REALISING FORMAL GROUPS

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ABSTRACT. We show that a large class of formal groups can be realised functorially by even periodic ring spectra.

1. INTRODUCTION

Let FG be the category of formal groups (of the sort usually considered in algebraic topology) over affine schemes. Thus, an object of FG consists of a pair (G, S), where S is an affine scheme, G is a formal group scheme over S, and a coordinate x can be chosen such that $\mathcal{O}_G \simeq \mathcal{O}_S[\![x]\!]$ as $\mathcal{O}_S$-algebras. A morphism from (G_0, S_0) to (G_1, S_1) is a commutative square

$$\begin{array}{ccc} G_0 & \xrightarrow{\tilde{p}} & G_1 \\ \downarrow & & \downarrow \\ S_0 & \xrightarrow[p]{} & S_1 \end{array}$$

such that the induced map $G_0 \to p^*G_1$ is an isomorphism of formal group schemes over S_0.

Next, recall that an *even periodic ring spectrum* is a commutative and associative ring spectrum E such that $E^1 = 0$ and E^2 contains a unit (which implies that $E \simeq \Sigma^2 E$ as spectra). Here we are using the usual notation $E^k = E^k(\text{point}) = \pi_{-k}E$. We write EPR for the category of even periodic ring spectra.

Given an even periodic ring spectrum E, we can form the scheme $S_E := \text{spec}(E^0)$ and the formal group scheme $G_E = \text{spf}(E^0\mathbb{C}P^\infty)$ over S_E. This construction gives rise to a functor $\Gamma\colon \text{EPR}^{\text{op}} \to \text{FG}$.

It is a natural problem to try to define a realisation functor $R\colon \text{FG} \to \text{EPR}^{\text{op}}$ with $\Gamma R(G, S) \simeq (G, S)$, or at least to do this for suitable subcategories of FG. For example, if we let LFG denote the category of Landweber exact formal groups, and put $\text{LEPR} = \{E \in \text{EPR} \mid \Gamma(E) \in \text{LFG}\}$, one can show that the functor $\Gamma\colon \text{LEPR}^{\text{op}} \to \text{LFG}$ is an equivalence; this is essentially due to Landweber, but details of this formulation are given in [4, Proposition 8.43]. Inverting this gives a realisation functor for LFG, and many well-known spectra are constructed using this. In particular, this gives various different versions of elliptic cohomology, based on various universal families of elliptic curves over rings such as $\mathbb{Z}[\frac{1}{6}, c_4, c_6]$.

In Definition 3.13, we will introduce a full subcategory category $\text{GFG} \subseteq \text{FG}$ of "good" formal groups. Among many other things, this contains all formal groups over fields of characteristic not equal to 2, and all formal groups over rings of the

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form $\mathbb{Z}[1/n]$ with n an even integer (this is proved as Theorem 3.14). Most formal groups in GFG are not Landweber exact. Our main result is as follows.

Theorem 1.1. *There is a realisation functor* $R\colon \mathrm{GFG} \to \mathrm{EPR}$*, with* $\Gamma R \simeq 1\colon \mathrm{GFG} \to \mathrm{GFG}$.

The results of [5] give all the required objects in EPR; the new content of the theorem is the analysis of morphisms. Here we explain the formal part of the construction; in Section 4 we will give additional details and prove that we have the required properties. The functor R actually arises as UV^{-1} for a pair of functors $\mathrm{GFG} \xleftarrow{V} \mathcal{E} \xrightarrow{U} \mathrm{EPR}$ in which V is an equivalence. The definition of $\mathcal{E}$ involves the periodic bordism spectrum $MP = \bigvee_{n\in\mathbb{Z}} \Sigma^{2n} MU$. We will verify in the appendix that this can be constructed as a strictly commutative ring, so we have a topological category Mod_0 of MP-modules. We write DMod_0 for the derived category, and EPA_0 for the category of even periodic commutative ring objects in DMod_0. The unit map $\eta\colon S \to MP$ gives a functor $\eta^*\colon \mathrm{EPA}_0 \to \mathrm{EPR}$, and the objects of the category $\mathcal{E}$ are the objects $E \in \mathrm{EPA}_0$ for which the associated formal group law is "stably realisable" in a sense to be explained later. The morphism set $\mathcal{E}(E_0, E_1)$ is a subset of $\mathrm{EPR}(\eta^* E_0, \eta^* E_1)$, the functor $V\colon \mathcal{E} \to \mathrm{GFG}$ is given by Γ, and the functor $U\colon \mathcal{E} \to \mathrm{EPR}$ is given by η^*. We say that a map $f\colon \eta^* E_0 \to \eta^* E_1$ in EPR is *good* if there is a commutative ring object A in the derived category of $MP \wedge MP$-modules together with maps $f'\colon E_0 \to (1 \wedge \eta)^* A$ and $f''\colon (\eta \wedge 1)^* A \to E_1$ in EPA_0 such that f'' is an equivalence and f is equal to the composite

$$\eta^* E_0 \xrightarrow{\eta^* f'} (\eta \wedge \eta)^* A \xrightarrow{\eta^* f''} \eta^* E_1.$$

The morphisms in the category $\mathcal{E}$ are just the good maps. To prove Theorem 1.1, we need to show that

(3) The composite of two good maps is good, so $\mathcal{E}$ really is a category.
(2) For any map $\Gamma(\eta^* E_0) \to \Gamma(\eta^* E_1)$ of good formal groups, there is a unique good map $\eta^* E_0 \to \eta^* E_1$ inducing it, so that V is full and faithful.
(1) For any good formal group (G, S) there is an object $E \in \mathrm{EPA}_0$ such that $\Gamma(\eta^* E) \simeq (G, S)$, so V is essentially surjective.

To prove statement (k), we need to construct modules over the k-fold smash power of MP. It will be most efficient to do this for all k, and use cosimplicial ideas to organize the functors between the various module categories.

Almost everywhere in this paper, we will assume that 2 has been inverted; we do not know whether there are any good formal groups in which this is not the case. We found in [5] that when 2 is not inverted, there are nontrivial obstructions to various realisation problems, which can only be calculated with considerable labour. We may return to this in future work.

2. Preliminaries

2.1. **Differential forms.** Let (G, S) be a formal group, and let $I \leq \mathcal{O}_G$ be the augmentation ideal. Recall that the cotangent space of G at zero is the module $\omega_G = I/I^2$. If x is a coordinate on G that vanishes at zero, then we write dx for the image of x in I/I^2, and note that ω_G is freely generated over $\mathcal{O}_S$ by dx. We define a graded ring $D(G,S)^*$ by

$$D(G,S)^k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \omega_G^{\otimes(-k/2)} & \text{if } k \text{ is even.} \end{cases}$$

Here the tensor products are taken over $\mathcal{O}_S$, and $\omega_G^{\otimes n}$ means the dual of $\omega_G^{\otimes|n|}$ when $n < 0$. Where convenient, we will convert to homological gradings by the usual rule: $D(G,S)_k = D(G,S)^{-k}$.

Now let E be an even periodic ring spectrum with $\Gamma(E) = (G,S)$. We then have $\mathcal{O}_G = E^0\mathbb{C}P^\infty$ and $I = \widetilde{E}^0\mathbb{C}P^\infty$ and one checks easily that the inclusion $S^2 = \mathbb{C}P^1 \to \mathbb{C}P^\infty$ gives an isomorphism $\omega_G = I/I^2 = \widetilde{E}^0 S^2 = E^{-2}$. Using the periodicity of E, we see that this extends to a canonical isomorphism $D(\Gamma(E))^* \simeq E^*$.

It also follows from this analysis (or from more direct arguments) that a map $f\colon E_0 \to E_1$ in EPR is a weak equivalence if and only if $\pi_0 f$ is an isomorphism.

2.2. **Groups and laws.** We define a category FGL as follows. The objects are pairs (R,F), where R is a commutative ring, and F is a formal group law over R. A morphism from (R_0,F_0) to (R_1,F_1) consists of a ring homomorphism $\phi\colon R_0 \to R_1$, together with a formal power series $f(x) \in R_1[\![x]\!]$ satisfying $f(0) = 0$ and $f'(0) \in R_1^\times$ and

$$(\phi_* F_0)(f(x), f(y)) = f(F_1(x,y)) \in R_1[\![x,y]\!].$$

Here $\phi_* F_0$ means the FGL over R_1 obtained by applying ϕ to the coefficients of F_0. The composition is given by $(\phi_1, f_1) \circ (\phi_0, f_0) = (\phi_1\phi_0, f_2)$, where $f_2(x) = (\phi_{1*} f_0)(f_1(x))$.

Given $(R,F) \in$ FGL we can make $R[\![x]\!]$ into a Hopf algebra over R (in a suitably completed sense) by defining

$$\psi(x) = F(x \otimes 1, 1 \otimes x) \in R[\![x \otimes 1, 1 \otimes x]\!] = R[\![x]\!]\widehat{\otimes} R[\![x]\!].$$

This makes the formal scheme $\operatorname{spec}(R) \times \widehat{\mathbb{A}}^1 = \operatorname{spf}(R[\![x]\!])$ into a formal group over $\operatorname{spec}(R)$. This construction gives a functor $Q\colon \mathrm{FGL}^{\mathrm{op}} \to \mathrm{FG}$, which is easily seen to be full and faithful.

If $(G,S) \in$ FG then we can choose a coordinate x on G that vanishes at zero, and we find that there is a unique FGL F over $\mathcal{O}_S$ such that

$$\psi(x) = F(x \otimes 1, 1 \otimes x) \in \mathcal{O}_{G^2} = \mathcal{O}_S[\![x \otimes 1, 1 \otimes x]\!].$$

We can regard x as a map $G \to \widehat{\mathbb{A}}^1$, and as such it induces an isomorphism $G \to Q(\mathcal{O}_S, F)$. It follows that Q is essentially surjective and thus an equivalence.

2.3. **Periodic bordism.** Consider the homology theory $MP_*(X) = MU_*(X) \otimes \mathbb{Z}[u,u^{-1}]$, where u has homological degree 2 (and thus cohomological degree -2). This is represented by the spectrum $MP = \bigvee_{n\in\mathbb{Z}} \Sigma^{2n} MU$, with an evident ring structure. It is well-known that MU is an E_∞ ring spectrum; see for example [3, Section IX]. It is also shown there that MU is an H_∞^2 ring spectrum, which means (as explained in [3, Remark VII.2.9]) that MP is an H_∞ ring spectrum; this is weaker than E_∞ in theory, but usually equivalent in practise. As one would expect, MP is actually an E_∞ ring spectrum; a proof is given in the appendix. It follows from [2, Proposition II.4.3] that one can construct a model for MP that is a strictly commutative ring spectrum (or "S-algebra"). We may also assume that it is a cofibrant object in the category of all strictly commutative ring spectra.

For typographical convenience, we write $MP(r)$ for the $(r+1)$-fold smash power $MP \wedge \ldots \wedge MP$, which is again a strictly commutative ring. The spectra $MP(r)$ fit together into a cosimplicial object in the usual way. In the category of strictly commutative ring spectra, the coproduct is the smash product. It follows formally that the smash product of cofibrant objects is cofibrant, so in particular the objects $MP(r)$ are all cofibrant.

The formal group $\Gamma(MP)$ has a canonical coordinate $\widetilde{x}$, giving a canonical formal group law $\widetilde{F}$ and an isomorphism $Q(\pi_0 MP, \widetilde{F}) = \Gamma(MP)$. By a well-known theorem of Quillen, this formal group law is universal: for any FGL F over any ring R, there is a unique map $c[F]\colon \pi_0 MP \to R$ carrying $\widetilde{F}$ to F. In this situation, we use the notation (R, F) to refer to R with the $\pi_0 MP$-algebra structure obtained from $c[F]$.

Next consider $MP_0 MP = \pi_0 MP(1)$. There are two maps $1 \wedge \eta, \eta \wedge 1\colon MP \to MP(1)$, giving rise to formal group laws $\widetilde{F}_0, \widetilde{F}_1$ over $\pi_0 MP(1)$. These arise from two different coordinates on the same formal group, so there is a canonical isomorphism $\widetilde{b}\colon \widetilde{F}_1 \to \widetilde{F}_0$ defined by a power series $\widetilde{b}(x) = \sum_{k \geq 0} \widetilde{b}_k x^{k+1} \in (\pi_0 MP(1))[\![x]\!]$. It is well-known that this is universal in the following sense: given any ring R and any isomorphism $b\colon F_1 \to F_0$ of formal group laws over R, there is a unique ring map $c[F_0 \xleftarrow{b} F_1]\colon \pi_0 MP(1) \to R$ carrying $\widetilde{F}_i$ to F_i and $\widetilde{b}$ to b. (This can be deduced from the corresponding result for MU, but note that in our periodic setting we get all isomorphisms, not just the strict ones.) We use the notation $(R, F_0 \xleftarrow{b} F_1)$ to refer to R with the $\pi_0 MP(1)$-algebra structure obtained from this map. It is clear by construction that

$$(\pi_0 MP \xrightarrow{\pi_0(1\wedge\eta)} \pi_0 MP(1) \xrightarrow{c[F_0 \xleftarrow{b} F_1]} R) = c[F_0]$$

$$(\pi_0 MP \xrightarrow{\pi_0(\eta\wedge 1)} \pi_0 MP(1) \xrightarrow{c[F_0 \xleftarrow{b} F_1]} R) = c[F_1].$$

The same circle of ideas shows that $\pi_0 MP(1) = \pi_0 MP[\widetilde{b}_0, \widetilde{b}_1, \ldots][\widetilde{b}_0^{-1}]$ (where $\pi_0 MP(1)$ is regarded as an algebra over $\pi_0 MP$ using the map $\eta \wedge 1$).

More generally, we find that $\pi_0 MP(r)$ is the universal example of a ring equipped with $r+1$ different formal group laws $(F_0, \ldots, F_r)$ and a chain of isomorphisms $F_0 \leftarrow \ldots \leftarrow F_r$ between them, and that $\pi_0 MP(r)$ is obtained from a polynomial algebra on countably many generators over $\pi_0 MP$ by inverting r of the generators.

One also checks that $\pi_1 MP(r) = 0$ for all r. The module $\pi_2 MP(r)$ is free of rank one over $\pi_0 MP(r)$, but there are a number of different natural choices of generator. The most natural way around this is to use the formalism of Section 2.1, and regard $\pi_2 MP(r)$ as $\omega_{G_{MP(r)}}$.

2.4. **Module categories.** We write Mod_r for the category of $MP(r)$-modules (in the strict sense, not the homotopical one). Note that a map $f\colon A_0 \to A_1$ of strictly commutative ring spectra gives a functor $f^*\colon \mathrm{Mod}_{A_1} \to \mathrm{Mod}_{A_0}$, which is just the identity on the underlying spectra (and thus preserves weak equivalences). It follows easily that for any two maps $A_0 \xrightarrow{f} A_1 \xrightarrow{g} A_2$, the functor $f^* g^*$ is actually equal (not just naturally isomorphic or naturally homotopy equivalent) to $(gf)^*$. Thus, the categories Mod_r fit together to give a simplicial category Mod_*.

Remark 2.1. For us, a *simplicial category* means a simplicial object in the category of categories. Elsewhere in the literature, the same phrase is sometimes used to refer to categories enriched over the category of simplicial sets, which is a rather different notion.

Next, we write DMod_r the derived category of Mod_r, as in [2, Chapter III]. As usual, there are two different models for a category such as DMod_r:

(a) One can take the objects to be the cofibrant objects in Mod_r, and morphisms to be homotopy classes of maps; or
(b) One can use all objects in Mod_r and take morphisms to be equivalence classes of "formal fractions", in which one is allowed to invert weak equivalences.

We will use model (b). This preserves the strong functorality mentioned previously, and ensures that DMod_* is again a simplicial category.

We also write EPA_r for the category of even periodic commutative ring objects in DMod_r, giving another simplicial category. (Note that periodicity is actually automatic, because $MP(r)$ is itself periodic.) We reserve the letter η for the unit map $S \to MP$. We generally write $\zeta\colon MP \to E$ for the unit map of an object in EPA_0, and ξ for unit maps in EPA_1.

If $E \in \mathrm{EPA}_0$ then the canonical coordinate $\widetilde{x}$ on $\Gamma(MP)$ gives a coordinate $\zeta_*\widetilde{x}$ on $\Gamma(\eta^*E)$ and thus a formal group law F over π_0E, with $c[F] = \pi_0\zeta\colon \pi_0MP \to \pi_0E$, and a canonical isomorphism $\Gamma(\eta^*F) \simeq Q(\pi_0E, F)$. A map $f\colon \eta^*E_0 \to \eta^*E_1$ in EPR gives a map $\Gamma(f)\colon \Gamma(\eta^*E_1) \to \Gamma(\eta^*E_0)$ in FG, corresponding to a map $(\pi_0f, b)\colon (\pi_0E_0, F_0) \to (\pi_0E_1, F_1)$ in FGL; the power series $b \in (\pi_0E_1)[\![t]\!]$ is characterised by the fact that $f_*x_0 = b(x_1)$ in $E_1^0\mathbb{C}P^\infty$.

3. Basic realisation results

Let R be a strictly commutative ring spectrum that is even and periodic, such that R_0 is an integral domain. The main examples will be $R = MP(r)$ for $r \geq 0$. Let $\mathcal{D}$ be the derived category of R-modules, and let $\mathcal{R}$ be the category of commutative ring objects $A \in \mathcal{D}$ such that $\pi_1A = 0$ and π_0A has no 2-torsion. We also write $\mathcal{R}_0$ for the category of commutative algebras over π_0R without 2-torsion. We say that an object $A_0 \in \mathcal{R}_0$ is *strongly realisable* if there exists an object $A \in \mathcal{R}$ and an isomorphism $A_0 \to \pi_0A$ such that for any $B \in \mathcal{R}$, the induced map $\mathcal{R}(A, B) \to \mathcal{R}_0(A_0, \pi_0B)$ is an isomorphism.

The results of [5] provide a good supply of strongly realisable rings, except that we need a little translation between the even periodic framework and the usual graded framework. Suppose that $A_0 \in \mathcal{R}_0$, and put $T = \mathrm{spec}(A_0)$. We have a unit map $\eta\colon \pi_0R \to A_0$ and thus a map $\mathrm{spec}(\eta)\colon T \to S_R$; we can pull back the formal group G_R along this to get a formal group $H := \mathrm{spec}(\eta)^*G_R$ over T. From this we get a map $\eta_*\colon R_* = D(G_R, S_R)_* \to D(H, T)_*$, which agrees with η in degree zero. Indeed, if we choose a generator u of R_2 over R_0, then η_* is just the map $R_0[u, u^{-1}] \to A_0[u, u^{-1}]$ obtained in the obvious way from η. It is easy to check that A_0 is strongly realisable (as defined in the previous paragraph) iff $D(H, T)_*$ is strongly realisable over R_* (as defined in [5]).

Definition 3.1. A *short ordinal* is an ordinal λ of the form $n.\omega + m$ for some $n, m \in \mathbb{N}$. A *regular sequence* in a ring R_0 is a system of elements $(x_\alpha)_{\alpha<\lambda}$ for some short ordinal λ such that x_α is not a zero-divisor in the ring $(S^{-1}R_0)/(x_\beta \mid \beta < \alpha)$.

An object $A_0 \in \mathcal{R}_0$ is a *localised regular quotient* (or LRQ) of R_0 if $A_0 = (S^{-1}R_0)/I$ for some subset $S \subset R_0$ and some ideal $I \leq S^{-1}R_0$ that can be generated by a regular sequence.

Remark 3.2. We have made a small extension of the usual notion of a regular sequence, to ensure that any LRQ of an LRQ of R_0 is itself an LRQ of R_0; see Lemma 3.9.

Proposition 3.3. *If A_0 is an LRQ of R_0 in which 2 is invertible, then it is strongly realisable.*

Proof. This is essentially [5, Theorem 2.6], translated into a periodic setting as explained above. Here we are using a slightly more general notion of a regular sequence, but all the arguments can be adapted in a straightforward way. The main point is that any countable limit ordinal has a cofinal sequence, so homotopy colimits can be constructed using telescopes in the usual way. Andrey Lazarev has pointed out a lacuna in [5]: it is necessary to assume that the elements x_α are all regular in $S^{-1}R_0$ itself, which is not generally automatic. However, we are assuming that R_0 is an integral domain so this issue does not arise. □

Proposition 3.4. *Suppose that*

- *A and B are strong realisations of A_0 and B_0*
- *The ring $A_0 \otimes_{R_0} B_0$ has no 2-torsion*
- *The natural map $A_0 \otimes_{R_0} B_0 \to (A \wedge_R B)_0$ is an isomorphism.*

Then $A \wedge_R B$ is a strong realisation of $A_0 \otimes_{R_0} B_0$.

Proof. This follows from [5, Corollary 4.5]. □

Proposition 3.5. *If $A_0 \in \mathcal{R}_0$ is strongly realisable, and B_0 is an algebra over A_0 that is free as a module over A_0, then B_0 is also strongly realisable.*

Proof. This follows from [5, Proposition 4.13]. □

Proposition 3.6. *Suppose that R_0 is a polynomial ring in countably many variables over $\mathbb{Z}$, that $A_0 \in \mathcal{R}_0$, and that $A_0 = \mathbb{Z}[1/n]$ as a ring (for some n). Then A_0 is an LRQ of R_0, and thus is strongly realisable if n is even.*

Proof. Choose a system of polynomial generators $\{x_k \mid k \geq 0\}$ for R_0 over $\mathbb{Z}$. Put $a_k = \eta(x_k) \in A_0 = \mathbb{Z}[1/n]$ and $y_k = x_k - a_k \in R_0[1/n]$. It is clear that $R_0[1/n] = \mathbb{Z}[1/n][y_k \mid k \geq 0]$, that the elements y_k form a regular sequence generating an ideal I say, and that $A_0 = R_0[1/n]/I$. □

Proposition 3.7. *Suppose that R_0 is a polynomial ring in countably many variables over $\mathbb{Z}$, that $A_0 \in \mathcal{R}_0$, and that A_0 is a field (necessarily of characteristic different from 2). Then A_0 is a free module over an LRQ of R_0, and thus is strongly realisable.*

Proof. For notational simplicity, we assume that A_0 has characteristic $p > 2$; the case of characteristic 0 is essentially the same.

Choose a set X of polynomial generators for R_0 over $\mathbb{Z}$. Let K be the subfield of A_0 generated by the image of η, or equivalently by $\eta(X)$. We can choose a subset $Y \subseteq X$ such that $\eta(Y)$ is a transcendence basis for K over $\mathbb{F}_p$. This means that the subfield L_0 of K generated by $\eta(Y)$ is isomorphic to the rational function field $\mathbb{F}_p(Y)$, and that K is algebraic over L_0. Put $S = \mathbb{Z}[Y] \setminus (p\mathbb{Z}[Y])$, so $L_0 =$

$(S^{-1}\mathbb{Z}[Y])/p$. Next, list the elements of $X \setminus Y$ as $\{x_1, x_2, \dots\}$, and let L_k be the subfield of K generated by $\{x_i \mid i \leq k\}$. (We will assume that $X \setminus Y$ is infinite; if not, the notation changes slightly.) As x_k is algebraic over L_{k-1}, there is a monic polynomial $f_k(t) \in L_{k-1}[t]$ with $L_k = L_{k-1}[x_k]/f_k(x_k)$. As L_{k-1} is a quotient of the ring $P_{k-1} := S^{-1}\mathbb{Z}[Y, x_1, \dots, x_{k-1}]$, we can choose a monic polynomial $g_k(t) \in P_{k-1}[t]$ lifting f_k, and put $z_k := g_k(x_k) \in P_k \subseteq S^{-1}R_0$. It is not hard to check that the sequence $(p, z_1, z_2, \dots)$ is regular in $S^{-1}R_0$, and that $(S^{-1}R_0)/(z_i \mid i > 0) = K$, so K is an LRQ of R_0. It is clear that A_0 is free over the subfield K. □

Definition 3.8. Let A_0 be a commutative algebra over $\pi_0 MP$ with no 2-torsion. Regard $\pi_0 MP$ as a subring of $\pi_0 MP(r)$ via any one of the $r+1$ obvious ring maps $\eta_0, \dots, \eta_r \colon MP \to MP(r)$. We say that A_0 is *stably realisable* if, given any map $\pi_0 MP(r) \to A_0$ extending the given map $\pi_0 MP \to A_0$, the resulting $\pi_0 MP(r)$-algebra is strongly realisable. (Using the symmetric group action on $MP(r)$, we see that this does not depend on which η_i we use.)

Lemma 3.9. *An LRQ of an LRQ is an LRQ.*

Proof. Suppose that $B = (S^{-1}A)/(x_\alpha \mid \alpha < \lambda)$ and $C = (T^{-1}B)/(y_\beta \mid \beta < \mu)$, where λ and μ are short ordinals, and the x and y sequences are regular in $S^{-1}A$ and $T^{-1}B$ respectively. Let T' be the set of elements of A that become invertible in $T^{-1}B$; clearly $S \subseteq T'$ and $T^{-1}B = ((T')^{-1}A)/(x_\alpha \mid \alpha < \lambda)$. As $(T')^{-1}A$ is a localisation of $S^{-1}A$ and localisation is exact, we see that x is a regular sequence in $(T')^{-1}A$ as well. After multiplying by suitable elements of T' if necessary, we may assume that y_β lies in the image of A (this does not affect regularity, as the elements of T' are invertible). We then put $z_\alpha = x_\alpha$ for $\alpha < \lambda$, and let $z_{\lambda+\beta}$ be any preimage of y_β in A for $0 \leq \beta < \mu$. This gives a regular sequence in $(T')^{-1}A$ indexed by $\lambda + \mu$, such that $C = ((T')^{-1}A)/(z_\gamma \mid \gamma < \lambda + \mu)$ as required. □

Proposition 3.10. *If A_0 is an LRQ of $\pi_0 MP$ in which 2 is invertible, then A_0 is stably realisable.*

Proof. We know from Section 2.3 that $\pi_0 MP(r)$ is a polynomial ring in countably many variables over $\pi_0 MP$, in which r of the variables have been inverted, so we can write

$$\pi_0 MP(r) = \pi_0 MP[x_1, x_2, \dots][x_1^{-1}, \dots, x_r^{-1}].$$

Suppose that $A_0 = (S^{-1}\pi_0 MP)/I$. Put

$$B_0 = A_0[x_1, x_2, \dots][x_1^{-1}, \dots, x_r^{-1}],$$

which is evidently an LRQ of $\pi_0 MP(r)$. Let $f \colon \pi_0 MP(r) \to A_0$ be any map extending the given map $\pi_0 MP \to A_0$, and put $a_k = f(x_k) \in A_0$, and $y_k = x_k - a_k \in B_0$. Clearly B_0 is a localisation of $A_0[y_k \mid k > 0]$, the sequence of y's is regular in B_0, and $B_0/(y_k \mid k > 0) = A_0$ as $\pi_0 MP(r)$-algebras. It follows that A_0 is an LRQ of an LRQ, and thus an LRQ, over $\pi_0 MP(r)$. It is thus strongly realisable as required. □

Corollary 3.11. *If A_0 is an algebra over $\pi_0 MP$ that is just $\mathbb{Z}[1/n]$ as a ring for some even integer n, then A_0 is stably realisable.* □

Proposition 3.12. *Let A_0 be an algebra over $\pi_0 MP$ that is a field of characteristic not equal to 2. Then A_0 is stably realisable.*

Proof. This is immediate from Proposition 3.7. □

It will be convenient to restate the above results in slightly different language.

Definition 3.13. Let F be a formal group law over a ring R. We say that F is strongly (resp. stably) realisable if R, regarded as an algebra over $\pi_0 MP$ by the classifying map of F, is strongly (resp. stably) realisable.

Let (G, S) be a formal group. We say that a coordinate x on G is strongly (resp. stably) realisable if the associated formal group law over $\mathcal{O}_S$ is strongly (resp. stably) realisable. We say that G is *good* if it admits a stably realisable coordinate. We write GFG for the category of good formal groups.

In these terms, our results give:

Theorem 3.14. *Suppose that R is a field of characteristic not equal to 2, or that $R = \mathbb{Z}[1/n]$ for some even integer n. Then every formal group law over R is stably realisable, so every formal group over* $\operatorname{spec}(R)$ *is good.* □

4. Proof of the main theorem

Let $\mathcal{E}$ denote the class of objects $E \in \mathrm{EPA}_0$ for which the formal group $\Gamma(\eta^* E)$ is good.

Proposition 4.1. *For any good formal group (G, S), there exists $E \in \mathcal{E}$ with $\Gamma(\eta^* E) \simeq (G, S)$.*

Proof. Choose a stably realisable coordinate x on G, and let F be the resulting formal group law over $\mathcal{O}_S$. As F is stably realisable, there exists $E \in \mathrm{EPA}_0$ together with an isomorphism $\phi\colon \pi_0 E \to \mathcal{O}_S$ of rings such that the composite

$$\pi_0 MP \xrightarrow{\pi_0 \zeta} \pi_0 E \xrightarrow{\phi} \mathcal{O}_S$$

is just $c[F]$. If we let F' be the formal group law over $\pi_0 E$ classified by $\pi_0 \zeta$ then $\Gamma(\eta^* E) = Q(\pi_0 E, F') \simeq Q(\mathcal{O}_S, F) \simeq (G, S)$ as required. □

Proposition 4.2. *Suppose we have objects $E_0, E_1 \in \mathcal{E}$, together with a map $p\colon \Gamma(\eta^* E_1) \to \Gamma(\eta^* E_0)$ in* GFG. *Then there is a unique good map $f\colon \eta^* E_0 \to \eta^* E_1$ such that $\Gamma(f) = p$.*

Proof. For $i = 0, 1$ we let x_i be the coordinate on $\Gamma(\eta^* E_i)$ supplied by the unit map $\zeta_i\colon MP \to E_i$ and let F_i be the corresponding formal group law over $\pi_0 E_i$. The map p gives us a map $\phi\colon \pi_0 E_0 \to \pi_0 E_1$ of rings together with an isomorphism $b\colon F_1 \to \phi_* F_0$ of formal group laws over $\pi_0 E_1$. We know that maps $MP \to \eta^* E_1$ in EPR biject with coordinates on $\Gamma(\eta^* E_1)$, so there is a unique map $\sigma\colon MP \to \eta^* E_1$ carrying the canonical coordinate $\widetilde{x}$ on $\Gamma(MP)$ to $b(x_1)$. We let β_0 be the composite

$$MP(1) \xrightarrow{\sigma \wedge \zeta_1} (\eta^* E_1) \wedge (\eta^* E_1) \xrightarrow{\text{mult}} \eta^* E_1$$

(so $\beta_0 \in \mathrm{EPR}(MP(1), \eta^* E_1)$) and we put

$$\beta = \pi_0 \beta_0 \colon \pi_0 MP(1) \to \pi_0 E_1.$$

It is clear that $\beta = c[\phi_* F_0 \xleftarrow{b} F_1]$.

We next introduce a category $\mathcal{B} = \mathcal{B}(E_0, p, E_1)$ as follows. The objects are triples (A, f', f'') where

(a) A is an object of EPA_1, with unit map $\xi\colon MP(1) \to A$.
(b) f' is a morphism $E_0 \to (1 \wedge \eta)^* A$ in EPA_0.
(c) f'' is an isomorphism $(\eta \wedge 1)^* A \to E_1$ in EPA_0.

(d) The composite

$$f = \theta(A, f', f'') := (\eta^* E_0 \xrightarrow{\eta^* f'} (\eta \wedge \eta)^* A \xrightarrow{\eta^* f''} \eta^* E_1)$$

satisfies $\pi_0 f = \phi \colon \pi_0 E_0 \to \pi_0 E_1$ and $f_* x_0 = b(x_1) \in E_1^0 \mathbb{C}P^\infty$.

The morphisms from (A, f', f'') to (B, g', g'') in $\mathcal{B}$ are the isomorphisms $u \colon A \to B$ in EPA_1 for which $((1 \wedge \eta)^* u) f' = g'$ and $g''((\eta \wedge 1)^* u) = f''$.

The maps of the form $\theta(A, f', f'')$ are precisely the good maps that induce p, and isomorphic objects of $\mathcal{B}$ have the same image under θ. It will thus suffice to show that all objects of $\mathcal{B}$ are isomorphic.

We first claim that for any object (A, f', f'') in $\mathcal{B}$, the composite

$$(MP \xrightarrow{\zeta_0} \eta^* E_0 \xrightarrow{f} \eta^* E_1) \in \mathrm{EPR}(MP, \eta^* E_1)$$

is equal to σ. Indeed, we certainly have $\zeta_{0*}(\widetilde{x}) = x_0$ and $f_*(x_0) = b(x_1)$ by condition (d), so $(f\zeta_0)_*(\widetilde{x}) = b(x_1)$ as required.

Next, we claim that the composite

$$\gamma_0 = (MP(1) \xrightarrow{\xi} (\eta \wedge \eta)^* A \xrightarrow{\eta^* f''} \eta^* E_1) \in \mathrm{EPR}(MP(1), \eta^* E_1)$$

is equal to β_0. Indeed, $MP(1)$ is just the coproduct in EPR of two copies of MP, so it will suffice to check that $\gamma_0 \circ (\eta \wedge 1) = \beta_0 \circ (\eta \wedge 1)$ and $\gamma_0 \circ (1 \wedge \eta) = \beta_0 \circ (1 \wedge \eta)$. By construction, we have $\beta_0 \circ (1 \wedge \eta) = \sigma = f\zeta_0$ and $\beta_0 \circ (\eta \wedge 1) = \zeta_1$. Now consider the following diagram, in which we have implicitly applied forgetful functors down to EPR.

$$\begin{array}{ccccc}
MP & \xrightarrow{1 \wedge \eta} & MP(1) & \xleftarrow{\eta \wedge 1} & MP \\
\zeta_0 \downarrow & & \xi \downarrow & & \downarrow \zeta_1 \\
E_0 & \xrightarrow[f']{} & A & \xrightarrow[f'']{} & E_1
\end{array}$$

As f' comes from a ring map $E_0 \to (1 \wedge \eta)^* A$ in EPA_0, it is certainly a map of MP-algebras in the naive, homotopical sense. In particular, it preserves units, which means that the left hand square commutes. Similarly, so does the right hand square. From the right hand square, we see that $\gamma_0 \circ (\eta \wedge 1) = \zeta_1$. From the left hand square, we see that $\gamma_0 \circ (1 \wedge \eta) = f\zeta_0 = \sigma$. It follows that $\gamma_0 = \beta_0$ as claimed, and so $\pi_0 \gamma_0 = \beta = c[\phi_* F_0 \xleftarrow{b} F_1]$. We deduce that $\pi_0 A$ is isomorphic as an algebra over $\pi_0 MP(1)$ to $(\pi_0 E_1, \phi_* F_0 \xleftarrow{b} F_1)$. As F_1 is a stably realisable formal group law, we see that this algebra is strongly realisable, so maps $A \to A'$ in EPA_1 biject with maps $\pi_0 A \to \pi_0 A'$ of $\pi_0 MP(1)$-algebras.

Now suppose we have another object (B, g', g'') in CB. We then have an isomorphism $v = (g'')^{-1} f'' \colon (\eta \wedge 1)^* A \to (\eta \wedge 1)^* B$ in EPA_1. If we let χ be the unit map for B, the previous paragraph tells us that following diagram commutes (with both composites equal to β):

$$\begin{array}{ccc}
\pi_0 MP(1) & \xrightarrow{\pi_0 \xi} & \pi_0 A \\
\pi_0 \chi \downarrow & & \downarrow \pi_0 f'' \\
\pi_0 B & \xrightarrow[\pi_0 g'']{} & \pi_0 E_1
\end{array}$$

It follows easily that $\pi_0 v$ is a map of $\pi_0 MP(1)$-algebras, so by strong realisablility it agrees with $\pi_0 u$ for a unique map $u \colon A \to B$ in EPA_1. As $\pi_0 u$ is an isomorphism, the same is true of u. We also claim that $(\eta \wedge 1)^*u = v$ as a map from $(\eta \wedge 1)^*A$ to $(\eta \wedge 1)^*B$ in EPA_0. Indeed, the $\pi_0 MP$-algebra $\pi_0((\eta \wedge 1)^*A)$ is isomorphic to $\pi_0 E_1$ and so is strongly realisable, so maps out of $(\eta \wedge 1)^*A$ are determined by their effect on π_0, and $\pi_0((\eta \wedge 1)^*u) = \pi_0 v$ by construction. This shows that $g'' \circ ((\eta \wedge 1)^*u) = f''$.

Finally, we claim that $((1 \wedge \eta)^*u) \circ f' = g'$ as maps from E_0 to $(1 \wedge \eta)^*B$ in EPA_0. As $\pi_0 E_0$ is strongly realisable, it again suffices to check the equation on π_0. From the definition of $\mathcal{B}$ we have

$$\pi_0(f'') \circ \pi_0(f') = \phi = \pi_0(g'') \circ \pi_0(g') \colon \pi_0 E_0 \to \pi_0 E_1,$$

and by construction we have $\pi_0 u = \pi_0(g'')^{-1}\pi_0(f'')$; it follows that $\pi_0(u) \circ \pi_0(f') = \pi_0(g')$ as required.

This shows that there is actually a unique isomorphism between any two objects of $\mathcal{B}$, so in particular the function θ is constant on $\mathcal{B}$, so there is a unique good map inducing p, as claimed. □

Lemma 4.3. *For any $E \in \mathcal{E}$, the identity map $1\colon \eta^*E \to \eta^*E$ is good.*

Proof. Note that the multiplication map $MP(1) = MP \wedge MP \to MP$ is a map of ring spectra (in the strict sense) and so induces a functor $\mu^*\colon \mathrm{EPA}_0 \to \mathrm{EPA}_1$ with $(1 \wedge \eta)^*\mu^*E = (\eta \wedge 1)^*\mu^*E = E$ on the nose. We can thus take $A = \mu^*E$ and $f' = f'' = 1_E$ to show that 1_E is good. □

Proposition 4.4. *Suppose we have objects $E_0, E_1, E_2 \in \mathcal{E}$ and good morphisms $\eta^*E_0 \xrightarrow{f} \eta^*E_1 \xrightarrow{g} \eta^*E_2$. Then the composite gf is also good.*

Proof. Let F_i be the formal group law for E_i, and let (ϕ, b) and (ψ, c) be the morphisms of formal group laws coming from f and g (so in particular $\phi = \pi_0 f$ and $\psi = \pi_0 g$). Choose objects $A, B \in \mathrm{EPA}_1$ and maps

$$f'\colon E_0 \to (1 \wedge \eta)^*A$$
$$f''\colon (\eta \wedge 1)^*A \xrightarrow{\simeq} E_1$$
$$g'\colon E_1 \to (1 \wedge \eta)^*B$$
$$g''\colon (\eta \wedge 1)^*B \xrightarrow{\simeq} E_2$$

exhibiting the goodness of f and g. Next, observe that we have a chain

$$\psi_*\phi_*F_0 \xleftarrow{\psi_* b} \psi_*F_1 \xleftarrow{c} F_2$$

of formal group laws over $\pi_0 E_2$. This makes $\pi_0 E_2$ into an algebra over $\pi_0 MP(2)$, and it is strongly realisable as such, because F_2 is stably realisable by assumption. We can thus choose an object $P \in \mathrm{EPA}_2$ and an isomorphism $w\colon \pi_0 P \to \pi_0 E_2$ of $\pi_0 MP(2)$-algebras. We now include $\pi_0 MP(1)$ in $\pi_0 MP(2)$ by the map $\eta \wedge 1 \wedge 1\colon MP(1) \to MP(2)$. The resulting algebra structure on $\pi_0 E_2$ classifies the chain $\psi_*F_1 \xleftarrow{c} F_2$. From the proof of Proposition 4.2, we see that the isomorphism $\pi_0 B \xrightarrow{\pi_0 g''} \pi_0 E_2$ is also a $\pi_0 MP(1)$-algebra map with respect to this structure. This means that we have an isomorphism $(\pi_0 g'')^{-1}w\colon \pi_0((\eta \wedge 1 \wedge 1)^*P) \to \pi_0 B$ of $\pi_0 MP(1)$-algebras, and by strong realisability, it comes from a unique isomorphism $v\colon (\eta \wedge 1 \wedge 1)^*P \to B$ in EPA_1.

Next, we have maps of algebras over $\pi_0 MP(1)$ as follows

$$\begin{aligned}\pi_0 A &\xrightarrow{\pi_0 f''} (\pi_0 E_1, \phi_* F_0 \xleftarrow{b} F_1) \\ &\xrightarrow{\psi} (\pi_0 E_2, \psi_*\phi_* F_0 \xleftarrow{\psi_* b} \psi_* F_1) \\ &\xrightarrow{w^{-1}} \pi_0((1\wedge 1\wedge \eta)^* P).\end{aligned}$$

As the first map is an isomorphism, we know that $\pi_0 A$ is strongly realisable, so there is a unique map $u\colon A \to (1\wedge 1\wedge\eta)^* P$ in EPA_1 inducing the composite map on π_0.

We now put

$$\begin{aligned}C &= (1\wedge\eta\wedge 1)^* P \in \mathrm{EPA}_1 \\ h' &= (E_0 \xrightarrow{f'} (1\wedge\eta)^* A \xrightarrow{(1\wedge\eta)^* u} (1\wedge\eta\wedge\eta)^* P = (1\wedge\eta)^* C) \\ h'' &= ((\eta\wedge 1)^* C = (\eta\wedge\eta\wedge 1)^* P \xrightarrow{(\eta\wedge 1)^* v} (\eta\wedge 1)^* B \xrightarrow{g''} E_2).\end{aligned}$$

As v and g'' are isomorphisms, the same is true of h''. We claim that after forgetting down to EPR, we have $h''h' = gf$; this will prove that gf is good as claimed. We certainly have $h''h' = g''vuf'$ and $gf = g''g'f''f'$ so it will suffice to show that $vu = g'f''\colon A \to B$ in EPR. For this, it will be enough to prove that the following diagram in EPA_0 commutes.

$$\begin{array}{ccc}(\eta\wedge 1)^* A & \xrightarrow{(\eta\wedge 1)^* u} & (\eta\wedge 1\wedge\eta)^* P \\ {\scriptstyle f''}\downarrow\simeq & & \simeq\downarrow{\scriptstyle (1\wedge\eta)^* v} \\ E_1 & \xrightarrow[g']{} & (1\wedge\eta)^* B.\end{array}$$

As this is a diagram in EPA_0 and $\pi_0((\eta\wedge 1)^* A) \simeq \pi_0 E_1$ is strongly realisable, it will be enough to check that the diagram commutes after applying π_0. By construction we have $\pi_0(u) = w^{-1}\circ\psi\circ\pi_0(f'')$ and $\psi = \pi_0(g) = \pi_0(g'')\circ\pi_0(g')$ and $\pi_0(v) = \pi_0(g'')^{-1}\circ w$. It follows directly that the above diagram commutes on homotopy, groups, so it commutes in EPA_0, so it commutes in EPR, so $gf = h''h'$ in EPR as explained previously. Thus, the map gf is good, as claimed. □

Proof of Theorem 1.1. We merely need to collect results together and explain the argument in the introduction in more detail. Lemma 4.3 and Proposition 4.4 show that we can make $\mathcal{E}$ into a category by taking the good maps from $\eta^* E_0$ to $\eta^* E_1$ as the morphisms from E_0 to E_1. Tautologically, we can define a faithful functor $U\colon \mathcal{E} \to \mathrm{EPR}$ by $U(E) = \eta^* E$ and $U(f) = f$. We then define $V = \Gamma U\colon \mathcal{E} \to \mathrm{FG}$; by the definition of $\mathcal{E}$, this actually lands in GFG. Proposition 4.1 says that V is essentially surjective, and Proposition 4.2 says that V is full and faithful. This means that V is an equivalence, so we can invert it and define $R = UV^{-1}\colon \mathrm{GFG} \to \mathrm{EPR}$. As $V = \Gamma U$ we have $\Gamma R = 1$, so R is the required realisation functor. □

Appendix A. The product on MP

In this appendix we verify that MP can be constructed as an E_∞ ring spectrum.

Let $\mathcal{U}$ be a complex universe. For any finite-dimensional subspace U of $\mathcal{U}$, we write $U_L = U\oplus 0 < \mathcal{U}\oplus\mathcal{U}$ and $U_R = 0\oplus U < \mathcal{U}\oplus\mathcal{U}$. We let $\mathrm{Grass}(U\oplus U)$ denote the Grassmannian of all subspaces of $U\oplus U$ (of all possible dimensions), and we write γ_U for the tautological bundle over this space, and $\mathrm{Thom}(U \oplus U)$ for the associated Thom space. If $U \leq U' < \mathcal{U}$ then we define $i\colon \mathrm{Grass}(U^2) \to \mathrm{Grass}((U')^2)$ by $i(A) = A \oplus (U' \ominus U)_R$. On passing to Thom spaces we get a map $\sigma\colon \Sigma^{U' \ominus U}\,\mathrm{Thom}(U^2) \to \mathrm{Thom}((U')^2)$. These maps can be used to assemble the spaces $\mathrm{Thom}(U^2)$ into a Σ-inclusion prespectrum indexed by the complex subspaces of $\mathcal{U}$. We write $T_{\mathcal{U}}$ for this prespectrum, and $MP_{\mathcal{U}}$ for its spectrification.

Now let $\mathcal{V}$ be another complex universe, so we have a prespectrum $T_{\mathcal{V}}$ over $\mathcal{V}$, and thus an external smash product $T_{\mathcal{U}} \wedge_{\mathrm{ext}} T_{\mathcal{V}}$ indexed on the complex subspaces of $\mathcal{U} \oplus \mathcal{V}$ of the form $U \oplus V$. The direct sum gives a map $\mathrm{Grass}(U^2) \times \mathrm{Grass}(V^2) \to \mathrm{Grass}((U \oplus V)^2)$ which induces a map $\mathrm{Thom}(U^2) \wedge \mathrm{Thom}(V^2) \to \mathrm{Thom}((U \oplus V)^2)$. These maps fit together to give a map $T_{\mathcal{U}} \wedge_{\mathrm{ext}} T_{\mathcal{V}} \to T_{\mathcal{U} \oplus \mathcal{V}}$, and thus a map $MP_{\mathcal{U}} \wedge_{\mathrm{ext}} MP_{\mathcal{V}} \to MP_{\mathcal{U} \oplus \mathcal{V}}$ of spectra over $\mathcal{U} \oplus \mathcal{V}$. Essentially the same construction gives maps

$$MP_{\mathcal{U}_1} \wedge_{\mathrm{ext}} \dots \wedge_{\mathrm{ext}} MP_{\mathcal{U}_r} \to MP_{\mathcal{U}_1 \oplus \dots \oplus \mathcal{U}_r}.$$

If $\mathcal{U}_1 = \dots \mathcal{U}_r = \mathcal{U}$, then this map is Σ_r-equivariant.

Now suppose instead that we have a complex linear isometry $f\colon \mathcal{U} \to \mathcal{V}$. This gives evident homeomorphisms $\mathrm{Thom}(U^2) \to \mathrm{Thom}((fU)^2)$, which fit together to induce a map $MP_{\mathcal{U}} \to f^* MP_{\mathcal{V}}$, which is adjoint to a map $f_* MP_{\mathcal{U}} \to MP_{\mathcal{V}}$. We next observe that this construction is continuous in all possible variables, including f. (This statement requires some interpretation, but there are no new issues beyond those that are well-understood for MU; the cleanest technical framework is provided by [1].) It follows that they fit together to give a map $\mathcal{L}_{\mathbb{C}}(\mathcal{U}, \mathcal{V}) \ltimes MP_{\mathcal{U}} \to MP_{\mathcal{V}}$ of spectra over $\mathcal{V}$.

We now combine this with the product structure mentioned earlier to get a map

$$\mathcal{L}_{\mathbb{C}}(\mathcal{U}^r, \mathcal{U}) \ltimes_{\Sigma_r} (MP_{\mathcal{U}} \wedge_{\mathrm{ext}} \dots \wedge_{\mathrm{ext}} MP_{\mathcal{U}}) \to MP_{\mathcal{U}}.$$

This means that $MP_{\mathcal{U}}$ has an action of the E_∞ operad of complex linear isometries, as required.

All that is left is to check that the spectrum $MP = MP_{\mathbb{C}^\infty}$ constructed above has the right homotopy type. As T is a Σ-inclusion prespectrum, we know that spectrification works in the simplest possible way and that MP is the homotopy colimit of the spectra

$$\Sigma^{-2n}\,\mathrm{Thom}(\mathbb{C}^n \oplus \mathbb{C}^n) = \bigvee_{k=-n}^{n} \Sigma^{-2n}\,\mathrm{Grass}_{k+n}(\mathbb{C}^n \oplus \mathbb{C}^n)^\gamma,$$

where $\mathrm{Grass}_d(V)$ is the space of d-dimensional subspaces of V. It is not hard to see that the maps of the colimit system preserve this splitting, so that MP is the wedge over all $k \in \mathbb{Z}$ of the spectra

$$X_k := \underset{\underset{n}{\longrightarrow}}{\mathrm{holim}}\, \Sigma^{-2n}\,\mathrm{Grass}_{k+n}(\mathbb{C}^n \oplus \mathbb{C}^n)^\gamma.$$

This can be rewritten as

$$X_k = \Sigma^{2k}\, \underset{\underset{n,m}{\longrightarrow}}{\mathrm{holim}}\, \Sigma^{-2(k+n)}\,\mathrm{Grass}_{k+n}(\mathbb{C}^m \oplus \mathbb{C}^n)^\gamma.$$

We can reindex by putting $n = i - k$ and $m = j + k$, and then pass to the limit in j. We find that

$$X_k = \Sigma^{2k}\, \underset{\underset{i}{\longrightarrow}}{\mathrm{holim}}\, \Sigma^{-2i}\,\mathrm{Grass}_i(\mathbb{C}^\infty \oplus \mathbb{C}^i)^\gamma.$$

It is well-known that $\mathrm{Grass}_i(\mathbb{C}^\infty \oplus \mathbb{C}^i)$ is a model for $BU(i)$, and it follows that $X_k = \Sigma^{2k} MU$, so $MP = \bigvee_k \Sigma^{2k} MU$ as claimed. We leave it to the reader to check that the product structure is the obvious one.

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