

Decomposition of the Diagonal Map

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Abstract

This paper presents a new method for using cup product information to draw conclusions about the Lusternik-Schnirelmann category of a space. The key idea is that of the Hopf set in X of a map $f : S^{n-1} \rightarrow L$; if $K = L \cup_f D^n \subseteq X$, then $\text{cat}_X(K) = \text{cat}_X(L)$ if and only if $*$ is in the Hopf set in X of f . The main result explicitly constructs elements of the Hopf set in X of f in terms of members of the Hopf set in X of the attaching maps of lower dimensional cells. Applications include: a calculation of the category of $Sp(2)$ without higher order cohomology operations; new, easily used upper bounds for Lusternik-Schnirelmann category that apply to any space; and new information about the category of the CW skeleta of loop spaces and free loop spaces on even-dimensional spheres.

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Introduction

It has long been recognized that cohomology classes hold information about Lusternik-Schnirelmann category. This fact manifested itself first in the classical cup length lower bound for the category of a space. Later, Schweitzer and Singhof were able to squeeze more information out of cohomology classes using higher order cohomology operations [12, 13]. More

recently, Fadell and Husseini introduced the notion of the category weight of a cohomology class [5]; other authors refined and extended this idea in [11] and [16]. In this paper we introduce a new way to obtain category information from cohomology classes.

The results in this paper hold equally well (with the same proof) for category and for weak category. To simplify the exposition, we restrict our attention to weak category in the introduction. All spaces are assumed to be connected.

Let X be a CW complex and $K \subseteq X$ a subcomplex. The *weak category of K in X* is the least integer k for which the composition

$$K \longrightarrow X \xrightarrow{\widehat{d}_{k+1}} X^{(k+1)}$$

is trivial, where $X^{(k+1)}$ is the $(k+1)$ -fold smash product of X with itself and $\widehat{d}_{k+1} : X \rightarrow X^{(k+1)}$ is the $(k+1)$ -fold reduced diagonal out of X . Suppose that $K = L \cup_f D^n$ and $\text{wcat}_X(L) \leq k$. Then there is a homotopy factorization of the form

$$\begin{array}{ccccc} K & \xrightarrow{i} & X & \xrightarrow{\widehat{d}_{k+1}} & X^{(k+1)} \\ q \downarrow & & & & \parallel \\ S^n & \xrightarrow{h} & & \longrightarrow & X^{(k+1)}. \end{array}$$

The map h is not usually uniquely determined. The set $\overline{\mathcal{H}}_{k+1}(f; X)$ of all h which make the diagram commute up to homotopy is called the *crude Hopf set of f in X* . Thus $\text{wcat}_X(K) \leq k$ if and only if $*$ $\in \overline{\mathcal{H}}_{k+1}(f; X)$.

Our main result shows how to use cohomology information to explicitly write down certain elements of $\overline{\mathcal{H}}_{k+1}(f; X)$ in terms of the crude Hopf sets of lower dimensional cells. For each dimension n we let u_n be the cohomology class represented by the composite

$$K \xrightarrow{q} K/K_{n-1} \longrightarrow K(\pi_n(K/K_{n-1}), n),$$

where K_{n-1} denotes the $(n-1)$ -skeleton of the CW complex K . Notice that the class u_n depends on the chosen CW decomposition of K . Write $K = K_{n-1} \cup_f (\bigvee_{k=1}^m D_k^n)$, $K_a = K_{a-1} \cup_g (\bigvee_{i=1}^s D_i^a)$ and $K_b = K_{b-1} \cup_{g'} (\bigvee_{j=1}^t D_j^b)$. If $a+b=n$ then there is a homomorphism

$$\bar{\theta} : \left[\bigvee_k S^n, \bigvee_{i,j} S_i^a \wedge S_j^b \right] \longrightarrow H^n(K; \pi_a(K/K_{a-1}) \otimes \pi_b(K/K_{b-1}))$$

whose image contains $u_a u_b$.

Theorem 6 *Let $K \subseteq X$ and suppose $\text{wcat}_X(K_{a-1}) \leq \kappa - 1$ and $\text{wcat}_X(K_{b-1}) \leq \lambda - 1$. Choose $\bar{h} \in \overline{\mathcal{H}}_{\kappa+1}(g; X)$ and $\bar{h}' \in \overline{\mathcal{H}}_{\lambda+1}(g'; X)$. Finally, choose $\bar{R} \in \bar{\theta}^{-1}(u_a u_b)$. Then*

$$(\bar{h}_i \wedge \bar{h}'_j) \circ \bar{R} \in \overline{\mathcal{H}}_{\kappa+\lambda}(f; X).$$

As mentioned above, the corresponding result for ordinary category also holds. We now turn to some consequences of this result. First we consider complexes of the form $X = S^a \cup_g D^b \cup_f D^{a+b}$. Our result overlaps with Singhof's main theorem in [14].

Theorem 8 *Let $X = S^a \cup_g D^b \cup_f D^{a+b}$. Then $H^*(X)$ has generators $u \in H^a(X)$, $v \in H^b(X)$ and $w \in H^{a+b}(X)$, and $uv = rw$ for some integer r . Let $h \in \overline{\mathcal{H}}_2(g; X)$.*

- (a) *If $r\Sigma^a h = 0$, then $\text{wcat}(X) \leq 2$.*
- (b) *If $b < 3a - 1$ then $\text{cat}(X) = \text{wcat}(X) = 3$ if and only if $rh \neq 0$.*

Theorem 8 applies to the space $Sp(2)$, showing that $\text{cat}(Sp(2)) = 3$. This was first proved by Schweitzer using secondary cohomology operations [12]. Recently, N. Iwase has proved that if X is a sphere bundle over a sphere (so $r = 1$ in our notation), then $\text{cat}(X) = 3$ if and only if $\Sigma^a h = 0$.

Next we apply the method to obtain new upper bounds for the (weak) category of arbitrary spaces.

Theorem 9 *Let $K \subseteq X$ be a subcomplex of dimension n . If $a + b = n$, then*

$$\text{wcat}_X(K) \leq \text{wcat}_X(K_{a-1}) + \text{wcat}_X(K_{b-1}) + 2.$$

If equality holds then there are classes $u \in H^a(K)$ and $v \in H^b(K)$ such that $uv \neq 0$. Also, $\text{wcat}_X(K_a) = \text{wcat}_X(K_{a-1}) + 1$ and $\text{wcat}_X(K_b) = \text{wcat}_X(K_{b-1}) + 1$, so $\text{wcat}_X(K) = \text{wcat}_X(K_a) + \text{wcat}_X(K_b)$.

This is a sort of converse to the classical cup product lower bound for the weak category of a space. Loosely speaking, the classical result states

that if there are lots of nontrivial cup products, then the weak category must be large. Theorem 9 shows that if the weak category is large, then there must be lots of nontrivial cup products. Theorem 9 also includes the classical result that if X is $(c - 1)$ -connected and kc -dimensional, then $\text{cat}(X) = k$ if and only if the k^{th} power of the fundamental class $u_c \in H^c(X, \pi_c(X))$ is nonzero.

A variety of methods have been used to detect category in cases where cup products have proven insufficient, e.g. Massey products, higher order operations and, more recently, refined cup products [10]. A common feature of these methods is that they each return elements whose dimension *exceeds* the sum of the dimensions of the elements on which it acts. Loosely speaking again, our results show that the cup product is the *only* category-detecting operation for which the dimension of the image is equal to the sum of the dimensions of the range.

When the cup products in question vanish we have the following improvement on the classical dimension divided by connectivity upper bound for the (weak) category of K in X .

Theorem 11 *Let $K \subseteq X$ be a subcomplex and suppose $\text{wcat}_X(K_{m-1}) < c$. Assume that $u_{2^k(m+1)-1}^2 = 0$ for $k = 0, 1, \dots, t$. Then*

$$\text{wcat}_X(K_{2^t(m+1)-1}) \leq 2^t c.$$

Theorem 11 implies that if X is $(m - 1)$ -connected and has trivial cup product structure, then the (weak) category of X_n is asymptotically bounded above by $\frac{n}{m+1}$. This applies, for example, to the free loop space on an even sphere. As a further consequence of Theorem 11, we derive new upper bounds for the (weak) category of the symplectic group $Sp(n)$.

Proposition 13 *If $n(2n + 1) \leq 63a + 31b + 15c + 7d$ with $a, b, c, d \in \mathbb{N}$, then*

$$\text{cat}(Sp(n)) \leq 16a + 8b + 4c + 2d.$$

If either $n(2n + 1) < 63a + 31b + 15c + 7d$ or one of b, c or d is at least 2, then the inequality is strict.

The analog for ordinary category is the same. This is an improvement on the best previously known upper bound, $\text{wcat}(Sp(n)) \leq \frac{1}{3}n(2n + 1)$.

In the particular case $n = 3$ we find that $\text{cat}(Sp(3))$ is either 4 or 5 (using Singhof's lower bound $\text{cat}(Sp(n)) \geq n + 1$ [14]). The best previously available upper bound, obtained from Ganea's theorem in [6], was $\text{cat}(Sp(3)) \leq 6$.

We conclude with some further applications of Theorem 11. In particular, we determine the (weak) category of certain skeleta of $X = \Omega S^{2n}$, showing that $\text{cat}_X(X_n)$ is equal to the cup length for all n . We end with a related calculation for the (weak) category of the skeleta of $\Lambda(S^{2n})$, the free loop space on an even dimensional sphere.

1 Category and Hopf Sets

In this section we define the (crude) Hopf set of the attaching map of a cell and we derive some basic results. Specifically, we relate the Hopf sets to the generalized Hopf invariants defined by Berstein and Hilton in [1] and find some conditions which guarantee that the (crude) Hopf set is a singleton.

Before we begin, we establish some notation. If G is an abelian group and $n \geq 2$, then $M_n(G)$ is the Moore space with homology G in dimension n . The cone on a Moore space $M_{n-1}(G)$ is denoted $C^n(G)$. All spaces in this paper are assumed to be connected and pointed; all maps and homotopies preserve base points.

Let X be a pointed CW complex. The k -fold **fat wedge** of X with itself is the subcomplex $T^k X = \{(x_1, \dots, x_k) \mid \text{at least one } x_i = *\} \subseteq X^k$. The quotient $X^k/T^k X$ is the k -fold smash product of X with itself, so we have a cofibration $T^k X \xrightarrow{l} X^k \xrightarrow{\wedge} X^{(k)}$.

Definition Let $K \subseteq X$ be a subcomplex. The (reduced) **category of K in X** , $\text{cat}_X(K)$, is the least integer k for which there is a lift up to

homotopy of $d_{k+1} \circ i$ through $T^{k+1}X$ in the diagram

$$\begin{array}{ccccc}
 & & & & T^{k+1}X \\
 & & & & \downarrow l \\
 K & \xrightarrow{i} & X & \xrightarrow{d_{k+1}} & X^{k+1} \\
 & & & \searrow \hat{d}_{k+1} & \downarrow \wedge \\
 & & & & X^{(k+1)}
 \end{array}$$

where d_{k+1} is the $(k+1)$ -fold diagonal. The **weak category of K in X** , $\text{wcat}_X(K)$, is the least integer k for which the composite $\hat{d}_{k+1} \circ i = \wedge \circ d_{k+1} \circ i : K \rightarrow X^{(k+1)}$ is nullhomotopic. When $K = X$, we simply write $\text{cat}(X)$ and $\text{wcat}(X)$.

The (weak) category of K in X is a special case of the more general concept of the category of a map $K \rightarrow X$. All of the results in this paper hold, with similar proofs, for the more general concept. Since our applications do not require it, however, we do not pursue maximum generality here.

The classical (and equivalent) definition is that the category of K in X is the least integer k such that K is a union of $k+1$ subcomplexes, each of which is contractible in X . From this it follows immediately that for any space X , $\text{cat}(X \cup CA) \leq \text{cat}(X) + 1$. The same is true for weak category [1].

Now suppose $K = L \cup_f C^n(G) \subseteq X$ is a subcomplex with $\text{wcat}_X(L) \leq k$. To decide whether or not $\text{wcat}_X(K) = k+1$ we must study the $(k+1)$ -fold reduced diagonal out of X . By assumption, $\hat{d}_{k+1}|_L \simeq *$. A choice of nullhomotopy gives a homotopy factorization of \hat{d}_{k+1} through $q : X \rightarrow M_n(G)$, so we have a homotopy commutative diagram of the form

$$\begin{array}{ccccc}
 K & \xrightarrow{i} & X & \xrightarrow{\hat{d}_{k+1}} & X^{(k+1)} \\
 q \downarrow & & & & \parallel \\
 M_n(G) & \xrightarrow{h} & & & X^{(k+1)}.
 \end{array}$$

Definition The **crude Hopf set of f in X** is the set of pointed homotopy classes

$$\overline{\mathcal{H}}_{k+1}(f; X) = \{h : M_n(G) \rightarrow X^{(k+1)} \mid h \circ q \simeq \hat{d}_{k+1}\}.$$

This set is nonempty if and only if $\text{wcat}_X(L) \leq k$.

We can perform a similar construction for ordinary category. If $\text{cat}_X(L) \leq k$ then $d_{k+1}|_L$ can be deformed into the fat wedge $T^{k+1}X$ and so there is a map $\delta \simeq d_{k+1}$ such that $\delta(L) \subseteq T^{k+1}X$. Thus we have a commutative diagram

$$\begin{array}{ccccc}
(K, *) & \longrightarrow & (X, *) & \xrightarrow{\delta} & (X, *)^{k+1} \\
\downarrow q & & & & \parallel \\
(K, L) & \longrightarrow & & & (X, *)^{k+1} \\
\uparrow E & & & \nearrow h & \\
(C^n(G), M_{n-1}(G)) & & & &
\end{array}$$

of pairs, where $E \in \pi_n(K, L; G)$ is the inclusion of the attached cell.

Definition The **Hopf set of f in X** is the set $\mathcal{H}_{k+1}(f; X)$ of all pointed homotopy classes $h : (C^n(G), M_{n-1}(G)) \rightarrow (X, *)^{k+1}$ such that $h = \delta \circ E$ for some $\delta \simeq d_{k+1}$ with $\delta(L) \subseteq T^{k+1}X$. This set is nonempty if and only if $\text{cat}(L) \leq k$.

The significance of the (crude) Hopf sets is established in the following proposition.

Proposition 1 *Let $K = L \cup_f C^n(G) \subseteq X$ as above.*

- (a) *If $\text{wcat}_X(L) \leq k$ then $\text{wcat}_X(K) \leq k$ if and only if $* \in \overline{\mathcal{H}}_{k+1}(f; X)$.*
- (b) *If $\text{cat}_X(L) \leq k$ then $\text{cat}_X(K) \leq k$ if and only if $* \in \mathcal{H}_{k+1}(f; X)$.*

Proof

We only prove (b). If $\text{cat}_X(K) \leq k$ then there is a map $\phi : K \rightarrow X^{k+1}$ which factors through $T^{k+1}X$ and which is homotopic to $d_{k+1}|_K$. By homotopy extension, we may extend ϕ to a map $\delta \simeq d_{k+1} : X \rightarrow X^{k+1}$. Since $\delta(K) \subseteq T^{k+1}X$, the corresponding map $\delta \circ i \circ E = * \in \mathcal{H}_{k+1}(f; X)$.

If $* \in \mathcal{H}_{k+1}(f; X)$ then there is a map $\delta \simeq d_{k+1} : X \rightarrow X^{k+1}$ such that $\delta \circ i \circ E : (C^n(G), M_{n-1}(G)) \rightarrow (X, *)^{k+1}$ is homotopic as a map of pairs to a map $e : C^n(G) \rightarrow T^{k+1}X$. By homotopy extension, $\delta|_L$ is homotopic to

a map $\delta' : L \longrightarrow T^{k+1}X$ such that $\delta'|_{M_{n-1}(G)} = e|_{M_{n-1}(G)}$. This homotopy and the given homotopy $\delta \circ E \simeq e$ can be pasted together to show that $\delta|_K$ is homotopic to a map $\phi : K \longrightarrow T^{k+1}X \hookrightarrow X^{k+1}$. \blacksquare

The Hopf set depends on the choice of subcomplex K , though we do not overload our notation with this. Let $f : M_{n-1}(G) \longrightarrow K$ and $i : K \hookrightarrow M$. Then $\overline{\mathcal{H}}_{k+1}(i \circ f; X) \subseteq \overline{\mathcal{H}}_{k+1}(f; X)$, assuming both sets are defined, and the containment may be strict (and similarly for \mathcal{H}_{k+1}). The general question of whether a member of the (crude) Hopf set of f extends to a larger subcomplex of X is an important one for our later work.

Lemma 2 *Let $L \subseteq K \subseteq X$ and write $L = L' \cup_f C^n(G)$.*

- (a) *There is at least one $h \in \overline{\mathcal{H}}_k(f; X)$ which extends to K/L' ; if $* \in \overline{\mathcal{H}}_k(f; X)$ then $*$ extends to K/L' .*
- (b) *There is at least one $h \in \mathcal{H}_k(f; X)$ which extends to (K, L') ; if $* \in \mathcal{H}_k(f; X)$ then $*$ extends to (K, L') .*

The proof is straightforward, and omitted.

Our terms crude Hopf set and Hopf set are justified by their relationship to the generalized Hopf invariants $\overline{H}_\phi(f)$ and $H_\phi(f)$ defined in [1]. We defer the definition of the (crude) Hopf invariants and the proof of Theorem 3 to the final section of the paper.

Theorem 3 *Let $K = L \cup_f C^n(G) \subseteq X$ with $\text{cat}_X(L) \leq k$. If $\phi : L \longrightarrow T^{k+1}X$ is a lift of the diagonal map. Then*

- (a) $\overline{H}_\phi(f; X) \in \overline{\mathcal{H}}_{k+1}(f; X)$;
- (a) $H_\phi(f; X) \in \mathcal{H}_{k+1}(f; X)$.

The Hopf invariants are independent of the choice of subcomplex K which contains $f(M_{n-1}(G))$. The following example illustrates the need to distinguish between (crude) Hopf invariants and (crude) Hopf sets.

Example A Let $X = S_1^3 \times S_2^3$. Give S_1^3 a CW decomposition of the form $S^3 = S^2 \cup D^3 \cup D^3$. The induced CW decomposition of X has the form $(S^3 \vee S^3) \cup_f D_1^6 \cup_g D_2^6$. If we attach D_2^6 first then the category

increases when we attach D_1^6 , so $*$ $\notin \mathcal{H}_2(f; X)$. Since the Hopf invariant of f is independent of the choice of subcomplex, we see that $H_\phi(f) \neq *$, even in the complex $(S^3 \vee S^3) \cup_f D^6$, where the category is not increased by the cellular attachment.

We have followed Berstein and Hilton in defining the (crude) Hopf sets for general cells $C^n(G)$. Their purpose was to apply the Hopf invariant to a homology decomposition of a space X . In our main results, however, we choose to work with CW decompositions. Therefore we restrict our attention to attaching standard cells D^n from now on.

We conclude this section by giving some conditions under which the crude Hopf set consists of a single member.

Theorem 4 *Let $K = L \cup_f D^n$ where L is $(p-1)$ -connected and $\text{wcat}_X(L) = k$. If either*

- (a) *f is stably trivial and $k \geq \frac{n+2}{2p} - 1$, or*
- (b) *$\dim(K) < (k+1)p - 1$*

then $\overline{\mathcal{H}}_{k+1}(f; X)$ is the singleton set $\{\overline{H}_\phi(f)\}$.

Proof

By Theorem 3 it suffices to show that $\overline{\mathcal{H}}_{k+1}(f; K)$ is a singleton set. The cofibration sequence $K \xrightarrow{q} S^n \xrightarrow{\partial} \Sigma L \xrightarrow{i} \Sigma K \xrightarrow{\Sigma q} \dots$ gives rise to a ladder of exact sequences of pointed sets

$$\begin{array}{ccccccc}
 [K, X^{(k+1)}] & \xleftarrow{q^*} & [S^n, X^{(k+1)}] & \xleftarrow{\partial^*} & [\Sigma L, X^{(k+1)}] & \xleftarrow{\quad} & \dots \\
 \downarrow & & \cong \downarrow & & \cong \downarrow & & \\
 [\Sigma^m K, X^{(k+1)}] & \xleftarrow{q^*} & [S^{n+m}, \Sigma^m X^{(k+1)}] & \xleftarrow{\partial^*} & [\Sigma^{m+1} L, \Sigma^m X^{(k+1)}] & \xleftarrow{\quad} & \dots
 \end{array}$$

The vertical maps are isomorphisms because $k \leq \frac{n+2}{2p} - 1$. The map ∂^* is trivial for m large enough because f is stably trivial. Since these pointed sets are in fact groups, it follows that q^* is injective. To prove (b), simply observe that the set $[\Sigma L, X^{(k+1)}]$ is trivial. ■

The following theorem from [2] allows us to apply this result to many Lie groups.

Theorem *Let $K = L \cup_f D^n$ be a compact connected n -dimensional manifold, with $\dim(L) < n$. If K is an H -space, then f is stably trivial.*

2 Decomposing the Diagonal

In this section we develop the key idea of the paper. Using the cup product structure in $H^*(X)$ with various coefficients we give an explicit formula for some members of the crude Hopf set of the attaching map of the n -cells of $K \subseteq X$ in terms of the the crude Hopf set of the attaching maps of lower dimensional cells.

Let K be a space with a given CW decomposition. For each n there is a fundamental cohomology class $u_n \in H^n(K; \pi_n(K/K_{n-1}))$, defined to be the composite

$$K \xrightarrow{q} K/K_{n-1} \xrightarrow{\quad} K(\pi_n(K/K_{n-1}), n),$$

$\xrightarrow{u_n}$

where unlabelled map above is the n^{th} Postnikov approximation. For example, if $K = M_n(\mathbb{Z}/p) = S^n \cup_p D^{n+1}$ then $u_n = 1 \in H^n(K; \mathbb{Z}/p) \cong \mathbb{Z}/p$ and $u_{n+1} = 1 \in H^{n+1}(K; \mathbb{Z}) \cong \mathbb{Z}/p$.

Let $K \subseteq X$ be an n -dimensional subcomplex and choose a and b such that $a+b = n$; write $K_a = K_{a-1} \cup_g (\bigvee_{i=1}^s D_i^a)$ and $K_b = K_{b-1} \cup_{g'} (\bigvee_{j=1}^t D_j^b)$. The inclusion $K_a/K_{a-1} \wedge K_b/K_{b-1} \hookrightarrow K/K_{a-1} \wedge K/K_{b-1}$ induces a surjection

$$\sigma : \bigoplus_{i,j} \mathbb{Z} \longrightarrow \pi_a(K/K_{a-1}) \otimes \pi_b(K/K_{b-1})$$

and a corresponding natural transformation σ_* of cohomology with coefficients. In the same way, a map $R \in [\bigvee_k S_k^n, \bigvee_{i,j} S_i^a \wedge S_j^b]$ induces a natural transformation $R_* : H^*(Z; \bigoplus_k \mathbb{Z}) \longrightarrow H^*(Z; \bigoplus_{i,j} \mathbb{Z})$. Define a map

$$\bar{\theta} : [\bigvee_k S_k^n, \bigvee_{i,j} S_i^a \wedge S_j^b] \longrightarrow H^*(K; \pi_a(K/K_{a-1}) \otimes \pi_b(K/K_{b-1}))$$

by the formula $\bar{\theta}(R) = \sigma_* R_*(u_n)$. Finally, observe that there is an isomorphism

$$\left[\bigvee_k (D_k^n, S_k^{n-1}), \bigvee_{i,j} (D_i^a, S_i^{a-1}) \wedge (D_j^b, S_j^{b-1}) \right] \cong \left[\bigvee_k S_k^n, \bigvee_{i,j} S_i^a \wedge S_j^b \right]$$

and a corresponding map

$$\left[\bigvee_k (D^n, S^{n-1}), \bigvee_{i,j} (D_i^a, S_i^{a-1}) \wedge (D_j^b, S_j^{b-1}) \right] \xrightarrow{\theta} H^n(K; \pi_a(K/K_{a-1}) \otimes \pi_b(K/K_{b-1})).$$

The key to our main result is the following technical lemma which establishes a useful commutative diagram.

Lemma 5 *The cohomology class $u_a u_b \in H^n(K; \pi_a(K/K_{a-1}) \otimes \pi_b(K/K_{b-1}))$ is in the image of $\bar{\theta}$ (and hence of θ). If $\bar{R} \in \bar{\theta}^{-1}(u_a u_b)$ then the diagram*

$$\begin{array}{ccc} K & \xrightarrow{\hat{d}} & K \wedge K \\ \downarrow q & & \downarrow q \wedge q \\ & & K/K_{a-1} \wedge K/K_{b-1} \\ & & \uparrow \\ \bigvee_k S^n & \xrightarrow{\bar{R}} & \bigvee_{i,j} (S^a)_i \wedge (S^b)_j \end{array}$$

commutes up to homotopy.

Proof

This follows from cellular approximation. ■

Armed with this lemma, we can prove our main result.

Theorem 6 *Let $K \subseteq X$ be as above. Choose maps $\bar{R} \in \bar{\theta}^{-1}(u_a u_b)$ and $R \in \theta^{-1}(u_a u_b)$.*

- (a) *Assume $\text{wcat}_X(K_{a-1}) \leq \kappa - 1$ and $\text{wcat}_X(K_{b-1}) \leq \lambda - 1$. Let $\bar{h} \in \bar{\mathcal{H}}_\kappa(g)$ and $\bar{h}' \in \bar{\mathcal{H}}_\lambda(g')$ be maps which extend to K/K_{a-1} and K/K_{b-1} , respectively. Then*

$$(\bar{h} \wedge \bar{h}') \circ \bar{R} \in \bar{\mathcal{H}}_{\kappa+\lambda}(f; X).$$

- (b) *Assume $\text{cat}_X(K_{a-1}) \leq \kappa - 1$ and $\text{cat}_X(K_{b-1}) \leq \lambda - 1$. Let $h \in \mathcal{H}_\kappa(g)$ and $h' \in \mathcal{H}_\lambda(g')$ be maps which extend to the pairs (K, K_{a-1}) and (K, K_{b-1}) , respectively. Then*

$$(h \times h') \circ R \in \mathcal{H}_{\kappa+\lambda}(f; X).$$

Proof

The diagram

$$\begin{array}{ccccc}
K & \xrightarrow{\widehat{d}} & K \wedge K & \longrightarrow & X \wedge X \xrightarrow{\widehat{d}_\kappa \wedge \widehat{d}_\lambda} X^{(\kappa+\lambda)} \\
\downarrow q & & \downarrow q \wedge q & & \parallel \\
& & K/K_{a-1} \wedge K/K_{b-1} & \longrightarrow & X^{(\kappa+\lambda)} \\
& & \uparrow & & \parallel \\
S_1^n \vee \cdots \vee S_m^n & \xrightarrow{\overline{R}} & \bigvee_{i,j} (S^a)_i \wedge (S^b)_j & \xrightarrow{\overline{h} \wedge \overline{h}'} & X^{(\kappa)} \wedge X^{(\lambda)}.
\end{array}$$

commutes up to homotopy by Lemma 5. Part (a) follows immediately from the definition of the crude Hopf set.

The proof of part (b) is similar. By definition of the Hopf set there are maps $\delta_\kappa \simeq d_\kappa : (X, *) \rightarrow (X, *)^\kappa$ with $\delta_\kappa(K_{a-1}) \subseteq T^\kappa X$ and $\delta_\lambda \simeq d_\lambda : (X, *) \rightarrow (X, *)^\lambda$ with $\delta_\lambda(K_{b-1}) \subseteq T^\lambda X$ so that the diagram

$$\begin{array}{ccccc}
(K, *) & \xrightarrow{d_2} & (K, *) \times (K, *) & \xrightarrow{\delta_\kappa \wedge \delta_\lambda} & (X, *)^{\kappa+\lambda} \\
\downarrow q & & \downarrow & & \parallel \\
(K, K_{n-1}) & \longrightarrow & (K, K_{a-1}) \times (K, K_{b-1}) & \longrightarrow & (X, *)^{\kappa+\lambda} \\
\uparrow E & & \uparrow & & \parallel \\
\bigvee_k (D^n, S^{n-1}) & \xrightarrow{R} & \bigvee_i D^a, \bigvee_j S^{a-1} \times \bigvee_j D^b, \bigvee_j S^{b-1} & \xrightarrow{h \times h'} & (X_a, *)^\kappa \times (X_b, *)^\lambda
\end{array}$$

commutes. Now $(h \times h') \circ R$ is in $\mathcal{H}_{\kappa+\lambda}(f; X)$ by definition. \blacksquare

The cup product $u_a u_b$ encodes all of the possible cup products of classes in dimension a with classes in dimension b . More precisely, write $\pi_a(K/K_{a-1}) = A_1 \oplus \cdots \oplus A_s$ and $\pi_b(K/K_{b-1}) = B_1 \oplus \cdots \oplus B_t$ where each summand is cyclic of prime power order, or infinite. Then $u_a = (u_a^1, \dots, u_a^s)$ and $u_b = (v_b^1, \dots, v_b^t)$, and $u_a u_b = (u_a^i v_b^j)$ where $u_a^i v_b^j \in H^n(K; A_i \otimes B_j)$. One can write down the maps \overline{R} explicitly in terms of these cup products.

It is perhaps interesting to consider what happens if we choose Hopf invariants in Theorem 6. It is not clear whether Hopf invariants always extend in the sense required in Theorem 6. In the cases where they do, the constructed elements can be understood using the following result.

Proposition 7 *Let $K \subseteq X$ as above. Let ϕ be a lift of $d_\kappa|_{K_{a-1}} : K_{a-1} \rightarrow X^\kappa$ through the fat wedge $T^\kappa X$ and let ψ be a lift of $d_\lambda|_{K_{b-1}}$ through the fat wedge $T^\lambda X$. Then*

- (a) $\overline{H}_\phi(D^a; X) \wedge \overline{H}_\psi(D^b; X) = \overline{H}_{\phi \times \psi}(D^{a+b}; X).$
- (b) $H_\phi(D^a; X) \times H_\psi(D^b; X) = H_{\phi \times \psi}(D^{a+b}; X).$

Here we are using the notation $H_\phi(D^m)$ to denote the Hopf invariant of the attaching map of the cell D^m . The proof of Proposition 7 is deferred to the final section.

3 The Category of Four-Cell Complexes

In this section we consider CW complexes X with CW decompositions of the form

$$X = S^a \cup_g D^b \cup_f D^{a+b},$$

with $b > a$. The cohomology generators $u \in H^a(X)$, $v \in H^b(X)$ and $w \in H^{a+b}(X)$ satisfy $uv = rw$ for some $r \in \mathbb{Z}$. We determine the (weak) category of certain spaces of this kind in terms of the crude Hopf set for the map g .

In the case X is a sphere bundle over a sphere, our result includes part 2 of Singhof's main result from [13]. Singhof's proof is much more complicated, using cohomology operations of arbitrarily high order.

Theorem 8 *With X as above, let $h \in \overline{\mathcal{H}}_2(g)$.*

- (a) *If $r\Sigma^a h = 0$, then $\text{wcat}(X) \leq 2$.*
- (b) *If $b < 3a - 1$, then $\overline{\mathcal{H}}_3(f) = \{r\Sigma^a h\}$, and so $\text{cat}(X) = \text{wcat}(X) = 3$ if and only if $rh \neq 0$.*

Proof

In applying Theorem 6 we must choose \overline{R} to be a map of degree r . Then we conclude that $r\Sigma^a h \in \overline{\mathcal{H}}_3(f)$, which proves part (a). Under the conditions of part (b) the crude Hopf set is a singleton by Theorem 4. That $\text{cat}(X) = \text{wcat}(X)$ in this case follows from Theorem 2.2 in [16]. \blacksquare

One of the sphere bundles over spheres to which our theorem applies is $Sp(2)$; thus we recover Schweitzer's calculation $\text{cat}(Sp(2)) = 3$ from [12] without secondary operations.

Example B Let $X = Sp(2)$, which has a CW decomposition

$$Sp(2) = S^3 \cup_g D^7 \cup_f D^{10}.$$

Since $7 < 3 \cdot 3 - 1$, Theorem 4 shows that $\overline{\mathcal{H}}_2(g; X)$ is the singleton $\{\overline{H}(f)\}$. By Proposition 17.2 in [3], $\overline{H}(g) = j_*\eta_6 \neq 0$ where $j : S^6 \hookrightarrow X^{(2)}$. Thus $\text{cat}(X_7) = \text{wcat}(X_7) = 2$. We conclude that $\overline{\mathcal{H}}_3(f; X) = \{j_*\eta_9\}$ where $j : S^9 \hookrightarrow X^{(3)}$, so

$$\text{cat}(Sp(2)) = \text{wcat}(Sp(2)) = 3.$$

4 Upper Bounds

Theorem 6 is well suited to finding upper bounds for the (weak) category of a space. In this section we show that the (weak) category of the CW-skeleta of a space can grow at most linearly with respect to dimension. Further, we show that linear growth can only happen in the presence of nontrivial cup products. If X has no cup products, then we obtain significantly stronger upper bounds on the (weak) category of X .

Consider the element $(\overline{h} \wedge \overline{h}') \circ \overline{R} \in \overline{\mathcal{H}}_{\kappa+\lambda}(f; X)$ constructed in Theorem 6. If either $\overline{R} \simeq *$ or $\overline{h} \wedge \overline{h}' \simeq *$ then $*$ $\in \overline{\mathcal{H}}_{\kappa+\lambda}(f; X)$ and $\text{wcat}(X) < \kappa + \lambda$. This is the key point in the proof of the following result.

Theorem 9 *Let $K \subseteq X$ be a subcomplex of dimension n . If $a + b = n$, then*

- (a) $\text{wcat}_X(K) \leq \text{wcat}_X(K_{a-1}) + \text{wcat}_X(K_{b-1}) + 2$;
- (b) *if equality holds in (a), then $u_a u_b \neq 0$. Also, $\text{wcat}_X(K_a) = \text{wcat}_X(K_{a-1}) + 1$ and $\text{wcat}_X(K_b) = \text{wcat}_X(K_{b-1}) + 1$, so $\text{wcat}_X(K) = \text{wcat}_X(K_a) + \text{wcat}_X(K_b)$.*

With the same hypotheses, we have the same conclusions for ordinary category:

- (c) $\text{cat}_X(K) \leq \text{cat}_X(K_{a-1}) + \text{cat}_X(K_{b-1}) + 2;$
- (d) *if equality holds in (c), then $u_a u_b \neq 0$. Also, $\text{cat}_X(K_a) = \text{cat}_X(K_{a-1}) + 1$ and $\text{cat}_X(K_b) = \text{cat}_X(K_{b-1}) + 1$, so $\text{cat}_X(K) = \text{cat}_X(K_a) + \text{cat}_X(K_b)$.*

Proof

We prove (a) and (b); (c) and (d) are similar. Write $\kappa = \text{wcat}_X(K_{a-1}) + 1$ and $\lambda = \text{wcat}_X(K_{b-1}) + 1$. By Lemma 2, $* \in \overline{\mathcal{H}}_{\kappa+1}(g; X)$ is a map which extends to all of K/K_{a-1} . Choosing $\overline{h} = *$, we have $(\overline{h} \wedge \overline{h}') \circ \overline{R} \simeq * \in \overline{\mathcal{H}}_{\kappa+\lambda+1}(f)$ by Theorem 6 so $\text{wcat}_X(K) \leq \kappa + \lambda$. If equality holds in (a) then none of \overline{h} , \overline{h}' or \overline{R} can be chosen to be $*$. If the cup product $u_a u_b$ were trivial then we could choose $\overline{R} \simeq *$. This proves part (b). ■

These theorems have useful and easily applied consequences for general spaces.

Corollary 10 *Let $K \subseteq X$ be a subcomplex and let a and b be any integers. Then*

- (a) $\text{wcat}_X(K_{a+b+1}) \leq \text{wcat}_X(K_a) + \text{wcat}_X(K_b) + 1;$
- (b) *If $\text{wcat}_X(K_{m-1}) < c$, then $\text{wcat}_X(K_{tm}) \leq tc$ for all $t \geq 0$;*
- (c) $\text{cat}_X(K_{a+b+1}) \leq \text{cat}_X(K_a) + \text{cat}_X(K_b) + 1;$
- (d) *If $\text{cat}_X(K_{m-1}) < c$, then $\text{cat}_X(K_{tm}) \leq tc$ for all $t \geq 0$.*

Next we turn to a more subtle application of Theorem 9.

Theorem 11 *Let $K \subseteq X$ and assume that $u_{2^s(m+1)-1}^2 = 0$ for $s = 0, 1, \dots, t$.*

- (a) *If $\text{wcat}_X(K_{m-1}) < c$ then $\text{wcat}_X(K_n) \leq \left\lceil \frac{n}{2^t(m+1)-1} \right\rceil 2^t c$ for all $n \geq 2^t(m+1) - 1$.*

(b) If $\text{cat}_X(K_{m-1}) < c$ then $\text{cat}_X(K_n) \leq \left\lceil \frac{n}{2^t(m+1)-1} \right\rceil 2^t c$ for all $n \geq 2^t(m+1) - 1$.

Proof

We prove part (a); the proof of (b) is identical. First we handle the case $n = 2^t(m+1) - 1$ by induction on t . The case $t = 0$ is obvious. For the inductive step we assume that $\text{wcat}_X(K_n) \leq 2^s c$ for $n = 2^s(m+1) - 1$ and $s < t$ and we need to prove that $\text{wcat}_X(K_{2n+1}) \leq 2^{s+1} c$. Since $u_{2^s(m+1)-1}^2 = 0$ we have $\text{wcat}_X(K_{2n}) < 2(2^s)c$ by Theorem 9. It follows that $\text{wcat}_X(K_{2n+1}) \leq 2^{s+1} c$.

Since $\text{wcat}_X(K_n)$ increases with n , the theorem will be established once we prove that $\text{wcat}_X(K_n) \leq \left\lceil \frac{n}{2^t(m+1)-1} \right\rceil 2^t c$ for $n = s(2^t(m+1) - 1)$, $s \geq 1$. We have already taken care of the case $s = 1$. The general case follows from Corollary 10. ■

As a consequence we see that the classical dimension over connectivity upper bound for category can be improved for spaces with sufficiently few cup products.

Corollary 12 *let X be an $(m-1)$ -connected space such that $u_{2^t(m+1)-1}^2 = 0$ for all $t \geq 0$. Then $\text{cat}_X(X_n)$ is asymptotically bounded above by $\frac{n}{m+1}$.*

It is worth mentioning that the condition $u_a u_b = 0$ is satisfied if there are no nonzero cup products of elements of dimension a with elements of dimension b in any coefficients. If the cohomology of X is torsion free, then it suffices to check this condition only for integral coefficients.

As a first application, we derive a new upper bound for the category of the symplectic groups $Sp(n)$.

Proposition 13 *If $n(2n+1) \leq 63a + 31b + 15c + 7d$ with $a, b, c, d \in \mathbb{N}$, then*

$$\text{cat}(Sp(n)) \leq 16a + 8b + 4c + 2d.$$

If either $n(2n+1) < 63a + 31b + 15c + 7d$ or one of b, c or d is at least 2, then the inequality is strict.

Proof

Write $X = Sp(n)$. From Example B we have $\text{cat}(X_7) = 2$ and $\text{cat}(X_6) = 1$ for $n \geq 2$. By Theorem 9 $\text{cat}_X(X_{15}) \leq 4$ and $\text{cat}_X(X_{14}) < 4$; $\text{cat}_X(X_{31}) \leq$

8 and $\text{cat}_X(X_{30}) < 8$; $\text{cat}_X(X_{63}) \leq 16$ and $\text{cat}_X(X_{62}) < 16$. Thus the result follows from parts (c) and (d) of Theorem 9. \blacksquare

In particular, $\text{cat}(Sp(3)) \leq 5$. Since Singhof has shown that $\text{cat}(Sp(n)) \geq n + 1$ [14], it follows that $\text{cat}(Sp(3))$ is either 4 or 5. The best previously available upper bound, using Ganea's theorem in [6], was $\text{cat}(Sp(3)) \leq 6$. Table 1 lists the upper bounds on $\text{cat}(Sp(n))$ given by Proposition 13 for $n \leq 30$.

Group G	cat(G) at most	Group G	cat(G) at most	Group G	cat(G) at most
$Sp(1)$	1	$Sp(11)$	65	$Sp(21)$	229
$Sp(2)$	3	$Sp(12)$	77	$Sp(22)$	251
$Sp(3)$	5	$Sp(13)$	89	$Sp(23)$	275
$Sp(4)$	9	$Sp(14)$	103	$Sp(24)$	299
$Sp(5)$	15	$Sp(15)$	119	$Sp(25)$	324
$Sp(6)$	20	$Sp(16)$	135	$Sp(26)$	351
$Sp(7)$	27	$Sp(17)$	151	$Sp(27)$	377
$Sp(8)$	35	$Sp(18)$	169	$Sp(28)$	405
$Sp(9)$	43	$Sp(19)$	189	$Sp(29)$	435
$Sp(10)$	53	$Sp(20)$	209	$Sp(30)$	465

TABLE 1: Upper Bounds for the Category of $Sp(n)$

Finally, we use our results in a number of examples. First, observe that for a given space X , $\text{cat}(X_n)$ depends strongly on the chosen CW decomposition of X . Nevertheless, the number $\text{cat}_X(X_n)$, however, depends only on the homotopy type of X ; thus the monotone sequence of numbers $\{\text{cat}_X(X_n) \mid n = 1, 2, \dots\}$ is a homotopy invariant of spaces. We determine the sequence for $X = \Omega S^{2n}$ and give upper and lower bounds on the sequence for $X = \Lambda(S^{2n})$.

Example C Consider the space $X = \Omega S^{2n}$. As is well known $(\Omega S^{2n})_{(p)} \simeq (S^{2n-1} \times \Omega S^{4n-1})_{(p)}$ for odd primes p . From this we see that the p -local category of X_m is given exactly by the cup length lower bound. We can calculate the p -local category of the skeleta explicitly: $\text{cat}_{(p)}(X_m) = \left\lfloor \frac{m}{4n-2} \right\rfloor$. On the other hand, before localization, the classical upper bound is $\text{cat}(X_m) \leq \left\lceil \frac{m}{2n-1} \right\rceil$, roughly twice as large. Using the techniques we have developed, we show by induction that $\text{cat}_X(X_m)$ is also given by cup

length for all m . Recall that the integral cohomology of X is given by $H^*(X) \cong \Lambda(a_{2n-1}) \otimes P(b_{4n-2})$, where $P(x)$ denotes the divided polynomial algebra on x . Since it is enough to calculate $\text{cat}_X(X_m)$ for $m = k(2n-1)$, assume that $\text{cat}_X(X_{k(2n-1)}) = l+1$, where $k = 2l+1$, and that $\text{cat}_X(X_{k(2n-1)}) > \text{cat}_X(X_{(k-1)(2n-1)}) = l$. Now we have $u_{k(2n-1)} = ab^l$ and $u_{2n-1} = a$. Since $u_{k(2n-1)} = a^2b^l = 0$, Theorem 9 shows that $\text{cat}_X(X_{(k+1)(2n-1)}) < \text{cat}_X(X_{k(2n-1)}) + \text{cat}_X(X_{2n-1}) = l+2$, in other words, $\text{cat}_X(X_{(k+1)(2n-1)}) = \text{cat}_X(X_{k(2n-1)})$. On the other hand, the cup length lower bound shows that $\text{cat}_X(X_{(k+2)(2n-1)}) \geq l+2$. This upper estimate must be an equality, and the induction is completed. Thus $\text{cat}_X(X_m) = \lfloor \frac{m}{4n-2} \rfloor$.

Example D In [4] it is shown the free loop space on an even sphere, $\Lambda(S^{2n})$, has infinite category, even though all cup products vanish in $H^*(\Lambda(S^{2n}))$ [9]. It is also shown that there are compact subsets with arbitrarily large category. Their proof shows that

$$\text{cat}_{\Lambda(S^{2n})}((\Omega S^{2n})_m) = \text{cat}_{\Omega S^{2n}}((\Omega S^{2n})_m),$$

so $\text{cat}_{\Lambda(S^{2n})}(\Lambda(S^{2n})_m)$ is asymptotically bounded below by $\frac{m}{4n-2}$. However, it is possible that the category of the skeleta of $\Lambda(S^{2m})$ grows much more quickly. We will obtain an upper bound. The first few nonzero homology groups of $\Lambda(S^{2n})$ are $H_{2n-1}(\Lambda(S^{2n})) \cong \mathbb{Z}$, $H_{2n}(\Lambda(S^{2n})) \cong \mathbb{Z}$ and $H_{6n-3}(\Lambda(S^{2n})) \cong \mathbb{Z}/2$. It follows that $\Lambda(S^{2n})$ can be given a CW decomposition of the form

$$\Lambda(S^{2n}) = (S^{2n-1} \vee S^{2n}) \cup D^{6n-3} \cup (\text{higher dimensional cells}).$$

Since $X = \Lambda(S^{2n})$ has no nontrivial cup products in any coefficients, we can apply Theorem 11 with $K = \Lambda(S^{2n})$ and $m = 6n-3$ (thus $\text{cat}_X(K_{m-1}) < 2$). The result is that

$$\text{cat}_X(\Lambda(S^{2n})_{2^t(6n-2)-1}) \leq 2^{t+1}.$$

for $t \geq 0$. Asymptotically, $\text{cat}_X(\Lambda(S^{2n})_m) \leq \frac{m}{3n-1}$ while the classical dimension divided by connectivity upper bound gives $\text{cat}_X(\Lambda(S^{2n})_m) \leq \frac{m}{2n-1}$.

5 Proof of Theorem 3 and Proposition 7

In this section we prove Theorem 3 and Proposition 7. The proof of Theorem 3 is a straightforward generalization of Schweitzer's proof in the case $X = K = S^p \cup D^n$ [12].

For any space Z we use p_s to denote the projection from Z^k or from $T^k Z$ onto the s^{th} factor; likewise, we use i_s to denote the inclusion of the s^{th} factor into either Z^k or $T^k Z$. We will be concerned with the diagonal maps relevant to the pair (K, L) ; the targets of these maps fit into the following commutative diagram of pairs

$$\begin{array}{ccccc} (L^{k+1}, T^{k+1}L) & \xrightarrow{m} & (K^{k+1}, T^{k+1}L) & \xrightarrow{\lambda} & (K^{k+1}, T^{k+1}K) \\ & & & & \downarrow j \\ & & & & (X^{k+1}, T^{k+1}X). \end{array}$$

(Note: A diagonal arrow labeled j also points from $(L^{k+1}, T^{k+1}L)$ to $(X^{k+1}, T^{k+1}X)$.)

We begin with the definition of the (crude) Hopf invariant. Let $K = L \cup_f C^n(G) \subseteq X$ where $f : M_{n-1}(G) \rightarrow L$. For each k the exact homotopy sequence for the pair $(L^k, T^k L)$ breaks up into split short exact sequences

$$0 \longrightarrow \pi_n(L^{k+1}, T^{k+1}L; G) \xrightarrow{\partial} \pi_{n-1}(T^{k+1}L; G) \xrightarrow{l_*} \pi_{n-1}(L^{k+1}; G) \longrightarrow 0$$

(Note: Curved arrows labeled α and β point from $\pi_{n-1}(T^{k+1}L; G)$ back to $\pi_n(L^{k+1}, T^{k+1}L; G)$ and $\pi_{n-1}(L^{k+1}; G)$ respectively.)

with splitting maps α and β defined by

$$\beta = \sum_{s=1}^{k+1} (i_s)_* (p_s)_* \quad \text{and} \quad \alpha = \partial^{-1}(1 - \beta l_*).$$

If $\text{cat}(L) \leq k$ then there is a map $\phi : L \rightarrow T^{k+1}L$ such that $l \circ \phi \simeq d_{k+1}$. Let $j : (L, *)^{k+1} \rightarrow (X, *)^{k+1}$ be the obvious inclusion. Following Berstein and Hilton [1], we make the following definition.

Definition The **Hopf ϕ -invariant** of f in X is

$$H_\phi(f; X) = j_* \alpha(\phi_*(f)) \in \pi_n(X^{k+1}, T^{k+1}X; G).$$

The **crude Hopf ϕ -invariant** of f in X is

$$\overline{H}_\phi(f; X) = \wedge_* H_\phi(f; X) \in \pi_n(X^{(k+1)}; G)$$

where $\wedge : (X^{k+1}, T^{k+1}X) \rightarrow (X^{(k+1)}, *)$ is the canonical quotient map.

Berstein and Hilton's definition was slightly different. They considered only the case $X = K$ and defined $H(f) = \alpha(\phi_*(f)) \in \pi_n(L^{k+1}, T^{k+1}L; G)$. When $X = K$, j_* is an isomorphism on $[K, -]$, so the difference is negligible. Our notation emphasizes the dependence on the choice of ϕ .

Proof of Theorem 3

We prove part (b); part (a) follows immediately on applying the map Λ_* .

Let $\delta : K \rightarrow K^{k+1}$ be a map homotopic to the diagonal such that $\delta|_L = \phi$. Let $E \in \pi_n(K, L)$ be the class of the top cell. We need to show that $j_*\delta_* : \pi_n(K, L; G) \rightarrow \pi_n(X^{k+1}, T^{k+1}X; G)$ carries E to some member of $\mathcal{H}_{k+1}(f; X)$. From here on, we suppress the coefficients in homotopy groups.

The proof amounts to staring at a large diagram, which we now construct. Let $\gamma = 1 - \beta l_*$ and $\bar{\gamma} = \partial^{-1}\gamma\partial : \pi_n(K^{k+1}, T^{k+1}L) \rightarrow \pi_n(K^{k+1}, T^{k+1}L)$. Equivalently,

$$\bar{\gamma} = 1 - \sum_{s=1}^{k+1} (\tilde{i}_s)_*(\tilde{p}_s)_*$$

where $\tilde{i} : (K, L) \rightarrow (K^{k+1}, T^{k+1}L)$ and $\tilde{p} : (K^{k+1}, T^{k+1}L) \rightarrow (K, L)$ are the maps induced by the inclusion and projection respectively.

CLAIM 1 $\lambda_*\bar{\gamma} = \lambda_*$.

PROOF We simply calculate

$$\lambda_*\bar{\gamma} = \lambda_*\left(1 - \sum_{s=1}^{k+1} (\tilde{i}_s)_*(\tilde{p}_s)_*\right) = \lambda_* - \sum_{s=1}^{k+1} \lambda_*(\tilde{i}_s)_*(\tilde{p}_s)_* = \lambda_*.$$

CLAIM 2 $\text{Im}(\bar{\gamma}) \subseteq \text{Im}(m_*)$.

PROOF From the exact sequence

$$\pi_*(L^{k+1}, T^{k+1}L) \xrightarrow{m_*} \pi_*(K^{k+1}, T^{k+1}L) \xrightarrow{n_*} \pi_*(K^{k+1}, L^{k+1})$$

we see that it is sufficient to show that $n_*\bar{\gamma} = 0$. This is another simple calculation:

$$\begin{aligned} n_*\bar{\gamma} &= n_*\partial^{-1}(1 - \beta l_*)\partial \\ &= \partial^{-1}(l_* - l_*\beta l_*)\partial \\ &= \partial^{-1}(l_* - l_*)\partial \\ &= 0. \end{aligned}$$

These maps all fit together into the following diagram

$$\begin{array}{ccccc}
& & & \pi_n(X^n, T^n X) & \\
& & (j\delta)_* & \nearrow & \\
& & & \lambda_* & \\
\pi_n(K, L) & \xrightarrow{\bar{\delta}_*} & \pi_n(K^{k+1}, T^{k+1} L) & \xrightarrow{\bar{\gamma}} & \pi_n(K^{k+1}, T^{k+1} L) & \xleftarrow{m_*} & \pi_n(L^{k+1}, T^{k+1} L) \\
\downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
\pi_{n-1}(L) & \xrightarrow{\phi_*} & \pi_{n-1}(T^{k+1} L) & \xrightarrow{\gamma} & \pi_{n-1}(T^{k+1} L) & & \pi_{n-1}(T^{k+1} L)
\end{array}$$

From the diagram,

$$\partial \bar{\gamma} \bar{\delta}_*(E) = \gamma \phi_* \partial(E) = \gamma \phi_*(f).$$

Since $\text{Im}(\bar{\gamma}) \subseteq \text{Im}(m_*)$ there is a map $h \in \pi_n(L^{k+1}, T^{k+1} L)$ such that $m_*(h) = \bar{\gamma} \bar{\delta}_*(E)$. Thus $\partial h = \gamma \phi_*(f)$, and so $j_*(h) = H_\phi(f; X)$. The calculation

$$(j\delta)_*(E) = \lambda_* \bar{\gamma} \bar{\delta}_*(E) = \lambda_*(m_*(h)) = H_\phi(f; X)$$

completes the proof. ■

Proof of Proposition 7

This follows from the formula $\gamma(\phi \times \psi)_*(x) = \gamma \phi_*(x) \times \gamma \psi_*(x)$. ■

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