On $\mathfrak{m}$-Primary Irreducible Ideals in $\mathbb{F}[x, y]$

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SUMMARY : *At the beginning of the last century F. S. Macaulay developed an elegant theory (see [4] Part IV) describing homogeneous ideals in polynomial rings. This theory makes the maximal-primary irriducible ideals $I \subset \mathbb{F}[z_1, \ldots, z_n]$ correspond to a single homogeneous inverse polynomial $\theta_I \in \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}]$. Macaulay's theory has recently attracted attention in connection with problems arising in invariant theory and algebraic topology (see e.g., the introduction to [7] and the references cited there). In this note we show how given an inverse binary form $\theta \in \mathbb{F}[x^{-1}, y^{-1}]$ one may explicitly write down generators of the corresponding maximal-primary irreducible ideal $I(\theta) \subset \mathbb{F}[x, y]$. As a bonus we obtain an elementary proof of the fact that such ideals are always generated by a regular sequence (see e.g., [13]).*

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AT THE beginning of the last century F. S. Macaulay developed an elegant theory describing homogeneous ideals in polynomial rings (see [4] Part IV). For an $\mathfrak{m}$-primary homogeneous[1] irriducible ideal $I \subset \mathbb{F}[z_1, \ldots, z_n]$ the description is in terms of a single homogeneous[2] inverse polynomial $\theta_I \in \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}]$. The **inverse polynomial** algebra $\mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}]$ is simply defined to be the algebra of polynomials in the formal variables $z_1^{-1}, \ldots, z_n^{-1}$ each of degree -1 and there is an $\mathbb{F}[z_1, \ldots, z_n]$-module structure on it. The module product is similar to the cap product between cohomology and homology, so we denote by a $\cap$ and define it on monomials by

$$z^E \cap z^{-F} = \begin{cases} z^{-F+E} & \text{if } F - E \in \mathbb{N}_0^n \\ 0 & \text{otherwise,} \end{cases}$$

where $E, F \in \mathbb{N}_0^n$. The $\cap$-product is extended to all elements of $\mathbb{F}[z_1, \ldots, z_n]$ and $\mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}]$ by bilinearity. Macaulay's Double Annihilator Theorem (see e.g., [7]) tells us that there is a bijective correspondence between nonzero elements of $\mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}]$ and maximal-primary irreducible ideals in $\mathbb{F}[z_1, \ldots, z_n]$. This correspondence associates to an inverse form θ its annihilator ideal $I(\theta)$.

This theory has recently attracted attention in connection with problems arising in invariant theory and algebraic topology (see e.g., the introduction to [7] and the references cited there as well as [6]). In this note we are concerned with the case $n = 2$ and show how given an inverse binary form $\theta \in \mathbb{F}[x^{-1}, y^{-1}]$ one may explicitly write down generators of the corresponding $\mathfrak{m}$-primary irreducible ideal $I(\theta) \subset \mathbb{F}[x, y]$. We do so by thinking of the quotient algebra $\mathbb{F}[x, y]/I(\theta)$ as the cohomology of the associated projective space bundle of a vector bundle on a projective space. As a bonus, our main result yields an elementary proof of the fact that $\mathfrak{m}$-primary ideals in $\mathbb{F}[x, y]$ are always generated by a regular sequence (see e.g., [13]).

The reader is refered to [10] for the algebra underlying of the projective bundle theorem and its uses in studying $\mathfrak{m}$-primary ideals in polynomial algebras. To formulate our results we assume a minimal amount of familiarity with Macaulay's theory of inverse systems, which can be obtained by doing exercises 21.6 and 21.7 in [1]. For the proofs however a short study Parts I and II of [7] (to which we also refer for unexplained notations and conventions) is almost indispensable.

§1. Stable Inverse Binary Forms

Let $\theta \in \mathbb{F}[x^{-1}, y^{-1}]_{-d}$ be a nonzero inverse binary form of degree $-d$ and write it as a sum of inverse binomials, viz.,

$$\theta = a_0 x^{-s} y^{-t} + a_1 x^{-s+1} y^{-t-1} + \cdots + a_s y^{-t-s},$$

where $s + t = d$, $s, t \in \mathbb{N}_0$, with s chosen as the largest integer e for which an inverse binomial $x^{-e} y^{-d+e}$ of degree $-d$ appears in θ with a nonzero coefficient. This condition defines $s, t \in \mathbb{N}_0$

[1] We always regard a polynomial algebras as graded with the variables of degree one and use $\mathfrak{m}$ to denote the (unique) graded maximal ideal.)

[2] We adhere to the conventions of J. C. Moore as far as graded objects go. This means only *homogeneous* elements are considered unless explicitly stated otherwise.

uniquely. Since Macaulay duals of proportional inverse forms coincide we may normalize θ so that $a_0 = 1 \in \mathbb{F}$, viz.,

$$\theta = x^{-s}y^{-t} + a_1x^{-s+1}y^{-t-1} + \cdots + a_sy^{-t-s} \in \mathbb{F}[x^{-1}, y^{-1}].$$

Introduce the polynomial[3]

$$a(z) = 1 + a_1z + \cdots + a_sz^s \in \mathbb{F}[z],$$

and note that there is a formal identity

$$(\text{⌘}) \qquad \theta(x^{-1}, y^{-1}) = x^{-s} \cdot y^{-t} \cdot a(z)\big|_{z=\frac{x}{y}} = x^{-s} \cdot y^{-t} \cdot a\left(\frac{x}{y}\right) \in \mathbb{F}[x^{-1}, y^{-1}].$$

Define

$$b(z) = 1 + b_1z + \cdots + b_sz^s \in \mathbb{F}[z]$$

by the requirement that it be the inverse of $a(z)$ in the quotient algebra $\mathbb{F}[z]/(z^{s+1})$, i.e., $a(z)b(z) = 1 \in \mathbb{F}[z]/(z^{s+1})$. Since $\overline{a}(z) = a_1z + \cdots a_sz^s$ is nilpotent in $\mathbb{F}[z]/(z^{s+1})$ and $a(z) = 1 + \overline{a}(z)$ this can be done by droping all powers of z higher than s from the formal series expansion in z of

$$\frac{1}{a(z)} = \frac{1}{1 + \overline{a}(z)} = 1 - \overline{a}(z) + \overline{a}(z)^2 + \cdots.$$

Note that in $\mathbb{F}[z]$ we then have an identity

$$(\text{✠}) \qquad b(z) \cdot a(z) = z^{s+1} \cdot j(z) \in \mathbb{F}[z]$$

for some polynomial $j(z) = j_0 + j_jz + \cdots \in \mathbb{F}[z]$ of degree at most $s - 1$.

We say that t is **large enough** if $t + 1 \geq s$ and then call θ **stable**. For a stable inverse binary form θ we may introduce the homogeneous form (called the *homoginizing form* in [10])

$$(\bigstar) \qquad h(x, y) = y^{t+1} + b_1xy^t + \cdots + b_sx^sy^{t+1-s} \in \mathbb{F}[x, y].$$

Note that in $\mathbb{F}[x, y]$ we have an identity

$$(\text{✭}) \qquad h(x, y) = y^{t+1}\left(1 + b_1(\frac{x}{y}) + \cdots + b_s\left(\frac{x}{y}\right)^s\right) = y^{t+1}b\left(\frac{x}{y}\right).$$

PROPOSITION 1.1**:** *Suppose that*

$$\theta = x^{-s}y^{-t} + a_1x^{-s+1}y^{-t-1} + \cdots + a_sy^{-t-s} \in \mathbb{F}[x^{-1}, y^{-1}]$$

is a stable inverse binary form in which t is large enough. Let $a(z)$, $b(z)$ and $h(x, y)$ be defined as above. Then the Macaulay dual ideal $I(\theta) \subset \mathbb{F}[x, y]$, i.e., the annhilator ideal of θ in $\mathbb{F}[x, y]$ with respect to the $\cap$-product, is generated by the two forms $f = x^{s+1}$ of degree $s + 1$ and h of degree $t + 1$ which are a regular sequence in $\mathbb{F}[x, y]$.

In the language of [10] this says that the $\mathfrak{m}$-primary irreducible ideal $I(\theta) \subset \mathbb{F}[x, y]$ defined by θ is a projective bundle ideal of bundle dimension $t + 1$ with base ideal $(x^{s+1}) \subset \mathbb{F}[x]$ and homoginizing form

$$h(x, y) = y^{t+1} + b_1xy^t + \cdots + b_sx^sy^{t+1-s} \in \mathbb{F}[x, y]$$

where

$$a(z) = 1 + a_1z + \cdots + a_sz^s \in \mathbb{F}[x],$$
$$b(z) = 1 + b_1z + \cdots + b_sz^s \in \mathbb{F}[z]$$

[3] The notation should show the dependence of $a(z)$, $b(z)$, and $h(x, y)$ on θ, e.g., $a_\theta(z)$, etc. However if only one θ is being discussed we drop this complication.

are inverse to each other in $\mathbb{F}[z]/(z^{s+1})$.

In the proof of Proposition 1.1 we make use of some elementary properties of Poincaré duality algebras and their relation to maximal primary ideals which may be found in [7] Part I. Recall that a **Poincaré duality algebra** of **formal dimension** d (denoted by f-dim$(H) = d$) over the field $\mathbb{F}$ is a commutative graded connected algebra over $\mathbb{F}$ such that the following conditions are satisfied:

(1) $H_i = 0$ for $i > d$.
(2) H_d is a 1-dimensional vector space over $\mathbb{F}$.
(3) An element $u \in H_i$ is nonzero if and only if there exists an element $u^\vee \in H_{d-i}$, called a **Poincaré dual** for u, such that the product $u \cdot u^\vee \neq 0 \in H_d$.

A nonzero element $[H] \in H_d$ is called a **fundamental class**.

Proof of Proposition 1.1: First of all note that $f, h \in I(\theta)$. For f this is immediate from the definition of $s \in \mathbb{N}_0$. For h one has to verify that $h \cap \theta = 0 \in \mathbb{F}[x^{-1}, y^{-1}]$. This we may do by computing $h \cap \theta$ as the ordinary product of the forms h and θ in the Laurent polynomial algebra $\mathbb{F}[x, y, x^{-1}, y^{-1}]$ and then dropping all monomials $x^i \cdot y^j$ from the result for which one or both of the exponents i or j is strictly positive. Using the identities (⌘), (✠), (★) and (✶) this yields

$$\begin{aligned} h \cdot \theta &= \left[y^{t+1} b\left(\frac{x}{y}\right)\right] \cdot \left[x^{-s} y^{-t} a\left(\frac{x}{y}\right)\right] \\ &= x^{-s} y \left[b\left(\frac{x}{y}\right) \cdot a\left(\frac{x}{y}\right)\right] = x^{-s} y \left[1 + \frac{x^{s+1}}{y^{s+1}} j\left(\frac{x}{y}\right)\right] \\ &= x^{-s} y \left[1 + \frac{x^{s+1}}{y^{s+1}} j(\left(\frac{x}{y}\right)\right] = x^{-s} \cdot y + x \cdot j\left(\frac{x}{y}\right) \cdot y^{-s}. \end{aligned}$$

Therefore $h \cap \theta = 0 \in \mathbb{F}[x^{-1}, y^{-1}]$ because $x^{-s} \cdot y$ has a positive exponent for y and $x \cdot j\left(\frac{x}{y}\right)$ is a sum of monomials $\frac{x^{i+1}}{y^i}$ where $i \geq 0$ each of which has a positive exponent for x.

Next note that the two forms f and h build a regular sequence in $\mathbb{F}[x, y]$ and hence $(f, h) \subset \mathbb{F}[x, y]$ is a $\mathfrak{m}$-primary irreducible ideal. The corresponding Poincaré duality quotient algebra $\mathbb{F}[x, y]/I(\theta)$ has formal dimension $\deg(f) + \deg(h) - 2 = (s + 1) + (t + 1) - 2 = s + t = d$ (see e.g., [7] §I.4 Example 1). The inclusion $(f, h) \subseteq I(\theta)$ therefore induces an epimorphism of algebras

$$\mathbb{F}[x, y]/(f, h) \longrightarrow \mathbb{F}[x, y]/I(\theta).$$

Since both these algebras are Poincaré duality algebras have formal dimension d this map must be an isomorphism (see e.g., [7] Lemma I.3.1). □

To verify the remark between the statement and proof of Proposition 1.1 we recall from [10] that $I \subset \mathbb{F}[z_1, \ldots, z_n, X]$ is called a **projective bundle ideal** with **base ideal** $J = I \cap \mathbb{F}[z_1, \ldots, z_n]$ if $\mathbb{F}[z_1, \ldots, z_n, X]/I$ is a free $\mathbb{F}[z_1, \ldots, z_n]/J$-module with respect to the module structure defined by the canonical inclusion $\mathbb{F}[V]/J \hookrightarrow \mathbb{F}[V, X]/I$. To verify that the ideal $I(\theta) \subset \mathbb{F}[x, y]$ defined by a stable form θ with t large enough is a projective bundle ideal of bundle dimension $t + 1$ with base ideal $(x^{s+1}) \subset \mathbb{F}[x]$ and homoginizing form $h(x, y) = y^{t+1} + b_1 x y^t + \cdots + b_s x^s y^{t+1-s}$ in $\mathbb{F}[x, y]$, note that since h is monic as a polynomial in y with coefficients in $\mathbb{F}[x]$ that $\mathbb{F}[x, y]/I(\theta)$ is free as a module over $\mathbb{F}[x]/(x^{s+1})$ as a result of the following lemma.

LEMMA 1.2: *Let A be a commutative graded connected algebra over a field and $A[X]$ the standard polynomial algebra over A (so X has degree 1). If $\alpha(X) \in A[X]$ is a monic polynomial of degree m in X then $A[X]/(\alpha(X))$ is a free A-module with basis $1, X, \ldots, X^{m-1}$.*

PROOF: One notes that homogeniety requires that the coefficients of $\alpha(X)$ except for the coefficient of X^m belong to the augmentation ideal of A. Hence $1 \otimes (X^i \cdot \alpha(X)) \equiv 1 \otimes X^{m+i}$ for $i = 0, 1, \ldots,$ in $\mathbb{F} \otimes_A A[X]$, so

$$1, X, \ldots, X^{m-1}, \alpha(X), X \cdot \alpha(X), \ldots,$$

project to an $\mathbb{F}$-vector space basis for the module of A-indecomposables $\mathbb{F} \otimes_A A[X]$. Since $A[X]$ is a free A-module they therefore are an A-basis for $A[X]$ by the graded Nakayama Lemma (see e.g. [8] Proposition 5.1.3). Since $\alpha(X), X \cdot \alpha(X), \ldots,$ span the ideal $(\alpha(X))$ as an A-module it follows that the inclusion $(\alpha(X)) \hookrightarrow A$ splits as a map of A-modules. Hence $A[X]/(\alpha(X))$ is also free as an A-module and $1, X, \ldots, X^{m-1}$ project to a basis for it. □

EXAMPLE 1: Let $\theta \in \mathbb{F}_2[x, y]$ be the Macaulay dual of the Dickson ideal $(\mathbf{d}_{2,0}, \mathbf{d}_{2,1}) \subset \mathbb{F}_2[x, y]$ (see e.g., [3], [7] §IV.3, or [6]). So

$$\theta = x^{-2}y^{-1} + x^{-1}y^{-2} \in \mathbb{F}_2[x^{-1}, y^{-1}].$$

Therefore $s = 2$, $t = 1$, and $t = 1 \geq 1 = s - 1$ is large enough, so θ is stable. We have

$$a(z) = 1 + z \in \mathbb{F}_2[z]$$

and

$$b(z) = \frac{1}{1+z} = 1 + z + z^2 \in \mathbb{F}_2[z].$$

Hence the bundle dimension is two, the homoginizing form is

$$y^2 + xy + x^2 \quad (= \mathbf{d}_{2,1}),$$

and Proposition 1.1 gives as generators for $I(\theta)$ the forms $f = x^3$ and $h = x^2 + xy + y^2 = \mathbf{d}_{2,1}$ (cf [6]). Since

$$\mathbf{d}_{2,0} = x^2y + xy^2$$
$$\mathbf{d}_{2,1} = x^2 + xy + y^2$$

one has

$$\mathbf{d}_{2,0} = x^3 + x(x^2 + xy + y^2) = f + xh$$

so indeed $(f, h) = (\mathbf{d}_{2,0}, \mathbf{d}_{2,1})$.

Note that in this case $t = s - 1$, so this ideal is barely stable. The ideal corresponding to

$$\theta^2 = x^{-4}y^{-2} + x^{-2}y^{-4} \in \mathbb{F}_2[x^{-1}, y^{-1}]$$

is no longer stable since $t = 2 \not\geq 3 = s - 1$, and therefore a form $h \in \mathbb{F}[x, y]$ for θ^2 cannot be defined. For a discussion of the relation between the ideals $I(\theta)$ and $I(\theta^2)$ in general see [11].

§2. General Inverse Binary Forms

The condition of stability is asymmetric; it depends on a choice of an ordered basis for the space of inverse linear forms. So one might wonder if by a suitable change of basis any inverse form can be made stable. This is not the case. The form $x^{-2} + x^{-1}y^{-1} + y^{-2} \in \mathbb{F}_2[x, y]$ is not stable because $s = 2$ but $t = 0$, so t is certainly not larger than $s - 1$. If it is rewritten

The group $GL(2, \mathbb{F}_2)$ is not all that big, and one only really need check the assertion for the basis where $u = x$ and $v = x + y$, which is a short computation.

as a sum of monomials in any other basis u^{-1}, v^{-1} for the inverse linear forms it becomes $u^{-2} + u^{-1}v^{-1} + v^{-2} \in \mathbb{F}_2[u^{-1}, v^{-1}]$, so remains unstable. So we proceed otherwise.

Suppose the inverse binary form

$$\theta = x^{-s}y^{-t} + a_1x^{-s+1}y^{-t-1} + \cdots + a_sy^{-t-s} \in \mathbb{F}[x^{-1}, y^{-1}]$$

is not stable. Since the stability condition depends only on s and t one sees that for suitably large $r \in \mathbb{N}$ the form

$$y^{-r}\theta = x^{-s}y^{-t-r} + a_1x^{-s+1}y^{-t-r-1} + \cdots + a_sy^{-t-s-r} \in \mathbb{F}[x^{-1}, y^{-1}]$$

is stable. As we next show the relation between the ideals $I(y^{-r}\theta)$ and $I(\theta)$ is quite simple so it can be used to construct generators for $I(\theta)$ from generators for $I(y^{-r}\theta)$. To formulate how this works it is convenient to introduce some terminology.

If H is a Poincaré duality algebra and $0 \neq u \in H$ a nonzero element, then $\mathrm{Ann}_H(u) \subset H$ is a $\mathfrak{m}$-primary irreducible ideal in H (see e.g., [7] Corollary I.2.3) so $H/\mathrm{Ann}_H(u)$ is again a Poincaré duality algebra called the **dual Poincaré algebra to u in H**. This is in analogy with the topological notion of dualizing a line bundle on a manifold.

Proposition 2.1: *Suppose that $\theta \in \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}]$ is a nonzero inverse form and $1 \le i \le n$ with $z_i \neq 0 \in \mathbb{F}[z_1, \ldots, z_n]/I(\theta)$. Then $I(z_i^{-1}\theta) \subset I(\theta)$ and the quotient algebra $\mathbb{F}[z_1, \ldots, z_n]/I(\theta)$ is dual to $z_i \in \mathbb{F}[z_1, \ldots, z_n]/I(z_i^{-1}\theta)$. In particular $I(\theta) = (I(z_i^{-1}\theta) : z_i) \subset \mathbb{F}[z_1, \ldots, z_n]$.*

Proof: Choose a very large integer k so that $z_1^k, \ldots, z_n^k$ belong to both $I(\theta)$ and $I(z_i^{-1}\theta)$. Let $u \in \mathbb{F}[z_1, \ldots, z_n]$ be a transition element (see [7] §I.2) for $I(z_i^{-1}\theta)$ over the ideal $(z_1^k, \ldots, z_n^k)$. Then (see [7] Theorem II.5.1)

$$z_i^{-1}\theta = u \cap z_1^{-(k+1)} \cdots z_n^{-(k+1)}$$

and by Noether's Involution Theorem (see [7] §I.2 or [4] §73) we obtain

$$(\text{⌘}) \qquad I(z_i^{-1}\theta) = \big((z_1^{-k}, \ldots, z_n^{-k}) : (u)\big).$$

Note that k being large we have

$$\theta = z_i \cap z_i^{-1}\theta = z_i \cap (u \cap z_1^{-(k+1)} \cdots z_n^{-(k+1)}) = z_iu \cap z_1^{-(k+1)} \cdots z_n^{-(k+1)},$$

so z_iu is a transition element for $I(\theta)$ over $(z_1^k, \ldots, z_n^k)$. Hence

$$(\text{⊕}) \qquad I(\theta) = \big((z_1^k, \ldots, z_n^k) : (z_iu)\big) = \big(\big((z_1^k, \ldots, z_n^k) : (u)\big) : (z_i)\big) = (I(z_i^{-1}\theta) : z_i),$$

and passing down to the quotient algebra $\mathbb{F}[z_1, \ldots, z_n]/I(z_i^{-1}\theta)$ we obtain that

$$\mathrm{Ann}_{\mathbb{F}[z_1, \ldots, z_n]/I(z_i^{-1}\theta)}(z_i) = I(\theta)/I(z_i^{-1}\theta).$$

From the formulae (⌘) and (⊕) it then follows that $I(z_i^{-1}\theta) \subset I(\theta)$. □

Proposition 2.2: *Let $I \subset \mathbb{F}[x, y]$ be a $\mathfrak{m}$-primary irreducible ideal with Macaulay dual*

$$\theta = x^{-s}y^{-t} + a_1x^{-s+1}y^{-t-1} + \cdots + a_sy^{-t-s} \in \mathbb{F}[x^{-1}, y^{-1}].$$

Then for sufficiently large $r \in \mathbb{N}_0$ the inverse form

$$y^{-r}\theta = x^{-s}y^{-t-r} + a_1x^{-s+1}y^{-t-r-1} + \cdots + a_sy^{-t-s-r} \in \mathbb{F}[x^{-1}, y^{-1}]$$

is stable, so $I(y^{-r}\theta) \subset \mathbb{F}[x, y]$ is a projective bundle ideal with generators

$$f_r = x^{s+1}, \quad h_r = y^{t+r+1-k}(y^k + b_1xy^{k-1} + \cdots + b_kx^k),$$

where

$$a(z) = 1 + a_1 z + \cdots a_s z^s \in \mathbb{F}[z],$$
$$b(z) = 1 + b_1 z + \cdots b_s z^s \in \mathbb{F}[z],$$

and $a(z) \cdot b(z) = 1 \in \mathbb{F}[z]/(z^{s+1})$. *Finally* $I(\theta) = (I(y^{-r}\theta) : y^r)$, *so every* $\mathfrak{m}$*-primary irreducible ideal in* $\mathbb{F}[x, y]$ *arises through successive dualization of a linear form from a projective bundle ideal defined by a stable inverse binary form.*

PROOF: The statement about stability is clear; one need only choose $r \in \mathbb{N}_0$ to be large enough so that $t + r + 1 - s$ is nonnegative. The remaining conclusuons follow from Propositions 1.1 and 2.1. □

§3. Generators for $\mathfrak{m}$-Primary Irreducible Ideals in $\mathbb{F}[x, y]$

What remains to discuss is how, given a regular ideal $I = (f, h) \subset \mathbb{F}[x, y]$ one finds generators for the ideal $(I : y^i)$, $i > 0$. The following lemmas take care of this.

LEMMA 3.1: *Suppose that* $f, h \in \mathbb{F}[x, y]$ *are a regular sequence. Then* y *divides at most one of* f *and* h.

PROOF: Suppose y were to divide both f and h, so we could write

$$f = yF, \quad h = yH.$$

Then

$$Fh = Fyh = fH \in (f),$$

but $\deg(F) < \deg(f)$ so $F \notin (f)$. This says that h is a zero divisor modulo (f) contrary to hypothesis. □

REMARK: In fact, if $f, h \in \mathbb{F}[x, y]$ are a regular sequence then they are relatively prime as the proof of Lemma 3.1 shows.

LEMMA 3.2: *Suppose that* $f, h \in \mathbb{F}[x, y]$ *are a regular sequence and* y *divides* h. *Let* $h' = h/y$. *Then* $f, h' \in \mathbb{F}[x, y]$ *is also a regular sequence.*

PROOF: We need to show that h' is not a zero divisor modulo (f). So suppose $Hh' \in (f)$. Then $Hh = Hyh' = y(Hh') \in (f)$ also. However, since f and h form a regular sequence this implies that $H \in (f)$, so indeed h' is not a zero divisor modulo (f). □

LEMMA 3.3: *Let* $f, h \in \mathbb{F}[x, y]$ *be a regular sequence. Suppose* y *divides* h *and set* $h' = h/y$. *Then* $\big((f, h) : y\big) = (f, h')$ *and* $f, h' \in \mathbb{F}[x, y]$ *is also a regular sequence.*

PROOF: Lemma 3.2 tells us that $f, h' \in \mathbb{F}[x, y]$ is a regular sequence and Noether's Involution Theorem (see e.g., [7] §I.2 or [4] §73) that $\big((f, h) : y\big)$ is a $\mathfrak{m}$-primary irreducible ideal. Inspection shows that $(f, h') \subseteq \big((f, h) : y\big)$ so there is an epimorphism

$$\mathbb{F}[x, y]/(fh') \xrightarrow{\varphi} \mathbb{F}[x, y]/\big((f, h) : y\big)$$

induced by the inclusion. By [7] §I.4 Example 1

$$\text{f-dim}\Big(\mathbb{F}[x, y]/(f, h')\Big) = \deg(f) + \deg(h') - 2 = \deg(f) + \deg(h) - 3.$$

From this same result and [7] Corollary I.2.3

$$\text{f-dim}\Big(\mathbb{F}[x, y]/\big((f, h) : y\big)\Big) = \deg(f) + \deg(h) - 2 - 1 = \deg(h) + \deg(h) - 3$$

also. The map φ is therefore an epimorphism between Poincaré duality algebras of the same formal dimension and hence an isomorphism (see e.g., [7] Lemma I.3.1). Therefore the inclusion $(f, h') \subseteq \big((f, h) : y\big)$ is an equality. $\square$

The last elementary lemma describes generators for $\big((f, h) : y\big)$ if $f, h \in \mathbb{F}[x, y]$ a regular sequence and neither f nor h is divisible by y. It is very much dependent on the number of variables being two and not more.

LEMMA 3.4: *Suppose that $f, h \in \mathbb{F}[x, y]$ is a regular sequence and y divides neither f nor h. Without loss of generality, let* $\deg(f) = a \le b = \deg(h)$. *Then y divides $h - x^{b-a}f$ and setting h' equal to the quotient one has $\big((f, h) : y\big) = (f, h')$.*

PROOF: The two pairs of forms f, h and $f, h - x^{b-a}f$ generate the same ideal so

$$\big((f, h) : y\big) = \big((f, h - x^{b-a}f) : y\big).$$

Lemma 3.3 applies to the pair $f, h - x^{b-a}f$ to yield the stated conclusion. $\square$

Combining Lemmas 3.1– 3.4 yields the following result.

THEOREM 3.5: *Suppose that $f, h \in \mathbb{F}[x, y]$ form a regular sequence. Then the ideal $\big((f, h) : y\big)$ is also generated by a regular sequence and generators for this ideal may be obtained as follows.*

(i) If y divides one of f or h (it cannot divide both by Lemma 3.1), say y divides h, set $h' = h/y$. Then $\big((f, h) : y\big) = (f, h')$.

(ii) If y divides neither f nor h, arrange that $\deg(f) = a \le \deg(h) = b$. *Then y divides $h - x^{b-a}f$. If we let h' be the quotient then $\big((f, h) : y\big) = (f, h')$.* $\square$

COROLLARY 3.6(W.V. Vasconcelos [13]): *A $\mathfrak{m}$-primary irreducible ideal $I \subset \mathbb{F}[x, y]$ is generated by a regular sequence.*

PROOF: Proposition 2.2 tells us that I arises by successive dualization of a linear form starting with a projective bundle ideal defined by a stable inverse binary form. By Proposition 1.1 such an ideal is regular. Theorem 3.5 assures us that dualization with respect to a linear form in two variables preserves regularity. $\square$

EXAMPLE 1: Consider an inverse binary form supported on two inverse binomials, say,

$$\theta = x^{-s}y^{-t} + x^{-s+r}y^{-t-r} \in \mathbb{F}[x^{-1}, y^{-1}]_{-d},$$

where $d = s + t$ and $r \in \mathbb{N}$. If t is large enough so θ is stable then[4]

$$a(z) = 1 + z^r \in \mathbb{F}[z]$$
$$b(z) = 1 + z^r + z^{2r} + \cdots + z^{r\left[\frac{s}{r}\right]} \in \mathbb{F}[z],$$

and $I(\theta)$ is generated by

$$f = x^{s+1}, \quad h = y^{t+1} + x^{\left[\frac{s}{r}\right]}y^{t+1-\left[\frac{s}{r}\right]} + \cdots + x^{\ell\left[\frac{s}{r}\right]}y^{t+1-\ell\left[\frac{s}{r}\right]},$$

where the integer ℓ is determined by the conditions $\ell\left[\frac{s}{r}\right] \le t + 1 < (\ell + 1)\left[\frac{s}{r}\right]$.

If θ is not stable, i.e., if $t < s - 1$, then to find generators for the ideal $I(\theta)$ we need to apply the algorithm of Theorem 3.5. To do so we start with the last stable case as a function of t,

[4] We write $\left[\frac{i}{j}\right]$ for the integral part of the fraction $\frac{i}{j}$.

namely $t = r\left[\frac{s}{r}\right] - 1$, for which the forms

$$f = x^{s+1}, \quad h = x^{r\left[\frac{s}{r}\right]} + y^{r\left[\frac{s}{r}\right]}$$

are generators of the ideal $I(\theta)$. Neither of these is divisible by y, and since $\left[\frac{s}{r}\right] \leq s < s+1$ the form of lower degree is h. So we introduce the alternate generator of larger degree

$$F = f - x^{s+1-\left[\frac{s}{r}\right]}h = x^{s+1-\left[\frac{s}{r}\right]}y^{\left[\frac{s}{r}\right]},$$

divide it by y, and use the result, which we denote by f', together with the form h of lower degree to obtain two generators

$$f' = x^{s+1-\left[\frac{s}{r}\right]}y^{\left[\frac{s}{r}\right]}, \quad h = x^{r\left[\frac{s}{r}\right]} + y^{r\left[\frac{s}{r}\right]}$$

for the ideal $I(\theta)$ with $t = r\left[\frac{s}{r}\right] - 2$, which is the first unstable case. As t continues to decrease the form f' remains divisible by y, so dividing by y the appropriate number of times gives one generator for the ideal, and the other remains h.

As an illustrative case consider the inverse binary form

$$\theta^2 = x^{-4}y^{-2} + x^{-2}y^{-4} \in \mathbb{F}_2[x^{-1}, y^{-1}]$$

which is the square of the Macaulay dual of the Dickson ideal $(\mathbf{d}_{2,0}, \mathbf{d}_{2,1}) \subset \mathbb{F}_2[x, y]$. For θ^2 one has $s = 4$ and $t = 2 \not\geq 3 = s - 1$, and θ^2 is not stable. The form $y^{-1}\theta^2$ however is stable, since

$$y^{-1}\theta^2 = x^{-4}y^{-3} + x^{-2}y^{-5},$$

and therefore

$$s = 4, \quad t = 2, \quad r = 2, \quad \left[\frac{s}{r}\right] = 2$$

$$a(z) = 1 + z^2 \in \mathbb{F}_2[z]$$

$$b(z) = 1 + z^2 + z^4 \in \mathbb{F}_2[z]$$

Applying the preceding discussion then yields as generators for the ideal $I(\theta^2)$ the two forms

$$f' = x^3y + xy^3, \quad h = x^4 + x^2y^2 + y^4.$$

Note by [11] (see also [7] Theorems II.6.6 and VI.5.3 or [2] §9) that $I(\theta^2) = (I(\theta)^{[2]} : xy)$ where $I(\theta)^{[2]}$ is the Frobenius square of $I(\theta)$. The ideal $I(\theta)^{[2]}$ is generated by $\mathbf{d}_{2,0}^2, \mathbf{d}_{2,1}^2$ and if one applies Theorem 3.5 to compute generators for $(I(\theta)^{[2]} : xy)$ from generators of $I(\theta)^{[2]}$ one receives exactly the same pair of forms.

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