

Poincaré Duality Algebras Mod Two

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AG-INVARIANTENTHEORIE

UNIVERSITY OF VIRGINIA

REVISION HISTORY: FIRST T_EXED AUGUST 2005 AND REVISED PERIODICALLY UNTIL THIS VERSION

LAST REVISION: SUMMER 2008

PRIVATE CIRCULATION: PRINTOUT OF PDA_QUOS.TEX ON 30.8.2008 @ 10:04

MATHEMATICS SUBJECT CLASSIFICATION : 13A02 Graded Commutative Algebra, 13A50 Invariant Theory

Typeset by *LS* T_EX

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A COMMUTATIVE graded¹ connected algebra H over a field \mathbb{F} is called a **Poincaré duality algebra of formal dimension d** (denoted by $\text{f-dim}(H) = d$) if the following conditions are satisfied:

- (1) $H_i = 0$ for $i > d$.
- (2) H_d is a 1-dimensional vector space over \mathbb{F} .
- (3) An element $u \in H_i$ is nonzero if and only if there exists an element $u^\vee \in H_{d-i}$, called a **Poincaré dual** for u , such that the product $u \cdot u^\vee \neq 0 \in H_d$.

A nonzero element $[H] \in H_d$ is called a **fundamental class**. Choosing a fundamental class enables us to define for $i = 0, \dots, d$ a nonsingular symmetric bilinear form

$$\langle - | - \rangle : H_i \times H_{d-i} \longrightarrow \mathbb{F}$$

by the requirement

$$u \cdot v = \langle u | v \rangle \cdot [H].$$

The notion of Poincaré duality algebras originated in the work of topologists on the cohomology of closed manifolds. Apart from the cosmetic difference of being graded commutative, instead of commutative and graded, the cohomology of a closed smooth manifold with field coefficients is a Poincaré duality algebra.

If a Poincaré duality algebra is generated by its homogeneous elements of degree one then it is said to be **standardly graded**. If H is a standardly graded Poincaré duality algebra then the dimension of its homogeneous component H_1 of degree one is called its **rank**. Standardly graded Poincaré duality algebras occur as quotient algebras of a (standardly graded) polynomial algebra by a maximal primary irreducible ideal (see e.g., [16] Lemma I.1.3 and Proposition I.1.5). Such ideals were studied in the work of F. S. Macaulay [13] who developed an elegant means of constructing them. The fact that these quotients are Poincaré duality algebras is a special case of a result of W. Gröbner [7].

In this note we study Poincaré duality algebras over the field \mathbb{F}_2 of two elements.² We obtain a complete classification of surfaces, i.e., Poincaré duality algebras of formal dimension two. To do so we determine the Grothendieck group of standardly graded surface algebras over an arbitrary field under the operation of connected sum. This group turns out to be finitely generated and mirrors faithfully the topological classification of closed surfaces. By contrast, for Poincaré duality algebras (standardly graded or not) of formal dimension strictly greater than two the Grothendieck group fails to be finitely generated.

We make a systematic study of standardly graded threefolds, i.e., Poincaré duality algebras of formal dimension three that are generated by their elements of degree one. The isomorphism classes of threefolds of rank at most three are in bijective correspondence with the orbits of the action of $\text{GL}(3, \mathbb{F}_2)$ on a 10-dimensional vector space, the space of catalecticant matrices. To determine the number of isomorphism classes we count the number of orbits using invariant

¹We warn the reader that we adhere to the grading conventions of J. C. Moore and therefore all elements and ideals in graded objects are to be considered as homogeneous unless explicitly stated to the contrary. Thus a graded object X in a category \mathcal{C} consists of a collection $\{X_i\}$ of objects of \mathcal{C} , one for each $i \in \mathbb{Z}$, called the homogeneous components of X of degree i .

²More generally, we denote by \mathbb{F}_q the finite field with $q = p^v$ elements, where p is any prime integer.

theory. As a byproduct we obtain a classification of arbitrary bilinear forms in up to three variables.

We determine all the standardly graded threefolds of rank at most three. There are 21 isomorphism classes. Twelve of these admit an unstable Steenrod algebra action, so could in theory be realized as the mod 2 cohomology of a closed manifold. We exhibit for each such example a corresponding manifold; most of these are obvious, but there is one example of a slightly exotic 3-manifold that is a torus bundle over a circle to which we devote some space.

For threefolds of higher rank we explain one of several ways to construct such algebras that are not connected sums using Macaulay's theory of inverse systems. There is an appendix, authored only by R.E. Stong, that in addition explains how this can be done by means of Steiner systems.³

This manuscript is organized into sections as follows:

- §1. Preliminaries
- §2. Formal Dimension Two : Surfaces
- §3. Formal Dimension Greater than Two
- §4. Threefolds of Rank Three I (Counting the Number of Isomorphism Classes)
- §5. Threefolds of Rank Three II (The Isomorphism Classes)
- §6. Threefolds of Rank Three III (Seperating Invariants)
- §7. Threefolds of Rank Three IV (A Torus Bundle over a Circle)
- §8. Indecomposable Threefolds of Arbitrary Rank
- §9. Comments by Larry Smith on the Appended Letter of R.E. Stong
- Appendix: Steiner Systems and Poincaré Duality Algebras by R.E. Stong

Most of the algebraic notations we employ are standard and can be found in [18] or [16]. A possible exception is the notation and terminology of catalecticants⁴ which comes from [16] Part VI Section 2. As background, we assume the reader has a passing understanding of the Steenrod algebra (e.g., as described in [19]), as well as the material in [16] Part I. Beginning with § 2 we will assume that the ground field is the Galois field \mathbb{F}_2 with two elements unless noted to the contrary. The Steenrod algebra of \mathbb{F}_2 will be denoted by \mathcal{A}^* .

³The letter from which the appendix is been taken has been commented by Larry Smith and edited with the assistance of Peter Landweber.

⁴This term was introduced by J.J. Sylvester.

§1. Preliminaries

If H is a standardly graded Poincaré duality algebra then we may write $H = \mathbb{F}[z_1, \dots, z_n]/I$ where $\mathbb{F}[z_1, \dots, z_n]$ is a graded polynomial algebra with generators z_1, \dots, z_n of degree one and $I \subset \mathbb{F}[z_1, \dots, z_n]$ is an \mathfrak{m} -primary irreducible ideal⁵ (see e.g., [16] Section I.1). If I contains no nonzero linear forms (which we assume to be the case unless explicitly noted to the contrary) then n is the rank of H and the formal dimension d of H is the smallest integer such that every monomial of degree $d + 1$ belongs to I .

If H', H'' are Poincaré duality algebras of formal dimension d their **connected sum**, denoted by $H' \# H''$, is defined by

$$(H' \# H'')_k = \begin{cases} \mathbb{F} \cdot 1 & \text{if } k = 0, \\ H'_k \oplus H''_k & \text{if } 0 < k < d, \\ \mathbb{F} \cdot [H' \# H''] & \text{if } k = d, \end{cases}$$

where fundamental classes $[H'] \in H'_d$ and $[H''] \in H''_d$ have been identified to a single element $[H' \# H''] \in H' \# H''$. The product of two elements in H' or H'' is as in H' respectively H'' modulo this identification, whereas H'_i and H''_j mutually annihilate each other if $0 < i, j < d$. If H' and H'' are standardly graded then so is $H' \# H''$.

Denote by $E(u_d)$ the exterior algebra with one generator u_d of degree d . If H is a Poincaré duality algebra of formal dimension d then a choice of a fundamental class $[H] \in H_d$ amounts to a map of algebras $E(u_d) \rightarrow H$. If H' and H'' are Poincaré duality algebras of formal dimension d with fundamental classes $[H']$ and $[H'']$ respectively then

$$\begin{array}{ccc} E(u_d) & \xrightarrow{\quad} & H'' \\ \downarrow & \lrcorner \bullet & \downarrow \\ H' & \xrightarrow{\quad} & H' \# H'' \end{array}$$

DIAGRAM 1.1: The Connected Sum as a Pushout

is a pushout diagram.

We will often be concerned with Poincaré duality algebras that cannot be written as a connected sum. So we introduce here the first of several criteria for a standardly graded Poincaré duality algebra to be decomposable with respect to the connected sum operation.

LEMMA 1.1: *Let H be a standardly graded Poincaré duality algebra of formal dimension d over an arbitrary ground field \mathbb{F} . If H can be written as a connected sum $H = H' \# H''$, then $H_1 = H'_1 \oplus H''_1$ with $H'_1 \cdot H''_1 = 0$. Conversely, if $H_1 = H'_1 \oplus H''_1$ with $H'_1 \cdot H''_1 = 0$, then $H = H' \# H''$ where H' is the subalgebra of H generated by H'_1 and H'' is the subalgebra of H generated by H''_1 .*

PROOF: The first assertion is clear from the definitions. For the second, suppose that $H_1 = H'_1 \oplus H''_1$ with $H'_1 \cdot H''_1 = 0$. Let H' be the subalgebra of H generated by H'_1 and H'' be the subalgebra of H generated by H''_1 . To show that $H = H' \# H''$ we first show that $H'_i = H'_i \oplus H''_i$ for $i = 1, \dots, d-1$. If $x \in H'_i$ then x is a sum of products of i elements of degree one, say

$$x = \sum_j x_{j,1} \cdots x_{j,i}, \quad x_{j,i} \in H_1.$$

⁵The usual notation for the maximal ideal of $\mathbb{F}[z_1, \dots, z_n]$ would be (z_1, \dots, z_n) or $\overline{\mathbb{F}[z_1, \dots, z_n]}$ both of which are rather long and the latter ugly. We therefore write \mathfrak{m} for this ideal.

Write $x_{j,i} = x'_{j,i} + x''_{j,i}$ with $x'_{j,i} \in H'_1$ and $x''_{j,i} \in H''_1$. Then a bit of rearranging of terms gives

$$\begin{aligned} x &= \sum_j (x'_{j,1} + x''_{j,1}) \cdots (x'_{j,i} + x''_{j,i}) \\ &= \sum_j ((x'_{j,1} \cdots x'_{j,i}) + (x''_{j,1} \cdots x''_{j,i})) \\ &= \sum_j (x'_{j,1} \cdots x'_{j,i}) + \sum_j (x''_{j,1} \cdots x''_{j,i}), \end{aligned}$$

where $\sum_j (x'_{j,1} \cdots x'_{j,i}) \in H'$ and $\sum_j (x''_{j,1} \cdots x''_{j,i}) \in H''$. Thus H'_i and H''_i together span H_i . To see that the sum $H_i = H'_i + H''_i$ is direct for $0 < i < d$, suppose that $x \in H'_i \cap H''_i$. By what we have already shown, for any element $u \in H_{d-i}$ we may write $u = u' + u''$ with $u' \in H'_{d-i}$ and $u'' \in H''_{d-i}$. Doing so one sees

$$x \cdot u = x \cdot u' + x \cdot u'' = 0 + 0,$$

since on the one hand u' annihilates x as $x \in H''$, and on the other hand u'' annihilates x as $x \in H'$. By Poincaré duality for H this means that $x = 0$ proving that the sum $H_i = H'_i + H''_i$ is direct for $i = 1, \dots, d-1$.

Finally we need to show that H' and H'' are Poincaré duality algebras. So let $0 \neq x' \in H'_i$, with $i \neq 0, d$. By Poincaré duality in H there is an element $x^\vee \in H_{d-i}$ with $x' \cdot x^\vee \neq 0 \in H_d$. Write $x^\vee = x'^\vee + x''^\vee$ with $x'^\vee \in H'_{d-i}$ and $x''^\vee \in H''_{d-i}$. Then

$$0 \neq x' \cdot x^\vee = x' \cdot x'^\vee + x' \cdot x''^\vee = x' \cdot x'^\vee + 0 = x' \cdot x'^\vee \in H'_d$$

showing that the product pairing $H'_i \times H'_{d-i} \rightarrow H'_d = \mathbb{F}$ is nonsingular. A similar argument applies to show that H'' is also a Poincaré duality algebra. \square

Rings of coinvariants provide a rich supply of standardly graded Poincaré duality algebras. One important family of such examples are the Dickson coinvariants. Using Lemma 1.1 one can show that these are $\#$ -indecomposable. Here is how.

EXAMPLE 1: Let $q = p^\nu$ where $p \in \mathbb{N}$ is a prime and denote by $\mathrm{GL}(n, \mathbb{F}_q)$ the full general linear group over the field \mathbb{F}_q with q elements. This group acts on the algebra of polynomial functions $\mathbb{F}_q[V]$ on the vector space $V = \mathbb{F}_q^n$ and the ring of invariants $\mathbf{D}(n) = \mathbb{F}_q[V]^{\mathrm{GL}(n, \mathbb{F}_q)}$ is well known to be a polynomial algebra called the Dickson algebra (see e.g., [18] §8.1). The **Dickson coinvariants** $\mathbb{F}_q[V]_{\mathrm{GL}(n, \mathbb{F}_q)} = \mathbb{F}_q \otimes_{\mathbf{D}(n)} \mathbb{F}_q[V]$ are therefore a Poincaré duality algebra (see e.g., [18] Theorem 6.5.1). According to the discussion of this algebra in [11] §2 whenever one chooses a basis z_1, \dots, z_n for the space V^* of linear forms then the monomial $u = z_1^{q^n - q^{n-1} - 1} \cdot z_2^{q^n - q^{n-2} - 1} \cdots z_n^{q^n - q^0 - 1}$ is a fundamental class for $\mathbb{F}_q[V]_{\mathrm{GL}(n, \mathbb{F}_q)}$. In particular the product $z_1 \cdot z_2 \cdots z_n$ is nonzero. If $\mathbb{F}_q[V]_{\mathrm{GL}(n, \mathbb{F}_q)}$ were $\#$ -decomposable then Lemma 1.1 would tell us that there is a direct sum decomposition $V^* = V'^* \oplus V''^*$ where $V'^* \neq 0 \neq V''^*$. If $z'_1, \dots, z'_{n'}$ were a basis for V'^* and $z''_1, \dots, z''_{n''}$ a basis for V''^* then since $z'_1 \cdot z''_1 = 0$ we would have a contradiction. Hence $\mathbb{F}_q[V]_{\mathrm{GL}(n, \mathbb{F}_q)}$ is $\#$ -indecomposable.

The operation $\#$ turns the isomorphism classes of Poincaré duality algebras of a fixed formal dimension d over a fixed ground field \mathbb{F} into a commutative torsion-free monoid. The standardly graded Poincaré duality algebras under connected sum form a submonoid. One of our purposes in this note is to study the corresponding Grothendieck groups.

REMARK: For formal dimensions zero, respectively one, there is only a single Poincaré duality algebra up to isomorphism. For formal dimension zero it is $H^*(\text{point}; \mathbb{F})$ and for formal dimension one it is $H^*(S^1; \mathbb{F})$. One has $\text{point} \# \text{point} = \text{point}$ and $S^1 \# S^1 = S^1$ so in both cases the sum of the Poincaré duality algebra with itself is itself.

In the sequel we will use the following notations for various topological spaces.

NOTATION: The unit sphere of \mathbb{R}^{n+1} is denoted by S^n and the projective space of \mathbb{R}^{n+1} by $\mathbb{R}\mathbb{P}(n)$. Note that $\mathbb{R}\mathbb{P}(n)$ is an n -dimensional manifold and is diffeomorphic to the orbit space of S^n by the antipodal map. If X is a topological space and $\xi \downarrow X$ an $(n+1)$ -dimensional real vector bundle, then $\mathbb{R}\mathbb{P}(\xi \downarrow X)$ denotes the total space of the corresponding bundle with fibre $\mathbb{R}\mathbb{P}(n)$.

§2. Formal Dimension Two : Surfaces

If H is a Poincaré duality algebra of formal dimension two then we call it a **surface algebra**. Poincaré duality tells us that with one exception it is generated by its elements of degree one: The exception is $H^*(S^2; \mathbb{F}_2)$. If H is standardly graded we may write it in the form $\mathbb{F}_2[z_1, \dots, z_n]/I$ where $I \subset \mathbb{F}_2[z_1, \dots, z_n]$ is an \mathfrak{m} -primary irreducible ideal containing no nonzero linear forms.

LEMMA 2.1: Suppose that $H = \mathbb{F}_2[z_1, \dots, z_n]/I$ is a standardly graded surface algebra. Then the \mathfrak{m} -primary irreducible ideal $I \subset \mathbb{F}_2[z_1, \dots, z_n]$ is \mathcal{A}^* -invariant.

PROOF: Without loss of generality we may suppose that I contains no nonzero linear forms. Since H has formal dimension two I must contain all forms of degree three or more. If $u \neq 0 \in I$ then for any $i > 0$ the element $\text{Sq}^i(u)$ must have at least degree three so is in I . \square

So Poincaré duality algebras of formal dimension two are unstable algebras over the Steenrod algebra. As such they have a Wu class (see e.g., [16] Section III.3) $\text{Wu}_1(H) \in H_1$ characterized by $\langle \text{Sq}^1(x) \mid [H] \rangle = \langle x \cdot \text{Wu}_1(H) \mid [H] \rangle$.

LEMMA 2.2: Let H be a standardly graded Poincaré duality algebra over \mathbb{F}_2 of formal dimension two with trivial Wu class. Then H is a connected sum of tori.⁶

PROOF: Write $H = \mathbb{F}_2[z_1, \dots, z_n]/I$ where $n = \text{rank}(H)$ and $I \subset \mathbb{F}_2[z_1, \dots, z_n]$ is an \mathfrak{m} -primary irreducible ideal. Since $\text{Wu}_1(H) = 0$ we have for any $z \in H_1$ that

$$z^2 = \text{Sq}^1(z) = \text{Wu}_1(H) \cdot z = 0 \cdot z = 0,$$

so $z_1^2, \dots, z_n^2 \in I$. As $z_1 \notin I$ its image in H is nonzero and it must have a Poincaré dual. Since $z_1^2 = 0$ a Poincaré dual of z_1 is not a multiple of z_1 so without loss of generality we may suppose that z_2 is a Poincaré dual to z_1 . The products between z_1 and z_2 must be as given in

\cdot	z_1	z_2
z_1	0	1
z_2	1	0

the adjacent table. So together z_1 and z_2 span a hyperbolic plane in the space H_1 of linear forms with respect to the product pairing $H_1 \times H_1 \rightarrow H_2 \cong \mathbb{F}$. There is a direct sum decomposition of H_1 into this hyperbolic plane and its annihilator (see e.g. [1] Theorem

3.5). We may suppose z_3, \dots, z_n chosen as a basis for the annihilator, so if we let H' , respectively H'' , denote the subalgebra of H generated by z_1 and z_2 , respectively z_3, \dots, z_n , then Lemma 1.1 shows that $H \cong H' \# H''$. Note that $\text{Wu}_1(H'') = 0$ because all the squares in

⁶A standardly graded Poincaré duality algebra is called a **torus** if it is isomorphic to $\mathbb{F}[z_1, \dots, z_n]/(z_1^2, \dots, z_n^2)$.

H'' are zero, hence we can repeat the preceding argument using H'' in place of H . Since H'' has rank $n - 2$ a simple induction completes the proof. \square

LEMMA 2.3: *Let H be a standardly graded Poincaré duality algebra over \mathbb{F}_2 of formal dimension two with nontrivial Wu class. Then one of the following holds.*

- (i) *If $Wu_1^2 \neq 0 \in H$ then $H = H^*(\mathbb{R}P(2) \#_j (\#(S^1 \times S^1)); \mathbb{F}_2)$, where $j = \text{rank}(H) - 1$.*
- (ii) *If $Wu_1^2 = 0$ then $H = H^*(\mathbb{R}P(2) \# \mathbb{R}P(2) \#_j (\#(S^1 \times S^1)); \mathbb{F}_2)$, where $j = \text{rank}(H) - 2$.*

PROOF: Write $H = \mathbb{F}_2[z_1, \dots, z_n]/I$ with I an \mathfrak{m} -primary irreducible ideal in the algebra $\mathbb{F}_2[z_1, \dots, z_n]$ not containing any nonzero linear forms. Since $Wu_1(H) \neq 0$ there is no loss in generality in supposing that $Wu_1(H) = z_1$.

Consider the case $0 \neq Wu_1^2 = z_1^2$. Let z_2, \dots, z_n be chosen so as to project to a basis for the annihilator of the image of z_1 in H with respect to the product pairing $H_1 \times H_1 \rightarrow H_2 = \mathbb{F}_2$. Then

$$H \cong \frac{\mathbb{F}_2[z_1]}{(z_1^3)} \# H''$$

where H'' is the subalgebra of H generated by z_2, \dots, z_n and $\mathbb{F}_2[z]/(z^3) \cong H^*(\mathbb{R}P(2); \mathbb{F}_2)$. Since $Wu_1(H) = z_1$ it follows that $Wu_1(H'') = 0$ and therefore H'' is a connected sum of tori by Lemma 2.2 proving (i).

Next suppose that $Wu_1(H)^2 = 0 \in H$. There must be a Poincaré dual z_1^\vee for the image of z_1 in H and, since $z_1^2 = Wu_1(H)^2 = 0$ no choice of z_1^\vee is a multiple of z_1 . So without loss of generality we may suppose the image of z_2

\cdot	z_1	z_2
z_1	0	1
z_2	1	1

is a Poincaré dual to the image of z_1 in H . Hence the product between these two elements in H is as given in the adjacent matrix.

The only entry needing explanation is the one for z_2^2 , which becomes clear if one notes

$$z_2^2 = \text{Sq}^1(z_2) = z_2 \cdot Wu_1(H) = z_2 z_1.$$

So again we look at the subspace of H_1 that annihilates the linear span of z_1 and z_2 . Neither z_1 nor z_2 belong to this subspace and it has dimension $n - 2$, since, by Poincaré duality the pairing $H_1 \times H_1 \rightarrow H_2 = \mathbb{F}_2$ is nonsingular. Therefore without loss of generality we may suppose that the annihilator of z_1 and z_2 in H_1 is spanned by z_3, \dots, z_n . Hence an application of Lemma 1.1 shows that $H \cong H' \# H''$ where $H' \subseteq H$ is the subalgebra generated by z_1, z_2 , and $H'' \subseteq H$ is the subalgebra generated by z_3, \dots, z_n .

Again, H'' has trivial Wu class since $Wu_1(H) = z_1 \in H'$, so by Lemma 2.2 the algebra

\cdot	x	y
x	1	0
y	0	1

H'' is a connected sum of tori. On the other hand H' is isomorphic to $H^*(\mathbb{R}P(2) \# \mathbb{R}P(2); \mathbb{F}_2)$. To see this simply make the change of basis $x = z_1 + z_2, y = z_2$, so that the product structure in the x, y basis is as pictured in the adjacent matrix. This shows that

the algebra H' is isomorphic to $\mathbb{F}_2[x, y]/(x^2 - y^2, xy) \cong H^*(\mathbb{R}P(2) \# \mathbb{R}P(2); \mathbb{F}_2)$ and completes the proof. \square

The following result mirrors cohomologically the topological classification of surfaces of which it is a consequence. However it is easy enough to prove directly by elementary means. An alternative formulation in the language of quadratic spaces may be found in [1] Chapter 3.

In the proof we make use of the notion of **catalecticant matrices** associated with a standardly graded Poincaré duality algebra H . If H has formal dimension d , then there is one such

matrix $cat(i, j)$ for each pair $i, j \in \mathbb{N}_0$ with $i + j = d$. To define these, one writes H as a quotient $\mathbb{F}[z_1, \dots, z_n]/I$. The algebra $\mathbb{F}[z_1, \dots, z_n]$ has a basis consisting of monomials in the variables which we choose to index by the elements $E \in \mathbb{N}_0^n$: For $E = (e_1, \dots, e_n) \in \mathbb{N}_0^n$ we set $z^E = z_1^{e_1} \dots z_n^{e_n}$ and let $|E|$ denote the sum $e_1 + \dots + e_n$ which is the degree of this monomial. Choose a fundamental class $[H] \in H_d$ for H . The catalecticant matrix $cat(i, j)$ has rows indexed by the monomials⁷ z^I with $|I| = i$ and columns by the monomials z^J with $|J| = j$. The (z^I, z^J) entry $c_{I,J}$ of $cat(i, j)$ is defined by the requirement that in H one has $z^I \cdot z^J = c_{I,J} \cdot [H]$. Thus $cat(i, j)$ encodes the product structure $H_i \times H_j \rightarrow H_d \cong \mathbb{F}$ of H . For more information, in particular the relation of these matrices to Macaulay's theory of inverse systems, see [16] Part VI.

LEMMA 2.4: *The algebras*

$$H^*(\mathbb{RP}(2)\#\mathbb{RP}(2)\#\mathbb{RP}(2)\#(\#_j(S^1 \times S^1)); \mathbb{F}_2)$$

and

$$H^*(\mathbb{RP}(2)\#(\#_{j+1}(S^1 \times S^1)); \mathbb{F}_2)$$

are isomorphic for any $j \in \mathbb{N}_0$.

PROOF: Clearly it is enough to consider the case $j = 0$ since the cases with $j > 0$ arise from the case with $j = 0$ by forming the connected sum with j copies of $H^*(S^1 \times S^1; \mathbb{F}_2)$.

Both the algebras $H^*(\mathbb{RP}(2)\#\mathbb{RP}(2)\#\mathbb{RP}(2); \mathbb{F}_2)$ and $H^*(\mathbb{RP}(2)\#(S^1 \times S^1); \mathbb{F}_2)$ are determined by a single catalecticant matrix $cat(1, 1)$ describing their product structure (see e.g., [16] Section VI.2). These matrices are as follows.

$cat(1, 1)$	$x \ y \ z$		$cat(1, 1)$	$u \ v \ w$
x	$1 \ 0 \ 0$		u	$1 \ 0 \ 0$
y	$0 \ 1 \ 0$		v	$0 \ 0 \ 1$
z	$0 \ 0 \ 1$		w	$0 \ 1 \ 0$
$H^*(\mathbb{RP}(2)\#\mathbb{RP}(2)\#\mathbb{RP}(2); \mathbb{F}_2)$			$H^*(\mathbb{RP}(2)\#(S^1 \times S^1); \mathbb{F}_2)$	

So one needs to show that the matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \in \mathrm{GL}(3, \mathbb{F}_2)$$

belong to the same orbit in $\mathrm{GL}(3, \mathbb{F}_2)$ under the operation of $\mathrm{GL}(3, \mathbb{F}_2)$ on $\mathrm{GL}(3, \mathbb{F}_2)$ given by

$$(\mathbf{T}, \mathbf{M}) \mapsto \mathbf{TMT}^{\mathrm{tr}} \quad \text{for } \mathbf{T}, \mathbf{M} \in \mathrm{GL}(3, \mathbb{F}_2).$$

In other words one is asking that the quadratic forms

$$\mathbb{F}_2^3 \times \mathbb{F}_2^3 \rightarrow \mathbb{F}_2$$

defined by the two catalecticant matrices be equivalent. If one rewrites the second quadratic form using as new ordered basis $u + v + w, u + v, u + w$ one obtains for the matrix of the

⁷So one needs to choose an ordering of the monomials.

quadratic form

$cat(1, 1)$	$u + v + w$	$u + v$	$u + w$
$u + v + w$	1	0	0
$u + v$	0	1	0
$u + w$	0	0	1

and the equivalence of the forms is demonstrated. \square

THEOREM 2.5: *The semigroup under connected sum of the surface algebras over \mathbb{F}_2 is generated by the two Poincaré duality algebras $H^*(S^1 \times S^1; \mathbb{F}_2)$ and $H^*(\mathbb{R}P(2); \mathbb{F}_2)$ subject to the single relation in the Grothendieck group⁸*

$$[H^*(\mathbb{R}P(2) \# \mathbb{R}P(2) \# \mathbb{R}P(2); \mathbb{F}_2)] = [H^*(\mathbb{R}P(2) \# (S^1 \times S^1); \mathbb{F}_2)].$$

PROOF: This follows from Lemmas 2.2, 2.3, and 2.4. \square

COROLLARY 2.6: *The Grothendieck group under connected sum of the surface algebras over \mathbb{F}_2 is isomorphic to \mathbb{Z} .*

PROOF: Denote the Grothendieck group by Λ and write $T = [H^*(S^1 \times S^1; \mathbb{F}_2)]$ and $P = [H^*(\mathbb{R}P(2); \mathbb{F}_2)]$ for the elements of Λ defined by the indicated algebras. Then by Theorem 2.5 T and P generate Λ and $3P = P + T$, so $T = 2P \in \Lambda$ and Λ is therefore isomorphic to \mathbb{Z} with P as generator. \square

REMARK: Needless to say the Grothendieck group under connected sum of the surface algebras over \mathbb{F}_2 is isomorphic to the Witt group of symmetric bilinear forms over \mathbb{F}_2 . This is because a Poincaré duality algebra H of formal dimension two is completely determined by the catalecticant matrix defined by the product pairing $H_1 \times H_1 \rightarrow H_2 = \mathbb{F}_2$.

§3. Formal Dimension Greater than Two

In this section we show that the Grothendieck group of standardly graded Poincaré duality algebras of formal dimension $d > 2$ is free abelian but not finitely generated (cf. § 2 for the case $d = 2$).

The following result is a sort of Krull–Schmidt Theorem for connected sums of standardly graded Poincaré duality algebras.⁹

PROPOSITION 3.1: *Suppose that H is a standardly graded Poincaré duality algebra of formal dimension $d > 2$ over an arbitrary field \mathbb{F} . If*

$$H'(1) \# \dots \# H'(r) = H = H''(1) \# \dots \# H''(s)$$

are two decompositions of H into connected sums of indecomposable Poincaré duality algebras (of necessity standardly graded and each of formal dimension d), then $r = s$ and after permutation $H'(i) \cong H''(i)$ for $i = 1, \dots, r = s$.

⁸The $[\]$ denotes equivalence class of the enclosed algebra in the Grothendieck group.

⁹One needs standardly graded here since the connected sum of any surface algebra H with $E(u_2) = H^*(S^2; \mathbb{F}_2)$ is H again.

PROOF: Let $H = H(1)\# \cdots \# H(t)$ be a decomposition of H into indecomposable Poincaré duality algebras. Then by Lemma 1.1

$$H_1 = H(1)_1 \oplus \cdots \oplus H(t)_1$$

and

$$H(i)_1 \cdot H(j)_1 = 0 \text{ for } i \neq j.$$

Every $x \in H_1$ can be written uniquely in the form

$$x = p_1(x) + \cdots + p_t(x)$$

with $p_i(x) \in H(i)_1$ for $i = 1, \dots, t$.

Suppose we also have a decomposition

$$H_1 = H'_1 \oplus H''_1$$

with

$$H'_1 \cdot H''_1 = 0.$$

Note that $p_i(H'_1), p_i(H''_1) \subseteq H(i)_1$ are linear subspaces that mutually annihilate each other: For if $x' \in H'_1$ and $x'' \in H''_1$ then, since $d > 2$,

$$\begin{aligned} 0 &= x' \cdot x'' = (p_1(x') + \cdots + p_t(x'))(p_1(x'') + \cdots + p_t(x'')) \\ &= \sum_{i', i''=1}^t p_{i'}(x') p_{i''}(x'') = p_1(x') p_1(x'') + \cdots + p_t(x') p_t(x'') \in H(1)_2 \oplus \cdots \oplus H(t)_2 \end{aligned}$$

because H'_1 and H''_1 mutually annihilate each other.

Since $H'_1 \oplus H''_1 = H_1$ it follows that for $1 \leq i \leq t$ one has $p_i(H'_1) + p_i(H''_1) = H(i)_1$. Moreover $H(i)$ is indecomposable, so by Lemma 1.1 $H(i)_1$ cannot be written as a nontrivial direct sum of subspaces that mutually annihilate each other. Therefore one of $p_i(H'_1)$ and $p_i(H''_1)$ is zero and the other the entire space. Again, applying Lemma 1.1, this says that there is a partition of $\{1, \dots, t\}$ into two disjoint subsets $\{i'_1, \dots, i'_k\}$ and $\{i''_1, \dots, i''_{t-k}\}$ such that

$$\begin{aligned} H' &= H(i'_1)\# \cdots \# H(i'_k) \\ H'' &= H(i''_1)\# \cdots \# H(i''_{t-k}). \end{aligned}$$

This proves the desired result in the case one of the decompositions has only two factors, and an easy induction then yields the general result. \square

This has the following consequence for the Grothendieck group.

COROLLARY 3.2: *The Grothendieck group of standardly graded Poincaré duality algebras of formal dimension $d > 2$ over an arbitrary field \mathbb{F} is a free abelian group with basis the equivalence classes of the indecomposables. \square*

Unfortunately, although the formal structure of the Grothendieck group of standardly graded Poincaré duality algebras of formal dimension $d > 2$ is straightforward, it seems difficult to find a minimal generating set. Certainly it is not finitely generated. For suppose that $H = \mathbb{F}[z_1, \dots, z_n]/I$ is a Poincaré duality algebra. If H were decomposable, then by Lemma 1.1 there would be a pair of complementary subspaces X, Y contained in the space of linear forms such that $X \cdot Y = 0$ in the quotient algebra H . In $\mathbb{F}[z_1, \dots, z_n]$ this would mean that $X \cdot Y \subseteq I$, so if the \mathfrak{m} -primary irreducible ideal $I \subset \mathbb{F}[z_1, \dots, z_n]$ contains no nonzero quadratic

forms, and such ideals exist if $\text{f-dim}(H) = d > 2$, at least one of such a pair of complementary subspaces of linear forms would have to be trivial. Hence H would be indecomposable. The following result provides other forms of indecomposables.

PROPOSITION 3.3: *If H' and H'' are standardly graded Poincaré duality algebras of formal dimensions d' , $d'' > 0$ then their tensor product $H' \otimes H''$ is a Poincaré duality algebra of formal dimension $d' + d''$ which is indecomposable with respect to the connected sum operation.*

PROOF: The first assertion is clear. So suppose one has $M \otimes N = H = P \# Q$ which is both a nontrivial tensor product of the Poincaré duality algebras M and N as well as a nontrivial connected sum of the Poincaré duality algebras P and Q . For $0 \neq m \in M_1$, m is uniquely expressible in the form $m = p + q$ with $p \in P_1$ and $q \in Q_1$. Also $p = m_p + n_p$, $q = m_q + n_q$ are uniquely expressible as sums of elements of M_1 and N_1 . Then $m = p + q = m_p + n_p + m_q + n_q$ gives $m = m_p + m_q$ and $n_p + n_q = 0$, so $m_q = m - m_p$ and $n_p = -n_q$. Next note

$$\begin{aligned} 0 &= pq = (m_p + n_p) \cdot (m_q + n_q) = (m_p + n_p) \cdot (m - m_p - n_p) \\ (\%) \quad &= m_p(m - m_p) - m_p n_p + n_p(m - m_p) - n_p n_p \\ &= m_p(m - m_p) + (m - 2m_p)n_p - n_p n_p \in M_1 \cdot M_1 + M_1 \cdot N_1 + N_1 \cdot N_1. \end{aligned}$$

One has a direct sum decomposition

$$(M \otimes N)_2 = M_2 \oplus (M_1 \otimes N_1) \oplus N_2$$

giving

$$m_p(m - m_p) = 0, \quad n_p n_p = 0$$

so from equation (%) we conclude that

$$0 = (m - 2m_p)n_p.$$

Note $0 = (m - 2m_p)n_p = (m - 2m_p) \otimes n_p \in M_1 \otimes N_1$ if and only if $n_p = 0$ or $m - 2m_p = 0$. In the case $m - 2m_p = 0$, then $m = 2m_p$ so

$$2m_p = m = m_p + n_p$$

which yields that $m_p = n_p$. Therefore

$$0 = (m - 2m_p)n_p = mn_p - 2n_p^2 = mn_p.$$

Since $m \neq 0$ and $mn_p = m \otimes n_p \in M_1 \otimes N_1$ we conclude $n_p = 0$ in this case also. Thus $p = m_p$ and likewise $q = m_q$, so $p, q \in M_1$ which gives $M_1 = (M_1 \cap P_1) \oplus (M_1 \cap Q_1)$. Similarly one has a direct sum decomposition $N_1 = (N_1 \cap P_1) \oplus (N_1 \cap Q_1)$.

Suppose that $M_1 \cap P_1 \neq 0$ and choose $m = m_p \in M_1 \cap P_1$. For any $n \in N_1$ write $n = n_p + n_q$ with $n_p \in N_1 \cap P_1$ and $n_q \in N_1 \cap Q_1$. Then $m_p n_q = 0$ since $P_1 \cdot Q_1 = 0$. Since $m_p \neq 0$ we must have $n_q = 0$. Thus $N_1 \cap Q_1 = 0$ and $N_1 = N_1 \cap P_1$.

Since $0 \neq N_1 = N_1 \cap P_1$ one deduces similarly that $M_1 = M_1 \cap P_1$. But this says that $Q_1 = 0$ contrary to the hypothesis that $H = P \# Q$ is a nontrivial connected sum. \square

We record one more #-indecomposability criterion here: There are others. To formulate it we require an idea we borrow from algebraic topology. Let A be a commutative graded algebra over a field and $X \subset A$ a graded subset. The \times -length¹⁰ of X is the smallest integer $c_X + 1$ such that the product of any $c_X + 1$ elements of X is zero in A if such an integer c_X exists, otherwise we say the \times -length of X is infinite.

¹⁰An algebraic topologist would probably call this the \cup -length (pronounced *cup length*). In topology \cup -length provides a lower bound for the *category* of a topological space, i.e., the number of open subsets, contractible in X , needed to cover it.

PROPOSITION 3.4: *Let H be a standardly graded Poincaré duality algebra of formal dimension d over an arbitrary field \mathbb{F} . Suppose there is a codimension one subspace $V \subset H_1$ of \times -length strictly less than d . Then either*

- (i) H is indecomposable with respect to the connected sum operation $\#$, or
- (ii) H has rank two and $H \cong \mathbb{F}[x, y]/(xy, x^d - y^d) \cong \mathbb{F}[x]/(x^{d+1}) \# (\mathbb{F}[y]/(y^{d+1}))$.

PROOF: Suppose that $H = H' \# H''$ is a nontrivial connected sum. Let the rank of H be r , that of H' be r' , and that of H'' be r'' so $r = r' + r''$. Recall¹¹ the formula from linear algebra relating the dimensions of two subspaces $U', U'' \subseteq U$, viz.,

$$\dim_{\mathbb{F}}(U' + U'') = \dim_{\mathbb{F}}(U') + \dim_{\mathbb{F}}(U'') - \dim_{\mathbb{F}}(U' \cap U'').$$

Apply this to $V, H'_1 \subset H_1$. After rearranging a bit one obtains

$$\dim_{\mathbb{F}}(V + H'_1) + \dim_{\mathbb{F}}(V \cap H'_1) = \dim_{\mathbb{F}}(V) + \dim_{\mathbb{F}}(H'_1) = r - 1 + r'.$$

On the other hand we have the inequality

$$\dim_{\mathbb{F}}(V + H'_1) + \dim_{\mathbb{F}}(V \cap H'_1) \leq r + \dim_{\mathbb{F}}(V \cap H'_1),$$

so

$$r + \dim_{\mathbb{F}}(V \cap H'_1) \geq r + r' - 1$$

whence we conclude that

$$\dim_{\mathbb{F}}(V \cap H'_1) \geq r' - 1.$$

Since $V \cap H'_1 \subset H_1$ the subalgebra of H generated by $V \cap H'_1$ has \times -length at most $d - 1$. If $\dim_{\mathbb{F}}(V \cap H'_1)$ were to equal r' then, since H' is a Poincaré duality algebra of formal dimension d , this would imply that H'_1 was trivial since no product of d elements of H'_1 could be nonzero. Hence $\dim_{\mathbb{F}}(V \cap H'_1) = r' - 1$. This tells us that $V \cap H'_1$ is a codimension one subspace of H'_1 whose \times -length is at most $d - 1$. By symmetry $V \cap H''_1 \subset H''_1$ is also a codimension one subspace of \times -length at most $d - 1$.

Putting these facts together says that $(V \cap H'_1) \oplus (V \cap H''_1) \subset V$ is a codimension one subspace. So we may choose a $v \in V$ that does not belong to this subspace. Write $v = v' + v''$ with $v' \in H'_1$ and $v'' \in H''_1$. Note that $v' \notin V \cap H'_1$: For if it were, then this would say $v'' = v - v' \in V \cap H''_1$ which implies that $v = v' + v''$ belongs to $(V \cap H'_1) \oplus (V \cap H''_1)$ contrary to how we chose v . Therefore $v' \notin V \cap H'_1$ and similarly $v'' \notin V \cap H''_1$.

Retaining these notations we next choose a basis $v_1, \dots, v_{r'-1}$ for $V \cap H'_1$. Note that v' extends this to a basis for H'_1 . Consider a product $v_{i_1} \cdots v_{i_k} \cdot (v')^{d-k}$ of d elements from this basis. One has, for $k > 0$,

$$0 = v_{i_1} \cdots v_{i_k} \cdot v^{d-k} = v_{i_1} \cdots v_{i_k} \cdot (v' + v'')^{d-k} = v_{i_1} \cdots v_{i_k} \cdot (v')^{d-k},$$

since $v'' \in H''_1$ annihilates $v_1, \dots, v_{r'-1} \in H'_1$, and $v_{i_1} \cdots v_{i_k} \cdot v^{d-k}$ is a product of d elements of V which has \times -length at most $d - 1$. Thus the only product of d elements of the basis $v_{i_1}, \dots, v_{i_k}, v'$ for H'_1 that is nonzero is $(v')^d$. Poincaré duality then forces that H' has rank one and is isomorphic to $\mathbb{F}[x]/(x^{d+1})$. Likewise $H'' \cong \mathbb{F}[y]/(y^{d+1})$. Finally one notes that $H = (\mathbb{F}[x]/(x^{d+1})) \# (\mathbb{F}[y]/(y^{d+1}))$ satisfies the hypotheses of the proposition – specifically the subspace spanned by $x + y$ in H_1 has codimension one and \times -length $d - 1$. \square

EXAMPLE 1: Consider the cohomology algebra

$$H^* \left((S^1 \times \cdots \times S^1) \times ((S^1 \times S^1) \# \cdots \# (S^1 \times S^1)); \mathbb{F}_2 \right).$$

$\xleftarrow{d-2}$ \xleftarrow{r}

¹¹ As usual \mathbb{F} denotes the ground field and $\dim_{\mathbb{F}}(\mathbf{-})$ the dimension of the vector space $\mathbf{-}$ over \mathbb{F} .

This is a standardly graded Poincaré duality algebra of formal dimension d and rank $d - 2 + 2r$ which has a codimension one subspace in H_1 of \times -length $d - 1$. To see this one considers the projection map

$$\begin{array}{ccc} S^1 \times \cdots \times S^1 \times ((S^1 \times S^1) \# \cdots \# (S^1 \times S^1)) & & \\ \longleftarrow d-2 \longrightarrow & \longleftarrow r \longrightarrow & \\ & \downarrow & \\ S^1 \times \cdots \times S^1 \times ((S^1 \times S^1) \# \cdots \# (S^1 \times S^1)) & & \\ \longleftarrow d-3 \longrightarrow & \longleftarrow r \longrightarrow & \end{array}$$

and takes the image of the induced map on the first cohomology modules.

For fixed d and distinct r these algebras are not isomorphic since they have different ranks. From Proposition 3.4 they are $\#$ -indecomposable for $d - 2 + 2r > 2$ and therefore one has the following.

FACT: *The Grothendieck group of standardly graded Poincaré duality algebras over a field of characteristic two and formal dimension three or more is not finitely generated.*

For yet other forms of $\#$ -indecomposables we refer to [21] and § 8.

The proofs in this section make clear that the case of formal dimension three might contain the key to systematically constructing indecomposable elements in the Grothendieck group of standardly graded Poincaré duality algebras of any formal dimension $d > 2$. So beginning with the next section we concentrate on the case $d = 3$ and $\mathbb{F} = \mathbb{F}_2$.

§4. Threefolds of Rank Three I (Counting the Number of Isomorphism Classes)

If $H = \mathbb{F}_2[x, y]/I$ is a Poincaré duality quotient algebra of $\mathbb{F}_2[x, y]$ of formal dimension two

$$\begin{array}{c|ccc} \text{cat}(1, 2) & x^2 & y^2 & xy \\ \hline x & a & b & c \\ y & c & d & b \end{array}$$

then one has the matrix of products which determines H up to isomorphism. Notice that the leftmost 2×2 submatrix, which is pictured below, defines a bilinear form $\varphi : H_1 \times H_1 \rightarrow \mathbb{F}_2$

by the rule $\varphi(u, v) = uv^2$ and this bilinear form determines $\text{cat}(1, 2)$ and hence H up to isomorphism. These bilinear forms were classified in [16] Section II.3 in slightly different language with the result that H is isomorphic to one of the following three examples.

$$\begin{array}{c|cc} \cdot & x^2 & y^2 \\ \hline x & a & b \\ y & c & d \end{array}$$

$$\begin{aligned} \mathbb{F}_2[x]/(x^3) &\cong H^*(\mathbb{R}\mathbb{P}(2); \mathbb{F}_2) \\ \mathbb{F}_2[x, y]/(xy, x^2 + y^2) &\cong H^*(\mathbb{R}\mathbb{P}(2) \# \mathbb{R}\mathbb{P}(2); \mathbb{F}_2) \\ \mathbb{F}_2[x, y]/(x^2, y^2) &\cong H^*(S^1 \times S^1; \mathbb{F}_2) \end{aligned}$$

The case of threefolds of formal dimension three is richer and more complicated. We devote the rest of this and the next few sections to their description.

A Poincaré duality algebra $H = \mathbb{F}_2[x, y, z]/I$ of formal dimension three and rank at most three

is completely determined by its matrix of products.¹²

$cat(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	a	b	c	d	g	j
y	d	e	f	b	j	h
z	g	h	i	j	c	f

Here $a, \dots, j \in H_3 = \mathbb{F}_2$. This matrix also defines a bilinear form $\varphi : H_1 \times H_1 \rightarrow \mathbb{F}_2$ by $\varphi(u, v) = uv^2$ after identifying H_3 with \mathbb{F}_2 . This form determines all but one of the entries of $cat(1, 2)$, namely it does not determine $xyz = j$. So one might approach the classification problem for threefolds of rank at most three by first classifying the bilinear forms φ , and then deal with there being two possible algebras for each equivalence class of such forms determined by the different values of j . The value of j itself turns out *not* to be an invariant of H , which makes the problem both more difficult and more interesting.

We employ Macaulay's Double Duality Theorem (see [16] Sections II.2 and VI.1) to put the isomorphism classes of standardly graded Poincaré duality algebras of formal dimension three and rank at most three into bijective correspondence with the orbits of the set of nonzero inverse cubic forms of the action of $GL(3, \mathbb{F}_2)$ on the inverse polynomial algebra¹³ $\mathbb{F}_2[x^{-1}, y^{-1}, z^{-1}]$. The action of $GL(3, \mathbb{F}_2)$ on the space of inverse cubic forms is **not** the linear action one would obtain by regarding $\mathbb{F}_2[x^{-1}, y^{-1}, z^{-1}]$ as a polynomial algebra in the variables x^{-1}, y^{-1}, z^{-1} and extending the natural $GL(3, \mathbb{F}_2)$ action on the three dimensional vector space \mathbb{F}_2^3 with basis x^{-1}, y^{-1}, z^{-1} to the polynomial algebra. Rather, the representation of $GL(3, \mathbb{F}_2)$ on the inverse cubic forms is the dual of the linear action of $GL(3, \mathbb{F}_2)$ on the cubic forms $\mathbb{F}_2[x, y, z]_3$. That these two representations of $GL(3, \mathbb{F}_2)$ are not isomorphic can best be seen by noting that $\mathbb{F}[x, y, z]_3^{GL(3, \mathbb{F}_2)} = 0$, whereas $\mathbb{F}_2[x^{-1}, y^{-1}, z^{-1}]_3^{GL(3, \mathbb{F}_2)}$ contains the nonzero invariant inverse cubic form $x^{-1}y^{-1}z^{-1}$ corresponding to the catalecticant matrix $cat(1, 2)$ all of whose entries are zero except for j which is 1. (See also [16] Section I.6 Example 1 and Section II.3 for further discussion of this point.)

The space of inverse cubic forms has dimension $\binom{5}{3} = 10$ so there are 1023 nonzero such forms. To show that the problem lies within reasonable bounds we make a count of how many orbits there are: It turns out there are 21. In the next section we will list representatives of these orbits and describe the corresponding ideals and Poincaré duality algebras.

NOTATION: For a finite set Y write $|Y|$ for the number of elements in Y . If the group G acts on a set X then X/G is the set of G -orbits and is called the **orbit space**. If $g \in G$ then X^g denotes the subset of X whose elements are fixed by g .

With the aid of the classical Cauchy–Frobenius Lemma we can replace the problem of counting the number of orbits of the action of $GL(3, \mathbb{F}_2)$ on the space of inverse cubic forms $\mathbb{F}_2[x^{-1}, y^{-1}, z^{-1}]_{-3}$ with a collection of problems in invariant theory. First recall that the formula of Cauchy–Frobenius converts the problem of counting orbits of a finite group acting on a finite set into computing the average number of fixed points of the elements of the group. Namely, this formula says (see e.g., [9] §1.1 or [22] volume II page 404): If X is a finite G

¹²For the notations and terminology of catalecticant matrices see [16] Section VI.2.

¹³Ordinary cubic forms are classified in [4]. The action of $GL(3, \mathbb{F}_2)$ involved in this classification is the dual inverse of the one used here.

set and G a finite group then

$$(*) \quad |X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

As it stands this formula is a bit intimidating: After all, in our case there would be 168 summands. However, there are various ways to simplify this expression to avoid redundant computation. For example, one can sum over one representative for each conjugacy class of elements of G since conjugate elements have the same number of fixed points. This simplification is the one we will use. We refer the reader to [9] which devotes several sections to other ways to transform formula (*).

To use these ideas as a means of counting the number of orbits of a finite group G acting linearly on $\mathbb{F}_q[z_1, \dots, z_n]_k$ one needs to assemble the following data.

- (1) A transversal g_1, \dots, g_t for the conjugacy classes of G .
- (2) For each conjugacy class, the number of elements it contains.
- (3) For each element g_i of the transversal, the Poincaré series of the ring of invariants $\mathbb{F}_q[z_1, \dots, z_n]^{\langle g_i \rangle}$ of the cyclic group $\langle g_i \rangle$ generated by g_i up to and including degree k .

We proceed to do this for $G = \mathrm{GL}(3, \mathbb{F}_2)$ in its tautological representation and $k = 3$.

From the equality (*), vector space duality, and the Jordan normal form (for more details see e.g., [16] Lemma I.6.3 and Proposition I.6.4) it follows that the number of orbits of $\mathrm{GL}(3, \mathbb{F}_2)$ acting on the space of inverse cubic forms is the same as the number of orbits of the action on the space of ordinary cubic forms.¹⁴ The Cauchy–Frobenius Formula (*) converts the orbit count into counting the number of fixed points of one representative for each conjugation class. These are problems in invariant theory since the number of fixed points of $g \in \mathrm{GL}(n, \mathbb{F}_q)$ on $\mathbb{F}_q[z_1, \dots, z_n]_k$ is q^d , where $d = \dim_{\mathbb{F}_q}(\mathbb{F}_q[z_1, \dots, z_n]_k^{\langle g \rangle})$ and $\mathbb{F}_q[z_1, \dots, z_n]^{\langle g \rangle}$ is the ring of invariants of the subgroup generated by g .

Since $\mathrm{GL}(3, \mathbb{F}_2)$ is the simple group of order 168 much of what we need concerning its structure can be extracted from [5]. The information about its conjugacy classes is summarized in Table 4.1 to follow. In the table $\omega \in \mathbb{F}_8^\times \cong \mathbb{Z}/7$ is a generator and we have identified $\mathrm{GL}(3, \mathbb{F}_2)$ with a subgroup of $\mathrm{GL}(3, \mathbb{F}_8)$ to write representatives for the conjugacy classes of elements of order seven.¹⁵

The next step is to compute the Poincaré series (at least up to degree three) of the rings of invariants of the cyclic groups generated by the representatives listed in the table.

CASE: χ_1 . The representative for this conjugacy class is the identity matrix, so the invariants are $\mathbb{F}_2[x, y, z]$ and the Poincaré series is

$$P_{\chi_1}(t) = P(\mathbb{F}_2[x, y, z], t) = \frac{1}{(1-t)^3} = 1 + 3t + 6t^2 + 10t^3 + \dots$$

The dimension of the fixed point set on the space of cubic forms is $d_{\chi_1} = 10$.

CASE: χ_2 . The involution that interchanges x with y represents the the conjugacy class χ_2 , so the ring of invariants is $\mathbb{F}_2[x + y, xy, z]$ and the Poincaré series

$$P_{\chi_2}(t) = P(\mathbb{F}_2[x, y, z]^{\mathbb{Z}/2}, t) = \frac{1}{(1-t)^2(1-t^2)} = 1 + 2t + 4t^2 + 6t^3 + \dots$$

¹⁴We emphasize this holds for the number of orbits and *not* their individual sizes. See e.g., Example 1 in Section I.6 of [16].

¹⁵These are sometimes called **Singer cycles**.

χ - class	order	# elements	representative
χ_1	1	1	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
χ_2	2	21	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
χ_3	3	56	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
χ_4	4	42	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$
χ_7^+	7	24	$\begin{bmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^4 \end{bmatrix}$
χ_7^-	7	24	$\begin{bmatrix} \omega^6 & 0 & 0 \\ 0 & \omega^5 & 0 \\ 0 & 0 & \omega^3 \end{bmatrix}$

 TABLE 4.1: Conjugacy Classes of $\text{GL}(3, \mathbb{F}_2)$

The dimension of the fixed point set on the space of cubic forms is $d_{\chi_2} = 6$.

CASE: χ_3 . The action of the cyclic group of order three generated by the representing matrix is as the alternating group A_3 whose invariants are a complete intersection algebra generated by the four forms e_1, e_2, e_3 , and ∇ where, for $i = 1, 2, 3$ the form e_i is the i -th elementary symmetric polynomial in x, y, z , and ∇ may be taken to be $x^2y + y^2z + z^2x$ (see e.g., [18] Chapter 4 Section 2 Example 2). The Poincaré series of this algebra is

$$P_{\chi_3}(t) = P(\mathbb{F}_2[x, y, z]^{A_3}, t) = \frac{1 + t^6}{(1-t)(1-t^2)(1-t^3)^2} 1 + t + 2t^2 + 4t^3 + \dots,$$

so the dimension of the fixed point set on the space of cubic forms is $d_{\chi_2} = 4$.

CASE: χ_4 . The action of the 4-cycle representing this conjugacy class is the full Jordan block of size 3×3 . The invariants were computed in [17] from which one sees that the Poincaré series is

$$P_{\chi_4}(t) = P(\mathbb{F}_2[x, y, z]^{\mathbb{Z}/4}, t) = \frac{1 + t^6}{(1-t)(1-t^2)(1-t^3)(1-t^4)} = 1 + t + 2t^2 + 3t^3 + \dots,$$

so the dimension of the fixed point set on the space of cubic forms is $d_{\chi_4} = 3$.

CASE: χ_7^+ . Since we are only interested in the Poincaré series (and that only up to degree three) we may extend the ground field from \mathbb{F}_2 to \mathbb{F}_8 . Doing so, we have a diagonal

action which sends monomials to monomials, viz., the representing matrix maps $x^a y^b z^c$ to $\omega^{a+2b+4c} x^a y^b z^c$. So we see that the ring of invariants has an \mathbb{F}_8 -basis consisting of monomials $x^a y^b z^c$ where $a + 2b + 4c \equiv 0 \pmod{7}$. If we interpret ω from the representing matrix in the Table 4.1 as a primitive 7-th root of unity in the complex numbers \mathbb{C} , we obtain a characteristic zero lift of the \mathbb{F}_8 -representation. By Molien's Theorem the Poincaré series is

$$\begin{aligned} P_{\chi_7^+}(t) &= P(\mathbb{F}_2[x, y, z]^{\mathbb{Z}/7}, t) = P(\mathbb{F}_8[x, y, z]^{\mathbb{Z}/7}, t) = P(\mathbb{C}[x, y, z]^{\mathbb{Z}/7}, t) \\ &= \frac{1}{(1 - \omega t)(1 - \omega^2 t)(1 - \omega^4 t)} = 1 + t^3 + \dots, \end{aligned}$$

and the dimension of the fixed point set on the space of cubic forms is $d_{\chi_7^+} = 1$.

CASE: χ_7^- . The representing matrix is the inverse of the representing matrix for χ_7^+ so the rings of invariants coincide and the dimension of the fixed point set on the space of cubic forms is $d_{\chi_7^-} = 1$.

For the number of orbits of $\mathrm{GL}(3, \mathbb{F}_2)$ acting on the space of cubic forms in $\mathbb{F}_2[x, y, z]$ we therefore obtain from the Cauchy–Frobenius Formula

$$\begin{aligned} |\mathbb{F}_2[x, y, z]_3 / \mathrm{GL}(3, \mathbb{F}_2)| &= \\ &= \frac{1}{168} (|\chi_1| \cdot d_{\chi_1} + |\chi_2| \cdot d_{\chi_2} + |\chi_3| \cdot d_{\chi_3} + |\chi_4| \cdot d_{\chi_4} + |\chi_7^+| \cdot d_{\chi_7^+} + |\chi_7^-| \cdot d_{\chi_7^-}) \\ &= \frac{1}{168} (1 \cdot 1024 + 21 \cdot 64 + 56 \cdot 16 + 42 \cdot 8 + 24 \cdot 2 + 24 \cdot 2) = \frac{3696}{168} = 22, \end{aligned}$$

where we have denoted the number of elements in the conjugacy class χ by $|\chi|$. Of these orbits one consists of the *zero* form of degree three, and this means we have proven the following result.

PROPOSITION 4.1: *There are 21 isomorphism classes of standardly graded Poincaré duality algebras of formal dimension three and rank at most three. \square*

The next step is to list representatives of the orbits and delineate the structure of the corresponding Poincaré duality algebras.

§5. Threefolds of Rank Three II (The Isomorphism Classes)

In the previous section we saw that there are exactly 21 distinct standardly graded Poincaré duality algebras of formal dimension three and rank at most three. We did so by using Macaulay's Double Duality Theorem which allowed us to reduce this problem to several invariant theoretic problems that put together gave us a formula for the number of orbits of $\mathrm{GL}(3, \mathbb{F}_2)$ acting on the space of inverse cubic forms $\mathbb{F}_2[x^{-1}, y^{-1}, z^{-1}]_{-3}$. Instead of trying to find representative inverse cubic forms for the orbits we choose a different tack to describe the 21 different standardly graded Poincaré duality algebras of formal dimension three and rank at most three.

Recall from § 4 (see [16] Part VI Section 2) that associated to each standardly graded Poincaré duality algebra $H = \mathbb{F}_2[x, y, z]/I$ of formal dimension three there is a catalecticant matrix $\mathrm{cat}_H(1, 2)$ encoding the products between linear and quadratic forms: It is a 3×6 matrix of

the form

$$(\ast) \quad \begin{array}{c|cccccc}
 \text{cat}_H(1, 2) & x^2 & y^2 & z^2 & xy & xz & yz \\
 \hline
 x & a & b & c & d & g & j \\
 y & d & e & f & b & j & h \\
 z & g & h & i & j & c & f
 \end{array}$$

where the entries a, \dots, j are from \mathbb{F}_2 , and take the value 1 precisely when the product of the linear form heading the row with the quadratic form heading the column is nonzero in the Poincaré duality algebra. This matrix determines H up to isomorphism. It depends of course on the choice of an ordered basis for the linear forms and an ordering of the monomials. If H' and H'' are standardly graded Poincaré duality algebras of formal dimension three and rank at most three then they are isomorphic if and only if there is an element $g \in \text{GL}(3, \mathbb{F}_2)$ such that $g \cdot \text{cat}_{H'}(1, 2) \cdot g^{\text{tr}} = \text{cat}_{H''}(1, 2)$.

Let us rephrase this a bit. Note that the set of all the matrices (\ast) forms a 10-dimensional vector space $\text{Cat}_{\mathbb{F}_2}(1, 2)$ over \mathbb{F}_2 which consists of the zero 3×6 matrix and the catalecticant matrices $\text{cat}_{\theta}(1, 2)$ where θ ranges over the nonzero inverse ternary cubic forms. The group $\text{GL}(3, \mathbb{F}_2)$ acts on the space $\text{Mat}_{\mathbb{F}_2}(3, 6)$ of 3×6 matrices over \mathbb{F}_2 preserving $\text{Cat}_{\mathbb{F}_2}(1, 2)$ by letting $g \in \text{GL}(3, \mathbb{F}_2)$ send \mathbf{M} into the matrix product $g \cdot \mathbf{M} \cdot g^{\text{tr}}$. This representation of $\text{GL}(3, \mathbb{F}_2)$ on $\text{Cat}_{\mathbb{F}_2}(1, 2)$ is isomorphic to the representation of $\text{GL}(3, \mathbb{F}_2)$ on the space of inverse cubic forms $\mathbb{F}_2[x^{-1}, y^{-1}, z^{-1}]_{-3}$. The way we choose to describe the 21 different isomorphism classes of standardly graded Poincaré duality algebras of formal dimension three and rank at most three is to give representing matrices for the orbits of the action of $\text{GL}(3, \mathbb{F}_2)$ on $\text{Cat}_{\mathbb{F}_2}(1, 2)$.

The leftmost 3×3 submatrix of $\text{cat}_H(1, 2)$ is the matrix of the bilinear form $\varphi_H : H_1 \times H_1 \rightarrow \mathbb{F}_2$ defined by $\varphi_H(u, v) = uv^2$. This bilinear form determines all but one of the entries of the matrix $\text{cat}_H(1, 2)$. Our strategy will be to first classify these bilinear forms, and then, to each equivalence class associate two catalecticant matrices corresponding to the possible values of $j \in \mathbb{F}_2$. We warn the reader in advance that the value of j is *not* an invariant of H (see the comments following Proposition 5.1). The first step is to determine the orbits of $\text{GL}(3, \mathbb{F}_2)$ acting on $\text{Mat}_{\mathbb{F}_2}(3, 3)$ by the **transposition action** given by letting $g \in \text{GL}(3, \mathbb{F}_2)$ send $\mathbf{M} \in \text{Mat}_{\mathbb{F}_2}(3, 3)$ into the matrix product $g \cdot \mathbf{M} \cdot g^{\text{tr}}$.

The next proposition tells us there are 12 such orbits. The information needed for its proof by means of the Cauchy–Frobenius formula is summarized in Table 5.1. We leave the verifications of the entries in this table to the reader, which amount to some exercises in matrix multiplication and linear algebra to find the linear subspace fixed by the representative of each conjugacy class, and then exponentiating to obtain the number of fixed points.

PROPOSITION 5.1: *There are 12 orbits of the action of $\text{GL}(3, \mathbb{F}_2)$ on $\text{Mat}_{\mathbb{F}_2}(3, 3)$ given by letting $g \in \text{GL}(3, \mathbb{F}_2)$ send $\mathbf{M} \in \text{Mat}_{\mathbb{F}_2}(3, 3)$ into the matrix product $g \cdot \mathbf{M} \cdot g^{\text{tr}}$.*

PROOF: We employ the Cauchy–Frobenius Formula (\ast) from § 4 and count the number of fixed points for one element out of each conjugacy class of elements in $\text{GL}(3, \mathbb{F}_2)$. A list of conjugacy classes and representatives for them is given in Table 4.1. So the number of orbits may be computed using Table 5.1 as follows:

$$\begin{aligned}
 |\text{Mat}_{\mathbb{F}_2}(3, 3)/\text{GL}(3, \mathbb{F}_2)| &= \frac{1}{168} (1 \cdot 512 + 21 \cdot 32 + 56 \cdot 8 + 42 \cdot 8 + 24 \cdot 1 + 24 \cdot 1) \\
 &= \frac{1}{168} (2016) = 12. \quad \square
 \end{aligned}$$

χ -class	Fixed Matrices	#Fixed Matrices
χ_1	all matrices	$2^9 = 512$
χ_2	$\begin{bmatrix} a & b & c \\ b & a & c \\ g & g & i \end{bmatrix}$	$2^5 = 32$
χ_3	$\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$	$2^3 = 8$
χ_4	$\begin{bmatrix} a & b & c \\ b+c & c & 0 \\ c & 0 & 0 \end{bmatrix}$	$2^3 = 8$
χ_7^+	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$2^0 = 1$
χ_7^-	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$2^0 = 1$

TABLE 5.1: Fixed Point Data

Since there are 22 orbits of $GL(3, \mathbb{F}_2)$ acting on the vector space $Cat_{\mathbb{F}_2}(1, 2)$ but 12 orbits on the space $Mat_{\mathbb{F}_2}(3, 3)$ it must be the case that choosing distinct values for j to extend an orbit of $Mat_{\mathbb{F}_2}(3, 3)$ to $Cat_{\mathbb{F}_2}(1, 2)$ can lead to the same orbit of $Cat_{\mathbb{F}_2}(1, 2)$. In other words, as already remarked the value of j is not an invariant of the orbit of the catalecticant matrix $(*)$ (see e.g., the discussion below of extending Orbit 1 from $Mat_{\mathbb{F}_2}(3, 3)$ by $j = 1$ to $Cat_{\mathbb{F}_2}(1, 2)$).

We list representatives for the orbits of $GL(3, \mathbb{F}_2)$ acting on $Mat_{\mathbb{F}_2}(3, 3)$ and their invariants in Table 5.2. We will make use of three invariants to distinguish the twelve orbits. These invariants are explained following the table. Table 4.1 shows there are six conjugacy classes in $GL(3, \mathbb{F}_2)$ and by contrast Table 5.2 shows there are only four orbits of $GL(3, \mathbb{F}_2)$ acting via the transposition action on the invertible matrices in $Mat_{\mathbb{F}_2}(3, 3)$ so the coincidences of the 2×2 case that occurred in [20] are not repeated in the 3×3 case, and the invariants of the conjugation action on $Mat_{\mathbb{F}_2}(3, 3)$ of $GL(3, \mathbb{F}_2)$ will be quite distinct from those of the transposition action.

There are three invariants used to distinguish the orbits of $GL(3, \mathbb{F}_2)$ on the space $Mat_{\mathbb{F}_2}(3, 3)$ as we explain next. A 3×3 matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \in Mat_{\mathbb{F}_2}(3, 3),$$

has two extensions to a 3×6 catalecticant matrix $(*)$, which depend on a choice of $j \in \mathbb{F}_2$. Such a 3×6 matrix defines a Poincaré duality algebra of formal dimension three and rank at

ORBIT 1: $\left\{ \begin{array}{l} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ s\text{-rank} = 3 \\ c\text{-invariant} = 4 \\ \text{symmetric} \end{array} \right.$	ORBIT 7: $\left\{ \begin{array}{l} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ s\text{-rank} = 2 \\ c\text{-invariant} = 2 \\ \text{asymmetric} \end{array} \right.$
ORBIT 2: $\left\{ \begin{array}{l} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ s\text{-rank} = 3 \\ c\text{-invariant} = 4 \\ \text{asymmetric} \end{array} \right.$	ORBIT 8: $\left\{ \begin{array}{l} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ s\text{-rank} = 2 \\ c\text{-invariant} = 4 \\ \text{asymmetric} \end{array} \right.$
ORBIT 3: $\left\{ \begin{array}{l} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ s\text{-rank} = 3 \\ c\text{-invariant} = 6 \\ \text{asymmetric} \end{array} \right.$	ORBIT 9: $\left\{ \begin{array}{l} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ s\text{-rank} = 2 \\ c\text{-invariant} = 0 \\ \text{symmetric} \end{array} \right.$
ORBIT 4: $\left\{ \begin{array}{l} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ s\text{-rank} = 3 \\ c\text{-invariant} = 2 \\ \text{asymmetric} \end{array} \right.$	ORBIT 10: $\left\{ \begin{array}{l} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ s\text{-rank} = 1 \\ c\text{-invariant} = 4 \\ \text{symmetric} \end{array} \right.$
ORBIT 5: $\left\{ \begin{array}{l} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ s\text{-rank} = 2 \\ c\text{-invariant} = 4 \\ \text{symmetric} \end{array} \right.$	ORBIT 11: $\left\{ \begin{array}{l} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ s\text{-rank} = 1 \\ c\text{-invariant} = 2 \\ \text{asymmetric} \end{array} \right.$
ORBIT 6: $\left\{ \begin{array}{l} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ s\text{-rank} = 2 \\ c\text{-invariant} = 6 \\ \text{asymmetric} \end{array} \right.$	ORBIT 12: $\left\{ \begin{array}{l} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ s\text{-rank} = 0 \\ c\text{-invariant} = 0 \\ \text{symmetric} \end{array} \right.$

 TABLE 5.2: $GL(3, \mathbb{F}_2)$ Orbits on $\text{Mat}_{\mathbb{F}_2}(3, 3)$

most three. The invariants we use are as follows: They do not depend on the choice of j .

s-Rank: This is the dimension of the image of the squaring map from linear to quadratic forms in the corresponding Poincaré duality algebra. As \mathbb{F}_2 is the ground field, the set of squares of linear forms is a vector subspace of the space of quadratic forms and the s -rank is its dimension. It is nothing but the ordinary rank of the leftmost 3×3 matrix of the corresponding catalecticant matrix, i.e., it is the rank of the bilinear form defined on the linear forms by $\varphi(u, v) = uv^2$.

c-Invariant: This is the number of nonzero cubes of linear forms in the corresponding Poincaré duality algebra and is independent of the choice of $j \in \mathbb{F}_2$.

Symmetry: The meaning of this is clear if applied to a 3×3 matrix, and for a catalecticant matrix it refers to the leftmost 3×3 submatrix and whether it is symmetric or not.

At this point our program is clear, if not easy. For each orbit of the $GL(3, \mathbb{F}_2)$ -action on $\text{Mat}_{\mathbb{F}_2}(3, 3)$ there are two catalecticant matrices differing in their j -values in (*) extending a 3×3 matrix representing one of the orbits in Table 5.2. These two extensions each define a Poincaré duality algebra, and these two algebras may differ or they may coincide: In fact, since there are only 21 distinct algebras and 23 choices¹⁶ for the catalecticant matrices

¹⁶Orbit 12 can only be extended by $j = 1$ since the zero matrix is not a catalecticant matrix.

obtained in this way there must be some coincidences.

We are going to need additional invariants to classify the 21 algebras. These new invariants must distinguish between algebras with the same s -rank, c -invariant, and symmetry/asymmetry property, i.e., between two algebras whose catalecticant matrices differ only in their j -value, if those algebras are not isomorphic: But, we cannot use j itself.

We will of course want to describe these algebras by giving a minimal set of generators for the corresponding ideals. Before starting we should perhaps look at what to expect. The Poincaré algebras we are seeking to classify are completely described by their catalecticant matrices $(*)$. If the rank of this matrix is 1, then one of the rows will be nonzero and the other rows will be zero or that row repeated. There will then be two linear relations and the algebra H will be isomorphic to $\mathbb{F}[u]/(u^4)$ which means there is an additional relation of degree 4. Thus the ideal is generated by forms of degrees 1, 1, and 4.

If the rank of $(*)$ is 2, then there will be a single relation of degree 1 and H will be isomorphic to $\mathbb{F}[u, v]/J$. By a result due to F. S. Macaulay (see [12], or [24] for a modern generalization) J will be a regular ideal, so will have two generators, say of degrees a and b . Since H has formal dimension 3 we must have $a + b - 2 = 3$ and, since we may suppose $1 < a \leq b$, there is only the solution $a = 2$ and $b = 3$. So the ideal is generated by forms of degrees 1, 2, and 3.

If the rank of $(*)$ is 3, then there are no linear relations so $\dim_{\mathbb{F}_2}(H_1) = 3$ and Poincaré duality implies that $\dim_{\mathbb{F}_2}(H_2) = 3$ also. Since $\dim_{\mathbb{F}_2}(\mathbb{F}_2[x, y, z]_2) = 6$ there must be 3 linearly independent quadratic relations, say f_1, f_2, f_3 . If the ideal is regular these will be a minimal generating set. If not, then in degree 3 the ideal they generate is spanned by the nine forms xf_i, yf_i, zf_i , where $i = 1, 2, 3$, but this subspace of the cubic forms has dimension at most 8, so there are linear relations between these nine forms. Note that $\mathbb{F}_2[x, y, z]_3$ has dimension 10. So one must add additional cubic forms to the set $\{f_1, f_2, f_3\}$ to raise the dimension of the homogeneous component of the ideal they generate in degree 3 to 9. Having done this one might need to add biquadratic forms in addition to force the homogeneous component of degree 4 of the ideal generated by the entirety of these forms to coincide with $\mathbb{F}_2[x, y, z]_4$. At this point we are done: We have found all the generators. Thus we would be looking for 3 quadratic generators, some cubic generators, and possibly some biquadratic generators. The case of $H^*(\mathbb{RP}(3)\#(\mathbb{RP}(2) \times \mathbb{RP}(1)); \mathbb{F}_2)$ (see orbit 13 in the tables that follow) requires all of these: In this case the ideal is generated by (x^4, y^3, z^2, xy, xz) . A final useful fact in the rank 3 case is that the minimal number of generators must be odd (see [25]). Having seen the patterns to expect we turn to the examples. In construction of the table we will make use of the following invariants in addition to those already introduced.

Dimension Sequences: We divide the seven nonzero elements $u \in H_1$ into two groups, those with $u^3 = 0$ and those with $u^3 \neq 0$. For each element $u \in H_1$ we compute the dimension of the image of left multiplication by u from H_1 to H_2 and obtain a list with seven entries which we divide into two parts, viz., $(-\cdots-)$, $(-\cdots-)$, those with $u^3 = 0$ and those with $u^3 \neq 0$, so the length of the second sequence is the c -invariant. We call these the **dimension sequences** and arrange the entries in nondecreasing order. If the two lists differ for two algebras with the same s -rank, c -invariant, and symmetry/asymmetry, i.e., for two algebras whose catalecticant matrices differ only in their j -value, then the algebras are not isomorphic. If the two lists coincide this may help us to find an automorphism that carries one catalecticant matrix into the other. Such an isomorphism must pair elements with the same cube and same $\dim_{\mathbb{F}_2}(u \cdot H_1)$.

Rank: For algebras with s -rank < 3 there may be only one choice of $j \in \mathbb{F}_2$ giving a threefold

of rank three.

\mathcal{A}^* -Action: The presence or absence of an unstable \mathcal{A}^* -algebra structure. The \mathcal{A}^* -algebra structure may help to find an isomorphism between algebras with distinct j values since the Wu classes must be preserved. It may also help us to distinguish them, if for example only one value of j leads to an unstable \mathcal{A}^* -algebra structure. Both situations arise. By Wu's Formula (see e.g. [23]), to check if a catalecticant matrix defines an algebra with an unstable \mathcal{A}^* -structure one writes under the catalecticant matrix the row vector

$$\text{Sq}^1(x^2, y^2, z^2, xy, xz, yz) = (0, 0, 0, \alpha_{xy}, \alpha_{xz}, \alpha_{yz})$$

where $\alpha_{uv} \neq 0$ precisely when $\text{Sq}^1(uv)$ is nonzero in the quotient algebra. This vector is a linear combination of rows of the catalecticant matrix if and only if the corresponding quotient algebra has an unstable \mathcal{A}^* -structure.

The following lemma tells us that in the symmetric case one always has an unstable \mathcal{A}^* -module structure.

LEMMA 5.2: *Let $H = \mathbb{F}_2[x, y, z]/I$ be a Poincaré duality algebra of formal dimension three with a symmetric catalecticant matrix. Then $I \subset \mathbb{F}_2[x, y, z]$ is an \mathcal{A}^* -invariant ideal and the algebra H has trivial Wu class.*

PROOF: Since I contains all forms of degree 4 or more we need only show that $\text{Sq}^1(h) \in I$ for every quadratic form $h \in I$. Let

$$h = ax^2 + by^2 + cz^2 + dxy + exz + fyz \in \mathbb{F}_2[x, y, z]$$

be any quadratic form. Then

$$\text{Sq}^1(h) = d(2xy + xy^2) + e(x^2z + xz^2) + f(y^2z + yz^2).$$

The symmetry of the catalecticant matrix tells us that

$$x^2y + xy^2, x^2z + xz^2, y^2z + yz^2 \in I$$

and therefore $\text{Sq}^1(h) \in I$ for any quadratic form. \square

We proceed to list the 21 different orbits of $\text{GL}(3, \mathbb{F}_2)$ on $\text{Cat}_{\mathbb{F}_2}(1, 2)$, the corresponding \mathfrak{m} -primary irreducible ideals in $\mathbb{F}_2[x, y, z]$, and Poincaré duality quotients. Due to the extensive amount of material included (and also omitted) we will not be able to give many details. In this list we use the notation $\lambda \downarrow \mathbb{R}\mathbb{P}(k)$ for the canonical line bundle over $\mathbb{R}\mathbb{P}(k)$ (remember $\mathbb{R}\mathbb{P}(1) = S^1$) and $\tau \downarrow \mathbb{R}\mathbb{P}(2)$ for the tangent bundle of $\mathbb{R}\mathbb{P}(2)$. The trivial k -plane bundle over the space X is denoted by $\mathbb{R}^k \downarrow X$.

Subgroups of $\text{GL}(3, \mathbb{F}_2)$ occurring in the list are Σ_4 , the symmetric group on 4 letters acting on \mathbb{F}_2^3 identified with the subrepresentation of the permutation representation on \mathbb{F}_2^4 spanned by the vectors with coordinate sum zero; Σ_3 as the permutation representation of \mathbb{F}_2^3 and its alternating subgroup A_3 ; $\text{Syl}_2(\text{GL}(3, \mathbb{F}_2))$ which is the 2-Sylow subgroup of $\text{GL}(3, \mathbb{F}_2)$ and is isomorphic to a dihedral group of order 8; $K \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ the elementary abelian 2-group of order 4 that occurs as the subgroup fixing a hyperplane pointwise; D_{12} a dihedral group of order 12 occurring as an extension of K by a three cycle; and various subgroups generated by a single involution, so isomorphic to $\mathbb{Z}/2$.

An inverse cubic form defining the i -th orbit is denoted by θ_i and the ideal it defines by $I(\theta_i) \subset \mathbb{F}_2[x, y, z]$.

Orbit 1 of $Cat_{\mathbb{F}_2}(1, 2)$

$cat_{\theta_1}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	1	0	0	0	0	0
y	0	1	0	0	0	0
z	0	0	1	0	0	0

symmetric

isotropy group = Σ_3

orbit size = 28

Orbit 1 of $Mat_{\mathbb{F}_2}(3, 3)$ extended by $j = 0$

rank = 3

s - rank = 3

c - invariant = 4

dimension sequences = (2, 2, 2), (1, 1, 1, 1, 3)

$\theta_1 = x^{-3} + y^{-3} + z^{-3}$

$I(\theta_1) = (z^3 - x^3, z^3 - y^3, xy, xz, yz)$

is an \mathcal{A}^* algebra realized by

$$H^*(\mathbb{RP}(3) \# \mathbb{RP}(3) \# \mathbb{RP}(3); \mathbb{F}_2)$$

Orbit 2 of $Cat_{\mathbb{F}_2}(1, 2)$

$cat_{\theta_2}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	1	0	0	0	0	1
y	0	1	0	0	1	0
z	0	0	1	1	0	0

symmetric

isotropy group = Σ_3

orbit size = 28

Orbit 1 of $Mat_{\mathbb{F}_2}(3, 3)$ extended by $j = 1$

rank = 3

s - rank = 3

c - invariant = 4

dimension sequences = (2, 2, 2), (1, 3, 3, 3)

$\theta_2 = x^{-3} + y^{-3} + z^{-3} + x^{-1}y^{-1}z^{-1}$

$I(\theta_2) = (x^2 + yz, y^2 + xz, z^2 + xy, x^2y, xy^2)$

is an \mathcal{A}^* algebra realized by

$$H^*(\mathbb{RP}(3) \# \mathbb{RP}(\tau \downarrow \mathbb{RP}(2)); \mathbb{F}_2)$$

We give some details to show the Poincaré duality algebra corresponding to this catalecticant matrix is realizable as the \mathbb{F}_2 -cohomology of the indicated topological space. First, change bases by setting $u = x + y + z$, $v = x + y$, $w = x + z$ so the catalecticant matrix becomes

$cat_{\theta'_2}(1, 2)$	u^2	v^2	w^2	uv	uw	vw
u	1	0	0	0	0	0
v	0	0	1	0	0	1
w	0	1	0	0	0	1

whose j value is zero. So this orbit contains matrices with distinct j -values showing that j is not an orbit invariant. Next, using this new basis and catalecticant matrix, note that the algebra is the connected sum of $\mathbb{F}_2[u]/(u^4)$ and the algebra of coinvariants $\mathbb{F}_2[v, w]_{GL(2, \mathbb{F}_2)}$ of the group $GL(2, \mathbb{F}_2)$, called the Dickson coinvariants of rank 2. The Dickson coinvariants are known (see e.g., [15]) to be realized as the cohomology of the projective space bundle associated to the tangent bundle $\tau \downarrow \mathbb{RP}(2)$. Denoting this projective space bundle by $\mathbb{RP}(\tau \downarrow \mathbb{RP}(2))$ the \mathbb{F}_2 -cohomology of the connected sum $\mathbb{RP}(3) \# \mathbb{RP}(\tau \downarrow \mathbb{RP}(2))$ realizes the algebra associated to this orbit as a cohomology algebra.

We will see further examples of such constructions involving projective space bundles in connection with realizing other Poincaré algebras as cohomology algebras. For the algebra in back of the projective bundle construction and its uses in constructing \mathfrak{m} -primary irreducible ideals in polynomial algebras see [21].

Orbit 3 of $Cat_{\mathbb{F}_2}(1, 2)$

$cat_{\theta_3}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	1	1	0	0	0	0
y	0	1	0	1	0	0
z	0	0	1	0	0	0

asymmetric

 isotropy group = A_3

orbit size = 56

 Orbit 2 of $Mat_{\mathbb{F}_2}(3, 3)$ extended by $j = 0$

rank = 3

 s -rank = 3

 c -invariant = 4

 dimension sequences = $(3, 3, 3), (1, 2, 2, 2)$
 $\theta_3 = x^{-3} + x^{-1}y^{-2} + y^{-3} + z^{-3}$
 $I(\theta_3) = (x^2 + xy + y^2, x^2y, xz, yz, z^3 + xy^2)$

 is not an \mathcal{A}^* algebra

The algebra corresponding to this orbit is the connected sum of $\mathbb{F}[z]/(z^4) = H^*(\mathbb{RP}(3); \mathbb{F}_2)$ and the rank two algebra with Macaulay inverse $x^{-3} + y^{-3} + x^{-1}y^{-2}$ which is not an \mathcal{A}^* -algebra.

 Orbit 4 of $Cat_{\mathbb{F}_2}(1, 2)$

$cat_{\theta_4}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	1	1	0	0	0	1
y	0	1	0	1	1	0
z	0	0	1	1	0	0

asymmetric

 isotropy group = A_3

orbit size = 56

 Orbit 2 of $Mat_{\mathbb{F}_2}(3, 3)$ extended by $j = 1$

rank = 3

 s -rank = 3

 c -invariant = 4

 dimension sequences = $(3, 3, 3), (3, 3, 3, 3)$
 $\theta_4 = x^{-3} + x^{-1}y^{-2} + y^{-3} + z^{-3} + x^{-1}y^{-1}z^{-1}$
 $I(\theta_4) = (x^2 + yz, x^2 + y^2 + xz, x^2 + y^2 + z^2 + xy, x^y, x^2z)$

 is not an \mathcal{A}^* algebra

 Orbit 5 of $Cat_{\mathbb{F}_2}(1, 2)$

$cat_{\theta_5}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	1	1	0	0	0	j
y	0	1	1	1	j	0
z	0	0	1	j	0	1

asymmetric

 isotropy group = $\mathbb{Z}/2$

orbit size = 84

 Orbit 3 of $Mat_{\mathbb{F}_2}(3, 3)$ extended by $j = 0$ or 1

rank = 3

 s -rank = 3

 c -invariant = 6

 dimension sequences = $(2), (2, 2, 2, 3, 3, 3)$
 $\theta_5 = x^{-3} + y^{-3} + z^{-3} + x^{-1}y^{-2} + y^{-1} + z^{-2}$
 $I(\theta_5) = (xz, z^2 + yz, x^3 + y^3, x^3 + z^3, z^3 + xy^2)$

 is not an \mathcal{A}^* algebra

Both values of j may be used to extend the matrix representing orbit 3 on $Mat_{\mathbb{F}_2}(3, 3)$ to a representative for orbit 5 on $Cat_{\mathbb{F}_2}(1, 2)$, demonstrating again that the value of j is not an orbit invariant. The generators for the ideal were computed for $j = 0$. The isotropy group of this orbit is $\mathbb{Z}/2$ generated by the involution

$$x \mapsto x + y, y \mapsto y, z \mapsto y + z$$

so the orbit contains 84 elements. Since the dimension sequences are $(2), (2, 2, 2, 3, 3, 3)$ an isomorphism between the algebra defined by $j = 0$ with the algebra defined by $j = 1$ must send the unique element u with $u^3 = 0$ to itself, so $x + z$ must be fixed. Further, the three elements in H_1 for $j = 0$ with $u^3 \neq 0$ and $\dim_{\mathbb{F}_2}(u \cdot H_1(j = 0)) = 2$, must go to the corresponding elements in H_1 for $j = 1$. Thus $\{x, z, y + z\}$ for $j = 0$ must go to $\{y, x + y, x + y + z\}$ for $j = 1$, and $\{x, z\}$ with sum $x + z$ must become $\{y, x + y + z\}$ also with sum $x + z$. A linear map implementing these correspondences is

$$x \mapsto y, y \mapsto z, z \mapsto x + y + z.$$

This ideal is not \mathcal{A}^* -invariant because $\text{Sq}^1(z^2 + yz) = y^2z + yz^2$ and $\theta_5(y^2z + yz^2) = 1 + 0 \neq 0$ in H , but $z^2 + yz \in I(\theta_5)$.

Orbit 6 of $\text{Cat}_{\mathbb{F}_2}(1, 2)$

$\text{cat}_{\theta_6}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	1	1	0	0	0	0
y	0	0	1	1	0	1
z	0	1	0	0	0	1

asymmetric

isotropy group = $\mathbb{Z}/2$

orbit size = 84

Orbit 4 of $\text{Mat}_{\mathbb{F}_2}(3, 3)$ extended by $j = 0$ or 1
rank = 3

s -rank = 3

c -invariant = 2

dimension sequences = $(2, 3, 3, 3, 3), (2, 3)$

$\theta_6 = x^{-3} + x^{-1}y^{-2} + y^{-1}z^{-2} + y^{-2}z^{-1}$

$I(\theta_6) = (xz, z^2 + xy, x^2 + y^2 + z^2 + yz)$

is not an \mathcal{A}^* algebra

A linear change of coordinates (cf. the discussion of the previous orbit for an explanation of how one arrives at such a map) sending the algebra for $j = 0$ isomorphically to the algebra for $j = 1$ is given by $x \mapsto x + z, y \mapsto x + y, z \mapsto z$.

Orbit 7 of $\text{Cat}_{\mathbb{F}_2}(1, 2)$

$\text{cat}_{\theta_7}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	1	0	0	0	0	0
y	0	1	0	1	0	0
z	0	0	0	0	0	0

symmetric

isotropy group = $\text{Syl}_2(\text{GL}(3, \mathbb{F}_2))$

orbit size = 21

Orbit 5 of $\text{Mat}_{\mathbb{F}_2}(3, 3)$ extended by $j = 0$

rank = 2

s -rank = 2

c -invariant = 4

dimension sequences = $(2, 2, 2), (3, 3, 3, 3)$

$\theta_7 = x^{-3} + y^{-3}$

$I(\theta_7) = (z, xy, x^3 + y^3)$

is an \mathcal{A}^* algebra realized by

$H^*(\mathbb{RP}(3) \# \mathbb{RP}(3); \mathbb{F}_2)$

Orbit 8 of $\text{Cat}_{\mathbb{F}_2}(1, 2)$

$\text{cat}_{\theta_8}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	1	0	0	0	0	1
y	0	1	0	0	1	0
z	0	0	0	1	0	0

symmetric

isotropy group = $\text{Syl}_2(\text{GL}(3, \mathbb{F}_2))$

orbit size = 21

Orbit 5 of $\text{Mat}_{\mathbb{F}_2}(3, 3)$ extended by $j = 1$

rank = 3

s -rank = 2

c -invariant = 4

dimension sequences = $(2, 2, 2), (3, 3, 3, 3)$

$\theta_8 = x^{-3} + y^{-3} + x^{-1}y^{-1}z^{-1}$

$I(\theta_8) = (x^2 + yz, y^2 + xz, z^2)$

is an \mathcal{A}^* algebra realized by

$H^*(\mathbb{RP}(\eta \downarrow (\mathbb{RP}(2) \# \mathbb{RP}(2)))) ; \mathbb{F}_2$

Here the 2-plane bundle $\eta \downarrow (\mathbb{RP}(2) \# \mathbb{RP}(2))$ has total Stiefel–Whitney class $1 + u + v + u^2$ where u, v are the nonzero elements of the first cohomology of the two connected sum components. The bundle η restricted to one copy of $\mathbb{RP}(2)$ in the connected sum is the tangent bundle, and restricted to the other copy the sum of the canonical line bundle with a trivial line bundle. It may be constructed by forming the bundle over the one point union $\mathbb{RP}(2) \vee \mathbb{RP}(2)$ of two copies of $\mathbb{RP}(2)$ which is τ over one copy and $\lambda \oplus \mathbb{R}$ over the other copy (the two being identified along a trivialization over the wedge point) and pulling this bundle back to $\mathbb{RP}(2) \# \mathbb{RP}(2)$ along the collapsing (or pinching) map $\mathbb{RP}(2) \# \mathbb{RP}(2) \rightarrow \mathbb{RP}(2) \vee \mathbb{RP}(2)$.

Orbit 9 of $Cat_{\mathbb{F}_2}(1, 2)$

$cat_{\theta_9}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	1	1	0	0	0	0
y	0	1	0	1	0	0
z	0	0	0	0	0	0

asymmetric

 isotropy group = D_{12}

orbit size = 14

 Orbit 6 of $Mat_{\mathbb{F}_2}(3, 3)$ extended by $j = 0$

rank = 2

 s -rank = 2

 c -invariant = 6

dimension sequences = (2), (3, 3, 3, 3, 3, 3)

 $\theta_9 = x^{-3} + y^{-3} + x^{-1}y^{-2}$
 $I(\theta_9) = (z, x^2 + xy + y^2, x^3 + y^3)$

 is not an \mathcal{A}^* algebra

 Orbit 10 of $Cat_{\mathbb{F}_2}(1, 2)$

$cat_{\theta_{10}}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	1	1	0	0	0	1
y	0	1	0	1	1	0
z	0	0	0	1	0	0

asymmetric

 isotropy group = D_{12}

orbit size = 14

 Orbit 6 of $Mat_{\mathbb{F}_2}(3, 3)$ extended by $j = 1$

rank = 2

 s -rank = 2

 c -invariant = 6

dimension sequences = (2), (3, 3, 3, 3, 3, 3)

 $\theta_{10} = x^{-3} + y^{-3} + x^{-1}y^{-2} + x^{-1}y^{-1}z^{-1}$
 $I(\theta_{10}) = (x^2 + xz + y^2, x^2 + yz, z^2)$

 is an \mathcal{A}^* algebra realized by

 $H^*(M^3; \mathbb{F}_2)$

 The mystery manifold M^3 is a torus bundle over a circle and is described in § 7.

 Orbit 11 of $Cat_{\mathbb{F}_2}(1, 2)$

$cat_{\theta_{11}}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	1	0	1	0	0	0
y	0	1	1	0	0	0
z	0	0	0	0	1	1

asymmetric

 isotropy group = $\mathbb{Z}/2$

orbit size = 84

 Orbit 7 of $Mat_{\mathbb{F}_2}(3, 3)$ extended by $j = 0$

rank = 3

 s -rank = 2

 c -invariant = 2

dimension sequences = (2, 2, 2, 3, 3), (2, 2)

 $\theta_{11} = x^{-3} + y^{-3}x^{-1}z^{-2} + y^{-1}z^{-2}$
 $I(\theta_{11}) = (x^2 + y^2 + z^2, xz + yz, xy)$

 is an \mathcal{A}^* algebra realized by

 $H^*(\mathbb{RP}(\lambda \# \mathbb{R}^2 \downarrow (\mathbb{RP}(2) \# \mathbb{RP}(2)); \mathbb{F}_2)$

 Orbit 12 of $Cat_{\mathbb{F}_2}(1, 2)$

$cat_{\theta_{12}}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	1	0	1	0	0	1
y	0	1	1	0	1	0
z	0	0	0	1	1	1

asymmetric

 isotropy group = $\mathbb{Z}/2$

orbit size = 84

 Orbit 7 of $Mat_{\mathbb{F}_2}(3, 3)$ extended by $j = 1$

rank = 3

 s -rank = 2

 c -invariant = 2

dimension sequences = (2, 2, 2, 3, 3), (3, 3)

 $\theta_{12} = x^{-3} + y^{-3}x^{-1}z^{-2} + y^{-1}z^{-2} + x^{-1}y^{-1}z^{-1}$
 $I(\theta_{12}) = (x^2 + y^2 + z^2, y^2 + xy + xz,$
 $x^2 + xy + yz, z^3, x^3 + y^3)$

 is not an \mathcal{A}^* algebra

Orbit 13 of $Cat_{\mathbb{F}_2}(1, 2)$

$cat_{\theta_{13}}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	1	0	1	0	0	0
y	0	1	0	0	0	0
z	0	0	0	0	1	0

asymmetric
isotropy group = 1
orbit size = 168

Orbit 8 of $Mat_{\mathbb{F}_2}(3, 3)$ extended by $j = 0$

rank = 3
 s - rank = 2
 c - invariant = 4
dimension sequences = $(1, 2, 3), (1, 2, 2, 3)$
 $\theta_{13} = x^{-3} + y^{-3} + x^{-1}z^{-2}$
 $I(\theta_{13}) = (x^2 + z^2, z^3, y^4, y(x + z), yz)$
is an \mathcal{A}^* algebra realized by
 $H^*(\mathbb{R}\mathbb{P}(3)\#(\mathbb{R}\mathbb{P}(2) \times \mathbb{R}\mathbb{P}(1)); \mathbb{F}_2)$

Orbit 14 of $Cat_{\mathbb{F}_2}(1, 2)$

$cat_{\theta_{14}}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	1	0	1	0	0	1
y	0	1	0	0	1	0
z	0	0	0	1	1	0

asymmetric
isotropy group = 1
orbit size = 168

Orbit 8 of $Mat_{\mathbb{F}_2}(3, 3)$ extended by $j = 1$

rank = 3
 s - rank = 2
 c - invariant = 4
dimension sequences = $(1, 2, 3), (1, 2, 2, 3)$
 $\theta_{14} = x^{-3} + y^{-3} + x^{-1}z^{-2} + x^{-1}y^{-1}z^{-1}$
 $I(\theta_{14}) = (x^2 + z^2, x^2 + yz, y^2 + xy + xz)$
is not an \mathcal{A}^* algebra

Orbit 15 of $Cat_{\mathbb{F}_2}(1, 2)$

$cat_{\theta_{15}}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	0	1	0	1	0	0
y	1	0	0	1	0	0
z	0	0	0	0	0	0

symmetric
isotropy group = Σ_4
orbit size = 7

Orbit 9 of $Mat_{\mathbb{F}_2}(3, 3)$ extended by $j = 0$

rank = 2
 s - rank = 2
 c - invariant = 0
dimension sequences = $(2, 2, 2, 2, 2, 2, 2), ()$
 $\theta_{15} = x^{-2}y^{-1} + x^{-1}y^{-2}$
 $I(\theta_{15}) = (x^2 + xy + y^2, x^2y + xy^2, z)$
is an \mathcal{A}^* algebra realized by
 $H^*(\mathbb{R}\mathbb{P}(\tau \downarrow \mathbb{R}\mathbb{P}(2)); \mathbb{F}_2)$

The corresponding Poincaré duality quotient algebra is $\mathbb{F}_2[x, y]_{GL(2, \mathbb{F}_2)}$, the rank two Dickson coinvariants.

Orbit 16 of $Cat_{\mathbb{F}_2}(1, 2)$

$cat_{\theta_{16}}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	0	1	0	1	0	1
y	1	0	0	1	1	0
z	0	0	0	1	0	0

symmetric
isotropy group = Σ_4
orbit size = 7

Orbit 9 of $Mat_{\mathbb{F}_2}(3, 3)$ extended by $j = 1$

rank = 3
 s - rank = 2
 c - invariant = 0
dimension sequences = $(2, 2, 2, 2, 2, 2, 2), ()$
 $\theta_{16} = x^{-2}y^{-1} + x^{-1}y^{-2}$
 $I(\theta_{16}) = (x^2 + xz, y^2 + yz, z^2)$
is an \mathcal{A}^* algebra realized by
 $H^*(\mathbb{R}\mathbb{P}(\zeta \downarrow \mathbb{R}\mathbb{P}(2)\#\mathbb{R}\mathbb{P}(2)); \mathbb{F}_2)$

Here the total Stiefel–Whitney class of the 2-plane bundle ζ is $1 + x + y$ where x and y are the nonzero elements of degree 1 in the first cohomology of the connected sum factors. So ζ is the pullback of $\lambda \oplus \mathbb{R} \downarrow \mathbb{R}\mathbb{P}(2)$ along the folding map $\mathbb{R}\mathbb{P}(2)\#\mathbb{R}\mathbb{P}(2) \rightarrow \mathbb{R}\mathbb{P}(2)$.

Orbit 17 of $Cat_{\mathbb{F}_2}(1, 2)$

$cat_{\theta_{17}}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	1	0	0	0	0	0
y	0	0	0	0	0	0
z	0	0	0	0	0	0

symmetric

 isotropy group = Σ_4

orbit size = 7

 Orbit 18 of $Cat_{\mathbb{F}_2}(1, 2)$

$cat_{\theta_{18}}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	1	0	0	0	0	1
y	0	0	0	0	1	0
z	0	0	0	1	0	0

symmetric

 isotropy group = Σ_4

orbit size = 7

 Orbit 19 of $Cat_{\mathbb{F}_2}(1, 2)$

$cat_{\theta_{19}}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	1	1	0	0	0	0
y	0	0	0	1	0	0
z	0	0	0	0	0	0

asymmetric

 isotropy group = $K \cong \mathbb{Z}/2 \times \mathbb{Z}/2$

orbit size = 42

 Orbit 20 of $Cat_{\mathbb{F}_2}(1, 2)$

$cat_{\theta_{20}}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	1	1	0	0	0	1
y	0	0	0	1	1	0
z	0	0	0	1	0	0

asymmetric

 isotropy group = $K \cong \mathbb{Z}/2 \times \mathbb{Z}/2$

orbit size = 42

 Orbit 10 of $Mat_{\mathbb{F}_2}(3, 3)$ extended by $j = 0$

rank = 1

 s - rank = 1

 c - invariant = 4

dimension sequences = (2, 2, 2), (3, 3, 3, 3)

 $\theta_{17} = x^{-3}$
 $I(\theta_{17}) = (x^4, y, z)$

 is an \mathcal{A}^* algebra realized by

$$H^*(\mathbb{RP}(3); \mathbb{F}_2)$$

 Orbit 10 of $Mat_{\mathbb{F}_2}(3, 3)$ extended by $j = 1$

rank = 3

 s - rank = 1

 c - invariant = 4

dimension sequences = (2, 2, 2), (3, 3, 3, 3)

 $\theta_{18} = x^{-3} + x^{-1}y^{-1}z^{-1}$
 $I(\theta_{18}) = (x^2 + yz, y^2, z^2)$

 is an \mathcal{A}^* algebra realized by

$$H^*(\mathbb{RP}(\lambda \oplus \lambda \downarrow (\mathbb{RP}(1) \times \mathbb{RP}(1))); \mathbb{F}_2)$$

 Orbit 11 of $Mat_{\mathbb{F}_2}(3, 3)$ extended by $j = 0$

rank = 2

 s - rank = 1

 c - invariant = 2

dimension sequences = (2, 2, 2, 2, 3), (3, 3)

 $\theta_{19} = x^{-3} + x^{-1}y^{-2}$
 $I(\theta_{19}) = (z, y^2, x^3)$

 is an \mathcal{A}^* algebra realized by

$$H^*(\mathbb{RP}(2) \times \mathbb{RP}(1); \mathbb{F}_2)$$

 Orbit 11 of $Mat_{\mathbb{F}_2}(3, 3)$ extended by $j = 1$

rank = 3

 s - rank = 1

 c - invariant = 2

dimension sequences = (2, 2, 2, 2, 3), (3, 3)

 $\theta_{20} = x^{-3} + x^{-1}y^{-2} + x^{-1}y^{-1}z^{-1}$
 $I(\theta_{20}) = (x^2 + y^2, x^2 + yz, z^2)$

 is an \mathcal{A}^* algebra realized by

$$H^*(S^1 \times (\mathbb{RP}(2) \# \mathbb{RP}(2)); \mathbb{F}_2)$$

This Poincaré duality algebra is also realized as the cohomology of $\mathbb{RP}(\xi \downarrow (S^1 \times S^1))$ which is the total space of the projective space bundle associated to a 2-plane bundle ξ over $S^1 \times S^1$ with nontrivial first Stiefel–Whitney class.

Orbit 21 of $Cat_{\mathbb{F}_2}(1, 2)$

$cat_{\theta_{21}}(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	0	0	0	0	0	1
y	0	0	0	0	1	0
z	0	0	0	1	0	0

symmetric

isotropy group = $GL(3, \mathbb{F}_2)$

orbit size = 1

Orbit 12 of $Mat_{\mathbb{F}_2}(3, 3)$ extended by $j = 1$

rank = 3

s - rank = 0

c - invariant = 0

dimension sequences = $(2, 2, 2, 2, 2, 2, 2), ()$

$\theta_{21} = x^{-1}y^{-1}z^{-1}$

$I(\theta_{21}) = (x^2, y^2, z^2)$

is an \mathcal{A}^* algebra realized by

$H^*(S^1 \times S^1 \times S^1; \mathbb{F}_2)$

In summary: This gives us 21 isomorphism classes of Poincaré duality algebras of formal dimension three and rank at most three. Of these examples:

- One example has rank one (orbit 17).
- Four examples have rank two (orbits 7, 9, 15, 19).
- Four examples of rank three are decomposable with respect to the connected sum operation (orbits 1, 2, 3, 13).
- Seven examples of rank three are indecomposable unstable \mathcal{A}^* -algebras (orbits 8, 11, 16, 18, 19, 20, 21).
- Five examples of rank three are indecomposable non- \mathcal{A}^* -algebras (orbits 4, 5, 6, 12, 14).
- All the #-decomposable algebras are defined by ideals with five generators.
- If both values of j in $(*)$ occur among the matrices in an orbit, then the isotropy group of that orbit is $\mathbb{Z}/2$ (orbits 5 and 6); however, the isotropy group can be $\mathbb{Z}/2$ and still only one value of j appear among the members of the orbit (orbits 11 and 12).

Finally, Table 5.3 summarizes the values of the invariants we have used in the classification of the orbits. In this table the dimension sequences give the value of the c -invariant as the length of the second dimension sequence; rank means the rank of the full 3×6 catalecticant matrix, and s -rank means the rank of the leftmost 3×3 submatrix.

§6. Threefolds of Rank Three III (Separating Invariants)

The elements of $\mathbb{F}_2[Cat_{\mathbb{F}_2}(1, 2)]^{GL(3, \mathbb{F}_2)}$ may be regarded as functions on the orbit space $Cat_{\mathbb{F}_2}(1, 2)/GL(3, \mathbb{F}_2)$, so it would be nice to compute this ring of invariants and select from it sufficiently many invariants to separate the orbits. Even using a two step approach, and first computing $\mathbb{F}_2[Mat_{\mathbb{F}_2}(3, 3)]^{GL(3, \mathbb{F}_2)}$, this seems at present an inordinately difficult problem. Instead we describe a procedure to find separating invariants by an indirect method and use it to determine invariants that separate the orbits of $GL(3, \mathbb{F}_2)$ on $Mat_{\mathbb{F}_2}(3, 3)$.

We will make use of the $GL(3, \mathbb{F}_2)$ -equivariant linear map

$$L : Cat_{\mathbb{F}_2}(1, 2) \longrightarrow Mat_{\mathbb{F}_2}(3, 3)$$

that assigns to a catalecticant matrix

$cat_H(1, 2)$	x^2	y^2	z^2	xy	xz	yz
x	a	b	c	d	g	j
y	d	e	f	b	j	h
z	g	h	i	j	c	f

$(*)$

POINCARÉ DUALITY ALGEBRAS MOD TWO

Orbit	rank	s-rank	c-invariant	dimension sequences	symmetric	\mathcal{A}^*	isotropy group
1	3	3	4	(2, 2, 2) (1, 1, 1, 3)	yes	yes	Σ_3
2	3	3	4	(2, 2, 2) (1, 3, 3, 3)	yes	yes	Σ_3
3	3	3	4	(3, 3, 3) (1, 2, 2, 2)	no	no	A_3
4	3	3	4	(3, 3, 3) (3, 3, 3, 3)	no	no	A_3
5	3	3	6	(2) (2, 2, 2, 3, 3, 3)	no	no	$\mathbb{Z}/2$
6	3	3	2	(2, 3, 3, 3, 3) (2, 3)	no	no	$\mathbb{Z}/2$
7	2	2	4	(2, 2, 2) (3, 3, 3, 3)	yes	yes	$\text{Syl}_2(\text{GL}(3, \mathbb{F}_2))$
8	3	2	4	(2, 2, 2) (3, 3, 3, 3)	yes	yes	$\text{Syl}_2(\text{GL}(3, \mathbb{F}_2))$
9	2	2	6	(2) (3, 3, 3, 3, 3, 3)	no	no	D_{12}
10	2	2	6	(2) (3, 3, 3, 3, 3, 3)	no	yes	D_{12}
11	3	2	2	(2, 2, 2, 3, 3) (2, 2)	no	yes	$\mathbb{Z}/2$
12	3	2	2	(2, 2, 2, 3, 3) (2, 2)	no	no	$\mathbb{Z}/2$
13	3	2	4	(1, 2, 3) (1, 2, 2, 3)	no	yes	$\{1\}$
14	3	2	4	(1, 2, 3) (1, 2, 2, 3)	no	no	$\{1\}$
15	2	2	0	(2, 2, 2, 2, 2, 2, 2) ()	yes	yes	Σ_4
16	3	2	0	(2, 2, 2, 2, 2, 2, 2) ()	yes	yes	Σ_4
17	1	1	4	(2, 2, 2) (3, 3, 3, 3)	yes	yes	Σ_4
18	3	1	4	(2, 2, 2) (3, 3, 3, 3)	yes	yes	Σ_4
19	2	1	2	(2, 2, 2, 2, 3) (3, 3)	no	yes	$\mathbb{Z}/2 \times \mathbb{Z}/2$
20	3	1	2	(2, 2, 2, 2, 3) (3, 3)	no	yes	$\mathbb{Z}/2 \times \mathbb{Z}/2$
21	3	0	0	(2, 2, 2, 2, 2, 2, 2) ()	yes	yes	$\text{GL}(3, \mathbb{F}_2)$

TABLE 5.3: Orbit Invariants

its leftmost 3×3 submatrix¹⁷ to separate the orbits of $\text{GL}(3, \mathbb{F}_2)$ on $\text{Cat}_{\mathbb{F}_2}(1, 2)$. The map \mathbf{L} induces a map

$$\mathbf{L}^* : \mathbb{F}_2[\text{Mat}_{\mathbb{F}_2}(3, 3)]^{\text{GL}(3, \mathbb{F}_2)} \longrightarrow \mathbb{F}_2[\text{Cat}_{\mathbb{F}_2}(1, 2)]^{\text{GL}(3, \mathbb{F}_2)}$$

that allows us to regard the elements of $\mathbb{F}_2[\text{Mat}_{\mathbb{F}_2}(3, 3)]^{\text{GL}(3, \mathbb{F}_2)}$ as functions on both of the orbit spaces $\text{Mat}_{\mathbb{F}_2}(3, 3)/\text{GL}(3, \mathbb{F}_2)$ and $\text{Cat}_{\mathbb{F}_2}(1, 2)/\text{GL}(3, \mathbb{F}_2)$. From the determination of the orbits $\text{Cat}_{\mathbb{F}_2}(1, 2)/\text{GL}(3, \mathbb{F}_2)$ we know this will not suffice to separate them – it is only a first step in that direction. The method we introduce does however allow one to finish the determination of separating invariants as we indicate briefly at the end of this section.

A 3×3 matrix $\mathbf{A} \in \text{Mat}_{\mathbb{F}_2}(3, 3)$ defines a bilinear form $\varphi_{\mathbf{A}} : \mathbb{F}_2^3 \times \mathbb{F}_2^3 \longrightarrow \mathbb{F}_2$ and conversely. The action of the group $\text{GL}(3, \mathbb{F}_2)$ on \mathbb{F}_2^3 permutes the nonzero elements of \mathbb{F}_2^3 so defines an inclusion $\text{GL}(3, \mathbb{F}_2) \xrightarrow{\theta} \Sigma_7$. Put the elements of \mathbb{F}_2^3 into a linear order and use it to define a linear map

$$\mathbf{T} : \text{Mat}_{\mathbb{F}_2}(3, 3) \longrightarrow \mathbb{F}_2^7 = W$$

by assigning to $\mathbf{A} \in \text{Mat}_{\mathbb{F}_2}(3, 3)$ the vector $(\varphi_{\mathbf{A}}(u_1, u_1), \dots, \varphi_{\mathbf{A}}(u_7, u_7))$, where u_1, \dots, u_7 are the distinct nonzero elements of \mathbb{F}_2^3 . Clearly the number $c_{\mathbf{A}}$ of nonzero coordinates of $\mathbf{T}(\mathbf{A})$, i.e., the number of elements whose square with respect to $\varphi_{\mathbf{A}}$ is nonzero, is an invariant of the orbit to which \mathbf{A} belongs. Note that for a catalecticant matrix \mathbf{C} with $\mathbf{L}(\mathbf{C}) = \mathbf{A}$ the number $c_{\mathbf{A}}$ coincides with the number of nonzero cubes in the corresponding Poincaré duality quotient algebra of $\mathbb{F}_2[x, y, z]$, and is the c -invariant used in § 5 to distinguish orbits in $\text{Cat}_{\mathbb{F}_2}(1, 2)$.

The group Σ_7 acts on \mathbb{F}_2^7 by permutation of the standard basis vectors so by means of θ we obtain an action of $\text{GL}(3, \mathbb{F}_2)$ on $W = \mathbb{F}_2^7$ making the map \mathbf{T} equivariant. Hence we have induced maps

$$\mathbb{F}_2[W]^{\Sigma_7} \subsetneq \mathbb{F}_2[W]^{\text{GL}(3, \mathbb{F}_2)} \xrightarrow{\mathbf{T}^*} \mathbb{F}_2[\text{Mat}_{\mathbb{F}_2}(3, 3)]^{\text{GL}(3, \mathbb{F}_2)}$$

which allow us to define the elements $\mathbf{T}^*(e_1), \dots, \mathbf{T}^*(e_7) \in \mathbb{F}_2[\text{Mat}_{\mathbb{F}_2}(3, 3)]^{\text{GL}(3, \mathbb{F}_2)}$, where $e_1, \dots, e_7 \in \mathbb{F}_2[W]^{\Sigma_7}$ are the elementary symmetric polynomials.

Denote by $w_1, \dots, w_7 \in W^*$ the standard dual basis. Recall that e_i is the sum of all the monomials in the Σ_7 -orbit of the monomial $w_1 \cdots w_i$. So, if $\mathbf{A} \in \text{Mat}_{\mathbb{F}_2}(3, 3)$ and the number of nonzero squares $\varphi_{\mathbf{A}}(u, u)$ for $u \in \mathbb{F}_2^3$ is $c_{\mathbf{A}}$, then it follows that the value of $\mathbf{T}(e_i)$ on \mathbf{A} is the number of ways that one can choose i elements from among the $c_{\mathbf{A}}$ nonzero elements $\varphi_{\mathbf{A}}(u, u)$ for $u \in \mathbb{F}_2^3$, i.e., $\varphi_{\mathbf{A}} = \binom{c_{\mathbf{A}}}{i}$ and this value¹⁸ is constant on the orbit of \mathbf{A} . From the table of orbits, Table 5.2, of $\text{Mat}_{\mathbb{F}_2}(3, 3)/\text{GL}(3, \mathbb{F}_2)$ we see that $c_{\mathbf{A}} \in \{0, 2, 4, 6\}$. Thus the odd symmetric polynomials evaluate to zero and one has the following table of values for the even ones.

Form \ $c_{\mathbf{A}}$	0	2	4	6
$\mathbf{T}^*(e_2)$	0	1	0	1
$\mathbf{T}^*(e_4)$	0	0	1	1
$\mathbf{T}^*(e_6)$	0	0	0	1

TABLE 6.1: Values of Even Symmetric Functions on Orbits

From this we see that the forms $\mathbf{T}^*(e_2), \mathbf{T}^*(e_4) \in \mathbb{F}_2[\text{Mat}_{\mathbb{F}_2}(3, 3)]^{\text{GL}(3, \mathbb{F}_2)}$ determine the c -invariant. So we have proven the following result.

¹⁷ Since the j -entry in (*) is not an orbit invariant there is no equivariant splitting for this map.

¹⁸ The value of a binomial coefficient $\binom{n}{k}$ with $k > n$ is defined to be zero.

PROPOSITION 6.1: *The c -invariant of a matrix $\mathbf{A} \in \text{Mat}_{\mathbb{F}_2}(3, 3)$, or a catalecticant matrix $\mathbf{C} \in \text{Cat}_{\mathbb{F}_2}(1, 2)$ with $\mathbf{L}(\mathbf{C}) = \mathbf{A}$, is determined by the two invariant forms $\mathbf{T}^*(e_2), \mathbf{T}^*(e_4) \in \mathbb{F}_2[\text{Mat}_{\mathbb{F}_2}(3, 3)]^{\text{GL}(3, \mathbb{F}_2)}$. Specifically one has the following table of values.*

c -invariant	condition
0	$\mathbf{T}^*(e_2)(\mathbf{A}) = 0 = \mathbf{T}^*(e_4)(\mathbf{A})$
2	$\mathbf{T}^*(e_2)(\mathbf{A}) = 1, \mathbf{T}^*(e_4)(\mathbf{A}) = 0$
4	$\mathbf{T}^*(e_2)(\mathbf{A}) = 0, \mathbf{T}^*(e_4)(\mathbf{A}) = 1$
6	$\mathbf{T}^*(e_2)(\mathbf{A}) = 1 = \mathbf{T}^*(e_4)(\mathbf{A})$

TABLE: Determination of the c -Invariant by Invariant Forms

So the number of nonzero cubes in the Poincaré duality algebra associated to \mathbf{C} is determined by the two invariant forms $\mathbf{T}^*(e_2), \mathbf{T}^*(e_4)$ by means of this table. \square

In addition to the c -invariant we also made use of the rank of a matrix and whether it is symmetric or not to determine its orbit. We deal next with the symmetry property, which we show is detected by a single invariant form. Let

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \in \text{Mat}_{\mathbb{F}_2}(3, 3).$$

If \mathbf{A} is not symmetric then there will be one, two, or three of the terms

$$b + d, c + g, f + h$$

which are nonzero. This means that exactly four of the terms

$$(\heartsuit) \quad (b + d), (c + g), (f + h), (b + d) + (c + g), (b + d) + (f + h), (c + g) + (f + h), \\ (b + d) + (c + g) + (f + h)$$

are nonzero. The action of $\text{GL}(3, \mathbb{F}_2)$ on $\text{Mat}_{\mathbb{F}_2}(3, 3)$ permutes these values. Define the map

$$\mathbf{S} : \text{Mat}_{\mathbb{F}_2}(3, 3) \longrightarrow \mathbb{F}_2^7 = W$$

by assigning to the matrix \mathbf{A} the vector whose coordinates are the terms listed in (\heartsuit) . The group Σ_7 acts on $W = \mathbb{F}_2^7$ by permutation of the coordinates and by means of the inclusion $\theta : \text{GL}(3, \mathbb{F}_2) \hookrightarrow \Sigma_7$ so does $\text{GL}(3, \mathbb{F}_2)$. The map \mathbf{S} is $\text{GL}(3, \mathbb{F}_2)$ -equivariant so induces a map

$$\mathbb{F}_2[W]^{\Sigma_7} \subsetneq \mathbb{F}_2[W]^{\text{GL}(3, \mathbb{F}_2)} \xrightarrow{\mathbf{S}^*} \mathbb{F}_2[\text{Mat}_{\mathbb{F}_2}(3, 3)]^{\text{GL}(3, \mathbb{F}_2)}.$$

PROPOSITION 6.2: *The form $\mathbf{S}^*(e_4) \in \mathbb{F}_2[\text{Mat}_{\mathbb{F}_2}(3, 3)]^{\text{GL}(3, \mathbb{F}_2)}$ detects if a matrix is symmetric or not. Specifically $\mathbf{S}^*(e_4)(\mathbf{A}) = 0$ if and only if $\mathbf{A} \in \text{Mat}_{\mathbb{F}_2}(3, 3)$ is symmetric.*

PROOF: If $\mathbf{A} \in \text{Mat}_{\mathbb{F}_2}(3, 3)$ is symmetric then it is in the kernel of \mathbf{S} so $\mathbf{S}^*(e_4)(\mathbf{A}) = 0$. If \mathbf{A} is not symmetric then exactly four of the seven values listed in (\heartsuit) are nonzero. Hence the only monomial in $\mathbf{S}^*(e_4)$ that can evaluate nonzero on \mathbf{A} is the one that selects these four coordinates so $\mathbf{S}^*(e_4)(\mathbf{A}) = 1$. \square

To complete the separation of the orbits of $\text{GL}(3, \mathbb{F}_2)$ on $\text{Mat}_{\mathbb{F}_2}(3, 3)$ by invariant forms it remains to find additional forms to determine the rank of a matrix $\mathbf{A} \in \text{Mat}_{\mathbb{F}_2}(3, 3)$. If, as before, $\varphi_{\mathbf{A}} : \mathbb{F}_2^3 \times \mathbb{F}_2^3 \longrightarrow \mathbb{F}_2$ is the bilinear form associated to $\mathbf{A} \in \text{Mat}_{\mathbb{F}_2}(3, 3)$, then putting the elements of $\mathbb{F}_2^3 \setminus \{0\}$ into a linear order allows us to define a linear map

$$\mathbf{M} : \text{Mat}_{\mathbb{F}_2}(3, 3) \longrightarrow \text{Mat}_{\mathbb{F}_2}(7, 7)$$

by assigning to \mathbf{A} the 7×7 matrix $[\varphi_{\mathbf{A}}(u, v)]_{u, v \in \mathbb{F}_2^3 \setminus \{0\}}$. The vector space $\text{Mat}_{\mathbb{F}_2}(7, 7)$ has dimension 49 and the symmetric group Σ_{49} acts on it by permutation of the standard basis elements $[\delta_{i, j}]_{1 \leq i, j \leq 7}$. The action of Σ_7 on \mathbb{F}_2^3 permutes the nonzero elements and induces an action of Σ_7 on $\text{Mat}_{\mathbb{F}_2}(7, 7)$ by simultaneous permutation of the rows and columns of a matrix. By means of the inclusion $\theta : \text{GL}(3, \mathbb{F}_2) \hookrightarrow \Sigma_7$ we therefore obtain an action of $\text{GL}(3, \mathbb{F}_2)$ on $\text{Mat}_{\mathbb{F}_2}(7, 7)$. Unravelling the definitions one finds that the linear map \mathbf{M} is $\text{GL}(3, \mathbb{F}_2)$ -equivariant so induces a map

$$\mathbf{M}^* : \mathbb{F}_2[\text{Mat}_{\mathbb{F}_2}(7, 7)]^{\Sigma_{49}} \longrightarrow \mathbb{F}_2[\text{Mat}_{\mathbb{F}_2}(3, 3)]^{\text{GL}(3, \mathbb{F}_2)}.$$

PROPOSITION 6.3: *Let $\sigma_i \in \mathbb{F}_2[\text{Mat}_{\mathbb{F}_2}(7, 7)]^{\Sigma_{49}}$ be the i -th elementary symmetric polynomial, $i = 1, \dots, 49$. Then*

$$\det, \mathbf{T}^*(e_2), \mathbf{T}^*(e_4), \mathbf{M}^*(\sigma_8) \in \mathbb{F}_2[\text{Mat}_{\mathbb{F}_2}(3, 3)]^{\text{GL}(3, \mathbb{F}_2)}$$

determine the rank of a matrix in $\text{Mat}_{\mathbb{F}_2}(3, 3)$ by means of the following table¹⁹

rank(\mathbf{A})	0	1	2	3
$\det(A)$	0	0	0	1
$\mathbf{M}^*(\sigma_8)(A)$	0	0	1	?
c -invariant	0	$\neq 0$	$\neq 0$	$\neq 0$

and the table from Proposition 6.1 to compute the c -invariant from $\mathbf{T}^*(e_2)(\mathbf{A})$ and $\mathbf{T}^*(e_4)(\mathbf{A})$.

PROOF: Clearly $\det \in \mathbb{F}_2[\text{Mat}_{\mathbb{F}_2}(3, 3)]^{\text{GL}(3, \mathbb{F}_2)}$ distinguishes the matrices of rank three from all the others, and on the basis of Propositions 6.1 and 6.2 one is left to distinguish three pairs of examples. To wit, one has the following list of problem pairs:

- orbits 5 and 10 which are symmetric with c -invariant 4 and have ranks 2 and 1 respectively,
- orbits 7 and 11 which are asymmetric with c -invariant 2 and ranks 2 and 1 respectively,
- orbits 9 and 12 which are symmetric with c -invariant 0 and have ranks 2 and 0 respectively.

The orbits 3 and 6 which are asymmetric and have c -invariant 6, but different ranks, are already distinguished by the determinant, since of the two of them only one has rank 3.

For $\mathbf{A} \in \text{Mat}_{\mathbb{F}_2}(3, 3)$ finding the entries of $\mathbf{M}(\mathbf{A})$ is routine. To fill in the missing entries of the first three rows you take sums of the first three columns, and then you get the bottom four rows by taking sums of the first three rows as indicated in the following schema.

$\mathbf{M}(\mathbf{A})$	x	y	z	$x + y$	$x + z$	$y + z$	$x + y + z$
x	a	b	c	\vdots	col ₁	col ₁	col ₂
y	d	e	f	\vdots	+	+	+
z	g	h	i	\vdots	col ₂	col ₃	col ₃
$x + y$	\dots	\dots	\dots	\vdots	col ₂	col ₃	col ₃
$x + z$			row ₁	+	row ₂		
$y + z$			row ₂	+	row ₃		
$x + y + z$			row ₁	+	row ₃		
		row ₁	+	row ₂	+	row ₃	

¹⁹The entry indicated by a ? plays no role.

Having done this for the twelve orbit representatives in Table 5.2 we can count the number $t_{\mathbf{A}}$ of nonzero entries in the matrix $\mathbf{M}(\mathbf{A})$. Since $\mathrm{GL}(3, \mathbb{F}_2)$ acts by permutation of the linear forms $x, y, z, x + y, x + z, y + z, x + y + z$ the number $t_{\mathbf{A}}$ is an invariant of the orbit of \mathbf{A} . We summarize the result of this computation in the following table.

Orbit of \mathbf{A}	1	2	3	4	5	6	7	8	9	10	11	12
nonzeros in $\mathbf{M}(\mathbf{A})$	28	28	28	28	24	24	24	24	24	16	16	0

From this table one sees that the rank of \mathbf{A} tells us $t_{\mathbf{A}}$ as recorded in the next table.

rank(\mathbf{A})	0	1	2	3
$t_{\mathbf{A}}$	0	16	24	28

The value of $\mathbf{M}^*(\sigma_i)(\mathbf{A})$ is given by $\binom{t_{\mathbf{A}}}{i}$ the number of ways to choose i nonzero entries in $\mathbf{M}(\mathbf{A})$. So $\mathbf{M}^*(\sigma_8)(\mathbf{A}) = \binom{16}{8} = 0$ if \mathbf{A} has rank 1 and $\mathbf{M}^*(\sigma_8)(\mathbf{A}) = \binom{24}{8} = 1$ if \mathbf{A} has rank 2 and therefore $\mathbf{M}^*(\sigma_8)$ can be used to separate the orbits of rank one matrices from those of rank two. This suffices to distinguish each of the pairs in our problem list. \square

COROLLARY 6.4: *If $\mathbf{A} \in \mathrm{Mat}_{\mathbb{F}_2}(3, 3)$ is a nonzero 3×3 matrix, then*

$$\mathrm{rank}(\mathbf{A}) = \begin{cases} 3 & \text{if and only if } \det(\mathbf{A}) \neq 0, \\ 2 & \text{if and only if } \det(\mathbf{A}) = 0 \text{ and } \mathbf{M}^*(\sigma_8) = 1, \text{ and} \\ 1 & \text{if and only if } \det(\mathbf{A}) = 0 \text{ and } \mathbf{M}^*(\sigma_8) = 0. \end{cases}$$

Hence the rank of a nonzero 3×3 matrix over the field \mathbb{F}_2 is determined by the two invariant forms $\det, \mathbf{M}^*(\sigma_8) \in \mathbb{F}_2[\mathrm{Mat}_{\mathbb{F}_2}(3, 3)]^{\mathrm{GL}(3, \mathbb{F}_2)}$. \square

Combining Propositions 6.1, 6.2, and 6.3 we see that the five invariant forms

$$\mathbf{T}^*(e_2), \mathbf{T}^*(e_4), \mathbf{S}^*(e_4), \det, \mathbf{M}^*(\sigma_8) \in \mathbb{F}_2[\mathrm{Mat}_{\mathbb{F}_2}(3, 3)]^{\mathrm{GL}(3, \mathbb{F}_2)}$$

suffice to determine the c -invariant, whether a matrix is symmetric or not, and its rank. Hence we have found separating forms for the orbit space $\mathrm{Mat}_{\mathbb{F}_2}(3, 3)/\mathrm{GL}(3, \mathbb{F}_2)$, to wit we have the following result.

THEOREM 6.5: *The five invariant forms*

$$\mathbf{T}^*(e_2), \mathbf{T}^*(e_4), \mathbf{S}^*(e_4), \det, \mathbf{M}^*(\sigma_8) \in \mathbb{F}_2[\mathrm{Mat}_{\mathbb{F}_2}(3, 3)]^{\mathrm{GL}(3, \mathbb{F}_2)}$$

separate the orbits of the transposition action of $\mathrm{GL}(3, \mathbb{F}_2)$ on $\mathrm{Mat}_{\mathbb{F}_2}(3, 3)$. \square

Note that by means of the map

$$\mathbf{L}^* : \mathbb{F}_2[\mathrm{Mat}_{\mathbb{F}_2}(3, 3)]^{\mathrm{GL}(3, \mathbb{F}_2)} \longrightarrow \mathbb{F}_2[\mathrm{Cat}_{\mathbb{F}_2}(1, 2)]^{\mathrm{GL}(3, \mathbb{F}_2)}$$

and Theorem 6.5 we have determined the c -invariant, symmetry or asymmetry, and the rank of a catalecticant matrix $\mathbf{C} \in \mathrm{Cat}_{\mathbb{F}_2}(1, 2)$. However these invariants do not suffice to separate the $\mathrm{GL}(3, \mathbb{F}_2)$ orbits. We need, for example, in addition the rank sequences, and whether or not the corresponding Poincaré duality quotient supports a Steenrod algebra action to determine the orbits. These are however both a question of rank (see § 5) and so it is possible to proceed in a way similar to that used in the proof of Proposition 6.3 to find additional invariant forms in $\mathbb{F}_2[\mathrm{Cat}_{\mathbb{F}_2}(1, 2)]^{\mathrm{GL}(3, \mathbb{F}_2)}$ to separate the orbits. Since the details offer no new features we omit them.

§7. Threefolds of Rank Three IV (A Torus Bundle over a Circle)

Of the seven standardly graded Poincaré duality algebras of rank three with an unstable Steenrod algebra action six have been identified as the cohomology of manifolds. They are all projective space bundles over either the torus $S^1 \times S^1$ or the Klein bottle $\mathbb{R}P(2) \# \mathbb{R}P(2)$. The missing example (orbit 10 of $Cat_{\mathbb{F}_2}(1, 2)$, which is orbit 6 of $Mat_{\mathbb{F}_2}(3, 3)$ extended by $j = 1$ to $Cat_{\mathbb{F}_2}(1, 2)$) cannot possibly be realized as a fibring over a surface. If one examines the catalecticant matrix

$$(\%) \quad \begin{array}{c|cccccc}
 cat_{\theta_{10}}(1, 2) & x^2 & y^2 & z^2 & xy & xz & yz \\
 \hline
 x & 1 & 1 & 0 & 0 & 0 & 1 \\
 y & 0 & 1 & 0 & 1 & 1 & 0 \\
 z & 0 & 0 & 0 & 1 & 0 & 0
 \end{array}$$

representing this orbit, one sees that $z^2 = 0$ and the algebra H corresponding to this matrix can be visualized as follows. As in [16] the entries on a given horizontal line in the diagram are a

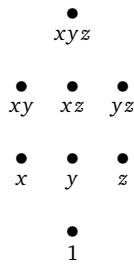


DIAGRAM 7.1: The Algebra for θ_{21}

basis for the homogeneous component of H of degree the number of lines above the line containing the unit 1 of the algebra. From this one finds the relations

$$x^2 = yz, \quad y^2 = xz + yz.$$

This shows that H is a free module over the subalgebra $\mathbb{F}[z]/(z^2) \subset H$ with basis the four elements $1, x, y, xy$. So H looks like the \mathbb{F}_2 -

cohomology of the total space M^3 of a fibring $S^1 \times S^1 \hookrightarrow M^3 \downarrow S^1$ which is totally nonhomologous to zero. With the help of some suggestions from Slava Kruskal we were able to find such a fibring which we explain next.

Consider the universal covering $\mathbb{R} \downarrow S^1$ which has deck transformation group \mathbb{Z} . If

$$(\%) \quad \mathbf{T} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{Z})$$

then we may use the matrix \mathbf{T} to define an action of \mathbb{Z} on the torus $S^1 \times S^1 = \mathbb{R} \times \mathbb{R} / \mathbb{Z} \times \mathbb{Z}$. We simply let \mathbb{Z} act on \mathbb{R}^2 in the usual way via the matrix $(\%)$ and note that this action preserves the integral lattice \mathbb{Z}^2 and so passes down to the orbit space. We may then form the associated torus bundle over the circle, viz.,

$$(\star) \quad \begin{array}{ccc}
 S^1 \times S^1 \hookrightarrow \mathbb{R} \times_{\mathbb{Z}} (S^1 \times S^1) = M^3 & & \\
 \downarrow & & \\
 \mathbb{R} / \mathbb{Z} & = & S^1.
 \end{array}$$

The fundamental group $\pi_1(M^3, *) = \pi_1$ is determined by an extension

$$(\star) \quad \begin{array}{ccccccc}
 1 \longrightarrow \pi_1(S^1 \times S^1, *) \longrightarrow \pi_1(M^3, *) \longrightarrow \pi_1(S^1, *) \longrightarrow 1 \\
 \parallel & & \parallel & & \parallel & & \\
 1 \longrightarrow \mathbb{Z} \times \mathbb{Z} & \longrightarrow & \pi_1 & \longrightarrow & \mathbb{Z} & \longrightarrow & 1
 \end{array}$$

which splits since \mathbb{Z} is a free group.

The manifold M^3 is acyclic since its universal covering space \tilde{M}^3 is $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$. To see this note

that there are maps $\alpha, \beta, \gamma : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$\begin{aligned}\alpha(t, x, y) &= (t, x + 1, y) \\ \beta(t, x, y) &= (t, x, y + 1) \\ \gamma(t, x, y) &= (t + 1, ax + by, cx + dy)\end{aligned}$$

with $\mathbb{R} \times (S^1 \times S^1)$ being obtained from $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ by dividing by the $(\mathbb{Z} \times \mathbb{Z})$ -action induced by α and β . The map γ induces an action of \mathbb{Z} on $\mathbb{R} \times (S^1 \times S^1)$ and the quotient space is M^3 . Hence M^3 is a $K(\pi, 1)$ -manifold with $\pi = \pi_1$ as its fundamental group. The maps α, β, γ which generate the deck transformation group of the universal covering $\tilde{M}^3 \downarrow M^3$ correspond in the usual way to generators of π_1 .

We are going to compute $H^*(M^3; \mathbb{F}_2)$ as a function of the entries of the matrix (\boxtimes) . We fix the notations introduced so far, so α, β, γ generate π_1 . By the universal coefficient theorem $H^1(M^3; \mathbb{F}_2) \cong \text{Hom}(\pi_1, \mathbb{Z}/2)$, and since $\mathbb{Z}/2$ is abelian, any homomorphism $\pi_1 \rightarrow \mathbb{Z}/2$ must vanish on the commutator subgroup $[\pi_1, \pi_1]$ of π_1 , so we start out by identifying $[\pi_1, \pi_1]$.

NOTATION: Denote by $\mathbb{Z}/2$ the cyclic group of order 2 written multiplicatively, i.e., identify $\mathbb{Z}/2$ with $\{\pm 1\}$.

The extension (\star) is determined by how γ conjugates α and β . One has

$$\begin{aligned}\gamma^{-1} \alpha^i \beta^j \gamma(t, x, y) &= \gamma^{-1} \alpha^i \beta^j (t + 1, ax + by, cx + dy) \\ &= \gamma^{-1}(t + 1, ax + by + i, cx + dy + j) = (t, x + p, y + q) \\ &= \alpha^p \beta^q(t, x, y),\end{aligned}$$

where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} i \\ j \end{bmatrix}.$$

Let $\varepsilon = \det(\mathbf{T})^{-1}$ be the inverse of the determinant of the matrix (\boxtimes) . By Cramer's rule for computing the inverse of a 2×2 matrix we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \varepsilon \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

From this we obtain for p and q the formulae

$$\begin{aligned}p &= d\varepsilon i - b\varepsilon j \\ q &= a\varepsilon j - c\varepsilon i.\end{aligned}$$

These formulae allow us to identify the commutator subgroup of π_1 . Namely, we have

$$\alpha^{-i} \beta^{-j} \gamma^{-1} \alpha^i \beta^j \gamma = \alpha^{(d\varepsilon-1)i - b\varepsilon j} \beta^{(a\varepsilon-1)j - c\varepsilon i},$$

so that the commutator subgroup lies in $\mathbb{Z} \times \mathbb{Z}$ generated by α and β and consists of the elements

$$\alpha^{(d\varepsilon-1)i - b\varepsilon j} \beta^{(a\varepsilon-1)j - c\varepsilon i}$$

where i and j are arbitrary integers. This means we can identify $[\pi_1, \pi_1]$ with the image of the homomorphism

$$\varphi : \mathbb{Z} \times \mathbb{Z} \xrightarrow{\mathbf{S}} \mathbb{Z} \times \mathbb{Z} \xrightarrow{\alpha \times \beta} \pi_1,$$

where $\mathbb{Z} \times \mathbb{Z}$ is embedded in π_1 by $\alpha \times \beta$ and \mathbf{S} is the matrix

$$(\ast) \quad \mathbf{S} = \begin{bmatrix} d\varepsilon - 1 & -b\varepsilon \\ -c\varepsilon & a\varepsilon - 1 \end{bmatrix}.$$

Therefore

$$\frac{\pi_1}{[\pi_1, \pi_1]} \cong \mathbb{Z} \times \left(\frac{\mathbb{Z} \times \mathbb{Z}}{\text{Im}(\varphi)} \right).$$

The first requirement on the fibring $S^1 \times S^1 \hookrightarrow M^3 \downarrow S^1$ is that the fundamental group of the base act trivially on the cohomology of the fibre. The fundamental group of the base S^1 is \mathbb{Z} with generator γ and

$$H^1(S^1 \times S^1; \mathbb{F}_2) = \text{Hom}(\pi_1(S^1 \times S^1, *), \mathbb{Z}/2) = \mathbb{F}_2 \times \mathbb{F}_2.$$

The element γ acts by the reduction of the matrix \mathbf{S} from $\text{GL}(2, \mathbb{Z})$ to $\text{GL}(2, \mathbb{F}_2)$. In order that this action be trivial one must therefore have

$$\alpha^i \beta^j = \alpha^{d\epsilon i - b\epsilon j} \beta^{a\epsilon j - c\epsilon i}$$

which yields the following conditions.

$$\begin{aligned} (\oplus) \quad & d\epsilon j - b\epsilon j \equiv i \pmod{2} \\ & a\epsilon j - c\epsilon i \equiv j \pmod{2} \end{aligned}$$

Since $\epsilon = \det(\mathbf{T}) = \pm 1$ this gives

$$d \equiv 1 \equiv a, \quad b \equiv c \pmod{2}$$

as the conditions for the fundamental group of the base to act trivially on the cohomology of the fibre of the bundle $S^1 \times S^1 \hookrightarrow M^3 \downarrow S^1$.

Using the identification of $H^1(M^3; \mathbb{F}_2)$ with $\text{Hom}(\pi_1, \mathbb{Z}/2)$ lets us define cohomology classes $x, y, z \in H^1(M^3; \mathbb{F}_2)$ by specifying them as homomorphisms $\pi_1 \rightarrow \mathbb{Z}/2$ as follows.

$$x \begin{cases} \alpha \mapsto -1 \\ \beta \mapsto 1 \\ \gamma \mapsto 1 \end{cases} \quad y \begin{cases} \alpha \mapsto 1 \\ \beta \mapsto -1 \\ \gamma \mapsto 1 \end{cases} \quad z \begin{cases} \alpha \mapsto 1 \\ \beta \mapsto 1 \\ \gamma \mapsto -1 \end{cases}$$

The cohomology class $z \in H^1(M^3; \mathbb{F}_2)$ comes from the base so $z^2 = 0$. The classes $x, y \in H^1(M^3; \mathbb{F}_2)$ restrict to become generators for the cohomology of the fibre $H^1(S^1 \times S^1; \mathbb{F}_2)$, so the fibration is totally nonhomologous to zero as required.

From this one sees that $H^*(M^3; \mathbb{F}_2)$ is a free module over $H^*(S^1; \mathbb{F}_2)$ with basis $1, x, y, xy$ so the product structure indicated in Diagram 7.1 indeed holds. In addition $z^2 = 0$. Thus the catalecticant matrix for $H^*(M^3; \mathbb{F}_2)$ agrees with (\otimes) in most of its entries. For the remaining entries, begin by writing

$$x^2 = kxz + lyz + mxy$$

and

$$y^2 = nxz + oyz + pxy,$$

where $k, l, m, n, o, p \in \mathbb{F}_2$. Since x^2 and y^2 restrict to zero on the fibre both m and p are zero, so one has

$$\begin{aligned} (\star) \quad & x^2 = kxz + lyz + mxy, \\ & y^2 = nxz + oyz + pxy, \\ & x^2 + y^2 = (x + y)^2 = (k + n)xz + (l + o)yz. \end{aligned}$$

Any homomorphism sending π_1 to $\mathbb{Z}/2$ must send the commutator subgroup $[\pi_1, \pi_1]$ to the unit element. Recall that $[\pi_1, \pi_1] = \text{Im}(\varphi)$ where $\varphi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ is given by the matrix

$$\mathbf{S} = \begin{bmatrix} d\epsilon - 1 & -b\epsilon \\ -c\epsilon & a\epsilon - 1 \end{bmatrix}$$

after identifying $\mathbb{Z} \times \mathbb{Z}$ with the subgroup of π_1 generated by α and β . Since we are assuming the fundamental group of the base of the fibration $S^1 \times S^1 \hookrightarrow M^3 \downarrow S^1$ acts trivially on the cohomology of the fibre the relations (\clubsuit) must hold, so $d\varepsilon - 1$, $-b\varepsilon$, $-c\varepsilon$, and $a\varepsilon - 1$ are all even and $\text{Im}(\varphi) \leq 2 \cdot \mathbb{Z} \times 2 \cdot \mathbb{Z}$. Moreover $\text{Im}(\varphi) = 2 \cdot \mathbb{Z} \times 2 \cdot \mathbb{Z}$ if and only if

$$\det \begin{bmatrix} \frac{d\varepsilon - 1}{2} & \frac{-b\varepsilon}{2} \\ \frac{-c\varepsilon}{2} & \frac{a\varepsilon - 1}{2} \end{bmatrix} = \pm 1,$$

which is equivalent to $\pi_1/[\pi_1, \pi_1] = \mathbb{Z} \times (\mathbb{Z}/2 \times \mathbb{Z}/2)$. This in turn can be the case precisely if none of the homomorphisms x , y , $x + y$ can be lifted from $\mathbb{Z}/2$ to $\mathbb{Z}/4$. From the Bockstein exact sequence

$$H^1(M^3; \mathbb{Z}/4) \xrightarrow{\text{reduction}} H^1(M^3; \mathbb{Z}/2) \xrightarrow{\text{Sq}^1} H^2(M^3; \mathbb{Z}/2)$$

we deduce as an equivalent condition that none of the classes

$$x^2 = \text{Sq}^1(x), \quad y^2 = \text{Sq}^1(y), \quad x^2 + y^2 = \text{Sq}^1(x + y)$$

should be zero, which means

$$\det \begin{bmatrix} d\varepsilon - 1 & -b\varepsilon \\ -c\varepsilon & a\varepsilon - 1 \end{bmatrix} = \pm 4.$$

Since $\varepsilon = \det(\mathbf{S}) = \pm 1$ this amounts to the following two conditions.

$$\begin{aligned} 1 - a\varepsilon - d\varepsilon + ad - bc &= \pm 4 \\ 1 - a\varepsilon - d\varepsilon + \varepsilon &= \pm 4 \end{aligned}$$

We also have the formulae (\star) for x^2 , y^2 , and $(x + y)^2$ from which it follows that none of them are zero if and only if x^2 and y^2 span the same 2-dimensional subspace of $H^2(M^3; \mathbb{F}_2)$ as xz and yz (cf. Diagram 7.1). So using formulae (\star) we conclude we must have

$$(\star) \quad 0 \neq \det \begin{bmatrix} k & l \\ n & o \end{bmatrix} = ko + ln.$$

Since $z^2 = 0$ we have $x^2z = kxz^2 + lyz^2 = 0$ and $y^2z = nxz^2 + oyz^2 = 0$. Then

$$x^3 = kx^2 + lxyz = lxyz$$

and

$$y^3 = nxyz + oy^2z = nxyz.$$

Also

$$x^2y = kxyz + ly^2z = kxyz$$

and

$$y^2x = nx^2z + oxyz = oxyz.$$

Since $xyz \in H^3(M^3; \mathbb{F}_2)$ is the fundamental class this gives us the following table of products in $H^*(M^3; \mathbb{F}_2)$.

$H^*(M^3; \mathbb{F}_2)$	x^2	y^2	z^2
x	l	o	0
y	k	n	0
z	0	0	0

TABLE 7.2: Partial Matrix of Products for $H^*(M^3; \mathbb{F}_2)$

We need still more information to complete this table to a catalecticant matrix for $H^*(M^3; \mathbb{F}_2)$ and thereby determine it. First note that this table tells us that

$$\begin{aligned} \text{Sq}^1(xy) &= x^2y + xy^2 = (k + o)xyz \\ \text{Sq}^1(xz) &= x^2z + xz^2 = 0 \\ \text{Sq}^1(yz) &= y^2z + yz^2 = 0. \end{aligned}$$

This means that the first Wu class $\text{Wu}_1(M^3)$ is nonzero if and only if $k + o \equiv 1 \pmod{2}$. Since $\text{Wu}_1(M^3) = w_1(M^3)$ is the first Stiefel–Whitney class of M^3 this is a question of whether M^3 is orientable or not.

To determine if M^3 is orientable or not, consider the action of $\pi_1(M^3)$ on the universal covering manifold $\tilde{M}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. The translations α and β preserve the orientation, whereas γ preserves the orientation for $\varepsilon = \det(\mathbf{S}) = 1$ and reverses it for $\varepsilon = \det(\mathbf{S}) = -1$. Thus M^3 is an orientable manifold for $\varepsilon = 1$ and a nonorientable manifold for $\varepsilon = -1$.

Notice also that the Wu class $\text{Wu}_1(M^3) = w_1(M^3)$ restricts trivially on the fibre, since the fibre $S^1 \times S^1$ is orientable. Therefore $\text{Wu}_1(M^3)$ is a multiple of z and it follows that $\text{Wu}_1(M^3) = z$ if $\varepsilon = -1$ and $\text{Wu}_1(M^3) = 0$ if $\varepsilon = 1$.

If $\varepsilon = 1$ then $\text{Wu}_1(M^3) = 0$ so Sq^1 maps $H^2(M^3; \mathbb{F}_2)$ trivially to $H^3(M^3; \mathbb{F}_2)$ and $(k + o)xyz = \text{Sq}^1(xy) = 0$. The matrix of products in Table 7.2 becomes

$H^*(M^3; \mathbb{F}_2)$	x^2	y^2	z^2
x	l	k	0
y	k	n	0
z	0	0	0

and is symmetric. Since the upper 2×2 block of $(*)$ is asymmetric we conclude that a manifold realizing orbit 10 cannot be orientable. To finish with this case, note that the determinant of the upper 2×2 block is nonzero by (\star) so there can only be the three possibilities listed next.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For $\varepsilon = -1$ the manifold is nonorientable, the Wu class $\text{Wu}_1(M^3) = z$, and Sq^1 maps $H^2(M^3; \mathbb{F}_2)$ nontrivially to $H^3(M^3; \mathbb{F}_2)$, so $(k + o)xyz = \text{Sq}^1(xy) = xyz$ and $k + o \equiv 1 \pmod{2}$. In this case exactly one of k and o is zero and the other is one so the matrix (\star) has determinant ln . We saw however that this determinant is 1, so both l and n are nonzero. Hence x^3 , y^3 , and $(x + y)^3$ are all nonzero. We can at this point fill in the catalecticant matrix for $H^*(M^3; \mathbb{F}_2)$ and find the following.

$H^*(M^3; \mathbb{F}_2)$	x^2	y^2	z^2	xy	xz	yz
x	1	1	0	$1 + o$	0	1
y	0	1	0	o	1	0
z	0	0	0	1	0	0

TABLE 7.3: Matrix of Products for $H^*(M^3; \mathbb{F}_2)$

In this catalecticant matrix $o = 0$ or 1 . Reversing the roles of x and y interchanges $o = 1$ with $o = 0$, and this would be a manifold whose cohomology realized the catalecticant matrix $(*)$ corresponding to orbit 10 of $\text{Cat}_{\mathbb{F}_2}(2, 1)$.

It only remains to exhibit a matrix \mathbf{S} with all the required properties for $H^*(M^3; \mathbb{F}_2)$ to realize the algebra of orbit 10 of $Cat_{\mathbb{F}_2}(1, 2)$ in the list of orbits in § 5, viz.,

$$\begin{aligned} a &\equiv 1 \equiv d, \quad b \equiv 0 \equiv c \pmod{2} \\ ad - bc &= -1 = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \varepsilon \in \mathbb{Z} \\ 1 - a\varepsilon - d\varepsilon + \varepsilon &= 1 + a + d - 1 = a + d = \pm 4 \in \mathbb{Z}. \end{aligned}$$

One such matrix is

$$\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}.$$

§8. Indecomposable Threefolds of Arbitrary Rank

We saw in § 3 that the Grothendieck group of standardly graded Poincaré duality algebras of formal dimension $d > 2$ is free abelian but not finitely generated. Specifically we showed in Example 1 of § 3 that for a fixed $d > 2$ the infinite family of such algebras,

$$H^* \left((S^1 \times \cdots \times S^1) \times ((S^1 \times S^1) \# \cdots \# (S^1 \times S^1)); \mathbb{F}_2 \right) \quad r \in \mathbb{N},$$

$\xleftarrow{d-2} \quad \xleftarrow{r}$

have formal dimension d , rank $d - 2 + 2r$, are pairwise nonisomorphic, and $\#$ -indecomposable for $d - 2 + 2r > 2$. We refer to these as the **standard** examples of $\#$ -indecomposables of a given formal dimension. Our purpose in this section is to provide constructions for nonstandard such examples of formal dimension $d = 3$. To do so we will avail ourselves of Macaulay's Double Annihilator Correspondence as explained in [16] Part VI Sections 1 and 2 which we next review in abridged form.

Fix an integer n and introduce the **inverse polynomial** algebra $\mathbb{F}_2[x_1^{-1}, \dots, x_n^{-1}]$ which we define to be the algebra of polynomials in the formal variables $x_1^{-1}, \dots, x_n^{-1}$ each of degree -1 . There is an $\mathbb{F}_2[x_1, \dots, x_n]$ -module structure on $\mathbb{F}_2[x_1^{-1}, \dots, x_n^{-1}]$, whose product we denote by \cap since it is similar to the cap product between cohomology and homology, and which is defined on monomials by

$$x^E \cap x^{-F} = \begin{cases} x^{-F+E} & \text{if } F - E \in \mathbb{N}_0^n \\ 0 & \text{otherwise,} \end{cases}$$

where $E, F \in \mathbb{N}_0^n$. The \cap -product is extended to all elements of $\mathbb{F}_2[x_1, \dots, x_n]$ and $\mathbb{F}_2[x_1^{-1}, \dots, x_n^{-1}]$ by bilinearity. Macaulay's Double Annihilator Theorem (loc. cit) tells us that there is a bijective correspondence between nonzero elements of $\mathbb{F}_2[x_1^{-1}, \dots, x_n^{-1}]$ of degree $-d$ and maximal primary irreducible ideals in $\mathbb{F}_2[x_1, \dots, x_n]$ whose Poincaré duality quotient has formal dimension d : This correspondence associates to an inverse form θ its annihilator ideal $I(\theta)$.

For $d = 3$ this means each inverse cubic form

$$\theta = \sum_{i,j,k \in S} x_i^{-1} x_j^{-1} x_k^{-1} \in \mathbb{F}_2[x_1^{-1}, \dots, x_n^{-1}],$$

where S is a subset of the set of unordered triples of elements of $\{1, \dots, n\}$, defines a 3-dimensional Poincaré duality algebra $H(\theta) = \mathbb{F}_2[x_1, \dots, x_n]/I(\theta)$. We are interested in finding sets of triples S so that $H(\theta)$ is $\#$ -indecomposable.²⁰

²⁰ A natural choice for such a form is the sum of all the inverse trinomials. For an analysis of the Poincaré duality algebras defined in this way see [21] §5.

Given an inverse cubic form $\theta = \sum_{i,j,k \in S} x_i^{-1} x_j^{-1} x_k^{-1} \in \mathbb{F}_2[x_1^{-1}, \dots, x_n^{-1}]$ the set of trinomials $x_i^{-1} x_j^{-1} x_k^{-1}$ that appear in this sum is the **support** of θ ; denoted by $\text{supp}(\theta)$. Note that the trinomials $x_r x_s x_t$ of $\mathbb{F}_2[x_1, \dots, x_n]$ that project to a fundamental class of $H(\theta)$ are precisely those for which the inverse trinomial $x_r^{-1} x_s^{-1} x_t^{-1}$ belongs to $\text{supp}(\theta)$. Hence if x_i does not occur in some trinomial in $\text{supp}(\theta)$ then there is no quadratic form q such that $x_i q \neq 0 \in H(\theta)$. Therefore $x_i = 0 \in H(\theta)$, so $x_i \in I(\theta)$, and from the point of view $H(\theta)$ it is as though x_i were not there, so we might as well discard it. So from this point on we assume that every variable x_i^{-1} occurs in some trinomial in $\text{supp}(\theta)$.

For $1 \leq i \leq j \leq n$ we say that the variables x_i and x_j are **linked** in θ , denoted by $x_i \stackrel{\theta}{\equiv} x_j$, if there are trinomials $t_1, \dots, t_s \in \text{supp}(\theta)$ such that

- (a) t_i and t_{i+1} have a common factor (are **directly linked**)
- (b) x_i^{-1} divides t_1 and x_j^{-1} divides t_s .

Linking is an equivalence relation²¹ and the equivalence classes are called the **components**. If there is only one equivalence class then θ is called **connected**. The following lemma is a direct consequence of the definitions and Lemma 1.1.

LEMMA 8.1: *Suppose that $\theta = \sum_{i,j,k \in S} x_i^{-1} x_j^{-1} x_k^{-1} \in \mathbb{F}_2[x_1^{-1}, \dots, x_n^{-1}]$ is an inverse cubic form. If θ is not connected then $H(\theta)$ is #-decomposable. \square*

So a #-indecomposable threefold corresponds to a connected inverse cubic form. In the sequel we will concentrate such forms and seek additional conditions that force the corresponding Poincaré duality quotient algebra to be #-indecomposable.

LEMMA 8.2: *Suppose that $\theta = \sum_{i,j,k \in S} x_i^{-1} x_j^{-1} x_k^{-1} \in \mathbb{F}_2[x_1^{-1}, \dots, x_n^{-1}]$ is an inverse cubic form satisfying the conditions*

- (i) every x_i^{-1} divides at least one trinomial in the support of θ , and
- (ii) every inverse binomial $x_i^{-1} x_j^{-1}$ with $i \neq j$ belongs to exactly one trinomial in the support of θ .

Then θ is connected.

PROOF: The conditions (i) and (ii) imply that any two inverse variables x_i^{-1} and x_j^{-1} belong to a trinomial in $\text{supp}(\theta)$ and so they are directly linked by that trinomial. \square

An inverse cubic form $\theta = \sum_{i,j,k \in S} x_i^{-1} x_j^{-1} x_k^{-1} \in \mathbb{F}_2[x_1^{-1}, \dots, x_n^{-1}]$ satisfying the conditions of Lemma 8.2 will be called **admissible**.

LEMMA 8.3: *If $\theta = \sum_{i,j,k \in S} x_i^{-1} x_j^{-1} x_k^{-1} \in \mathbb{F}_2[x_1^{-1}, \dots, x_n^{-1}]$ is an admissible inverse cubic form then the corresponding Poincaré duality algebra $H(\theta)$ has rank n .*

PROOF: This is because x_i^{-1} occurs in some inverse trinomial $t = x_i^{-1} x_j^{-1} x_k^{-1}$ of the support of θ and hence by the remark preceding Lemma 8.1 $x_i x_j x_k$ represents a fundamental class of $H(\theta)$. Denote by x_i^* the binomial $x_j x_k$. Suppose that we had a linear relation

$$(*) \quad a_1 x_1 + \dots + a_n x_n = 0$$

in $H(\theta)$. Taking the product with x_i^* and using that t is the only inverse trinomial in $\text{supp}(\theta)$ divisible by $x_j^{-1} x_k^{-1}$ we get $0 = a_i x_i x_i^* = a_i x_i x_j x_k \in H(\theta)$. Since $x_i x_j x_k$ is a fundamental class this forces $a_i = 0$ and since $1 \leq i \leq n$ was arbitrary the linear relation $(*)$ is trivial; whence $x_1, \dots, x_n \in H(\theta)_1$ are linearly independent. \square

²¹ Assuming as we did above that every variable x_i for $1 \leq i \leq n$ occurs in some trinomial in the support of θ .

PROPOSITION 8.4: *Suppose that $\theta = \sum_{i,j,k \in S} x_i^{-1} x_j^{-1} x_k^{-1} \in \mathbb{F}_2[x_1^{-1}, \dots, x_n^{-1}]$ is an admissible inverse cubic form satisfying the condition that no trinomial in the support of θ is divisible by the square of one of the variables x_i^{-1} for $i = 1, \dots, n$. Then the ideal $I(\theta) \subset \mathbb{F}_2[x_1, \dots, x_n]$ is generated by quadratic forms.*

PROOF: We begin by assembling several facts and establishing a bit of notation. First of all, since $H(\theta)$ has rank n by Lemmas 8.2 and 8.1, the homogeneous components of $H(\theta)$ in degrees 1 and 2 both have dimension n . Since x_1, \dots, x_n is a basis for $H(\theta)_1$ it follows in the notation of the proof of Lemma 8.3 that x_1^*, \dots, x_n^* is a basis for $H(\theta)_2$.

Next let J be the ideal in $\mathbb{F}_2[x_1, \dots, x_n]$ generated by the quadratic forms in $I(\theta)$ and set $Q = \mathbb{F}[x_1, \dots, x_n]/J$. We are going to show that $I(\theta) = J$. Since $I(\theta)$ and J are identical in degree 2 so are $H(\theta)$ and Q . This means that x_1^*, \dots, x_n^* is also a basis for Q_2 . Therefore the products $x_r x_s^*$ span Q in degree 3. Finally, remember that $x_1^2, \dots, x_n^2 \in I(\theta)$ since no inverse trinomial in the support of θ is divisible by the square of one of the variables x_i^{-1} for $i = 1, \dots, n$. So $x_1^2, \dots, x_n^2 \in J$ also.

Since neither $I(\theta)$ nor J contain any linear forms they also agree in degree one. To show that $I(\theta) = J$ we need to show they also agree in degree three. Since the natural map $Q \rightarrow H(\theta)$ induced by the inclusion $J \subseteq I(\theta)$ is an epimorphism this will be the case if and only if Q_3 is 1-dimensional. Since Q is generated as an algebra by its homogeneous component of degree one, x_1, \dots, x_n is a basis for Q_1 , and x_1^*, \dots, x_n^* is a basis for Q_2 , the products $x_r x_s^*$ span Q_3 , so we must show these products in fact span only a 1-dimensional subspace of Q_3 .

First consider a product with equal indices, viz., $x_r x_r^*$. Given $1 \leq i, j \leq n$ there is a unique $1 \leq k = k(i, j) \leq n$ such that $x_i^{-1} x_j^{-1} x_k^{-1}$ is in the support of θ . Hence $x_i^* = x_j x_k$ and $x_j^* = x_i x_k$, so $x_i x_i^* = x_i(x_j x_k) = x_j(x_i x_k) = x_j x_j^*$, and in Q_3 all products $x_r x_r^*$ are the same.

Next consider the case of a product $x_i x_j^*$ where $1 \leq i \neq j \leq n$ with distinct indices. Then there is a unique inverse trinomial $x_i^{-1} x_j^{-1} x_k^{-1}$ in the support of θ so $x_j^* = x_i x_k$. Hence $x_i x_j^* = x_i(x_i x_k) = x_i^2 x_k = 0 \in Q_3$ since $I(\theta)$, and also J , contains all squares x_1^2, \dots, x_n^2 .

Finally notice that Q_4 , and hence Q_i for $i \geq 4$ are zero. For if not then there would have to be a nonzero monomial in Q_4 . But such a monomial $x_i x_j x_k x_\ell$ can be written as the product of $x_i x_j x_k$ and x_ℓ . Since as already shown all trinomials are either zero or equal in Q_3 we could rewrite this as $(x_\ell^* x_\ell) x_\ell = x_\ell^* x_\ell^2 = 0$, since all squares belong to $I(\theta)$ and hence also to J .

In summary, we have shown the map $Q \rightarrow H(\theta)$ is an isomorphism, and since it was induced by the inclusion $J \subseteq I(\theta)$ we must have $J = I(\theta)$, so $I(\theta)$ is generated by the quadratic forms it contains. \square

A square free inverse cubic form (i.e., one whose support contains no trinomials of the form $x_i^{-2} x_j^{-1}$) such as considered in Proposition 8.4, defines a Poincaré duality algebra in which all squares are zero. One may therefore regard such an algebra as a quotient of the exterior algebra $E(x_1, \dots, x_n)$. If we were to do so, then we could also do away with the inverse polynomial algebra and set Macaulay's theory up so that there is a bijective correspondence between irreducible ideals and principal ideals in $E(x_1, \dots, x_n)$ given by taking annihilators. Although this is an attractive alternative we prefer to retain the inverse polynomial setup for its generality and compatibility with the existing literature.

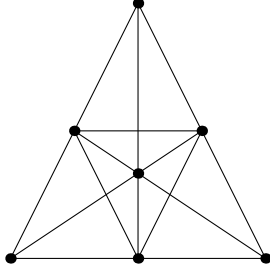
COROLLARY 8.5: *Suppose that $\theta = \sum_{i,j,k \in S} x_i^{-1} x_j^{-1} x_k^{-1} \in \mathbb{F}_2[x_1^{-1}, \dots, x_n^{-1}]$ is an admissible inverse cubic form satisfying the condition that no trinomial in the support of θ is divisible*

by the square of one of the variables x_i^{-1} for $i = 1, \dots, n$. Then the algebra $H(\theta)$ is #-indecomposable.

PROOF: The ideal of a decomposable threefold always requires generators of degree three.²² For let $H = H' \# H''$ be a decomposable threefold. Let $H' = \mathbb{F}_2[x'_1, \dots, x'_{n'}]/I'$ and $H'' = \mathbb{F}_2[x''_1, \dots, x''_{n''}]/I''$. If $x'_i x'_j x'_k$ represents a fundamental class of H' and $x''_i x''_j x''_k$ represents a fundamental class of H'' then their difference, viz., $x'_i x'_j x'_k - x''_i x''_j x''_k \in \mathbb{F}_2[x'_1, \dots, x'_{n'}, x''_1, \dots, x''_{n''}]$ is zero in H and part of a minimal generating set for the ideal $I \subset \mathbb{F}_2[x'_i x'_j x'_k, x''_i x''_j x''_k]$ whose quotient algebra is H as follows from Diagram 1.1 in § 1. \square

We still have the problem of supplying examples of inverse cubic forms satisfying the conditions required in Corollary 8.5. In three variables there is only one, namely $x^{-1}y^{-1}z^{-1}$ and the Poincaré duality algebra it defines is $H^*(S^1 \times S^1 \times S^1; \mathbb{F}_2)$ (see Orbit 21 in § 5). There are none in four, five, or six variables (see Lemma 8.6 to follow), so seven variables would be the next place to look, and here is one such.

EXAMPLE 1: This example is based on the barycentric subdivision of a triangle as pictured



adjacent. So $n = 7$. The seven inverse variables $x_1^{-1}, \dots, x_7^{-1}$ are indexed by the seven vertices of the barycentric subdivision as pictured. The monomials in the support of θ are the products of the variables corresponding to vertices on any of the six lines containing three vertices, and the product of the vertices forming the inner triangle. This is easily seen to be admissible with no trinomial in $\text{supp}(\theta)$ divisible

by the square of an inverse variable. This example is not isomorphic to the standard #-indecomposable example $H^*(S^1 \# (S^1 \times S^1) \# (S^1 \times S^1) \# (S^1 \times S^1); \mathbb{F}_2)$ that we constructed in Example 1 of section 3. To see this one computes the dimension of the principal ideals generated by the linear forms: For the standard example this is always 2 but for the example derived from the barycentric subdivision of the triangle the dimension of the principal ideal $(x_i + x_j)$ for $i \neq j$ is 4. (In fact there are only two #-indecomposable threefolds of rank seven and this is the *other* one.)

Before looking at further examples we note that there is a restriction²³ on n , the number of variables, imposed by the conditions of Corollary 8.5.

LEMMA 8.6: Suppose that $\theta = \sum_{i,j,k \in S} x_i^{-1} x_j^{-1} x_k^{-1} \in \mathbb{F}_2[x_1^{-1}, \dots, x_n^{-1}]$ is an admissible inverse cubic form. Then n must be odd and $n \equiv 0$ or $1 \pmod{3}$.

PROOF: To see this note that there are $\binom{n}{2}$ inverse quadratic monomials $x_i^{-1} x_j^{-1}$ and each inverse trinomial $x_i^{-1} x_j^{-1} x_k^{-1}$ in $\text{supp}(\theta)$ contains 3 such pairs, viz., $x_i^{-1} x_j^{-1}$, $x_j^{-1} x_k^{-1}$, $x_i^{-1} x_k^{-1}$. So the number of trinomials appearing in θ , i.e., the cardinality of the set S , is $\frac{1}{3} \binom{n}{2} = \frac{n(n-1)}{6}$ so $n(n-1)$ is divisible by 6. Since certainly one of the two numbers n , $n-1$ is even this gives that $n \equiv 0$ or $1 \pmod{3}$.

²²The argument that follows actually shows that the ideal defining a #-decomposable standardly graded Poincaré duality algebra of formal dimension d always requires a generator of degree d .

²³In fact it will follow from the appended letter of R.E. Stong that these conditions are sufficient for the existence of a square free admissible inverse cubic form as was shown by Kirkman [10]. See the discussion in §15.4 of [8].

For each x_i^{-1} there are $n - 1$ inverse quadratic forms $x_i^{-1}x_j^{-1}$ with $j \neq i$. Each of these occurs in exactly one inverse trinomial $x_i^{-1}x_j^{-1}x_k^{-1}$ in the support of θ . Thus the number of trinomials in $\text{supp}(\theta)$ is $\frac{n-1}{2}$ (interchange the roles of j and k) so $\frac{n-1}{2}$ must be an integer, and hence n is odd. \square

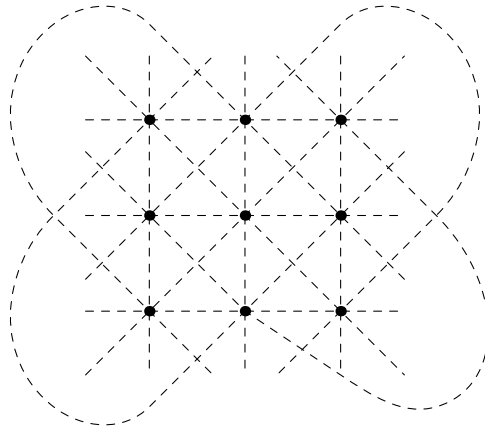
So the smallest rank of a nonstandard #-indecomposable threefold defined by a square free admissible inverse cubic form is seven and we saw an example of such a form already. The next possible rank of a form fulfilling the conditions of Corollary 8.5 in view of Lemma 8.6 is nine. The classical configuration of inflection points on a nonsingular cubic curve (see e.g., [3] pages 377 et seq.) provide a means of constructing such an example.

EXAMPLE 2: Fix a singularity free plane cubic curve $C \subset \mathbb{C}\mathbb{P}(2)$, say

$$C = \left\{ [z_0, z_1 z_2] \mid z_0^3 + z_1^3 + z_2^3 + z_0 z_1 z_2 = 0 \right\}.$$

This curve is elliptic and admits an abelian group structure: Roughly speaking, if p, q belong to C they determine a line in $\mathbb{C}\mathbb{P}(2)$ and one first defines a third point on C as the remaining point²⁴ of intersection of that line with the curve. The abelian group structure is derived from this operation (loc. cit.).

A nonsingular cubic curve has nine inflection points. If $p, q, r \in C$ are inflection points then $p, q,$ and r are collinear if and only if their sum in the group structure is zero. In this way we get twelve lines passing through triples of collinear inflection points on C . The *pattern* of points on a one of these lines as illustrated in the accompanying graphic is either horizontal, vertical, or one of the six patterns involved in the rule for determinants of 3×3 matrices.



We use the inflection points of C to index nine variables and the twelve lines to index twelve trinomials: Each trinomial is the product of the variables indexed by the inflection points in the line. Taking these trinomials to be the support of a square free inverse cubic form in nine variables we get one fulfilling the conditions of Corollary 8.5 of rank nine.

In the graphic the points could be interpreted as the nine *points* of the vector space \mathbb{F}_3^2 and the twelve *lines* as the lines in this vector space, the essential issue being that each line contains exactly three points. Using larger dimensional vector spaces leads to further examples of forms satisfying the conditions of Corollary 8.5. The incidence relations between the lines and points used to index trinomials and variables in these examples are reminiscent of finite geometries. This is not an accident as the appendix to this manuscript shows.

²⁴ There is the possibility that p or q has multiplicity 2 as a point of intersection.

In addition to these examples, Example 1 is the first of an infinite family of similar examples with ranks $2^m - 1$ in which the variables are indexed by the nonzero elements of the vector space \mathbb{F}_2^m . Here is how this works.

EXAMPLE 3: Let $m \in \mathbb{N}$ and consider two distinct nonzero vectors $v', v'' \in \mathbb{F}_2^m \setminus \{0\}$. Then v' and v'' are linearly independent and the 2-dimensional subspace they span consists of the four points $\{v', v'', v' + v'', 0\}$. The $n = 2^m - 1$ variables x_v are to be indexed by the nonzero vectors $v \in \mathbb{F}_2^m \setminus \{0\}$. The trinomials in the support of the inverse cubic form γ_m in these variables are indexed by the two dimensional subspaces $U = \{v', v'', v' + v'', 0\}$, where $v' \neq v''$ are nonzero, and the trinomial $t_U = x_{v'}x_{v''}x_{v'+v''} \in \text{supp}(\gamma_m)$. So there are $\frac{n(n-1)}{6}$ trinomials in the support of γ_m . (The minimal size of $\text{supp}(\theta)$ for a connected inverse cubic form in n variables is $\frac{n-1}{2}$.) This example can also be thought of as having trinomials indexed by the code words of weight three in the Hamming code JH_m (see e.g., [6] chapter 3 §3.1).

§9. Comments by Larry Smith on the Appended Letter of R.E. Stong

The authors speculated for a while that examples illustrating Corollary 8.5 could only exist if the number of variables was $2^m - 1$, for some $m \in \mathbb{N}$, because we had the Examples 1 and 3 of §8 and no others at the start. However, it soon became apparent that this was not the case, as we both (re)discovered basically the same example in nine variables described in two different ways in Example 2 above. Other examples followed.

Then Bob realized that there was a way to organize almost all of our examples around the classical notion of a *Steiner Triple System*. Although he informed me of this and sent me details and examples he did not write out a systematic account until a month before he died. He sent copies of the letter (which took him several days to write) containing detailed proofs to Peter Landweber and myself. It is a classical example of Bob's style and tenacity in approaching detailed computational problems, not to mention the elegance of the final result. This was entirely his discovery, so in agreement with Bob's family and Peter Landweber this account is appended to the current manuscript. To retain as much of Bob's character as possible it is formatted as closely to the original as the differences between mediums allows. It would also have been appropriate to reproduce it in a digitalization of Bob's lovely handwriting, but despite my asking him several times to provide me with large enough drawings to code such a font, he never did.²⁵

The only Poincaré duality algebras considered in Bob's letter are generated by their homogeneous components of degree one: This is almost never explicitly mentioned in the letter. A few notational dissimilarities with the rest of the manuscript occur, e.g., Bob writes the grading index as an exponent instead of as a subscript, calls the Poincaré duality algebra P instead of H , writes the projective space $\mathbb{RP}(k)$ as \mathbb{RP}^k , as well as a number of other minor differences. Footnotes have been added to clarify some few cases where confusion could arise. In a very few instances I have changed notation or an index that seemed an obvious misprint to me. I take full responsibility for all errors and no credit for any of the results.

The shortness of the individually dated blocks of text in this letter shows how difficult it must have been for him to write these details. (In recent years I received three or four letters a week, each 4 – 7 pages long).

²⁵This footnote is typeset in a digitalization of my father's calligraphy.

Appendix

Steiner Systems and Poincaré Duality Algebras

A Letter from R.E. Stong to Peter Landweber and Larry Smith

11 March 2008

⋮

My interest comes from correspondence with Larry that used Steiner systems²⁶ to build Poincaré duality algebras. In looking back at that material what I find is completely incoherent. It is a jumble of examples.²⁷ I decided to try to present the material a bit more meaningfully – and decided to write it for both of you.

—x—

So what is a Steiner system? A **Steiner system** $S(t, k, v)$, $2 \leq t < k < v \in \mathbb{N}$ is a set $V = \{1, 2, \dots, v\}$ and a collection \mathcal{B} of k -element subsets B of V (called **blocks**). A block $B \in \mathcal{B}$ will be written $\{i_1, i_2, \dots, i_k\}$ or $\{i_1 < i_2 < \dots < i_k\}$ if order is given. The condition that is satisfied is that for every t -element subset $\{i_1, \dots, i_t\}$ of V there is exactly one block B of \mathcal{B} , $\{i_1, \dots, i_t, j_1, \dots, j_{k-t}\}$ containing $\{i_1, \dots, i_t\}$.

Being given a set $V = \{1, 2, \dots, v\}$ and a collection \mathcal{B} of k -element subsets of V one has defined a Poincaré duality algebra $P = \mathbb{F}_2[x_1, \dots, x_v]/I(\theta)$ given by a Macaulay class $\theta = \sum x_{i_1}^{-1} \dots x_{i_k}^{-1}$ in $\mathbb{F}_2[x_1^{-1}, \dots, x_v^{-1}]$. Here θ is the sum of those monomials $x_{i_1}^{-1} \dots x_{i_k}^{-1}$ for which $\{i_1, i_2, \dots, i_k\} = B \in \mathcal{B}$. This is the same as saying that the product $x_{i_1} \dots x_{i_k}$ is a nonzero element of P^k if and only if $\{i_1, i_2, \dots, i_k\} = B \in \mathcal{B}$.

P is a nice little exterior algebra.²⁸ It is a quotient of $E = \mathbb{F}_2[x_1, \dots, x_v]/(x_1^2, \dots, x_v^2)$ and with $x_i^2 = 0$ and working over the field \mathbb{F}_2 one has $x^2 = 0$ for all $x \in P^1 = \text{Span}\{x_1, \dots, x_v\}$. It is k -dimensional²⁹ with $P^k \cong \mathbb{F}_2$.

EXAMPLE: If $\mathcal{B} = \{B\}$ consists of a single block $B = \{1 < 2 < \dots < k\}$ then $P = \mathbb{F}_2[x_1, \dots, x_v]/(x_1^2, \dots, x_k^2, x_{k+1}, \dots, x_v) \cong H^*(\underbrace{\mathbb{R}P^1 \times \dots \times \mathbb{R}P^1}_{k \text{ copies}}; \mathbb{F}_2)$. This is a Poincaré duality algebra of rank k ; rank $P = \dim P^1$.

EXAMPLE: If $\mathcal{B} = \{\{1, 2, \dots, k\}, \{k+1, \dots, 2k\}\}$ consists of 2 blocks, $2k = v$, then $P = Q \# R$ is the connected sum of two subalgebras Q generated by x_1, \dots, x_k and R generated by x_{k+1}, \dots, x_{2k} . Here $P^j = Q^j \oplus R^j$ for $1 \leq j < k$ and $P^0 = Q^0 = R^0$ with $Q^1 \cdot R^1 = 0$.

These are nice enough little Poincaré algebras, but not terribly nice.

The question becomes what properties P has if \mathcal{B} is a Steiner system. For example, if \mathcal{B} is a Steiner system is rank $P = v$? If \mathcal{B} is a Steiner system, is P indecomposable under connected sum? These are properties P may or may not have.

²⁶L.S.: Steiner systems are special types of block designs. Background can be found in [8] and there are a number of interesting exercises in [2] §14.

²⁷L.S. We were behaving a bit like 19-th century naturalists on a newly discovered tropical island so to speak at the time, indulging ourselves in examining each and every new flower (i.e., example) in detail and loosing track of searching for an organizing principal.

²⁸L.S.: What Bob means here is that all squares are zero in P , i.e., that it is a quotient algebra of the exterior algebra $E(x_1, \dots, x_n) = \mathbb{F}_2[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$ as he explains in the next sentence.

²⁹L.S.: Clearly Bob meant of formal dimension k . I have changed this accordingly in the sequel.

—×—

If P is the Poincaré duality algebra defined by \mathcal{B} , then the dual pairing $P^t \times P^{k-t} \rightarrow P^k \cong \mathbb{F}_2$ is described by knowing the pairs of elements $(x_T, x_S) \in P^t \times P^{k-t}$ with $\theta(x_T x_S) \neq 0$. Here³⁰ we write $x_T = x_{i_1} \cdots x_{i_t}$ if $T = \{i_1, \dots, i_t\}$, we write $x_S = x_{j_1} \cdots x_{j_{k-t}}$ if $S = \{j_1, \dots, j_{k-t}\}$.

If \mathcal{B} is a Steiner system and $x_T \in P^t$, there is exactly one $B \in \mathcal{B}$ for which $B \supset T$ and exactly one $x_S \in P^{k-t}$ for which $B = T \cup S$. For $\{(x_{T'}, x_{S'}) \in P^t \times P^{k-t} \mid \theta(x_{T'} x_{S'}) \neq 0\}$ there is exactly one pair $(x_T, x_{S'})$ with $\theta(x_T x_{S'}) \neq 0$ and the $x_{S'}$ is x_S .

The dual pairing $P^t \times P^{k-t} \rightarrow P^k = \mathbb{F}_2$ may also be described as an isomorphism $P^t \rightarrow \text{Hom}(P^{k-t}, \mathbb{F}_2)$. This homomorphism sends x_T to the homomorphism that sends $x_{S'}$ to $\theta(x_T x_{S'})$. For x_T , this sends x_S to 1 and $x_{S'}$ to zero if $S' \neq S$. That homomorphism is usually denoted by x_S^* (i.e., if written in the ‘basis’ $x_{S'}$ it is the dual basis element).

12 March 2008

The dual pairing $P^t \times P^{k-t} \rightarrow P^k = \mathbb{F}_2$ admits a third description as an isomorphism $P^{k-t} \rightarrow \text{Hom}(P^t, \mathbb{F}_2)$ but this homomorphism is harder to understand.

In the case $k-t \geq t$, $x_T \in P^t$ gives the block $B = T \cup S$ and there is a unique block containing S . Thus $\theta(x_T x_S) = 1$ and $\theta(x_{T'} x_S) = 0$ if $T' \neq T$. Then $x_S \in P^{k-t}$ defines the homomorphism $x_T^* : P^t \rightarrow \mathbb{F}_2$ with $x_T^*(x_T) = 1$, $x_T^*(x_{T'}) = 0$ for $T' \neq T$.

If $k-t < t$, there may be classes $x_{T'} \in P^t$, $T' \neq T$ for which $\theta(x_{T'} x_S) \neq 0$. One has an element $x_T^* \in \text{Hom}(P^t, \mathbb{F}_2)$ given by $x_T^*(x_T) = 1$, $x_T^*(x_{T'}) = 0$ for $T' \neq T$, but this may not be an element of $\text{Hom}(P^t, \mathbb{F}_2)$. If $x_T^* \in \text{Hom}(P^t, \mathbb{F}_2)$ then x_T^* cannot be any class of the form $x_{S'} \in P^{k-t}$.

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To try to see what is happening, let’s consider the finite projective plane with 7 elements over the field \mathbb{F}_2 . This is a set of triples $(x, y, z) \neq (0, 0, 0)$ with $x, y, z \in \mathbb{F}_2$ and the lines of this plane are the triples satisfying a homogeneous equation $ax + by + cz = 0$, $a, b, c \in \mathbb{F}_2$ not all zero.

If you identify (x, y, z) in the plane with the integer $x + 2y + 4z$ the projective plane becomes $\{1, 2, 3, 4, 5, 6, 7\}$ and the lines become blocks

$$(a, b, c) = \begin{cases} (1, 0, 0) \leftrightarrow \{2, 4, 6\} \\ (0, 1, 0) \leftrightarrow \{1, 4, 5\} \\ (0, 0, 1) \leftrightarrow \{1, 2, 3\} \\ (1, 1, 0) \leftrightarrow \{3, 4, 7\} \\ (1, 0, 1) \leftrightarrow \{2, 5, 7\} \\ (0, 0, 1) \leftrightarrow \{1, 6, 7\} \\ (1, 1, 1) \leftrightarrow \{3, 5, 6\} \end{cases}$$

This describes the projective plane as a Steiner triple system $S(2, 3, 7)$.

One now has a homomorphism $P^2 \rightarrow \text{Hom}(P^1, \mathbb{F}_2)$ which assigns to $x_i x_j$ the homomorphism x_k^* where $x_i x_j x_k$ is a block of the Steiner triple system. One has

$$\begin{aligned} x_1^* &= (x_2 x_3) = (x_4 x_5) = (x_6 x_7) \\ x_2^* &= (x_1 x_3) = (x_4 x_6) = (x_5 x_7) \\ x_3^* &= (x_1 x_2) = (x_4 x_7) = (x_5 x_6) \\ x_4^* &= (x_1 x_5) = (x_2 x_4) = (x_3 x_7) \\ x_5^* &= (x_1 x_4) = (x_2 x_7) = (x_3 x_6) \\ x_6^* &= (x_1 x_7) = (x_2 x_4) = (x_3 x_5) \\ x_7^* &= (x_1 x_6) = (x_5 x_7) = (x_3 x_4) \end{aligned}$$

³⁰L.S.: Here Bob is thinking of θ as a linear form defined on the homogeneous component of degree k of $\mathbb{F}_2[x_1, \dots, x_n]$.

This gives $\dim \operatorname{Hom}(P^1, \mathbb{F}_2) = 7 = \nu$ and one has $\operatorname{rank} P = \nu$.

Knowing that $\operatorname{rank} P = 7$ is sufficient to say that P is indecomposable.

If P is decomposable, one has $P = Q \# R$ which means $P^1 = Q^1 \oplus R^1$, $P^2 = Q^2 \oplus R^2$, $P^3 = Q^3 \oplus R^3$. Because $Q, R \subset P$ one has $x^2 = 0$ for all $x \in Q^1$ or all $x \in R^1$. Because $\mathfrak{f}\text{-dim } Q = \mathfrak{f}\text{-dim } R = 3$ one must have $\dim Q^1 \geq 3$ and $\dim R^1 \geq 3$ and then $7 = \dim P^1 = \dim Q^1 + \dim R^1$ so one of them must be 3 and the other is 4. So let's choose $\dim Q^1 = 3$, $\dim R^1 = 4$.

Now one has

FACT: *There is no Poincaré algebra M having $x^2 = 0$ for all $x \in M^1$ with $\mathfrak{f}\text{-dim } M = n$ and $\operatorname{rank} M = n + 1$.*

PROOF: Suppose $x_1 \cdots x_n \neq 0$. Then $M^1 = \operatorname{Span}\{x_1, \dots, x_n\} \oplus V$ where $V = \{y \in M^1 \mid yx_1 \cdots \widehat{x}_j \cdots x_n = 0 \ \forall j = 1, \dots, n\}$. (Here \widehat{x}_j denotes that x_j is omitted.) Specifically, for $x \in M^1$, $x = \bar{x} + (x + \bar{x})$ where $\bar{x} = \sum_{j=1}^n (xx_1 \cdots \widehat{x}_j \cdots x_n)x_j \in \operatorname{Span}\{x_1, \dots, x_n\}$ and $x + \bar{x} \in V$ since

$$(x + \bar{x})x_1 \cdots \widehat{x}_j \cdots x_n = xx_1 \cdots \widehat{x}_j \cdots x_n + \sum_{k=1}^n (xx_1 \cdots \widehat{x}_k \cdots x_n)x_k x_1 \cdots \widehat{x}_j \cdots x_n$$

and

$$x_k x_1 \cdots \widehat{x}_j \cdots x_n = \begin{cases} 0 & j \neq k \\ x_1 \cdots x_n \neq 0 & j = k. \end{cases}$$

Now for $\operatorname{rank} M = n + 1$, V must be one dimensional and one may let $V = \operatorname{Span}\{u\}$. Then M^{n-1} is spanned by the classes $x_1 \cdots \widehat{x}_j \cdots x_n$ and $ux_1 \cdots \widehat{x}_i \cdots \widehat{x}_j \cdots x_n$ ($1 \leq i < j \leq n$) and for each of these classes α one has $u\alpha = 0$. By duality one must have $u = 0$, but $\operatorname{Span}\{u\} = V$ is one dimensional. \square

The fact says the algebra R cannot exist so P is indecomposable

— \times —

So we have checked out one example³¹ and it satisfied $\operatorname{rank} P = \nu$ and P is indecomposable.

It seems to me that the argument we just did for the projective plane actually works for any Steiner triple system $S(2, 3, \nu)$.

Being given a 2-element set $\{i, j\} \subset V$ there is a unique block $B = \{i, j, k\}$ containing $\{i, j\}$. Thus $\theta(x_i x_j x_{k'}) = \begin{cases} 0 & k \neq k' \\ 1 & k = k' \end{cases}$ and that says the homomorphism $P^2 \rightarrow \operatorname{Hom}(P^1, \mathbb{F}_2)$ sends $x_i x_j$ to x_k^* .

On the other hand, given $k \in V$ there is an $i \neq k$ in V and the 2-element set $\{i, k\}$ gives a third element $\{i, j, k\}$ or a block $B \in \mathcal{B}$. Then $x_i x_j \in P^2$ gives $x_k^* \in \operatorname{Hom}(P^1, \mathbb{F}_2)$.

Thus pairs $x_i x_j \in P^2$ give classes $x_k^* \in \operatorname{Hom}(P^1, \mathbb{F}_2)$ and every x_k^* comes from some $x_i x_j \in P^2$. This says $\dim \operatorname{Hom}(P^1, \mathbb{F}_2) = \nu$, which says $\operatorname{rank} P = \nu$. Thus one has

FACT: *For a Steiner triple system $S(2, 3, \nu)$ the associated Poincaré algebra P has $\operatorname{rank} P = \nu$.*

— \times —

Now consider a Steiner system $S(k, t, \nu)$ with $2 \leq t < k < \nu$. One then has $2 \leq t \leq k - 1 < k < \nu$ but this is not a Steiner system $S(k - 1, k, \nu)$. For a given $k - 1$ element set $\{i_1, \dots, i_{k-1}\} \subset V$ there may be no block $\{i_1, \dots, i_{k-1}, i_k\}$ in \mathcal{B} containing $\{i_1, \dots, i_{k-1}\}$. If however there is such a block, it is unique. If $\{i_1, \dots, i_{k-1}, i'\}$, $\{i_1, \dots, i_{k-1}, i''\}$ are blocks in \mathcal{B} both contain $\{i_1, \dots, i_t\}$ and hence they coincide.

³¹L.S.: We knew about this example since the winter of 2005 – 2006 in the terms described here using the projective plane over \mathbb{F}_2 , as well as in terms of the barycentric subdivision of the 2-simplex (see § 8 Example 1). The connection with Steiner triple systems was discovered by Bob in February of 2006 in mulling over Example 3 in § 8 and the vast number of other examples we had using square free admissible Macaulay classes as described in § 8.

Now consider $j \in \{1, 2, \dots, v\}$. There is a block $\{j, i_1, \dots, i_{k-1}\} = B \in \mathcal{B}$. Specifically $\{j, i_1, \dots, i_{t-1}\}$ can be taken to be a t -set containing j and there is then a block containing $\{j, i_1, \dots, i_{t-1}\}$ in \mathcal{B} . By uniqueness $\{i_1, \dots, i_{k-1}, j\} = B \in \mathcal{B}$ and for $j' \neq j$, $\{i_1, \dots, i_{k-1}, j'\} \notin \mathcal{B}$. One then has $\theta(x_{i_1} \cdots x_{i_{k-1}} \cdot x_{j'}) = \begin{cases} 0 & j' \neq j \\ \neq 0 & j' = j \end{cases}$ and under the homomorphism $P^{k-1} \rightarrow \text{Hom}(P^1, \mathbb{F}_2)$, the element $x_{i_1} \cdots x_{i_{k-1}}$ is sent to $x_j^* \in \text{Hom}(P^1, \mathbb{F}_2)$. We then have

FACT: For a Steiner system $S(t, k, v)$, $2 \leq t < k < v$ the associated Poincaré duality algebra has rank v .

Fantastic! After many days of hacking around, we finally have an argument that seems to work. And it isn't really a difficult argument.

Now, of course, we would like to consider the question: *Is the Poincaré duality algebra P of a Steiner system always indecomposable?* We really don't have much evidence for that – we just know one example – and we are wanting something nice to happen.³²

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So now we want to look at indecomposability of P .

One starting point would be to consider another example. The next smallest Steiner system is $S(2, 3, 9)$. This can be defined as the affine plane over the field \mathbb{F}_3 . A point in the plane is a pair (x, y) with $x, y \in \mathbb{F}_3$ and the lines in the plane are the sets of solutions of linear equations $ax + by = c$ with $a, b, c \in \mathbb{F}_3$, $(a, b) \neq (0, 0)$.

One can also write this down as blocks in $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and the blocks are:

$$\begin{array}{lll} (1, 2, 3) & (4, 5, 6) & (7, 8, 9) \\ (1, 4, 7) & (1, 5, 9) & (1, 6, 8) \\ (2, 4, 9) & (2, 5, 8) & (2, 6, 7) \\ (3, 4, 8) & (3, 5, 7) & (3, 6, 9) \end{array}$$

If $P = Q \# R$ is a connected sum then Q and R are subalgebras satisfying $x^2 = 0$ for all $x \in Q^1$ or $x \in R^1$. Because $\text{f-dim } Q = \text{f-dim } R = 3$ one has $\dim Q^1 \geq 3$, $\dim R^1 \geq 3$ and $9 = \dim Q^1 + \dim R^1$. Because there is no Poincaré duality algebra M satisfying $x^2 = 0$ for all $x \in M^1$ with $\text{f-dim } M = n$ and $\text{rank } M = n + 1$ neither Q nor R can have rank 4, so one of Q and R must have rank 3 and the other has rank 6. Let's choose $\text{rank } Q = 3$ and $\text{rank } R = 6$.

Then Q^1 has a basis $\{x, y, z\}$ and $xyz \neq 0 \in Q^3$. The annihilator of x in P^1 is then the space $\text{Span}\{x\} + P^1$ and is 7-dimensional. If P is decomposable, there must then be a class $x \in P^1$ with $\dim xP^1 = 2$.

Is there something else we can try?

Later

I went to look at the old correspondence with Larry and happily found a result.³³

LEMMA: Let M be a Poincaré duality algebra satisfying $u^2 = 0$ for all $u \in M^1$. If there is a class $x \in M^1$ with $\dim xM^1 = \text{rank } M - 1$ then M is indecomposable.

³²L.S.: In retrospect we had lots of examples. We just hadn't thought to reformulate them in the language of Steiner systems. In the text beginning next Bob *rediscovers* the example (see § 8 Example 2) of inflection points on a nonsingular elliptic curve in the language of Steiner systems.

³³Compare this with Proposition 3.4.

PROOF: Suppose $M = Q\#R$. For $x \in M^1$, $x = q + r$ with $q \in Q^1$ and $r \in R^1$ and for $z = q' + r'$, $xz = qq' + rq' + qr' + rr' = qq' + rr'$ since rq' and qr' belong to $Q^1 \cdot R^1 = 0$. Now $Q^1 \rightarrow Q^2 : q' \mapsto qq'$ has kernel containing q , so $\dim qQ^1 \leq \dim Q^1 - 1$ and $R^1 \rightarrow R^2 : r' \mapsto rr'$ has kernel containing r so $\dim rR^1 \leq \dim R^1 - 1$. Then $xM^1 \subset qQ^1 + rR^1$ and this subspace $qQ^1 + rR^1$ has dimension at most $\dim Q^1 - 1 + \dim R^1 - 1 = \dim M^1 - 2$, contrary to hypothesis. \square

Now consider a Steiner triple system $S(2, 3, \nu)$. For any $j > 1$, $\{1, j\}$ is a 2-element subset of V and there is a block $B = \{1, j, k\} \in \mathcal{B}$. Then $\theta(x_i x_j x_k) \neq 0$ and $x_i x_j \in P^1$ is $x_k^* \in \text{Hom}(P^1, \mathbb{F}_2) \cong P^2$. The elements x_ℓ^* , $\ell > 1$ are linearly independent, so the elements $x_i x_j$ are linearly independent and $\dim x_1 P^1 = \nu - 1$. This gives the following result.

FACT: *If P is the Poincaré duality algebra of a Steiner triple system $S(2, 3, \nu)$ then P is indecomposable.*

Now consider a Steiner system $S(2, k, \nu)$ with $k > 3$. For any $j > 1$, $\{1, j\}$ is a 2-element subset of V and there is a unique block $\{1, j, i_1, \dots, i_{k-2}\} = B \in \mathcal{B}$. For any i'_1, \dots, i'_{k-2} not equal to i_1, \dots, i_{k-2} , $\{1, j, i'_1, \dots, i'_{k-2}\}$ is not a block in \mathcal{B} . Thus

$$\theta(x_1 x_j x_{i'_1} \cdots x_{i'_{k-2}}) = \begin{cases} 0 & \{i'_1, \dots, i'_{k-2}\} \neq \{i_1, \dots, i_{k-2}\} \\ \neq 0 & \{i'_1, \dots, i'_{k-2}\} = \{i_1, \dots, i_{k-2}\} \end{cases}$$

If one has a linear combination $\sum \alpha_j x_1 x_j = 0$ in P^2 then multiplying by $x_{i_1} \cdots x_{i_{k-2}}$ and applying θ gives

$$0 = \theta\left(\sum \alpha_j x_1 x_j x_{i_1} \cdots x_{i_{k-2}}\right) = \sum \alpha_j \theta(x_1 x_j x_{i_1} \cdots x_{i_{k-2}}) = \alpha_j.$$

Thus the classes $x_1 x_j$ are linearly independent and $\dim x_1 P^1 = \nu - 1$. This gives

FACT: *If P is the Poincaré duality algebra of a Steiner system $S(2, k, \nu)$ then P is indecomposable.*

—x—

That is rather curious. We've gotten the result for Steiner systems $S(t, k, \nu)$ with $t = 2$. There doesn't seem an obvious way to approach $t > 2$. (Of course, the point may be that indecomposability is basically a degree 2 property.)

—x—

Let's consider a Steiner system $S(t, k, \nu)$ with $2 < t < k < \nu$ and let's suppose that $k - t \geq t$. For any t -element set $\{i_1, \dots, i_t\} = T$ there is a unique block $B = T \cup S$, $S = \{j_1, \dots, j_{k-t}\}$ with $B \in \mathcal{B}$. Since $k - t \geq t$ there is a unique block $B' = T' \cup S$ containing $\{j_1, \dots, j_{k-t}\}$ and hence a unique block containing S . This implies that the homomorphism $E^t \rightarrow P^t$ is an isomorphism.³⁴ Because $k - t \geq t$ this implies that $E^j \rightarrow P^j$ is an isomorphism for all $j \leq t$, and in particular for $j = 2$. Because the classes $x_1 x_j$ with $1 < j$ are linearly independent in E^2 they must then be linearly independent in P^2 and $\dim x_1 P^1 = \text{rank } P^1 - 1$. Thus we have

FACT: *If P is the Poincaré duality algebra of a Steiner system $S(t, k, \nu)$ with $k - t \geq t$, then P is indecomposable.*

(Note. This includes the case $S(2, k, \nu)$ with $k > 3$ since $k - 2 \geq 2$, so we have eliminated the restriction $t = 2$. That was part of a more general result.)

³⁴L.S.: Clearly E is the exterior algebra $E(x_1, \dots, x_n)$, as in the text dated 11 March 2008, and also as in that text Bob is regarding P as a quotient algebra of E .

R.E. STONG

16 March 2008

I haven't made any progress on these questions for a couple of days. . . .
So I will stop here. I will try to go on later.

Sincerely,
Bob

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