \setminus (U \cap S)$. So their number is

$$2^{|S|} - 1 - (2^{|U| - |U \cap S|} - 1) = 2^{|S|} - 2^{|S| - |U \cap S|}$$

which is even, and hence zero in $\mathbb{F}$ (which remember was assumed to have characteristic 2). This says that the matrix $\mathbf{M}(j,\ n-j)$ with rows and columns indexed as described above has the form indicated in the next table.

$\mathbf{M}(j,\ n-j)$	X^{n-j}	small sets			large sets		
X^j	1	0	$\cdots$	0	0	$\cdots$	0
	1	1	0	0	0	$\cdots$	0
small sets	$\vdots$	0	$\ddots$	0	0	$\ddots$	0
	1	0	0	1	0	$\cdots$	0
	$\vdots$	0	$\cdots$	0	1	0	0
large sets		0	$\ddots$	0	0	$\ddots$	0
	1	0	$\cdots$	0	0	0	1

The matrix $\mathbf{M}(j,\ n-j)$ is therefore nonsingular so the monomials indexing the rows are linearly independent in $P(n)$. Hence we have proven the first assertion of the following result.

Lemma 4.3: *For $j \le 2n$ we have*

$$\dim_F(P(n)_j) = 1 + \binom{n}{1} + \cdots + \binom{n}{j}$$

and therefore the epimorphism

$$E(z_1, \ldots, z_n)[X] \longrightarrow P(n)$$

is an isomorphism through degree $\left[\frac{n}{2}\right]$.

Proof: One only need note that $E(z_1, \ldots, z_n)[X]$ also has dimension $1 + \binom{n}{1} + \cdots + \binom{n}{j}$ for $j \le \left[\frac{n}{2}\right]$. □

From Lemma 4.3 we can deduce the interesting result that the minimum number of generators of the ideal $I(\theta_n)$ exceeds the number of variables by roughly $n!$. This shows that there are $\mathfrak{m}$-primary irreducible ideals not arising from connected sums[8] where the number of generators is arbitrarily larger than the number of variables or the formal dimension of the corresponding Poincaré duality quotient algebra.

Proposition 4.4: *The number of linearly independent forms of degree $\frac{n}{2}$ for n even or $\frac{n-1}{2}$ for n odd needed to generate the ideal $I(\theta_n)$ is*

$$\binom{n+1}{\frac{n}{2}+1} \quad \text{if } n \text{ is even, or}$$

$$\binom{n}{\frac{n+1}{2}} \quad \text{if } n \text{ is odd.}$$

Therefore the minimum number of generators of the ideal $I(\theta_n)$ exceeds the rank of $P(n)$ by one less than the preceding numbers.

Proof: Suppose that $n = 2k$ is even. Since $P(n)$ is a Poincaré duality algebra the homogeneous components $P(n)_{k-1}$ and $P(n)_{k+1}$ must have the same dimension. By Lemma 4.3 the dimension of $P(n)_{k-1}$ is

$$1 + \binom{n}{1} + \cdots + \binom{n}{k}.$$

[8] Lemma 4.3 and decomposability criteria from [12] tell us that $P(n)$ is not representable as a nontrivial connected sum.

On the other hand the Poincaré series of the algebra $E(z_1, \dots, z_n)[X]$ is given by

$$P(E(z_1, \dots, z_n)[X], t) = \frac{(1+t)^n}{1-t}$$

so the homogeneous component of $E(z_1, \dots, z_n)[X]$ of degree $k+1$ has dimension

$$1 + \binom{n}{1} + \cdots + \binom{n}{k+1}$$

This means that the kernel of the natural map $E(z_1, \dots, z_n)[X] \longrightarrow P(n)$ must have dimension at least

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

as claimed. The case $n = 2k+1$ is similar and left to the reader. Finally, if one recalls that $I(\theta_n)$ requires the n quadratic generators $z_1^2, \dots, z_n^2$ the statement about the excess of generators over the rank follows. □

We next note that the ideals $I(\theta_n)$ are compatible with the action of the Steenrod algebra and compute their conjugate Wu classes (see e.g., [1] or [14] pp 98 – 99 and 122 – 123 for the case of the prime field and [3], [8] Section III.3, for arbitrary Galois fields).

PROPOSITION 4.5: *Let $\mathbb{F} = \mathbb{F}_q$ be the Galois field with q elements,*

$$\theta_n = \varphi(X) \cap (\sigma_n \cdot X^{-n}) \in \mathbb{F}_q[z_1{-}1, \dots, z_n^{-1}, X^{-1}],$$

and $I(\theta_n) \subset \mathbb{F}_q[z_1, \dots, z_n, X]$ the $\mathfrak{m}$-primary irreducible ideal it defines. Then $I(\theta_n)$ is a $\mathcal{P}^*$-invariant ideal and the conjugate Wu classes of the unstable $\mathcal{P}^*$ quotient algebra $\mathbb{F}_q[z_1, \dots, z_n, X]/I(\theta_n)$ are given by the formula

$$\chi\mathrm{Wu} = (1 + X^{q-1})^n \cdot \prod_{i=1}^{n} (1 + X^{q-1} - X^{q-2} z_i + \cdots - X z_i^{q-2} + z_i^{q-1}).$$

PROOF: By [8] Theorem VI.6.2 one has $\mathcal{P}(\theta_n) = \chi\mathrm{Wu}(P(n)) \cdot \theta_n$, where $\mathcal{P} = 1 + \mathcal{P}^1 + \cdots + \mathcal{P}^k + \cdots$ is the formal sum of the reduced power operations. To compute $\mathcal{P}(\theta_n)$ we note that by the mixed Cartan formula (see the discussion preceding Theorem VI.6.2 in [8])

$$(\maltese) \qquad \mathcal{P}(\theta_n) = \mathcal{P}(\varphi_n(X)) \cap \big(\mathcal{P}(\sigma_n) \cdot \mathcal{P}(X^{-n})\big)$$

so we turn to a computation of the individual factors.

The form

$$\varphi_n(X) = e_n + e_{n-1}X + \cdots e_1 X^{n-1} + X^n = \prod_{i=1}^{n} (X + z_i)$$

so is a product of linear forms and hence a Thom class. Therefore one obtains

$$(\star) \qquad \begin{aligned} \mathcal{P}(\varphi_n(X)) &= \prod_{i=1}^{n} \mathcal{P}(X + z_i) = \prod_{i=1}^{n} (X + z_i + X^q + z_i^q) \\ &= \prod_{i=1}^{n} \big((X + z_i) + (X + z_i)(X^{q-1} - X^{q-2} z_i + \cdots - X z_i^{q-2} + z_i^{q-1})\big) \\ &= \prod_{i=1}^{n} (X + z_i) \prod_{i=1}^{n} (1 + X^{q-1} - X^{q-2} z_i + \cdots - X z_i^{q-2} + z_i^{q-1}) \\ &= \varphi_n(X) \prod_{i=1}^{n} (1 + X^{q-1} - X^{q-2} z_i + \cdots - X z_i^{q-2} + z_i^{q-1}). \end{aligned}$$

The action of the total reduced power operation $\mathcal{P}$ on the inverse monomial X^{-n} is given by (see e.g., [8] Proposition 6.5.2 and the discussion following its proof)

$$(\text{✶}) \qquad \mathcal{P}(X^{-n}) = X^{-n}(1 + X^{-1})^{n}.$$

The action of $\mathcal{P}$ on the inverse monomial σ_n is trivial, i.e.,

$$(\text{❇}) \qquad \mathcal{P}(\sigma_n) = \sigma_n.$$

Substituting formulae (★), (✶), and (❇) into formula (✠) yields

$$\begin{aligned}\mathcal{P}(\theta_n) &= \varphi_n(X)\prod_{i=1}^{n}(1 + X^{q-1} - X^{q-2}z_i + \cdots - Xz_i^{q-2} + z_i^{q-1}) \cap (\sigma_n \cdot X^{-n} \cdot (1 + X^{-1})^n) \\ &= \big((1+X)^n(1 + X^{q-1} - X^{q-2}z_i + \cdots - Xz_i^{q-2} + z_i^{q-1})\varphi_n(X)\big) \cap (\sigma_n \cdot X^{-n})\end{aligned}$$

from which the desired conclusion follows by [8] Theorem VI.6.2. □

Further intereting families of $\mathfrak{m}$-primary irriducible ideals are provided by starting with other families of classical polynomials, such as the Dickson polynomials, and, or, varying the homoginization schema employed. It is not our intention to persue this further here.

§5. The Poincaré Duality Algebra Dual to an Element

The idea of pulling off sections discussed in § 4 is a special case of another construction called dualizing. The motivation comes from topology again. If $\lambda \downarrow M$ is a line bundle over a closed smooth manifold with first Stiefel–Whitney class w_1 then the **manifold dual to** w_1 is obtained as follows: Choose a classifying map $f_\lambda : M \longrightarrow \mathbb{RP}(\ell)$ for some large integer ℓ which is transverse regular to $\mathbb{RP}(\ell - 1)$. The manifold N dual to w_1 is then the preimage $f_\lambda^{-1}(\mathbb{RP}(\ell - 1))$. It can be thought of as the set of zeros of a generic section to λ. The normal bundle of N in M is λ.

Algebraically this corresponds to the following simple construction. Let H be a Poincaré duality algebra and $0 \neq u \in H$. Then the **dual of** u **in** H is the quotient algebra $H/\mathrm{Ann}_H(u)$. The algebra $H/\mathrm{Ann}_H(u)$ is a Poincaré duality algebra of formal dimension f-dim(H) – deg(u) (see e.g., [8] Corollaries I.2.2 – I.2.4). Since H is a Poincaré duality algebra the trivial ideal $(0) \subset H$ is irreducible so the Noether Involution Theorem tells us that $\mathrm{Ann}_H(\mathrm{Ann}_H(u)) = u$ for any nonzero element $u \in H$ and that an ideal $J \subset H$ is irreducible if and only if $\mathrm{Ann}_H(J)$ is a principal ideal.

Pulling off sections in the sense of § 4 amounts to dualizing powers of X in $\mathbb{F}[V, X]/(J, h(X))$. Here is why.

PROPOSITION 5.1: *Let $I(\theta) \subset \mathbb{F}[z_1, \ldots, z_n]$ be the $\mathfrak{m}$-primary irreducible ideal defined by the inverse form $\theta \in \mathbb{F}[z_1, \ldots, z_n]$. If $u \neq 0 \in \mathbb{F}[z_1, \ldots, z_n]/I(\theta)$ then the inverse form $u \cap \theta$ is a Macaulay dual for the ideal K in $\mathbb{F}[z_1, \ldots, z_n]$ defining the dual to u in $\mathbb{F}[z_1, \ldots, z_n]/I(\theta)$.*

PROOF: Let $J = I(u \cap \theta) \subset \mathbb{F}[z_1, \ldots, z_n]$. Then J is $\mathfrak{m}$-primary and irreducible. The corresponding Poincaré duality quotient algebra has formal dimension $|\deg(\theta)| - \deg(u)$. If $f \in J$ than $\theta(fu) = (u \cap \theta)(f) = 0$ so $fu \in I(\theta)$. This means there is an induced epimorphism

$$\varphi : \mathbb{F}[z_1, \ldots, z_n]/J \longrightarrow \Big(\mathbb{F}[z_1, \ldots, z_n]/(I(\theta)\Big)\Big/\mathrm{Ann}_{\mathbb{F}[z_1,\ldots,z_n]/(I(\theta)}(u) = \mathbb{F}[z_1, \ldots, z_n]/K.$$

The algebra $\mathbb{F}[z_1, \ldots, z_n]/I(\theta)$ also has formal dimension $|\deg(\theta)| - \deg(u)$ by [8] Corollary I.2.3 and hence φ must be an isomorphism by loc. cit Corollary I.2.4, making the inclusion $I(u \cap \theta) \subseteq K$ an equality. $\square$

Despite being simple the dualizing construction can have surprising consequences. Again, we illustrate this with some unusual examples. We choose $\mathbb{F}_2$ as ground field in these examples to simplify the arithmetic.

EXAMPLE 1: Consider the inverse form[9] $\theta_s = (x^{-1} + y^{-1})^s \in \mathbb{F}_2[x^{-1}, y^{-1}]$ for $s \in \mathbb{N}$. To compute the corresponding ideal $I(\theta_s)$ we introduce the auxillary form

$$\theta_s y^{-t} = x^{-s}y^{-t} + \cdots + y^{-(s+k)}$$

which is of the type conidered in Example 1 in § 2. Write $s = 2^a + b$ with $0 \le b < 2^a$, and let $c = 2^a - b$. As in that example the ideal $I(\theta_s y^{-t})$ is generated by the two forms x^{s+1} and $y^{t+1-k}(y^k + \alpha_1 xy^{k-1} + \cdots \alpha_k x^k)$ where $\alpha_1, \ldots, \alpha_k$ are determined by

$$\begin{aligned} 1 + \alpha_1 x + \cdots + \alpha_k x^k &= \frac{1}{(1+x)^s} = \frac{(1+x)^{2^{s+1}}}{(1+x)^{2^a+b)}} = (1+x)^{2^a-b} = (1+x)^c \\ &= 1 + \binom{c}{1}x + \cdots + \binom{c}{c}x^c. \end{aligned}$$

In the case that $t = c - 1$ this tells us that the ideal $I(\theta_s y^{c-1})$ is generated by the two forms x^{s+1} and $(x+y)^c$. Next, note that

$$x^{s+1} = x^{2^a+b+1}x^{b+1}\big((x+y)^{2^a} + (x+y)^{2^a}\big)$$

so the ideal $I(\theta_s y^{c-1})$ is also generated by the two forms $(x+y)^c$ and $x^{b+1}y^{2^a}$. If we dualize y^{c-1} in the quotient algebra $\mathbb{F}_2[x, y]/I(\theta_s y^{c-1})$ we obtain $I(\theta_s)$ and [11] Proposition 2.2 tells us this is the ideal generated by $(x+y)^c$ and $(xy)^{b+1}$

If $s = 2^t - 1$ then $c = 1$ so the resulting ideal $I(\theta_s)$ contains the linear form $x + y$. If in addition s is divisible by r say, and $(2^t - 1)/r$ is not itself a power of two minus one,then the ideal $I(\theta_{s/r})$ contains no nonzero linear forms as the value of c for $(2^t - 1)/r$ is not one. Since θ_s is the r-th power of $\theta_{s/r}$ we obtain lots of examples of inverse forms θ for which the ideal $I(\theta^r)$ is not contained in $I(\theta)$. This should be contrasted with the results of [11] that imply $I(\theta^{2^i}) \subset I(\theta)$ for any nonzero inverse form θ in characteristic two and any $i \in \mathbb{N}_0$. For more about the interaction of the Frobenius map with irreducible ideals and their Macaulay inverses see [8] §II.6 and [2] §9.

EXAMPLE 2: Consider the ideal $I = (e_1, \ldots, e_n)$ in $\mathbb{F}_2[z_1, \ldots, z_n]$ generated by the elementary symmetric polynomilas $e_1, \ldots, e_n$. The Macaulay inverse for this ideal is

$$\theta = \sum_{\sigma \in \Sigma_n} z_{\sigma(1)}^{0} z_{\sigma(2)}^{-1} \cdots z_{\sigma(n)}^{-(n-1)},$$

see [4] §4 Example 2. By Sharp's Theorem (cf [8] Theorem II.6.5) the Frobenius square $I^{[2]}$ is $\mathfrak{m}$-primary, irreducible, and by [8] Theorem II.6.6 has as Macaulay inverse the form $z_1^{-1} \cdots z_n^{-1}\theta^2$. If we dualize $z_1 \cdots z_n \in H = \mathbb{F}_2[z_1, \ldots, z_n]/I^{[2]}$ then Proposition 5.1 tells us that

$$(z_1 \cdots z_n) \cap (z_1^{-1} \cdots z_n^{-1}\theta^2 = \theta^2 = \sum_{\sigma \in \Sigma_n} z_{\sigma(1)}^{0} z_{\sigma(2)}^{-2} \cdots z_{\sigma(n)}^{-2(n-1)}$$

[9] We have again chosen $\mathbb{F}_2$ as ground field to avoid problems of signs.

is the Macaulay dual for the ideal of $\mathbb{F}_2[z_1, \dots, z_n]$ which is the kernel of the natural epimorphism of $\mathbb{F}_2[z_1, \dots, z_n]$ onto $H/\mathrm{Ann}_H(z_1 \cdots z_n)$. The ideal $I(\theta^2)$ contains $I^{[2]} = (e_1^2, \dots, e_n^2)$ and by [11] Corollary 4.2 $I(\theta^2) = (I^{[2]} : z_1 \cdots z_n)$. Since $z_1 \cdots z_n$ divides e_n it follows from loc. cit. Lemma 2.1 that $(I^{[2]} : z_1 \cdots z_n) = (e_1^2, \dots, e_{n-1}^2, e_n)$. Therefore the dual of $z_1 \cdots z_n = e_n$ in $\mathbb{F}_2[z_1, \dots, z_n]/I^{[2]}$ turns out to be $I(\theta^2) = (e_1^2, \dots, e_{n-1}^2, e_n)$, and more generally $I(\theta^{2^r}) = (e_1^{2^r}, \dots, e_{n-1}^{2^r}, e_n^{2^r-1})$, The element $z_1 \cdots z_n$ is a Thom class so by [8] Theorem III.1.4 all these ideals are $\mathscr{A}^*$-invariant.

Example 3: There is a variation of the preceding example that arises because the algebra $\mathbb{F}_2[z_1, \dots, z_n]/(e_1, \dots, e_n)$ is not really a rank n algebra: It has rank $n-1$ since $e_1 = z_1 + \cdots + z_n$. The images of $e_2, \dots, e_n$ in the quotient algebra $\mathbb{F}_2[z_1, \dots, z_{n-1}] = \mathbb{F}_2[z_1, \dots, z_n]/(e_1)$ are denoted by $w_2, \dots, w_n$. One has

$$w_i = \begin{cases} \overline{e}_i + \overline{e}_1\overline{e}_{i-1} & \text{for } i = 2, \dots, n-1 \text{ and} \\ \overline{e}_1\overline{e}_{n-1} & \text{for } i = n, \end{cases}$$

where $\overline{e}_1, \dots, \overline{e}_n \in \mathbb{F}_2[z_1, \dots, z_{n-1}]$ are the elementary symmetric polynomials in the variables $z_1, \dots, z_{n-1}$. A Macaulay inverse for the ideal generated by $w_2, \dots, w_n$ is

$$\psi = \sum_{\sigma \in \Sigma_{n-1}} z_{\sigma(1)}^{-1} \cdots z_{\sigma(n-1)}^{-(n-1)}$$

(see [8] Section VI.4). So if we dualize $z_1 \cdots z_{n-1}$ in $\mathbb{F}_2[z_1, \dots, z_{n-1}]/(w_2, \dots, w_n)$ we find by the same reasoning as in Example 2 that it is the quotient algebra of $\mathbb{F}_2[z_1, \dots, z_{n-1}]$ by the ideal $I(\psi^2) = (\overline{e}_2^2 + \overline{e}_1^4, \dots, \overline{e}_{n-1}^2\overline{e}_1^2\overline{e}_{n-2}^2, \overline{e}_1^2\overline{e}_{n-1})$. Note that the generator of maximal degree, viz. $\overline{e}_1^2\overline{e}_{n-1}$ is not a polynomial in the generators of the original ideal $(w_2, \dots, w_n) = I(\psi)$, a phenomen not seen in examples before. Again $I(\psi^{2^r})$ is a family of $\mathscr{A}^*$-invariant ideals.

§6. Applications

In this section we collect a number of examples and applications of projective bundle ideals. To begin, recall that for two Poincaré duality algebras H' and H'' of the same formal dimension their **connected sum**, denoted by $H' \# H''$ is defined in the following way:

$$(H' \# H'')_k = \begin{cases} \mathbb{F} \cdot [H' \# H''] & \text{if } k = d \\ H'_k \oplus H''_k & \text{if } 0 < k < d \\ 1 \cdot \mathbb{F} & \text{if } k = 0. \end{cases}$$

The products of two elements in either H' or H'' are as before, modulo the identification of the three fundamental classes $[H']$, $[H' \# H'']$, $[H'']$. The product of an element of H' of positive degree and of H'' of positive degree is zero. The operation # turns the isomorphism classes of Poincaré duality algebras of a fixed formal dimension d over a fixed ground field $\mathbb{F}$ into a commutative torsion free monoid.

One says that a Poincaré duality algebra H is **#-decomposable** if there are two nontrivial Poincaré duality algebras H' and H'' such that $H \cong H' \# H''$, otherwise one says that H is **#-indecomposable**. The #-indecomposables are generators of the Grothendieck group of the monoid of Poincaré duality algebras of a given formal dimension under the connected sum operation. The Grothendieck group of standardly graded Poincaré duality algebras of formal dimension d is of interest as a means of classification since every standardly graded Poincaré duality algebra can be written in an essentially unique way as a connected sum of #-indecomposable Poincaré duality algebras (see [12]). It seems however very difficult to write down a minimal generating set if $d > 2$. There are simply too many #-indecomposables to account for as is implied by the following result. To formulate it we need to introduce another concept borrowed from topology.

Let A be a commutative graded algebra over a field and $X \subset A$ a graded subset. The **×-length**[10] of X is the smallest integer $c_X + 1$ such that the product of any $c_X + 1$ elements of X is zero in A if such an integer c_X exists, otherwise we say the ×-length of X is infinite.

PROPOSITION 6.1: *Let H be a standardly graded Poincaré duality algebra of formal dimension d. Suppose there is a codimension one subspace $V \subset H_1$ of ×-length strictly less than d. Then either*

(i) H is indecomposable with respect to the connected sum operation #, or

(ii) H has rank two and $H \cong \mathbb{F}[x, y]/(xy, x^d - y^d) = \big(\mathbb{F}[x](x^{d+1})\#\mathbb{F}[y]/(y^{d+1}\big)$.

PROOF: Suppose that $H = H'\#H''$ is a nontrivial connected sum. Let the rank of H be r, that of H' be r', and that of H'' be r'', so $r = r' + r''$. Recall[11] the formula from linear algebra relating the dimensions of two subspaces $U', U'' \subseteq U$, viz.,

$$\dim_{\mathbb{F}}(U' + U'') = \dim_{\mathbb{F}}(U') + \dim_{\mathbb{F}}(U'') - \dim_{\mathbb{F}}(U' \cap U'').$$

Apply this to $V, H_1' \subset H_1$. After rearranging a bit one obtains

$$\dim_{\mathbb{F}}(V + H_1') + \dim_{\mathbb{F}}(V \cap H_1') = \dim_{\mathbb{F}}(V) + \dim_{\mathbb{F}}(H_1') = r - 1 + r'.$$

On the otherhand we have the inequality

$$\dim_{\mathbb{F}}(V + H_1') + \dim_{\mathbb{F}}(V \cap H_1') \leq r + \dim_{\mathbb{F}}(V \cap H_1'),$$

so

$$r + \dim_{\mathbb{F}}(V \cap H_1') \geq r + r' - 1$$

whence we conclude that

$$\dim_{\mathbb{F}}(V \cap H_1') \geq r' - 1.$$

Since $V \cap H_1' \subset V$ the subalgebra of H generated by $V \cap H_1'$ has ×-length at most $d - 1$. If $\dim_{\mathbb{F}}(V \cap H_1')$ were to equal r' then, since H' is a Poincaré duality algebra of formal dimension d, this would imply that H_1' was trivial since no product of d elements of H_1' could be nonzero. Hence $\dim_{\mathbb{F}}(V \cap H_1') = r' - 1$. This tells us that $V \cap H_1'$ is a codimension subspace of H_1'. By symmetry $V \cap H_1'' \subset H_1''$ is also a codimension one subspace of ×-length at most $d - 1$.

Putting these facts together says that $(V \cap H_1') \oplus (V \cap H_1'') \subset V$ is a codimension one subspace. So we may choose a $v \in V$ that does not belong to this subspace. Write $v = v' + v''$ with $v' \in H_1'$ and $v'' \in H_1''$. Note that $v' \notin V \cap H_1'$: For if it were, then this would say $v'' = v - v' \in V \cap H_1''$ which implies that $v = v' + v''$ belongs to $(V \cap H_1') \oplus (V \cap H_1'')$ contrary to how we chose v. Therefore $v' \notin V \cap H_1'$ and similarly $v'' \notin V \cap H_1''$.

Retaining these notations we next choose a basis $v_1, \ldots, v_{r'-1}$ for $V \cap H_1'$. Note that v' extends this to a basis for H_1'. Consider a product $v_{i_1} \cdots v_{i_k} \cdot (v')^{d-k}$ of d elements from this basis. One has

$$v_{i_1} \cdots v_{i_k} \cdot (v')^{d-k} = v_{i_1} \cdots v_{i_k} \cdot (v' + v'')^{d-k} = v_{i_1} \cdots v_{i_k} \cdot v^{d-k} = 0$$

for $k > 0$, since $v'' \in H_1''$ annihilates $v_1, \ldots, v_{r'-1} \in H_1'$, and $v_{i_1} \cdots v_{i_k} \cdot v^{d-k}$ is a product of d elements of V which has ×-length at most $d - 1$. Thus the only product of d elements of the basis $v_{i_1}, \ldots, v_{i_k}, v'$ for H_1' that is nonzero is $(v')^d$. Poincaré duality then forces that H' has

[10] An algebraic topologist would probably call this the ∪-length (pronounced *cup length*). In topology ∪-length provides a lower bound for the *category* of a topological space, i.e., the number of contractible subsets needed to cover a space.

[11] As usual $\mathbb{F}$ denotes the ground field.

rank one and is isomorphic to $\mathbb{F}[x]/(x^{d+1})$. Likewise $H'' \cong \mathbb{F}[y]/(y^{d+1}$. Finally one notes that $H = \big(\mathbb{F}[x]/(x^{d+1})\big)\#\big(\mathbb{F}[y]/(y^{d+1})\big)$ satisfies the hypotheses of the propossition: Namely the subspace spanned by $x + y$ in H_1 has codimension one and $\times$-length $d-1$. □

COROLLARY 6.2: *Let $I \subset \mathbb{F}[V, X]$ be a projective bundle ideal with a Poincaré duality quotient algebra of formal dimension m. Then either*
(i) $\mathbb{F}[V, X]/I$ is #-indecomposable, or
(ii) $\mathbb{F}[V, X] \cong \mathbb{F}[x, y]/(xy, x^{m+1} - y^{m+1})$.

PROOF: Let $J = \mathbb{F}[V] \cap I$ be the base ideal and $\mathbb{F}[V]/J$ have formal dimension d. Consider the coexact sequence

$$\mathbb{F} \longrightarrow \mathbb{F}[V]/J \longrightarrow \mathbb{F}[V, X] \longrightarrow \mathbb{F}[X]/(X^{k+1}) \longrightarrow \mathbb{F}.$$

By Lemma 2.1 the formal dimension of $\mathbb{F}[V, X]/I$ is $d + k$, so the codimension one subspace V of $\big(\mathbb{F}[V, X]/I\big)_1$ has $\times$-length at most $d < m = d + k$ and the result follows from Proposition 6.1. □

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