Algebraic geometry over model categories

A general approach to derived algebraic geometry

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Abstract

For a (semi-)model category M, we define a notion of a "homotopy" Grothendieck topology on M, as well as its associated model category of stacks. We use this to define a notion of geometric stack over a symmetric monoidal base model category; geometric stacks are the fundamental objects to "do algebraic geometry over model categories". We give two examples of applications of this formalism. The first one is the interpretation of DG-schemes as geometric stacks over the model category of complexes and the second one is a definition of étale K-theory of E_∞-ring spectra.

This first version is very preliminary and might be considered as a detailed research announcement. Some proofs, more details and more examples will be added in a forthcoming version.

Key words: Stacks, model categories, E_∞-algebras, DG-schemes.

Contents

1 Introduction

By definition, a scheme is obtained by gluing together affine schemes for the Zariski topology. Therefore, algebraic geometry is a theory which is based on the two fundamental notions of *affine scheme* and *Grothendieck topology.* It was observed already a long time ago that these two notions still make sense in more general contexts, and that *schemes* can be defined in very general settings. This has led to the theory of *relative algebraic geometry*, which allows one to *do algebraic geometry* over well behaved symmetric monoidal base categories (see [De1, De2, Ha]); usual algebraic geometry corresponds then to the "absolute" case where the base category is the category of $\mathbb{Z}$-modules.
The goal of the present work is to start a program to develop *algebraic geometry relatively to symmetric monoidal ∞-categories.* Our motivations for starting such a program come from several questions in algebraic geometry and algebraic topology and will be clarified in the two entries *Examples and applications* and *Relations with other works* of this introduction.

It is well known that model categories give rise in a natural way to ∞-*categories.* Indeed, B. Dwyer and D. Kan defined a simplicial localization process, which starting from a model category M, constructs a simplicial category LM, the simplicial localization of M (see [D-K1]). As simplicial categories may be viewed as ∞-categories for which i-morphisms are invertible up to $(i+1)$-morphisms for all $i > 1$ (see [H-S, §2]), this suggests that model categories *are* a certain kind of ∞-categories. In the same way, symmetric monoidal model categories (as defined in [Ho, §4]) *are* a certain kind of symmetric monoidal ∞-categories (e.g. in the sense of [To1]). As a first step in our program we would like to present in this paper a setting to *do algebraic geometry relatively to symmetric monoidal model categories.* For this, we will concentrate on defining a category of *geometric stacks over a base symmetric monoidal model category*, whose construction will be the main purpose of this work.

Review of usual algebraic geometry

In order to explain our approach, we shall first present in detail a construction of the usual category of schemes, or more generally of algebraic stacks and of n-geometric stacks (see [S1]), emphasizing the categorical ingredients needed in each step, in such a way that the generalization that will follow should look fairly natural.

The starting point is the category Aff of *affine schemes.* By definition, we will take Aff to be the opposite of the category of commutative and unital rings. We consider its Yoneda embedding

$$h : Aff \longrightarrow Aff^{\wedge},$$

where $Aff^{\wedge}$ is the category of presheaves of simplicial sets on Aff (i.e. $Aff^{\wedge} := \mathrm{SPr}(Aff)$) and h maps an affine scheme X to $h_X := Hom(-,X)$ (here a set is always considered as a constant simplicial set). From a categorical point of view, the embedding $h : Aff \longrightarrow Aff^{\wedge}$ is obtained by *formally adding homotopy colimits to* Aff (see [Du2] for more details on this point of view). This process is relevant to our situation, as *gluing objects in* Aff will be done by taking certain formal homotopy colimits of objects in Aff

(i.e. taking homotopy colimits in $Aff^{\wedge}$ of object in Aff). The category $Aff^{\wedge}$ is in a natural way a model category, where equivalences are defined objectwise, and the functor h induces a Yoneda embedding on the level of the homotopy categories

$$h : Aff \longrightarrow Ho(Aff^{\wedge}).$$

Throughout this introduction, the homotopy category $Ho(\mathcal{C})$ of a category $\mathcal{C}$ with a distinguished class of morphisms w will denote the category obtained from $\mathcal{C}$ by formally inverting all morphisms in w; when $\mathcal{C}$ is a model category, we will implicitly assume that w is the set of weak equivalences and when $\mathcal{C}$ does not come naturally equipped with a model category structure we will consider it as a trivial model category where w consists of all the isomorphisms.

The next step is choosing a *Grothendieck topology* on Aff, that will be used to glue affine schemes. For the purpose of schemes, the Zariski topology is enough, but étale or even faithfully flat and quasi-compact (for short ffqc) topologies also proves very useful in order to define more general objects as algebraic spaces or algebraic stacks. We will choose here to work with the ffqc topology though the construction will be valid for any Grothendieck topology on Aff. The ffqc topology makes Aff into a Grothendieck site and therefore one can consider its category of (∞-)*stacks*, denoted by $Ho(Aff^{\sim,\mathrm{ffqc}})$. For us, the category $Ho(Aff^{\sim,\mathrm{ffqc}})$ is the full sub-category of $Ho(Aff^{\wedge})$ consisting of simplicial presheaves satisfying the descent condition for ffqc hyper-covers [1]. The category $Ho(Aff^{\sim,\mathrm{ffqc}})$ is precisely the homotopy category of a certain model category structure on $Aff^{\wedge}$, and the category $Aff^{\wedge}$ together with this model structure will be called the model category of stacks, and denoted by $Aff^{\sim,\mathrm{ffqc}}$. Finally, it is known that the topology ffqc is sub-canonical, or in other words that the Yoneda embedding $h : Aff \longrightarrow Ho(Aff^{\wedge})$ factors through $Ho(Aff^{\sim,\mathrm{ffqc}})$. Therefore we have an induced fully faithful functor

$$h : Aff \longrightarrow Ho(Aff^{\sim,\mathrm{ffqc}}).$$

A stack in the essential image of the functor h will be called by extension an *affine scheme*.

Let us consider now a simplicial object $X_* : \Delta^{op} \longrightarrow Aff^{\sim,\mathrm{ffqc}}$ (i.e. X_* is a bi-simplicial presheaf) and suppose that X_* satisfies the following three conditions¿

1. The simplicial object X_* is a *Segal groupoid* in $Aff^{\sim,\mathrm{ffqc}}$ (see Def. 3.3.1);

2. The image of each X_n in $Ho(Aff^{\sim,ffqc})$ is a disjoint union of affine schemes;

3. The two morphisms (*source* and *target*) $X_1 \rightrightarrows X_0$ are faithfully flat and affine morphisms (this makes sense since the X_n's are disjoint union of affine schemes).

To such a groupoid we associate its homotopy colimit $|X_*| \in Ho(Aff^{\sim,\mathrm{ffqc}})$, which can be defined to be the stack associated to the diagonal of the bi-simplicial presheaf X_*. It is not difficult to check that the full sub-category of $Ho(Aff^{\sim,\mathrm{ffqc}})$, consisting of objects isomorphic to some $|X_*|$, with X_* satisfying conditions (1), (2) and (3), is equivalent to the homotopy category of *algebraic stacks* (in the sense of Artin, see [La-Mo]) having an affine diagonal. In particular, it contains the category of separated schemes as a full

[1]For us the word *stack* will always mean a *stack of* ∞*-groupoids*, adopting the point of view of [S2] and [To2], according to which *stacks of* ∞*-groupoids* are modelled by simplicial presheaves.

sub-category. Iterating this constructions as in [S1], one can also construct the homotopy category of *geometric n-stacks* (which for n big enough contains the homotopy category of general algebraic stacks as a full sub-category).
This is precisely the construction we will imitate in defining our category of geometric stacks over a symmetric monoidal model category.

Geometric stacks over symmetric monoidal model categories

The previous construction of the homotopy category of algebraic stacks is purely categorical. Indeed, it starts with the symmetric monoidal category $(\mathbb{Z}-mod, \otimes)$, of $\mathbb{Z}$-modules. The category Aff of affine schemes is then the opposite of the category of commutative and unital monoids in the symmetric monoidal category $(\mathbb{Z}-mod, \otimes)$, which is a categorical notion. Furthermore, the notion of a topology on Aff is also categorical. Our goal is to extend this categorical construction to the case where $(\mathbb{Z}-mod, \otimes)$ is replaced by a general *symmetric monoidal model category* $\mathcal{C}$ (in the sense of [Ho, §4]). Of course, we want to keep track of the *homotopical* information contained in $\mathcal{C}$ and we will therefore require our constructions to be invariant when replacing $\mathcal{C}$ by a Quillen equivalent symmetric monoidal model category.

Let us start with a base *symmetric monoidal model category* $(\mathcal{C}, \otimes)$ and try to imitate the construction of algebraic stacks we have presented. The first step is to find a reasonable analog of the category of commutative and unital rings. It has been known since a long time by topologists that the correct analog of commutative rings in a *homotopical* context is the notion of E_∞*-algebra* (see for example [E-K-M-M, Hin, Sp]). This notion is a generalization of the notion of commutative monoid adapted to the case of symmetric monoidal model categories. In particular it makes sense to consider the category $E_\infty - Alg(\mathcal{C})$, of E_∞-*algebras* in $\mathcal{C}$. Furthermore, it is proved in [Ber-Moe, Hin, Sp] that $E_\infty - Alg(\mathcal{C})$ carries a natural model category structure[2]. Then, by analogy with the case of usual algebraic geometry, we simply define the model category $\mathcal{C} - Aff$, of *affine stacks over* $\mathcal{C}$, to be the opposite of the model category of E_∞-algebras in $\mathcal{C}$. It is reasonable to denote by $Spec\, A$ the object of $\mathcal{C} - Aff$ corresponding to a E_∞-algebra A. The reader should note that if the model structure on $\mathcal{C}$ is trivial (i.e. equivalences are isomorphisms), then $\mathcal{C} - Aff$ is nothing else than the usual category of commutative and unital monoids in $\mathcal{C}$, together with the trivial model structure. In particular, if $\mathcal{C} = \mathbb{Z} - mod$ (endowed with the trivial model structure), the category $\mathcal{C} - Aff$ is the usual category of affine schemes.

Our next step is to define an analog of the *Yoneda embedding* for $\mathcal{C} - Aff$. More generally, the problem is to find a good analog of the Yoneda embedding for a model category M. Of course, as an abstract category M possesses the usual Yoneda embedding, but this construction is not suited for our purposes as it is not an invariant of the Quillen equivalence class of M (for example, it does not induces an embedding of the homotopy category $Ho(M)$). To solve this problem, we define a model category $M^\wedge$ which takes into account and depends on the model structure on M. The underlying category of $M^\wedge$ is as

[2]To be very precise, at this point one needs the weaker notion of *semi-model* category, but we will neglect this technical subtlety in this introduction.

usual the category of simplicial presheaves $\mathrm{SPr}(M)$; however, the model category structure we consider on $M^{\wedge}$ is such that its fibrant objects are exactly objectwise fibrant simplicial presheaves $F : M^{op} \longrightarrow SSet$ sending weak equivalences in M to weak equivalences of simplicial sets. Technically, $M^{\wedge}$ is defined as the left Bousfield localization of the objectwise model structure with respect to the equivalences in M (see Def. 2.1.1). The construction $M \mapsto M^{\wedge}$ has then the property of sending Quillen equivalences to Quillen equivalences (see Prop. 2.1.5). Furthermore, using mapping spaces in the model category M, we construct a functor $\underline{h} : M \longrightarrow M^{\wedge}$, which roughly speaking sends an objects x to the simplicial presheaf $y \mapsto Map_M(y, x)$, $Map_M(-,-)$ denoting the mapping space. This functor can be right derived into a fully faithful functor $\mathbb{R}\underline{h} : Ho(M) \longrightarrow Ho(M^{\wedge})$, which will be our *"homotopical" Yoneda embedding* for the model category M (see Thm. 2.1.13).

To go further one has to introduce a good notion of *Grothendieck topology* on $\mathcal{C} - Aff$ and an associated notion of *stack*. As in the previous step, we approach, more generally, the problem of defining what is a homotopy meaningful Grothendieck topology τ on a general model category M and what is the associated model category of stacks $M^{\sim,\tau}$, in such a way that for trivial model structures (i.e. when the weak equivalences are exactly the isomorphisms) one finds back the usual notions. For this, we introduce a notion of *"homotopy" Grothendieck topology* on a model category using the point of view of pre-topologies (see Def. 2.2.1). The idea is to give the usual *data* of τ-coverings at the level of the homotopy category $Ho(M)$ and require the usual *conditions* of stability with respect to *isomorphisms* and *composition* in $Ho(M)$ itself while the requirement of stability under *fibred products* in $Ho(M)$ is replaced with the requirement of stability under *homotopy fibred products*. Therefore, the data of coverings for a topology τ on M are defined in $Ho(M)$ while the conditions these data have to satisfy are given at the "higher level" in M itself. This is completely natural from an homotopic point of view and one obtains almost, but not exactly, a usual Grothendieck topology on $Ho(M)$. We may call the pair (M, τ) a *model site*. A stack is then naturally defined as an object in $Ho(M^{\wedge})$ satisfying a reasonable descent condition with respect to τ-hypercoverings (see Def. 2.3.1). We actually define a model category of stacks $M^{\sim,\tau}$ as a certain left Bousfield localization of the model category $M^{\wedge}$ with respect to a set S_τ of maps in $M^{\wedge}$ determined by the topology.

Let us come back to our model category of affine stacks $\mathcal{C} - Aff$. We suppose that we have chosen a topology τ on $\mathcal{C} - Aff$ which is sub-canonical, in the sense that the Yoneda embedding $\mathbb{R}\underline{h}$ factors through $Ho(\mathcal{C} - Aff^{\sim,\tau}) \hookrightarrow Ho(\mathcal{C} - Aff^{\wedge})$. Therefore, $\mathbb{R}\underline{h}$ induces a full embedding $\mathbb{R}\underline{h} : Ho(\mathcal{C} - Aff) \longrightarrow Ho(\mathcal{C} - Aff^{\sim,\tau})$, and objects in the essential image of this functor will naturally be called *affine stacks over* $\mathcal{C}$. The definition of geometric stacks over $\mathcal{C}$ is then straightforward. One consider simplicial objects $X_* : \Delta^{op} \longrightarrow \mathcal{C} - Aff^{\sim,\tau}$, which are Segal groupoids such that X_0 is a disjoint union of affine stacks and with $X_1 \rightrightarrows X_0$ affine τ-coverings. The homotopy colimit $|X_*| \in Ho(\mathcal{C} - Aff^{\sim,\tau})$ of such a simplicial object will be called a *1-geometric stack over* $\mathcal{C}$ for the topology τ. Iterating this construction as in [S1], one also defines *n-geometric stacks over* $\mathcal{C}$. The sub-category of $Ho(\mathcal{C} - Aff^{\sim,\tau})$, consisting of n-geometric stacks for some n, will be our setting to *do algebraic geometry over the symmetric monoidal model category* $\mathcal{C}$.

The following table offers a synthesis of our construction showing how it parallels the classical constructions in algebraic geometry.

Algebraic Geometry over $\mathbb{Z}$-mod	**Algebraic Geometry over a model category**
Base Category : $(\mathbb{Z}-mod, \otimes)$	Base Category : $\mathcal{C} = (\mathcal{C}, \otimes)$
$alg =$ Commutative algebras in $(\mathbb{Z}-mod, \otimes)$	$Alg = E_\infty -$ algebras in $\mathcal{C}$
$Aff = alg^{op} =$ Affine Schemes over $(\mathbb{Z}-mod, \otimes)$	$\mathcal{C} - Aff := (Alg)^{opp} =$ Affine stacks over $\mathcal{C}$
$Aff^\wedge$	$\mathcal{C} - Aff^\wedge$
Yoneda embedding : $Aff \hookrightarrow Aff^\wedge$	"Homotopy" Yoneda embedding : $Ho(\mathcal{C} - Aff) \hookrightarrow Ho(\mathcal{C} - Aff^\wedge)$
τ : Grothendieck topology onAff	τ : "Homotopy" topology on $\mathcal{C} - Aff$
Category of stacks : $Ho(Aff^{\sim,\tau})$	Category of "homotopy" stacks : $Ho(\mathcal{C} - Aff^{\sim,\tau})$
Algebraic stacks in $Ho(Aff^{\sim,\tau})$: $\lvert X_* \rvert$	Geometric stacks in $Ho(\mathcal{C} - Aff^{\sim}_{\tau})$: $\lvert X_* \rvert$

Examples and applications

The construction outlined above of the category of n-geometric stacks over a symmetric monoidal model category $\mathcal{C}$ has found his motivations in various questions coming from algebraic geometry, algebraic topology an the recent rich interplay between them . Among them, we describe below those which were the most influential for us. The first two are investigated in this paper.

1. *Extended or derived moduli problems.* Some of the moduli spaces arising in Algebraic Geometry turns out to be non smooth (e.g. the moduli stack of vector bundles over a variety of dimension greater than one) and this maybe considered as a non-natural phenomenon. To overcome this difficulty, the current general philosophy (see [Ko],[Ka], [Ci-Ka1]) teach us to consider the usual moduli spaces considered so far as a *truncation* of an *extended* or *derived* moduli space. The non smoothness would then arise from the fact that one is only considering this truncation instead of the whole object. The usual approach to these extended moduli spaces is through DG-schemes (e.g. [Ka, Ci-Ka1]). However, it was already noticed that the homotopy category of DG-schemes might be not very well suited for the functorial point of view on derived algebraic geometry. Quoting [Ci-Ka2],

"Similarly to the case of the usual algebro-geometric moduli spaces, it would be nice to characterize RHilb and RQuot in terms of the representability of some functors. This is not easy, however, as the functors should be considered on the derived category of dg-schemes (with quasi-isomorphisms inverted) and for morphisms in this localized category there is currently no explicit description. The issue should be probably addressed in a wider foundational context for dg-schemes in our present sense by means of gluing maps which are only quasi-isomorphisms on pairwise intersections, satisfying cocycle conditions only up to homotopy on triple intersection etc."

Our personal way of understanding the "issue" referred to in this quotation is by stating that *DG-schemes should be interpreted as geometric stacks over the symmetric monoidal ∞-category of complexes.* As a first evidence for this, we will produce a functor

$$\Theta : Ho(DG - Sch) \longrightarrow Ho(C(k) - Aff^{\sim,\mathrm{ffqc}}),$$

where $C(k)$ is the symmetric monoidal model category of complexes of k-modules (for any commutative and unital ring k), $Ho(DG - Sch)$ is the homotopy category of DG-schemes over k [3] and ffqc is a certain extension of the faithfully flat and quasi-compact topology from usual k-algebras to E_∞-algebras in $C(k)$. We prove furthermore that Θ takes values in the category of geometric stacks and we conjecture it is fully faithful.

In a forthcoming version, we will also give an interpretation in our setting of the notion of *injective resolution of BG* defined in [Ka].

2. *Brave New Algebraic Geometry.* Since the recent progress in stable algebraic topology that led to a satisfying theory of spectra as a monoidal model category (see [E-K-M-M], [Ho-Sh-Sm], [Ly]), it has become clear that one is actually able to do usual commutative algebra on commutative monoid objects in these categories, the so called *brave new rings.* It seems therefore natural to try to embed this brave new commutative algebra in a *brave new algebraic geometry* i.e. in a kind of algebraic geometry over (structured) spectra. This could give new insights in Elliptic Cohomology and Topological Modular Forms, theories for which the interplay between geometry and topology already proved rich and powerful (see for example [G-H], [Hop], [AHS], [Str]).

   As an example of application of our theory to this circle of ideas, we will use the category of stacks over the symmetric monoidal model category of symmetric spectra in order to define the notion of *étale K-theory of an E_∞-ring spectrum.* We are not sure to deeply understand the issue of such a construction but it certainly gives an answer to a question pointed out to us by P.A. Ostvær. To be more precise, we will define an étale topology on $Sp^\Sigma - Aff$, the model category of affine stacks over the model category of symmetric spectra. Then, sending each E_∞-ring spectrum to its K-theory space (as defined for example in [E-K-M-M, §VI]) gives rise to a simplicial

[3] Using E_∞-algebra structures, the definition of DG-schemes given in [Ci-Ka1] can be generalized over an arbitrary ring k (see Def. 4.1.3).

presheaf

$$\begin{array}{rccc} K: & Sp^{\Sigma}-Aff & \longrightarrow & SSet \\ & Spec\,A & \longrightarrow & K(A), \end{array}$$

and therefore to an object $K \in Sp^{\Sigma} - Aff^{\sim,\text{ét}}$. This object is in general not fibrant (because it does not satisfy the descent condition for étale hypercoverings) and therefore we define for $Spec\,A \in Sp^{\Sigma} - Aff$, $\mathbb{K}_{\text{ét}}(A) := RK(Spec\,A)$, where RK is a fibrant replacement of K in the model category $Sp^{\Sigma} - Aff^{\sim,\text{ét}}$. The space $\mathbb{K}_{\text{ét}}(A)$ comes equipped with a natural localization morphism $K(A) \longrightarrow \mathbb{K}_{\text{ét}}(A)$.

3. *Higher Tannakian duality.* In the preliminary manuscript [To1], 1-Segal (or simplicial) Tannakian categories were introduced in order to extend to higher homotopy groups the algebraic theory of fundamental groups. The general idea was to replace in the usual Tannakian formalism the base symmetric monoidal category of vector spaces by the symmetric monoidal ∞-category of complexes. Furthermore, as relative algebraic geometry has found interesting applications in the Tannakian formalism (see [De1]), it should not be surprising that algebraic geometry over the ∞-category of complexes is relevant to higher Tannakian theory. As an example of this principle, we will use our notion of geometric stacks over the symmetric monoidal model category of complexes over some ring k in order to define the notion of *affine ∞-gerbes*.

   We start with the symmetric monoidal model category $C(k)$ of complexes over k, together with a Grothendieck topology τ on $C(k) - Aff$ that will be assumed to be sub-canonical. In practice, the choice of the topology τ is a very important issue, but we will avoid going into these kind of considerations here. We consider $Gp(C(k) - Aff^{\sim,\tau})$, the category of group objects in the model category of stacks. For each $G \in Gp(C(k) - Aff^{\sim,\tau})$, one can form its classifying simplicial presheaf $BG \in Ho(C(k) - Aff^{\sim,\tau})$. In the case where the underlying stack of G is affine and the morphism $G \longrightarrow *$ is a τ-covering, the classifying stack BG is a 1-geometric stack. Stacks of the form BG for G satisfying the above conditions will be called *neutral affine gerbes over $C(k)$*, or *neutral affine ∞-gerbes over k* (this definition depends on the topology τ). As the usual neutral Tannakian duality study neutral affine gerbes (see [Sa]), neutral affine gerbes over $C(k)$ are the basic object of study of higher Tannakian duality. In the future, the higher Tannakian formalism will be developed consistently as a certain kind of *algebraic geometry over $C(k)$*.

Relations with other works

The first work we would like to mention is K. Behrend's recent work on differential graded schemes (see [Be]). We learned about it in one of his talks during fall 2000 at the MPI in Bonn. Though our interpretation of DG-schemes, as geometric stacks over the model category of complexes, is similar to his own approach, the two works seem totally independent and it is not clear to us how the two approaches can be compared and to which extent they are really equivalent.

There are also some relations with several works on E_{∞}-algebras, in which already some standard geometrical constructions were investigated, as for example the cotangent complex, the tangent Lie algebra, the K-theory and Hochschild cohomology spectrum, André-Quillen cohomology etc. (see [E-K-M-M, Hin, G-H]). We are quite convinced that all these constructions can be generalized naturally to our setting of geometric stacks over general model categories and will allow in future to talk about the cotangent complex, the Lie algebra, the cohomology or K-theory, André-Quillen cohomology etc. of a general geometric stack.

We have already mentioned at the beginning of this introduction that our approach is a first approximation of what we think algebraic geometry over symmetric monoidal ∞-categories should be. Using the theory of Segal categories introduced by Z. Tamsamani and C. Simpson ([Ta, H-S]), it is possible to develop such a theory without any considerations on model categories. However, in order to compare construction of the category of geometric stacks presented in this paper to a purely ∞-categorical construction, one needs very strong version of *strictification results* (as for example in [H-S, §18]). These results are already partially proved, and are part of the foundational development of the theory of higher categories. There is no doubt that the combined two approaches, together with a comparison theorem allowing to pass from the world of ∞-categories to the world of model categories, will be a very powerful tool, allowing much more naturality and manageability. As an example, let us mention that a first consequence of such a unified theory would be a ∞-categorical interpretation of the theory of E_∞-algebras as *commutative monoids in symmetric monoidal ∞-categories*. Such considerations appeared already in T. Leisnter's work on up-to-homotopy monoid structures ([Le]) and in the first author's preprint [To1].

As we have already stressed, there are some applications of our theory to the conjectural higher Tannakian formalism described in [To1]. In particular, the theory of affine stacks and schematic homotopy types of [To2], as well as its application to non-abelian Hodge theory in [Ka-Pa-To], can be interpreted in terms of algebraic geometry over the model category of complexes (at least in characteristic zero).

Finally, as explained in his letter [M], Y. Manin's idea of a "secondary quantization of algebraic geometry" seems to be part of algebraic geometry over the symmetric monoidal model category of motives (for example that defined in [Sp]) but our ignorance of this subject does not allow us to say more. However, using our notion of geometric stacks over a model category, we are able to define an interesting candidate for the motive of an algebraic stack (in the sense of Artin) as a 1-*geometric stack over the model category of motives*. This construction, which was suggested to us by the lecture of [M], might be closely related to the subject of secondary quantized algebraic geometry and will be hopefully investigated in a future work.

Organization of the paper

In the first Section of the paper we develop the theory of "homotopical" Grothendieck topologies over model categories and the associated theory of stacks. We first define the Yoneda embedding of a model category, then we introduce the notion of topology and construct the model category of stacks.

In the second Section, we define and investigate the notion of geometric stack over a symmetric monoidal model category. For this, we apply the theory of stacks developed in the first Section to the model category of E_∞-algebras in a base symmetric monoidal model category. We define inductively the notion of n-geometric stack and give a characterization by means of Segal groupoids as explained in this introduction.

Finally, in the third Section, we give two applications of the present theory of algebraic geometry over model categories. We first explain how DG-schemes may be interpreted as geometric stacks over the model category of complexes and we conclude by defining of the étale K-theory space of an E_∞-ring spectra.

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Notations and conventions:

Throughout all this work, $\mathbb{U}$ and $\mathbb{V}$ will be two universes, with $\mathbb{U} \in \mathbb{V}$, and we will assume that $\mathbb{U}$ contains the set of natural integers, $\mathbb{N} \in \mathbb{U}$. We will use the expression $\mathbb{U}$-set (resp. $\mathbb{U}$-group, $\mathbb{U}$-simplicial set, ...) to denote sets (resp. groups, resp. simplicial sets ...) belonging to $\mathbb{U}$. The corresponding categories will be denoted by $\mathbb{U}-Set$, $\mathbb{U}-Gp$, $\mathbb{U}-SSet$ .... The words *set* (resp. *group*, resp. *simplicial set* ...) will always refer to sets (resp. groups, resp. simplicial sets ...) belonging to the universe $\mathbb{V}$. The corresponding categories will simply be denoted by Set, Gp, $SSet$ ....

We will make the following exceptions when referring to categories. A $\mathbb{U}$-category (resp. a $\mathbb{V}$-category) will refer to a category C such that for every pair of objects (X, Y) in C, the set $Hom(X, Y)$ belongs to $\mathbb{U}$ (resp. $\mathbb{V}$). By convention, all categories will be $\mathbb{V}$-categories. We will say that a category is $\mathbb{U}$-small (resp. $\mathbb{V}$-small) if it belongs to $\mathbb{U}$ (resp. to $\mathbb{V}$).

Our references for model categories are [Ho, Hi]. For the weaker notion of semi-model category we refer to [Sp]. An opposite category of a semi-model category will again be called a semi-model category. We will not make a difference between the original notion and its dual.

For any simplicial semi-model category M, we will denote by $\underline{Hom}_M$ its simplicial Hom set. It will also be denoted simply by $\underline{Hom}$ when the reference to M is clear. The derived version of these simplicial Hom will be denoted by $\mathbb{R}\underline{Hom}$ (see [Ho, Thm. 4.3.2]). The set of morphisms in the homotopy category $Ho(M)$ will be denoted by $[-,-]_M$, or simply by $[-,-]$ when the reference to M is clear.

For a general semi-model category M, its mapping complexes will be denoted by Map_M (or Map when the reference to M is clear), and will always be considered in the homotopy category of simplicial sets (see [Ho, 5.5.4], [Hi, §18], [Sp, I.2]). The homotopy fibred products in $Ho(M)$ will be denoted by $x \times^h_z y$. In the same vein, the homotopy cofibered products will be denoted by $x \coprod^h_z y$ (see [Hi, §11]).

By the expression $\mathbb{V}$-*cellular model categories* (resp. $\mathbb{V}$-*combinatorial) model categories*) we mean a model category satisfying the conditions of definitions [Hi] (resp. [Sm]), expect that all sets have to be understood as $\mathbb{V}$-sets (in particular the ordinals appearing in the definition belong to $\mathbb{V}$).

As usual, the standard simplicial category will be denoted by Δ. For any simplicial object $F \in \mathcal{C}^{\Delta^{op}}$ in a category $\mathcal{C}$, we will use the notation $F_n := F([n])$. Similarly, for any co-simplicial object $F \in \mathcal{C}^{\Delta}$, we will use the notation $F_n := F([n])$.

2 Stacks over model categories

In this first section, we will present a theory of stacks over (semi-)model categories (we will be using [Sp, §2] as a reference for semi-model categories). For this, we will start by defining the Yoneda embedding of a model category, whose idea essentially goes back to some fundamental work of B. Dwyer and D. Kan (see [D-K2]). We do not claim any originality in this first paragraph, and the results stated are probably well known. Then, we introduce the notion of a Grothendieck topology on a model category, which as far as we know is a new notion. The definition we give is very close to the usual one, and essentially one only needs to replace in the usual definition isomorphisms by equivalences and fibred products by homotopy fibred products. Using this notion, we define homotopy hypercovers, which are a straightforward generalization of hypercovers in Grothendieck's sites, and use them to define a model category of stacks over a model category endowed with a topology.
In this first version of the paper, we have not detailed the standard properties of the model category of stacks, but we have included some statements concerning homotopy sheaves and computations of homotopy fibred products. These exactness properties are fundamental to do elementary manipulations in the model category of stacks. Finally, we end the section by discussing the functoriality properties of the given constructions.

Setting. Throughout this section we will consider a semi-model category M, together with a sub-semi-model category $M_{\mathbb{U}} \subset M$. By this we mean that a morphism in $M_{\mathbb{U}}$ is an equivalence (resp. a fibration, resp. a cofibration) if and only if it is an equivalence (resp. a fibration, resp. a cofibration) in M. Moreover, we will suppose that $M_{\mathbb{U}}$ is stable under the functorial factorization in M (i.e. the functorial factorization functor of M can be chosen such that the factorization of a morphism in $M_{\mathbb{U}}$ stays in $M_{\mathbb{U}}$).

We will also assume that $M_{\mathbb{U}}$ is a $\mathbb{U}$-category, which is furthermore a $\mathbb{V}$-small category. The typical example of such a situation the reader should keep in mind is when M is the model category of $\mathbb{V}$-simplicial sets (respectively, $\mathbb{V}$-simplicial groups, complexes of $\mathbb{V}$-abelian groups, ...) and $M_{\mathbb{U}}$ is the sub-category of $\mathbb{U}$-simplicial sets (respectively, $\mathbb{U}$-simplicial groups, complexes of $\mathbb{U}$-abelian groups, ...).

We will make a systematic use of the left Bousfield localization technique for which we refer to [Hi, Ch. 3, 4]. In particular, the following elementary result will be used very frequently and we state it here merely for reference's convenience.

Proposition 2.0.1 *([Hi, Prop. 3.6.1]) Let M be a model $\mathbb{V}$-category which is left proper and $\mathbb{V}$-cellular (see [Hi, 14.1]) or $\mathbb{V}$-combinatorial (see [Sm]). If S is a $\mathbb{V}$-set of morphisms in M and $L_S(M)$ denotes the left Bousfield localization of M with respect to S, then an object W in $L_S(M)$ is fibrant iff it is fibrant in M and is S-local i.e. for any map $f : A \to B$ in S, the induced map between homotopy mapping spaces $f^*(W) : Map_M(B, W) \to Map_M(A, W)$ is an equivalence in $SSet$.*

Note that we use here a slightly different notion of local objects from that used in [Hi, Def. 3.2.4]: in Hirschhorn's terminology fibrant objects in $L_S(M)$ are exactly what he calls S-local objects in M.

2.1 The Yoneda embedding for semi-model categories

In this first paragraph we will construct the analog of the Yoneda embedding for semi-model categories. We will start with the more general situation of a $\mathbb{V}$-small category C together with a set of morphism S in C and define a model category $(C,S)^{\wedge}$. The model category $(C,S)^{\wedge}$ has to be thought as a homotopy analog of the category of presheaves of sets $D^{\wedge}$ on a small category D, and is constructed as a certain left Bousfield localization of the model category of simplicial presheaves on C. This model category will be shown to be functorial in (C,S) and will only depend on the *weak equivalence class* of (C,S) (see Prop. 2.1.5). Then, if (C,S) is the semi-model category $M_{\mathbb{U}} \subset M$ together with its equivalences, we will define a Quillen adjunction

$$Re : M_{\mathbb{U}}^{\wedge} \longrightarrow M \qquad M_{\mathbb{U}}^{\wedge} \longleftarrow M : \underline{h}.$$

This adjunction will be shown to induce a fully faithful functor

$$\mathbb{R}\underline{h} : Ho(M_{\mathbb{U}}) \longrightarrow Ho(M_{\mathbb{U}}^{\wedge}),$$

which will be our final Yoneda embedding.

Let us start with the general situation of a $\mathbb{V}$-small category C, together with a subset of morphisms S in C. Let $SPr(C)$ be the category of $\mathbb{V}$-simplicial presheaves on C, which by [Hi, Thm. 13.8.1] is a model category where fibrations and equivalences are defined objectwise. This model category is furthermore proper and simplicial and the corresponding simplicial Hom will simply be denoted by $\underline{Hom}$. Recall that the simplicial structure on $SPr(C)$ is the data for any simplicial set K and any $F \in SPr(C)$, of simplicial presheaves $K \times F$ and F^K defined by the following formulas

$$(K \times F)(x) := K \times F(x) \qquad (F^K)(x) := \underline{Hom}_{SSet}(K, F(x)).$$

These formulas allow one to define the simplicial set of morphisms between two simplicial presheaf F and G by the formula

$$\underline{Hom}(F,G)_n := Hom(\Delta^n \times F, G).$$

As usual, one has the natural adjunction isomorphisms

$$\underline{Hom}_{SPr(C)}(K \times F, G) \simeq \underline{Hom}_{SSet}(K, \underline{Hom}(F,G)) \simeq \underline{Hom}_{SPr(C)}(F, G^K),$$

for any simplicial set K and simplicial presheaves $F, G \in SPr(C)$.

The model category $SPr(C)$ is also $\mathbb{V}$-cellular and $\mathbb{V}$-combinatorial, therefore the left Bousfield localization techniques of [Hi] or [Sm] can be used to invert any $\mathbb{V}$-set of maps.

Let $h_{-} : C \longrightarrow SPr(C)$ be the functor mapping an object $x \in C$ to the simplicial presheaf it represents. In other words, $h_x : C^{op} \longrightarrow SSet$ send y to the constant simplicial set $Hom(y,x)$ of morphisms from y to x in C. We will denote by h_S the image of the set of morphism S by the functor h. The reader should note that h_S is a $\mathbb{V}$-set.

Definition 2.1.1 • *The simplicial model category $(C,S)^{\wedge}$ is the left Bousfield localization of $SPr(C)$ along the set of maps h_S. When the set of map S is clear from the context, we will simply write $C^{\wedge}$ for $(C,S)^{\wedge}$.*

- *The derived simplicial Hom of $(C,S)^\wedge$ will be denoted by*

$$\mathbb{R}_S\underline{Hom}(-,-) : Ho((C,S)^\wedge)^{op} \times Ho((C,S)^\wedge) \longrightarrow Ho(SSet).$$

Important remark. By definition, for $F, G \in (C,S)^\wedge$, one has $\mathbb{R}_S\underline{Hom}(F,G) \simeq \mathbb{R}\underline{Hom}(F,RG)$, where RG is a fibrant model for G in $(C,S)^\wedge$. This implies that in general, if G is not fibrant in $(C,S)^\wedge$, then the natural morphism

$$\mathbb{R}\underline{Hom}(F,G) \longrightarrow \mathbb{R}_S\underline{Hom}(F,G)$$

is not an isomorphism. This is why we need to mention the set S in the notation $\mathbb{R}_S\underline{Hom}$.

We will call a simplicial presheaf $F \in SPr(C)$ *h_S-local*, if for any $h_x \longrightarrow h_y$ in h_S the induced morphism

$$\mathbb{R}\underline{Hom}(h_y,F) \longrightarrow \mathbb{R}\underline{Hom}(h_x,F)$$

is an isomorphism. The reader is warned that this is a slightly different notion of local object with respect to [Hi, Def. 3.2.4].

Now, the Yoneda lemma implies that one has $\mathbb{R}\underline{Hom}(h_y,F) \simeq F(y)$; therefore, the previous morphism is isomorphic, in the homotopy category of simplicial sets, to the transition morphism $F(y) \longrightarrow F(x)$. This implies that F is an h_S-local object if and only if for any morphism $x \longrightarrow y$ in C which is in S, the induced morphism $F(y) \longrightarrow F(x)$ is an equivalence. Using this and proposition 2.0.1 one finds immediately the following result.

Lemma 2.1.2 *An object $F \in (C,S)^\wedge$ is fibrant if and only if it satisfies the following two conditions:*

1. *For every object $x \in C$, the simplicial set $F(x)$ is fibrant (i.e. F is fibrant as an object in $SPr(C)$);*
2. *For any morphism $x \longrightarrow y$ in S, the induced morphism $F(y) \longrightarrow F(x)$ is an equivalence.*

Proof: It is a straightforward application of proposition 2.0.1. □

The previous lemma implies that the homotopy category $Ho((C,S)^\wedge)$ can be naturally identified to the the full sub-category of $Ho(SPr(C))$ consisting of simplicial presheaves $F : C^{op} \longrightarrow SSet$ sending morphisms of S to equivalences in $SSet$. Furthermore, the fibrant replacement functor in $(C,S)^\wedge$ induces a functor $r : Ho(SPr(C)) \longrightarrow Ho((C,S)^\wedge)$, which is a retraction of the natural inclusion $Ho((C,S)^\wedge) \hookrightarrow Ho(SPr(C))$.

Let (C,S) and (D,T) be two $\mathbb{V}$-small categories with distinguished subsets of morphisms and $f : C \longrightarrow D$ a functor which sends S into T. The functor induces a direct image functor on the categories of simplicial presheaves

$$f_* : SPr(D) \longrightarrow SPr(C),$$

defined by $f_*(F)(x) := F(f(x))$, for $F \in SPr(D)$ and $x \in C$. This functor has a left adjoint

$$f^* : SPr(C) \longrightarrow SPr(D),$$

characterized by the property that $f^*(h_x) \simeq h_{f(x)}$, for any $x \in C$.

Lemma 2.1.3 *The adjunction*

$$f^* : (C,S)^\wedge \longrightarrow (D,T)^\wedge \qquad (C,S)^\wedge \longleftarrow (D,T)^\wedge : f_*$$

is a Quillen adjunction.

Proof: It is clear that the functor f_* preserves levelwise equivalences and fibrations; therefore, the adjunction (f^*, f_*) is a Quillen adjunction between the model categories $SPr(C)$ and $SPr(D)$. Using the general properties of left Bousfield localization of model categories (see [Hi, Ch. 3, 4]), it is then enough to prove that the functor f_* sends fibrant objects in $(D,T)^\wedge$ to fibrant objects in $(C,S)^\wedge$. But this is clear by lemma 2.1.2 and the fact that $f(S) \subset T$. $\square$

Definition 2.1.4 *A functor $f : (C,S) \longrightarrow (D,T)$ between two $\mathbb{V}$-small categories with subsets of morphisms is a* weak equivalence *if it satisfies the following three conditions:*

- $f(S) \subset T$;
- *There exists a functor $g : D \longrightarrow C$ with $g(T) \subset S$ and natural transformations*

  $$fg \Leftarrow A \Rightarrow Id \qquad gf \Leftarrow B \Rightarrow Id,$$

  with A (resp. B) an endofunctor of D (resp. of C);
- *For any object $y \in D$ (resp. $x \in C$), the induced morphisms*

  $$fg(y) \longleftarrow A(y) \longrightarrow y \qquad (resp. \quad gf(x) \longleftarrow B(x) \longrightarrow x)$$

  are in T (resp. in S).

Proposition 2.1.5 *Let $f : (C,S) \longrightarrow (D,T)$ be a weak equivalence between $\mathbb{V}$-small categories with subsets of morphisms and $g : D \longrightarrow C$ be a functor like in defintion 2.1.6. Then, the two Quillen adjunctions*

$$f^* : (C,S)^\wedge \longrightarrow (D,T)^\wedge \qquad (C,S)^\wedge \longleftarrow (D,T)^\wedge : f_*,$$

$$g^* : (D,T)^\wedge \longrightarrow (C,S)^\wedge \qquad (D,T)^\wedge \longleftarrow (C,S)^\wedge : g_*,$$

are Quillen equivalences.

Proof: We will prove that the induced functor

$$\mathbb{L}f^* : Ho((C,S)^\wedge) \longrightarrow Ho((D,T)^\wedge)$$

is an equivalence of categories, with quasi-inverse $\mathbb{L}g^*$. This will be enough to show that (f^*, f_*) and (g^*, g_*) are Quillen equivalences.

If $F \in Ho((C,S)^{\wedge})$, let us prove that the natural morphisms

$$\mathbb{L}g^*\mathbb{L}f^*(F) \longleftarrow \mathbb{L}B^*(F) \longrightarrow F$$

are isomorphisms in $Ho((C,S)^{\wedge})$. One can find a simplicial object L_* of $(C,S)^{\wedge}$, such that for any $[n] \in \Delta$, L_n is isomorphic to a coproduct of simplicial presheaves of the form h_x with $x \in C$, together with an isomorphism in $Ho((C,S)^{\wedge})$

$$F \simeq hocolim_{[n]\in\Delta} L_n.$$

Then, since $\mathbb{L}g^*$, $\mathbb{L}f^*$ and $\mathbb{L}B^*$ commute with homotopy colimits (because g^*, f^* and B^* are left Quillen functors), one is reduced to the case where $F = h_x$, for some $x \in C$. But, as objects of the form h_z are always cofibrant, one has

$$\mathbb{L}g^*\mathbb{L}f^*(h_x) \simeq h_{gf(x)} \qquad \mathbb{L}B^*(h_x) \simeq h_{B(x)},$$

and it remains to show that the natural morphism $h_{gf(x)} \longleftarrow h_{B(x)} \longrightarrow h_x$ is an isomorphism in $Ho((C,S)^{\wedge})$. But this is true by definition of the model structure on $(C,S)^{\wedge}$ and by the fact that the morphisms

$$gf(x) \longleftarrow B(x) \longrightarrow x$$

belong to S.

In the same way, we prove that for any $F \in Ho((D,T)^{\wedge})$, the morphisms

$$\mathbb{L}f^*\mathbb{L}g^*(F) \longleftarrow \mathbb{L}A^*(F) \longrightarrow F$$

are isomorphisms in $Ho((C,S)^{\wedge})$. $\square$

Remark. In their paper [D-K1], Dwyer and Kan proved that the model category $(C,S)^{\wedge}$ is an invariant up to Quillen equivalences of the simplicial localization category $L(C,S)$. Proposition 2.1.5 is only a special case of this result.

We now come back to the basic setting of this section i.e. to our semi-model categories $M_{\mathbb{U}} \subset M$. The set of equivalences in $M_{\mathbb{U}}$ will be denoted by $\mathbf{W}_{\mathbb{U}}$.

Definition 2.1.6 *Let $M_{\mathbb{U}}^c$ (resp. $M_{\mathbb{U}}^f$, resp. $M_{\mathbb{U}}^{cf}$) be the sub-category of $M_{\mathbb{U}}$ consisting of cofibrant (resps. fibrant, resp. cofibrant and fibrant) objects. We will note*

$$M_{\mathbb{U}}^{\wedge} := (M_{\mathbb{U}}, \mathbf{W}_{\mathbb{U}})^{\wedge} \qquad (M_{\mathbb{U}}^c)^{\wedge} := (M_{\mathbb{U}}^c, \mathbf{W}_{\mathbb{U}} \cap M_{\mathbb{U}}^c)^{\wedge}$$

$$(M_{\mathbb{U}}^f)^{\wedge} := (M_{\mathbb{U}}^f, \mathbf{W}_{\mathbb{U}} \cap M_{\mathbb{U}}^f)^{\wedge} \qquad (M_{\mathbb{U}}^{cf})^{\wedge} := (M_{\mathbb{U}}^{cf}, \mathbf{W}_{\mathbb{U}} \cap M_{\mathbb{U}}^{cf})^{\wedge}.$$

These are simplicial model categories and the corresponding derived simplicial Hom will be denoted by

$$\mathbb{R}_w\underline{Hom}(-,-) \qquad \mathbb{R}_{w,c}\underline{Hom}(-,-) \qquad \mathbb{R}_{w,f}\underline{Hom}(-,-) \qquad \mathbb{R}_{w,cf}\underline{Hom}(-,-).$$

Lemma 2.1.7 *The natural inclusions*

$$i_c : M^c_{\mathbb{U}} \subset M_{\mathbb{U}} \qquad i_f : M^f_{\mathbb{U}} \subset M_{\mathbb{U}} \qquad i_{cf} : M^{cf}_{\mathbb{U}} \subset M_{\mathbb{U}},$$

induce equivalences of categories

$$\mathbb{R}(i_c)_* : Ho(M^{\wedge}_{\mathbb{U}}) \simeq Ho((M^c_{\mathbb{U}})^{\wedge}) \qquad \mathbb{R}(i_f)_* : Ho(M^{\wedge}_{\mathbb{U}}) \simeq Ho((M^f_{\mathbb{U}})^{\wedge})$$

$$\mathbb{R}(i_{cf})_* : Ho(M^{\wedge}_{\mathbb{U}}) \simeq Ho((M^{cf}_{\mathbb{U}})^{\wedge}).$$

These equivalences are furthermore compatible with derived simplicial Hom, in the sense that there exist natural isomorphisms

$$\mathbb{R}_{w,c}\underline{Hom}(\mathbb{R}(i_c)_*(-), \mathbb{R}(i_c)_*(-)) \simeq \mathbb{R}_w\underline{Hom}(-,-)$$

$$\mathbb{R}_{w,f}\underline{Hom}(\mathbb{R}(i_f)_*(-), \mathbb{R}(i_f)_*(-)) \simeq \mathbb{R}_w\underline{Hom}(-,-)$$

$$\mathbb{R}_{w,cf}\underline{Hom}(\mathbb{R}(i_{cf})_*(-), \mathbb{R}(i_{cf})_*(-)) \simeq \mathbb{R}_w\underline{Hom}(-,-).$$

Proof: It is an application of proposition 2.1.5. Let us prove for example that $\mathbb{R}(i_c)_*$ is an equivalence. For this, let $Q : M_{\mathbb{U}} \longrightarrow M^c_{\mathbb{U}}$ be a cofibrant replacement functor (see [Ho, p. 5]). By definition, there exist natural transformations

$$Qi_c \Rightarrow Id \qquad i_cQ \Rightarrow Id,$$

such that for any $x \in M_{\mathbb{U}}$ the induced morphisms $Qi_c(x) \longrightarrow x$, $i_cQ(x) \longrightarrow x$ are equivalences in $M_{\mathbb{U}}$. Proposition 2.1.5 then implies that the derived functor $\mathbb{R}(i_c)_*$ is an equivalence, which preserves derived simplicial Hom (as any Quillen equivalence does).

The same proof (applied to the opposite category $M^{op}_{\mathbb{U}}$) implies that $\mathbb{R}(i_f)_*$ is an equivalence. Finally, to prove that $\mathbb{R}(i_{cf})_*$ is an equivalence one applies proposition 2.1.5 first to a cofibrant replacement functor $Q : M_{\mathbb{U}} \longrightarrow M^c_{\mathbb{U}}$ and then to the restriction of a fibrant replacement functor $R : M^c_{\mathbb{U}} \longrightarrow M^{cf}_{\mathbb{U}}$. $\square$

Lemma 2.1.7 is useful to establish functorial properties of the homotopy category $Ho(M^{\wedge}_{\mathbb{U}})$. Indeed, if $f : M_{\mathbb{U}} \longrightarrow N_{\mathbb{U}}$ is a functor whose restriction to $M^{cf}_{\mathbb{U}}$ preserves equivalences, then f induces well defined functors

$$\mathbb{R}f_* : Ho(N^{\wedge}_{\mathbb{U}}) \longrightarrow Ho((M^{cf}_{\mathbb{U}})^{\wedge}) \simeq Ho(M^{\wedge}_{\mathbb{U}}),$$

$$\mathbb{L}f^* : Ho(M^{\wedge}_{\mathbb{U}}) \simeq Ho((M^{cf}_{\mathbb{U}})^{\wedge}) \longrightarrow Ho(N^{\wedge}_{\mathbb{U}}).$$

The functor $\mathbb{R}f_*$ is clearly right adjoint to $\mathbb{L}f^*$.

For example, let $f : M_{\mathbb{U}} \longrightarrow N_{\mathbb{U}}$ be a right Quillen functor. Then, the restriction of f on $M^{cf}_{\mathbb{U}}$ preserves equivalences and therefore induces a well defined functor

$$\mathbb{R}f_* : Ho(N^{\wedge}_{\mathbb{U}}) \longrightarrow Ho(M^{\wedge}_{\mathbb{U}}).$$

The same argument applies to a left Quillen functor $g : M_{\mathbb{U}} \longrightarrow N_{\mathbb{U}}$, which by restriction to the subcategory of cofibrant and fibrant objects induces a well defined functor

$$\mathbb{R}g^* : Ho(N^{\wedge}_{\mathbb{U}}) \longrightarrow Ho(M^{\wedge}_{\mathbb{U}}).$$

Definition 2.1.8 *Let $M_{\mathbb{U}}$ and $N_{\mathbb{U}}$ be two $\mathbb{V}$-small semi-model categories and $f : M_{\mathbb{U}} \longrightarrow N_{\mathbb{U}}$ be a functor whose restriction to $M_{\mathbb{U}}^{cf}$ preserves equivalences. The previously defined functor*

$$\mathbb{R}f_* : Ho(N_{\mathbb{U}}^{\wedge}) \longrightarrow Ho(M_{\mathbb{U}}^{\wedge})$$

will be called the inverse image *functor. Its left adjoint*

$$\mathbb{L}f^* : Ho(M_{\mathbb{U}}^{\wedge}) \longrightarrow Ho(N_{\mathbb{U}}^{\wedge})$$

will be called the direct image *functor.*

Remark. The reader should be warned that the direct and inverse image functors' construction is not functorial in f. In other words, if one does not add some hypotheses on the functor f, then in general $\mathbb{R}f_* \circ \mathbb{R}g_*$ is not isomorphic to $\mathbb{R}(g \circ f)_*$. However, one has the following easy proposition, which ensures in many cases the functoriality of the previous construction.

Proposition 2.1.9

1. *Let $M_{\mathbb{U}}$, $N_{\mathbb{U}}$ and $P_{\mathbb{U}}$ be $\mathbb{V}$-small semi-model categories and*

   $$M_{\mathbb{U}} \xrightarrow{f} N_{\mathbb{U}} \xrightarrow{g} P_{\mathbb{U}}$$

   be two functors preserving fibrant objects and equivalences between them. Then, there exist natural isomorphisms

   $$\mathbb{R}(g \circ f)_* \simeq \mathbb{R}f_* \circ \mathbb{R}g_* : Ho((P_{\mathbb{U}})^{\wedge}) \longrightarrow Ho((M_{\mathbb{U}})^{\wedge}),$$

   $$\mathbb{L}(g \circ f)^* \simeq \mathbb{L}g^* \circ \mathbb{L}f^* : Ho((M_{\mathbb{U}})^{\wedge}) \longrightarrow Ho((P_{\mathbb{U}})^{\wedge}).$$

   These isomorphisms are furthermore associative and unital in the arguments f and g.

2. *Let $M_{\mathbb{U}}$, $N_{\mathbb{U}}$ and $P_{\mathbb{U}}$ be $\mathbb{V}$-small semi-model categories and*

   $$M_{\mathbb{U}} \xrightarrow{f} N_{\mathbb{U}} \xrightarrow{g} P_{\mathbb{U}}$$

   be two functors preserving cofibrant objects and equivalences between them. Then, there exist natural isomorphisms

   $$\mathbb{R}(g \circ f)_* \simeq \mathbb{R}f_* \circ \mathbb{R}g_* : Ho((P_{\mathbb{U}})^{\wedge}) \longrightarrow Ho((M_{\mathbb{U}})^{\wedge}),$$

   $$\mathbb{L}(g \circ f)^* \simeq \mathbb{L}g^* \circ \mathbb{L}f^* : Ho((M_{\mathbb{U}})^{\wedge}) \longrightarrow Ho((P_{\mathbb{U}})^{\wedge}).$$

   These isomorphisms are furthermore associative and unital in the arguments f and g.

Proof: The proof is the same as that of the usual property of composition for derived Quillen functors (see [Ho, Thm. 1.3.7]), and is left to the reader. □

Examples of functors as in the previous proposition are given by right or left Quillen functors. Therefore, given

$$M_{\mathbb{U}} \xrightarrow{f} N_{\mathbb{U}} \xrightarrow{g} P_{\mathbb{U}}$$

a pair of right Quillen functors, one has

$$\mathbb{R}(g \circ f)_* \simeq \mathbb{R}f_* \circ \mathbb{R}g_* \qquad \mathbb{L}(g \circ f)^* \simeq \mathbb{L}g^* \circ \mathbb{L}f^*.$$

In the same way, if f and g are left Quillen functors, one has

$$\mathbb{R}(g \circ f)_* \simeq \mathbb{R}f_* \circ \mathbb{R}g_* \qquad \mathbb{L}(g \circ f)^* \simeq \mathbb{L}g^* \circ \mathbb{L}f^*.$$

Proposition 2.1.10 *If $f : M_{\mathbb{U}} \longrightarrow N_{\mathbb{U}}$ is a (right or left) Quillen equivalence between $\mathbb{V}$-small semi-model categories, then the induced functors*

$$\mathbb{L}f^* : Ho(M_{\mathbb{U}}^{\wedge}) \longrightarrow Ho(N_{\mathbb{U}}^{\wedge}) \qquad Ho(M_{\mathbb{U}}^{\wedge}) \longleftarrow Ho(N_{\mathbb{U}}^{\wedge}) : \mathbb{R}f_*,$$

are equivalences, quasi-inverse of each others.

Proof: Let us prove the proposition in the case where f is a right Quillen functor. The case where f is left Quillen is proved similarly.

Let $g : N_{\mathbb{U}} \longrightarrow M_{\mathbb{U}}$ be the left adjoint to f; let us show that $\mathbb{L}g^*$ is quasi-inverse to $\mathbb{L}f^*$. For this, let us consider the following two functors

$$f : M_{\mathbb{U}}^{f} \longrightarrow N_{\mathbb{U}} \qquad RgQ : \; N_{\mathbb{U}} \xrightarrow{Q} N_{\mathbb{U}}^{c} \xrightarrow{g} M_{\mathbb{U}} \xrightarrow{R} M_{\mathbb{U}}^{f},$$

where Q is a cofibrant replacement functor and R is a fibrant replacement functor. The natural transformations $Id \Rightarrow R$, $Q \Rightarrow Id$ and $gf \Rightarrow Id$, induce natural transformations

$$RgQf \Leftarrow gQf \Rightarrow gf \Rightarrow Id.$$

By hypothesis, for any $x \in M_{\mathbb{U}}^{f}$, the induced morphisms

$$RgQf(x) \longleftarrow gQf(x) \longrightarrow x$$

are equivalences. The same proof as in proposition 2.1.5 shows that $\mathbb{L}g^*\mathbb{L}f^*$ is isomorphic to the identity. The dual argument then shows that $\mathbb{L}f^*\mathbb{L}g^*$ is isomorphic to the identity. $\Box$

To finish this paragraph, we will define a Quillen adjunction

$$Re : M_{\mathbb{U}}^{\wedge} \longrightarrow M \qquad M_{\mathbb{U}}^{\wedge} \longleftarrow M : \underline{h}$$

and show that the functor $\underline{h}$ induces a fully faithful embedding on the level of homotopy categories (see Thm. 2.1.13).

¿From now on, let $(\Gamma : M_{\mathbb{U}} \longrightarrow M_{\mathbb{U}}^{\Delta}, i)$ be a fixed *cofibrant resolution functor* (see [Hi, 17.1.3]). This means that for any object $x \in M_{\mathbb{U}}$, $\Gamma(x)$ is a co-simplicial object in $M_{\mathbb{U}}$, which is cofibrant for the Reedy model structure on $M_{\mathbb{U}}^{\Delta}$, together with a natural weak equivalence $i(x) : \Gamma(x) \longrightarrow c(x)$, $c(x)$ being the constant co-simplicial object in $M_{\mathbb{U}}$ at x. In the case that the semi-model category M is simplicial, one can use the standard cofibrant resolution functor $\Gamma(x) := \Delta^* \otimes x$.

At the level of model categories, the construction of the functor $\underline{h}$ will depend on the choice of Γ, but after passing to the homotopy categories it will be shown that possibly different choices give the same Yoneda embedding.

We define the functor $\underline{h}_{-} : M \longrightarrow SPr(M_{\mathbb{U}})$, by sending each $x \in M$ to the simplicial presheaf

$$\begin{array}{rccc} \underline{h}_x : & M_{\mathbb{U}}^{op} & \longrightarrow & SSet \\ & y & \mapsto & Hom(\Gamma(y), x), \end{array}$$

where $\Gamma(y)$ is the cofibrant resolution of y induced by the functor Γ. To be more precise, the presheaf of n-simplices of $\underline{h}_x$ is given by the formula

$$(\underline{h}_x)_n(-) := Hom(\Gamma(-)_n, x).$$

Lemma 2.1.11 *The functor $\underline{h} : M \longrightarrow SPr(M_{\mathbb{U}})$ is a right Quillen functor.*

Proof: The fact that $\underline{h}$ is right Quillen is a direct verification and is proved in detail in [Du2, 9.5]. □

Corollary 2.1.12 *The adjunction*

$$Re : M_{\mathbb{U}}^{\wedge} \longrightarrow M \qquad M_{\mathbb{U}}^{\wedge} \longleftarrow M : \underline{h}$$

is a Quillen adjunction.

Proof: By the general properties of Bousfield localization of model categories (see [Hi, Ch. 3, 4]) and by lemma 2.1.11 it is enough to show that the functor $\underline{h}$ preserves fibrant objects. But, by definition of $\underline{h}$ this follows immediately from the standard properties of mapping spaces (see [Hi, §18]) and lemma 2.1.2. □

Remark. The reader should notice that if (Γ', i') is another cofibrant resolution functor, then the two derived functor $\mathbb{R}h_{-}$ and $\mathbb{R}h'_{-}$ obtained using respectively Γ and Γ' are naturally isomorphic. Therefore, our construction does not depend on the choice of Γ once one is passed to the homotopy category.

The main result of this first paragraph is the following one, that will play the role of the Yoneda embedding in our theory.

Theorem 2.1.13 *For any object $x \in Ho(M)$ which is isomorphic to an object in $Ho(M_{\mathbb{U}})$, the adjunction morphism*

$$\mathbb{L}Re\mathbb{R}\underline{h}_x \longrightarrow x$$

is an isomorphism in $Ho(M)$.

Equivalently, the restriction of $\mathbb{R}\underline{h} : Ho(M) \longrightarrow Ho(M_{\mathbb{U}}^{\wedge})$ to the full subcategory of objects isomorphic to an object of $Ho(M_{\mathbb{U}})$, is fully faithful.

Proof: Let x be a fibrant and cofibrant object in $M_{\mathbb{U}}$ and $x \longrightarrow x_*$ a simplicial resolution of x in $M_{\mathbb{U}}$ (see [Hi, 17.1.2]). We consider the following two simplicial presheaves

$$\begin{array}{rccc} h_{x_*} : & (M_{\mathbb{U}}^{c})^{op} & \longrightarrow & SSet \\ & y & \mapsto & Hom(y, x_*), \end{array}$$

$$\begin{array}{cccc} \underline{h}_{x_*} : & (M^c_{\mathbb{U}})^{op} & \longrightarrow & SSet \\ & y & \mapsto & Hom(\Gamma(y), x_*). \end{array}$$

The augmentation $\Gamma(-) \longrightarrow c(-)$ and co-augmentation $x \longrightarrow x_*$ induce a commutative diagram in $(M^{cf}_{\mathbb{U}})^{\wedge}$

$$\begin{array}{ccc} h_x & \xrightarrow{a} & \underline{h}_x \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle d} \\ h_{x_*} & \xrightarrow{c} & \underline{h}_{x_*}. \end{array}$$

By the properties of mapping spaces (see [Hi, §18]), both morphisms c and d are equivalences in $SPr(M^c_{\mathbb{U}})$. Furthermore, the morphism $h_x \longrightarrow h_{x_*}$ is isomorphic in $Ho(SPr(M^c_{\mathbb{U}}))$ to the induced morphism $h_x \longrightarrow hocolim_{[n]\in\Delta} h_{x_n}$. As each morphism $h_x \longrightarrow h_{x_n}$ is an equivalence in $(M^c_{\mathbb{U}})^{\wedge}$, this implies that d is an equivalence in $(M^c_{\mathbb{U}})^{\wedge}$. We deduce from this that the natural morphism $h_x \longrightarrow \underline{h}_x$ is an equivalence in $(M^c_{\mathbb{U}})^{\wedge}$.

Let us show how this implies that for any $x \in M_{\mathbb{U}}$, the natural morphism $h_x \longrightarrow \mathbb{R}\underline{h}_x$ is an isomorphism in $M^{\wedge}_{\mathbb{U}}$. Indeed, if F is a fibrant object in $M^{\wedge}_{\mathbb{U}}$ and F_c is its restriction to $M^c_{\mathbb{U}}$, then one has

$$\mathbb{R}\underline{Hom}_{M^{\wedge}_{\mathbb{U}}}(\mathbb{R}\underline{h}_x, F) \simeq \mathbb{R}\underline{Hom}_{(M^c_{\mathbb{U}})^{\wedge}}(\underline{h}_{RQ(x)}, F_c) \simeq \mathbb{R}\underline{Hom}_{(M^c_{\mathbb{U}})^{\wedge}}(h_{RQ(x)}, F_c) \simeq F(RQ(x))$$

$$\simeq \mathbb{R}\underline{Hom}_{M^{\wedge}_{\mathbb{U}}}(h_{RQ(x)}, F) \simeq \mathbb{R}\underline{Hom}(h_x, F),$$

where $RQ(x)$ is a fibrant and cofibrant model for x in $M_{\mathbb{U}}$. This shows, by the Yoneda lemma for $Ho(M^{\wedge}_{\mathbb{U}})$, that $h_x \longrightarrow \underline{h}_x$ is an equivalence in $M^{\wedge}_{\mathbb{U}}$.

Now, let $x \in M^f_{\mathbb{U}}$ and let us consider the natural morphism in $Ho(M)$,

$$Re(h_x) \longrightarrow \mathbb{L}Re(\underline{h}_x) \longrightarrow x.$$

As $h_x \longrightarrow \underline{h}_x$ is an equivalence and h_x is cofibrant in $M_{\mathbb{U}}$, the first morphism $Re(h_x) \longrightarrow \mathbb{L}Re(\underline{h}_x)$ is an isomorphism in $Ho(M)$. Therefore, to finish the proof of the theorem, it remains to show that $Re(h_x) \longrightarrow x$ is an isomorphism in $Ho(M)$. But, by adjunction, for any fibrant object $y \in M$, one has

$$[Re(h_x), y]_M \simeq [h_x, \underline{h}_y]_{M^{\wedge}_{\mathbb{U}}} \simeq \pi_0(\underline{Hom}(\Gamma(x), y)) \simeq [x, y]_M,$$

showing that $Re(h_x) \longrightarrow x$ is indeed an isomorphism in $Ho(M)$. □

To finish this paragraph, let us notice that for any object $x \in M_{\mathbb{U}}$, the natural morphism $i : \Gamma(-) \longrightarrow c(-)$ induces in the obvious manner a morphism in $M^{\wedge}$, $h_x \longrightarrow \underline{h}_x$.

Corollary 2.1.14 *For any object $x \in M_{\mathbb{U}}$, the natural morphism*

$$h_x \longrightarrow \underline{h}_x$$

is an equivalence in the model category $M^{\wedge}_{\mathbb{U}}$.

Proof: This follows immediately from the proof of the previous theorem. □

2.2 Grothendieck topologies on semi-model categories

In this paragraph, we present the notion of a Grothendieck topology on a semi-model category. The definition is quite natural as it is formally obtained by replacing isomorphisms by equivalences and fibred products by homotopy fibred products in the usual definition.

Recall that for any diagram $x \xrightarrow{a} z \xleftarrow{b} y$ in a semi-model category M, one can define a homotopy fibred product $x \times^h_z y \in Ho(M)$ (see [Hi, §11]). Explicitly, it is defined by

$$x \times^h_z y := x' \times_{y'} z',$$

where $x' \xrightarrow{a'} z' \xleftarrow{b'} y'$ is an equivalent diagram such that the two morphisms a' and b' are fibrations and the objects x', y' and z' are fibrant. The object $x \times^h_z y$ only depends, up to a natural isomorphism in $Ho(M)$, on the equivalence class of the diagram $x \xrightarrow{a} z \xleftarrow{b} y$. Furthermore it only depends, up to a non-natural isomorphism, on the isomorphism class of the image of the diagram $x \xrightarrow{a} z \xleftarrow{b} y$ in $Ho(M)$. In other words, for any diagram, $x \xrightarrow{a} z \xleftarrow{b} y$ in $Ho(M)$, the isomorphism class of the object $x \times^h_z y \in Ho(M)$ is well defined. In the same way, the isomorphism classes of the two projections $x \times^h_z y \longrightarrow x$ and $x \times^h_z y \longrightarrow y$ are well defined.

Definition 2.2.1 *A* topology τ *on a* $\mathbb{V}$*-small semi-model* $\mathbb{U}$*-category* M *is the data for any object* $x \in M$*, of a* $\mathbb{V}$*-set* $Cov_\tau(x)$ *of* $\mathbb{U}$*-small family of objects in* $Ho(M)/x$*, called* covering families *of* x*, satisfying the following three conditions:*

- (Stability) *For all* $x \in M$ *and any isomorphism* $y \to x$ *in* $Ho(M)$*, the family* $\{y \to x\}$ *is in* $Cov_\tau(x)$*.*
- (Composition) *If* $\{u_i \to x\}_{i\in I} \in Cov_\tau(x)$*, and for any* $i \in I$*,* $\{v_{ij} \to u_i\}_{j\in J_i}$*, the family* $\{v_{ij} \to x\}_{i\in I, j\in J_i}$ *is in* $Cov_\tau(x)$*.*
- (Homotopy base change) *For any* $\{u_i \to x\}_{i\in I} \in Cov_\tau(x)$*, and any morphism in* $Ho(M)$*,* $y \to x$*, the family* $\{u_i \times^h_x y \to y\}_{i\in I}$ *is in* $Cov_\tau(y)$*.*

A $\mathbb{V}$*-small semi-model* $\mathbb{U}$*-category* M *together with a topology* τ *will be called a (*$\mathbb{V}$*-small)* semi-model *(*$\mathbb{U}$*-)*site.

Remark. For any semi-model category M, one can form its *homotopy* 2textit-category $\mathcal{D}^{\leq 2}(M)$ (see [Ga-Zi] and [Sp, §2]). This 2-category is a fine enough invariant of M to be able to recover the homotopy fibred products. It is then easy to check that the data of a topology on M only depends on the 2-category $\mathcal{D}^{\leq 2}(M)$, up to a 2-equivalence. It seems to us that this is the reason why the homotopy 2-category of differential graded algebras is used in [Be]. We warn the reader that however the homotopy category of stacks we will define in Def. 2.4.1 depends on more than just $\mathcal{D}^{\leq 2}(M)$, as *higher homotopies in* M enter in the definition.

Before going further in the study of topologies on semi-model categories, we would like to present three examples.

- *Trivial model structure.* Let M be a $\mathbb{V}$-small $\mathbb{U}$-category with the trivial model structure (i.e. equivalences are isomorphisms and all morphisms are fibrations and cofibrations). Then, $Ho(M) = M$ and the homotopy fibred products are just fibred products. Therefore, a topology on the model category M in the sense of definition 2.2.1 is the same thing as a usual Grothendieck topology on the category M.

- *Topological spaces.* Let us take as M the model category of $\mathbb{U}$-topological spaces, Top, and let us define a topology τ in the following way. A family of morphism in $Ho(Top)$, $\{X_i \to X\}_{i\in I}$, $I \in \mathbb{U}$, is defined to be in $Cov_\tau(X)$ if the induced map $\coprod_{i\in I} \pi_0(X_i) \longrightarrow \pi_0(X)$ is surjective. The reader will check immediately that this defines a topology on Top in the sense of definition 2.2.1.

- *Negatively graded CDGA (see [Be]).* Let k be a field of characteristic zero and $M = CDGA_k^{op}$ be the opposite model category of commutative and unital differential graded k-algebras in negative degrees which belong to $\mathbb{U}$ (see for example [Hin] for the description of its model structure). Let τ_0 be one of the usual topologies defined on k-schemes (e.g. Zariski, Nisnevich, étale, ffpf or ffqc). Let us define a topology τ on $CDGA_k^{op}$ in the sense of Def. 2.2.1, as follows. A family of morphisms in $Ho(CDGA_k)$, $\{B \to A_i\}_{i\in I}$, $I \in \mathbb{U}$, is defined to be in $Cov_\tau(B)$ if it satisfies the two following conditions:

  1. The induced family of morphisms of affine k-schemes

  $$\{Spec\, H^0(A_i) \to SpecH^0(B)\}_{i\in I}$$

  is a τ-covering.

  2. For any $i \in I$, one has $H^*(A_i) \simeq H^*(B) \otimes_{H^0(B)} H^0(A_i)$.

  The reader can check as an exercise that this actually defines a topology on the model category $CDGA_k^{op}$. We will come back to this very important example in the last section of the paper.

2.3 Homotopy hypercovers

Using Reedy model structures ([Ho, 5.2]) on the category of simplicial objects in a model category, we generalize the definition of hypercovers to the case of (semi-)model sites.

Let $M_\mathbb{U}$ be a $\mathbb{V}$-small semi-model $\mathbb{U}$-category and let us consider $sM_\mathbb{U}$ the category of simplicial objects in $M_\mathbb{U}$. By definition, the category $M_\mathbb{U}$ has all $\mathbb{U}$-limits and all $\mathbb{U}$-colimits so that the category $sM_\mathbb{U}$ is naturally enriched in $\mathbb{U}-SSet$. Recall that for $K \in \mathbb{U}-SSet$ and $x_* \in sM_\mathbb{U}$, one has by definition

$$\begin{array}{cccc} K \times x_* : & \Delta^{op} & \longrightarrow & M_\mathbb{U} \\ & [n] & \mapsto & \coprod_{K_n} x_n. \end{array}$$

For x_* and y_* objects in $sM_\mathbb{U}$, we define

$$\begin{array}{cccc} \underline{Hom}(x_*, y_*) : & \Delta^{op} & \longrightarrow & \mathbb{U}-SSet \\ & [n] & \mapsto & Hom_{sM_\mathbb{U}}(\Delta^n \otimes x_*, y_*). \end{array}$$

Finally, the exponential object y_*^K, for $K \in \mathbb{U}-SSet$ and $y_* \in sM_{\mathbb{U}}$, is characterized by the adjunction isomorphism $Hom(K \otimes x_*, y_*) \simeq Hom(x_*, y_*^K)$, for all $x_* \in sM_{\mathbb{U}}$.

The category $sM_{\mathbb{U}}$ is endowed with its Reedy structure described in [Ho, Thm. 5.2.5] and [Sp, Prop. 2.7], which makes it into a $\mathbb{V}$-small semi-model $\mathbb{U}$-category. Let us recall that equivalences in $sM_{\mathbb{U}}$ are defined to be levelwise equivalences. Recall also, that a morphism $f : x_* \longrightarrow y_*$ is defined to be a fibration if for all n the morphism induced by the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$,

$$x_*^{\Delta^n} \longrightarrow x_*^{\partial\Delta^n} \times_{y_*^{\partial\Delta^n}} y_*^{\Delta^n},$$

is a fibration in $M_{\mathbb{U}}$.

For a $\mathbb{U}$-simplicial set K, the functor

$$\begin{array}{cccc} (-)^K : & sM_{\mathbb{U}} & \longrightarrow & M_{\mathbb{U}} \\ & x_* & \mapsto & (x_*^K)_0, \end{array}$$

which sends a simplicial object x_* to the 0-th level of the exponential object x_*^K, is a right Quillen functor. Its right derived functor will be denoted by

$$(-)^{\mathbb{R}K} : Ho(sM_{\mathbb{U}}) \longrightarrow Ho(M_{\mathbb{U}}).$$

We will consider objects of $M_{\mathbb{U}}$ as constant simplicial objects via the constant simplicial functor $M_{\mathbb{U}} \longrightarrow sM_{\mathbb{U}}$. In particular, for $x \in M_{\mathbb{U}}$ and $K \in \mathbb{U}-SSet$, we will consider the object $x^{\mathbb{R}K} \in Ho(M_{\mathbb{U}})$.

Definition 2.3.1 *Let $x \in M_{\mathbb{U}}$ be an object in a semi-model site $(M_{\mathbb{U}}, \tau)$. A* homotopy τ-hypercover *of x in $M_{\mathbb{U}}$, is a simplicial object $u_* \in Ho(sM_{\mathbb{U}})$, together with a morphism $u_* \longrightarrow x$ in $Ho(sM_{\mathbb{U}})$, such that for any $n \geq 0$, the natural morphism*

$$u_*^{\mathbb{R}\Delta^n} \longrightarrow u_*^{\mathbb{R}\partial\Delta^n} \times^h_{x^{\mathbb{R}\partial\Delta^n}} x^{\mathbb{R}\Delta^n}$$

is a τ-covering in $M_{\mathbb{U}}$.

2.4 The model category of stacks

In this paragraph we will use the notion of homotopy hypercover defined previously in order to construct the model category of stacks over a model site. Our construction is based on a recent result of D. Dugger identifying the model category of simplicial presheaves of [Ja] as the left Bousfield localization of the model category of simplicial presheaves for the trivial topology by *formally inverting hypercovers* (see [Du1]). By definition, our model category of stacks over a model site (M, τ) will be the left Bousfield localization of the model category $M^\wedge$ by formally inverting τ-hypercovers. In this first version of the paper, we will also state without proof a generalization of Dugger's theorem by introducing the notion of homotopy sheaves in our setting. This result is fundamental to control elementary manipulations in the model category of stacks (as for example, homotopy fibred products).

We come back to the basic setting of the present section, i.e. to an inclusion of semi-model categories $M_{\mathbb{U}} \subset M$, together with the associated Yoneda embedding

$$\mathbb{R}\underline{h} : Ho(M_{\mathbb{U}}) \longrightarrow Ho(M_{\mathbb{U}}^{\wedge})$$

defined in the first paragraph. We will suppose that $M_{\mathbb{U}}$ is endowed with a topology τ in the sense of definition 2.2.1.

We define two $\mathbb{V}$-sets of morphisms in $M_{\mathbb{U}}^{\wedge}$ in the following way. For this, recall the functor $h : M_{\mathbb{U}} \longrightarrow M_{\mathbb{U}}^{\wedge} = SPr(M_{\mathbb{U}})$, which maps an object $x \in M_{\mathbb{U}}$ to the constant simplicial presheaf it represents.

For any $\mathbb{U}$-set I and any family of cofibrant objects $\{x_i\}_{i\in I} \in (M_{\mathbb{U}}^{c})^I$, we consider the following natural morphism in $SPr(M_{\mathbb{U}})$

$$\coprod_{i\in I} h_{x_i} \longrightarrow h_{\coprod_{i\in I} x_i}.$$

When I varies in the set of $\mathbb{U}$-sets and the x_i's vary in the set of object in $M_{\mathbb{U}}^{c}$, we find a $\mathbb{V}$-set of morphisms in $M_{\mathbb{U}}^{\wedge}$.

$$S_{sum} := \{\coprod_{i\in I} h_{x_i} \longrightarrow h_{\coprod_{i\in I} x_i} \mid I \in \mathbb{U}, \{x_i\}_{i\in I} \in (M_{\mathbb{U}}^{c})^I\}.$$

Now, for any fibrant object $x \in M_{\mathbb{U}}^{f}$, let $HHC(x)$ be the $\mathbb{V}$-set of simplicial objects $u_* \in s(M/x)$, whose image in $Ho(sM)/x$ is a homotopy τ-hypercover of x in $M_{\mathbb{U}}$ (see Def. 2.3.1). For any $u_* \in HHC(x)$, $[n] \mapsto h_{u_n}$ is a simplicial presheaf on $M_{\mathbb{U}}$ defined by the following formula:

$$\begin{array}{cccc} h_{u_*} : & M_{\mathbb{U}}^{op} & \longrightarrow & SSet \\ & y & \mapsto & ([n] \mapsto h_{u_n}(y)). \end{array}$$

The augmentation $u_* \longrightarrow x$ gives then a morphism of simplicial presheaves $h_{u_*} \longrightarrow h_x$. When x varies in $M_{\mathbb{U}}^{f}$ and u_* varies in $HHC(x)$, we find a $\mathbb{V}$-set of morphisms in $M_{\mathbb{U}}^{\wedge}$

$$S_{hhc} := \{h_{u_*} \longrightarrow h_x \mid x \in M_{\mathbb{U}}^{f}, u_* \in HHC(x)\}.$$

Definition 2.4.1 *The* simplicial model category of stacks on $M_{\mathbb{U}}$ for the topology τ *is the left Bousfield localization of the simplicial model category $M_{\mathbb{U}}^{\wedge}$ along the $\mathbb{V}$-set of morphisms $S_\tau := S_{sum} \cup S_{hhc}$. It will be denoted by $M_{\mathbb{U}}^{\sim,\tau}$, or simply by $M_{\mathbb{U}}^{\sim}$ when the topology τ is clear.*

The derived simplicial Hom of $M_{\mathbb{U}}^{\sim,\tau}$ will be denoted by

$$\mathbb{R}_{w,\tau}\underline{Hom}(-,-) : Ho(M_{\mathbb{U}}^{\sim,\tau})^{op} \times Ho(M_{\mathbb{U}}^{\sim,\tau}) \longrightarrow Ho(SSet).$$

The following characterization of fibrant objects in $M_{\mathbb{U}}^{\sim,\tau}$ is an immediate application of the general criterion in Prop. 2.0.1.

Lemma 2.4.2 *An object $F \in M_{\mathbb{U}}^{\sim,\tau}$ is fibrant if and only if it satisfies the following four conditions:*

1. *For any $x \in M_{\mathbb{U}}$, the simplicial set $F(x)$ is fibrant;*

2. *For any equivalence $y \to x$ in M, the induced morphism $F(x) \longrightarrow F(y)$ is an equivalence of simplicial sets;*

3. *For any $\mathbb{U}$-set I and any family of cofibrant objects $\{x_i\}_{i\in I}$ in $M_{\mathbb{U}}$, the natural morphism of simplicial sets*

$$F(\coprod_{i\in I} x_i) \longrightarrow \prod_{i\in I} F(x_i)$$

is an equivalence;

4. *For any fibrant object $x \in M_{\mathbb{U}}^f$ and any simplicial object $u_* \in s(M_{\mathbb{U}}/x)$, whose image in $Ho(sM_{\mathbb{U}})/x$ is a homotopy τ-hypercover, the natural morphism in $Ho(SSet)$*

$$F(x) \longrightarrow holim_{[n]\in\Delta} F(u_n)$$

is an isomorphism.

Proof: It is a direct application of proposition 2.0.1. □

¿From the previous lemma we immediately deduce that the homotopy category $Ho(M_{\mathbb{U}}^{\sim,\tau})$ can be identified with the full subcategory of $Ho(SPr(M_{\mathbb{U}}))$ of simplicial presheaves satisfying conditions (2), (3) and (4) of lemma 2.4.2. Furthermore, the natural inclusion $Ho(M_{\mathbb{U}}^{\sim,\tau}) \longrightarrow Ho(SPr(M_{\mathbb{U}}))$ has a left adjoint which is a retraction. This retraction will be denoted by $a : Ho(SPr(M_{\mathbb{U}})) \longrightarrow Ho(M_{\mathbb{U}}^{\sim,\tau})$; note that a^2 is naturally isomorphic to a.

Definition 2.4.3

- *A* stack *on $M_{\mathbb{U}}$ for the topology τ is an object $F \in Ho(SPr(M_{\mathbb{U}}))$ such that the natural morphism $F \longrightarrow a(F)$ is an isomorphism.*

- *For any $F \in Ho(SPr(M_{\mathbb{U}}))$, the* stack associated *to F is the stack $a(F)$.*

- *The topology τ is* sub-canonical *if for any $x \in Ho(M_{\mathbb{U}})$, the object $\mathbb{R}\underline{h}_x \in Ho(M_{\mathbb{U}}^{\wedge})$ is stack.*

Remarks:

- If $M_{\mathbb{U}}$ is endowed with the trivial model structure, then the model category $M_{\mathbb{U}}^{\sim,\tau}$ is Quillen equivalent to the model category of simplicial presheaves defined by J.F. Jardine in [Ja]. This is proved in [Du1]. We will also state a more general result which, for not necessarily trivial model structures, identifies the equivalences in $M_{\mathbb{U}}^{\sim,\tau}$ as *local equivalences* (see Thm. 2.5.5). Note however, that the model structure we use is not the one defined in [Ja] but rather its projective analog described in [H-S, §5] and [Bl].

- When the topology τ is trivial, then the model category $M_{\mathbb{U}}^{\sim,\tau}$ is equivalent to $M_{\mathbb{U}}^{\wedge}$. In particular, a stack for the trivial topology is a simplicial presheaf $F : M_{\mathbb{U}}^{op} \longrightarrow SSet$ which preserves equivalences.

- When the topology τ is sub-canonical, one obtains a fully faithful functor

$$\mathbb{R}\underline{h} : Ho(M_{\mathbb{U}}) \longrightarrow Ho(M_{\mathbb{U}}^{\sim,\tau}),$$

which embeds the homotopy theory of $M_{\mathbb{U}}$ into the homotopy theory of stacks over $M_{\mathbb{U}}$.

The following criterion for the topology τ to be sub-canonical can be deduced immediately from lemma 2.4.2.

Corollary 2.4.4 *A topology τ on a semi-model category $M_{\mathbb{U}}$ is sub-canonical if and only if for every homotopy τ-hypercover $u_* \longrightarrow x$ in $M_{\mathbb{U}}$, the natural morphism*

$$hocolim_{[n]\in\Delta^{op}} u_n \longrightarrow x$$

is an isomorphism in $Ho(M_{\mathbb{U}})$.

Proof: It is a direct application of lemma 2.4.2 and of the universal property of homotopy colimits. □

2.5 Exactness properties of the model category of stacks

In this paragraph we will present a more classical definition of weak equivalences in $M_{\mathbb{U}}^{\sim,\tau}$, closer to the one used in [Ja]. The comparison theorem 2.5.5 is an (easy) extension of a result of D. Dugger. Therefore, we will not give all details and the interested reader may consult [Ja, Du1] for further materials.

In the whole paragraph a topology τ is fixed on $M_{\mathbb{U}}$.

Let us start by saying a few words on the notion of sheaves of sets in our setting of semi-model sites. As usual, any $\mathbb{V}$-set will be considered as a constant $\mathbb{V}$-simplicial set (note that such simplicial sets are always fibrant in $SSet$) and therefore any functor $M_{\mathbb{U}}^{op} \longrightarrow Set$ will be regarded as an object in $SPr(M_{\mathbb{U}})$.

Definition 2.5.1 *1. A* sheaf of sets on the model site *$(M_{\mathbb{U}}, \tau)$ is a functor $F : M_{\mathbb{U}}^{op} \longrightarrow Set$ which is stack when considered as an object in $Ho(SPr(M_{\mathbb{U}}))$.*

2. A morphism of sheaves *(of sets) $F \longrightarrow F'$ is a natural transformation.*

The category of sheaves (of sets) on the model site $(M_{\mathbb{U}}, \tau)$ will be denoted by $Sh(M_{\mathbb{U}}, \tau)$.

By definition, a functor $F : M_{\mathbb{U}}^{op} \longrightarrow Set$ is a sheaf when it satisfies the following three conditions:

1. For every equivalence $x \to y$ in $M_{\mathbb{U}}$, the induced morphism $F(y) \longrightarrow F(x)$ is an isomorphism;

2. For every $\mathbb{U}$-small family of objects in $M_{\mathbb{U}}$, $\{x_i\}_{i\in I}$, $I \in \mathbb{U}$, the induced morphism

$$F(\coprod_{i\in I}^{h} x_i) \longrightarrow \prod_{i\in I} F(x_i)$$

is an isomorphism;

3. For any fibrant object $x \in M_{\mathbb{U}}^{f}$ and any simplicial object $u_* \in s(M_{\mathbb{U}}/x)$, whose image in $Ho(sM_{\mathbb{U}})/x$ is a homotopy τ-hypercover, the natural morphism in $Ho(SSet)$

$$F(x) \longrightarrow lim_{[n]\in\Delta} F(u_n) \simeq Ker\,(F(u_0) \rightrightarrows F(u_1))$$

is an isomorphism.

Remark. By the universal property of the homotopy category, the category of sheaves on $(M_{\mathbb{U}}, \tau)$ is naturally equivalent to a full sub-category of the category of presheaves of sets $Set^{Ho(M_{\mathbb{U}})^{op}}$. However, as the topology τ on the semi-model category $M_{\mathbb{U}}$ does *not* induce in general a topology on $Ho(M_{\mathbb{U}})$, the category $Sh(M_{\mathbb{U}}, \tau)$ is a priory not a category of sheaves in the usual sense.

Lemma 2.5.2 *The natural functor $Sh(M_{\mathbb{U}}, \tau) \longrightarrow Ho(SPr(M_{\mathbb{U}}))$ factors through the full sub-category of stacks $Ho(M_{\mathbb{U}}^{\sim,\tau})$ and it is fully faithful.*

Proof: Let $F \in Sh(M_{\mathbb{U}}, \tau)$ be a sheaf and let us consider it as an object in the model category $M_{\mathbb{U}}^{\sim,\tau}$. The sheaf conditions together with lemma 2.4.2 show that F is a fibrant object in $M_{\mathbb{U}}^{\sim,\tau}$ and in particular that its image in $Ho(SPr(M_{\mathbb{U}}))$ is a stack.

Furthermore, as a sheaf is always a fibrant object in $M_{\mathbb{U}}^{\sim\tau}$, one checks immediately that for two sheaves F and G, the set of morphisms $[F, G]$ in $Ho(M_{\mathbb{U}}^{\sim,\tau})$ is isomorphic to the set of natural transformations between F and G. □

Let us consider $(M_{\mathbb{U}}, triv)$, the semi-model site with trivial topology. Then, the category $Sh(M_{\mathbb{U}}, triv)$ is equivalent to the category of functors $F : M_{\mathbb{U}}^{op} \longrightarrow Set$ sending equivalences to isomorphisms. In particular, there exists a natural fully faithful functor

$$Sh(M_{\mathbb{U}}, \tau) \longrightarrow Sh(M_{\mathbb{U}}, triv).$$

Lemma 2.5.3 *The natural functor*

$$Sh(M_{\mathbb{U}}, \tau) \longrightarrow Sh(M_{\mathbb{U}}, triv)$$

has a left adjoint $a_0 : Sh(M_{\mathbb{U}}, triv) \longrightarrow Sh(M_{\mathbb{U}}, \tau)$. Moreover, the functor a_0 is left exact (i.e. it commutes with finite limits).

Sketch of Proof: The idea is to imitate the usual associated sheaf functor construction, replacing fibred products by homotopy fibred products. The proof of the left exactness of a_0 is very similar to the usual one. $\square$

Remark. We have denoted a_0 the associated sheaf functor in order to make a difference with the associated stack functor a. However, we will show (see Thm. 2.5.5) that they coincide when applied to a sheaf, considered as an object in $Ho(M_{\mathbb{U}}^{\wedge})$. Note also that a_0 is only defined for presheaves $M_{\mathbb{U}}^{op} \longrightarrow Set$ sending equivalences in $M_{\mathbb{U}}$ to isomorphisms (i.e. for sheaves for the trivial topology on $M_{\mathbb{U}}$).

In the next definition, $\pi_n(K)$, for $K \in SSet$ and $n \geq 0$ denote the set of homotopy classes of morphisms $\Delta^n \longrightarrow K$ which sends $\partial\Delta^n$ to a point. More precisely,

$$\pi_n(K) := \pi_0 \left(\mathbb{R}\underline{Hom}(\Delta^n, K) \times^h_{\mathbb{R}\underline{Hom}(\partial\Delta^n, K)} K_0 \right),$$

where $\mathbb{R}\underline{Hom}(\Delta^n, K) \longrightarrow \mathbb{R}\underline{Hom}(\partial\Delta^n, K)$ is induced by the restriction to $\partial\Delta^n \subset \partial\Delta^n$ and $K_0 \longrightarrow \mathbb{R}\underline{Hom}(\partial\Delta^n, K)$ is adjoint to the natural projection $K_0 \times \partial\Delta^n \longrightarrow K_0 \longrightarrow K$.

The natural projection

$$\mathbb{R}\underline{Hom}(\Delta^n, K) \times^h_{\mathbb{R}\underline{Hom}(\partial\Delta^n, K)} K_0 \longrightarrow K_0$$

induces morphisms $\pi_n(K) \longrightarrow K_0$, which make the $\pi_n(K)$ for $n > 0$ (resp. for $n > 1$) group objects (resp. abelian group objects) over the set of 0-simplices K_0.

Definition 2.5.4 *1. Let $F \in SPr(M_{\mathbb{U}})$ be a stack for the trivial topology on $M_{\mathbb{U}}$ (i.e. F sends equivalences in $M_{\mathbb{U}}$ to equivalences in $SSet$). The* homotopy groups presheaves *of F are defined by*

$$\begin{array}{cccc} \pi_n^{pr}(F): & M_{\mathbb{U}}^{op} & \longrightarrow & Set \\ & x & \mapsto & \pi_n(F(x)). \end{array}$$

They come equipped with a natural projection $\pi_n^{pr}(F) \longrightarrow F_0$. The associated sheaves of $\pi_n^{pr}(F)$ are denoted by

$$\pi_n^{\tau}(F) := a_0(\pi_n^{pr}(F)).$$

*2. Let $F, F' \in SPr(M_{\mathbb{U}})$ be two stacks for the trivial topology on $M_{\mathbb{U}}$. A morphism $F \longrightarrow F'$ in $SPr(M_{\mathbb{U}})$ is called a π_*^{τ}-*equivalence *if for all $n \geq 0$ the following square is cartesian in $Sh(M_{\mathbb{U}}, \tau)$*

$$\begin{array}{ccc} \pi_n^{\tau}(F) & \longrightarrow & \pi_n^{\tau}(F') \\ \downarrow & & \downarrow \\ a_0(F_0) & \longrightarrow & a_0(F'_0). \end{array}$$

The main theorem of this paragraph is the following generalization of the main result proved in [Du1]. Its proof will not be given in this version of the paper.

Theorem 2.5.5 *Let F and F' be stacks for the trivial topology on $M_{\mathbb{U}}$ (i.e. they preserve equivalences) and $f : F \longrightarrow F'$ be a morphism in $SPr(M_{\mathbb{U}})$. Then, f is an equivalence in $M_{\mathbb{U}}^{\sim,\tau}$ if and only if it is a π_*^{τ}-equivalence.*

Besides its own interest, the previous theorem implies the following corollary which will be crucial for the study of geometric stacks in the next paragraph.

Corollary 2.5.6 *Let*

$$\begin{array}{ccc} F & \longrightarrow & F_1 \\ \downarrow & & \downarrow \\ F_2 & \longrightarrow & F_0 \end{array}$$

be a homotopy cartesian diagram in $SPr(M_{\mathbb{U}})$. If F_1, F_2 and F_0 are stacks for the trivial topology on $M_{\mathbb{U}}$, then the natural morphism

$$F \longrightarrow F_1 \times^h_{F_0} F_2$$

is an isomorphism in $Ho(M_{\mathbb{U}}^{\sim,\tau})$.

In other words, the associated stack functor a, when restricted to the full sub-category of stacks for the trivial topology, commutes with homotopy fibred products.

Proof: It is an application of theorem 2.5.5, lemma 2.5.3, the long exact sequence in homotopy for a homotopy fibred product of simplicial sets and an extended version of the five lemma. $\square$

Remark. The functor $a : Ho(SPr(M_{\mathbb{U}})) \longrightarrow Ho(M_{\mathbb{U}}^{\sim,\tau})$ will not commute with homotopy fibred products in general. Indeed, suppose that τ is the trivial topology and let x be an object of $M_{\mathbb{U}}$. The object $h_x \in SPr(M_{\mathbb{U}})$ represented by x, is such that $a(h_x) \simeq \underline{h}_x$. Therefore, if a would commute with homotopy fibred products, the natural functor $M_{\mathbb{U}} \longrightarrow Ho(M_{\mathbb{U}})$ would send fibred products to homotopy fibred products, which is not the case in general.

2.6 Functoriality

We finish this first section by the standard functoriality properties of the category of stacks i.e. with direct and inverse images functors.

We have seen in §1.1 that any functor $f : M_{\mathbb{U}} \longrightarrow N_{\mathbb{U}}$ between $\mathbb{V}$-small semi-model categories, whose restriction to $M_{\mathbb{U}}^{cf}$ preserves equivalences, gives rise to a pair of adjoint functors

$$\mathbb{L}f^* : Ho(M_{\mathbb{U}}^{\wedge}) \longrightarrow Ho(N_{\mathbb{U}}^{\wedge}) \qquad Ho(M_{\mathbb{U}}^{\wedge}) \longleftarrow Ho(N_{\mathbb{U}}^{\wedge}) : \mathbb{R}f_*.$$

Definition 2.6.1 *Let $f : M_{\mathbb{U}} \longrightarrow N_{\mathbb{U}}$ be a functor between two $\mathbb{V}$-small semi-model categories with topologies τ_M and τ_N, respectively. Let us suppose that the restriction of f to $M_{\mathbb{U}}^{cf}$ preserves equivalences. Then, f is said to be* continuous *if the inverse image functor*

$$\mathbb{R}f_* : Ho(N_{\mathbb{U}}^{\wedge}) \longrightarrow Ho(M_{\mathbb{U}}^{\wedge})$$

preserves the categories of stacks.

It is immediate to check that if f is a continuous functor, then the functor

$$\mathbb{R}f_* : Ho(N_{\mathbb{U}}^{\sim,\tau_N}) \longrightarrow Ho(M_{\mathbb{U}}^{\sim,\tau_M})$$

has a left adjoint

$$\mathbb{L}(f^*)^{\sim} : Ho(M_{\mathbb{U}}^{\sim,\tau_M}) \longrightarrow Ho(N_{\mathbb{U}}^{\sim,\tau_N}).$$

Explicitly, it is defined by the formula

$$\mathbb{L}(f^*)^{\sim}(F) := a(\mathbb{L}f^*(F)),$$

for $F \in Ho(M_{\mathbb{U}}^{\sim,\tau_M}) \subset Ho(M_{\mathbb{U}}^{\wedge})$, a being the associated stack functor.

Proposition 2.6.2 *Let $(M_{\mathbb{U}}, \tau_M)$, $(N_{\mathbb{U}}, \tau_N)$ and $(P_{\mathbb{U}}, \tau_P)$ be $\mathbb{V}$-small semi-model sites.*

1. *Let*

$$M_{\mathbb{U}} \xrightarrow{f} N_{\mathbb{U}} \xrightarrow{g} P_{\mathbb{U}}$$

be two continuous functors preserving fibrant objects and equivalences between them. Then, there exist natural isomorphisms

$$\mathbb{R}(g \circ f)_* \simeq \mathbb{R}f_* \circ \mathbb{R}g_* : Ho((P_{\mathbb{U}})^{\sim,\tau_P}) \longrightarrow Ho((M_{\mathbb{U}})^{\sim,\tau_M}),$$

$$\mathbb{L}((g \circ f)^*)^{\sim} \simeq \mathbb{L}(g^*)^{\sim} \circ \mathbb{L}(f^*)^{\sim} : Ho((M_{\mathbb{U}})^{\sim,\tau_M}) \longrightarrow Ho((P_{\mathbb{U}})^{\sim,\tau_P}).$$

These isomorphisms are furthermore associative and unital in the arguments f and g.

2. *Let*

$$M_{\mathbb{U}} \xrightarrow{f} N_{\mathbb{U}} \xrightarrow{g} P_{\mathbb{U}}$$

be two continuous functors preserving cofibrant objects and equivalences between them. Then, there exist natural isomorphisms

$$\mathbb{R}(g \circ f)_* \simeq \mathbb{R}f_* \circ \mathbb{R}g_* : Ho((P_{\mathbb{U}})^{\sim,\tau_P}) \longrightarrow Ho((M_{\mathbb{U}})^{\sim,\tau_M}),$$

$$\mathbb{L}((g \circ f)^*)^{\sim} \simeq \mathbb{L}(g^*)^{\sim} \circ \mathbb{L}(f^*)^{\sim} : Ho((M_{\mathbb{U}})^{\sim,\tau_M}) \longrightarrow Ho((P_{\mathbb{U}})^{\sim,\tau_P}).$$

These isomorphisms are furthermore associative and unital in the arguments f and g.

Proof: It is immediate from proposition 2.1.9 and the basic properties of the functor a. $\square$

The following criterion gives some examples of continuous functors.

Lemma 2.6.3 *Let $f : M_{\mathbb{U}} \longrightarrow N_{\mathbb{U}}$ be a right Quillen functor between two $\mathbb{V}$-small semi-model categories with topologies τ_M and τ_N, respectively. Suppose that f satisfies the following two conditions:*

1. *For any $x \in Ho(M_{\mathbb{U}})$ and any covering family $\{u_i \to x\}_{i\in I} \in Cov_{\tau_M}(x)$, the induced family*
$$\{\mathbb{R}f(u_i) \to \mathbb{R}f(x)\}_{i\in I}$$
is in $Cov_{\tau_N}(\mathbb{R}f(x))$.

2. *The functor $\mathbb{R}f : Ho(M_{\mathbb{U}}) \longrightarrow Ho(N_{\mathbb{U}})$ commutes with coproducts.*

Then, the functor f is continuous.

Proof: Let $F \in Ho(N_{\mathbb{U}}^{\sim,\tau_N})$ be a stack and let us prove that $\mathbb{R}f^*$ is a stack. For this, we use lemma 2.4.2. The reader should notice that conditions (1) and (2) are always satisfied by $\mathbb{R}f_*(F)$, if they are by F.

Recall that by definition of the functor $\mathbb{R}f_*$, for $F \in Ho(N_{\mathbb{U}}^{\sim})$ and $x \in M_{\mathbb{U}}$, one has a natural isomorphism $\mathbb{R}f_*(F)(x) \simeq F(\mathbb{R}f(x))$ in $Ho(M_{\mathbb{U}})$.

Now, by condition (2) on f, one has

$$\mathbb{R}f_*(F)(\coprod_{i\in I}^{h} x_i) \simeq \mathbb{R}\prod_{i\in I} \mathbb{R}f_*(F)(x_i),$$

for any family of objects $\{x_i\}_{i\in I}$ in $Ho(N_{\mathbb{U}})$. This show that $\mathbb{R}f_*(F)$ satisfies (3) of lemma 2.4.2. Furthermore, as f is right Quillen it commutes with the functors $(-)^{\mathbb{R}K}$ (introduced just before definition 2.3.1), for any simplicial set K. Using this and condition (1) on f, one checks immediately that $\mathbb{R}f_*(F)$ satisfy condition (4) of lemma 2.4.2. □

3 Stacks over E_∞-algebras

In this second section we present the construction of the category of *geometric stacks over a base symmetric monoidal model category*, as sketched in the Introduction. For this, we will start by recalling the homotopy theory of E_∞-algebras and modules over them in general symmetric monoidal model categories. The references for this part are the foundational papers [E-K-M-M], [Kr-Ma], [Hin] and especially [Sp] where the general case is studied (in particular the case where the monoid axiom does not hold). We will then apply our theory of stacks to (semi-)model categories of E_∞-algebras to give a definition of geometric stacks. In the last paragraph we give the groupoid approach to the construction of geometric stacks.

Setting. Throughout this section we will consider a left proper $\mathbb{V}$-cofibrantly generated symmetric monoidal model category $\mathcal{C}$. The unit of $\mathcal{C}$ will be denoted by $\mathbf{1}$ and will always be assumed to be a cofibrant object. We also assume $\mathcal{C}$ satisfies assumption [Sp, 9.6], i.e. that the domains of the generating cofibrations of $\mathcal{C}$ are cofibrant.

We will also consider $\mathcal{C}_{\mathbb{U}} \subset \mathcal{C}$ a sub-monoidal model category. By this we mean that $\mathcal{C}_{\mathbb{U}}$ is stable under the monoidal structure and is a sub-model category as already explained at the beginning of section 1. We will assume that $\mathcal{C}_{\mathbb{U}}$ is a $\mathbb{U}$-cofibrantly generated model category which is $\mathbb{V}$-small, and that the domains and codomains of the generating cofibrations and trivial cofibrations in $\mathcal{C}$ belong to $\mathcal{C}_{\mathbb{U}}$.

Finally, we assume that $\mathcal{C}$ is an algebra over the model category $SSet$ (of $\mathbb{V}$-simplicial sets) or over $C(\mathbb{Z})$ (the category of complexes of $\mathbb{V}$-abelian groups). The sub-model category $\mathcal{C}_{\mathbb{U}}$ is then assumed to be stable under external products by $\mathbb{U}$-simplicial sets or by complexes of $\mathbb{U}$-abelian groups.

3.1 Review of operads and E_∞-algebras

The main reference for this paragraph is [Sp, §1 − 10], as well as the foundational papers [E-K-M-M, Hin, Kr-Ma]. The reader may also consult [Ber-Moe].

Recall first that an *operad* $\mathcal{O}$ in $\mathcal{C}$ is the data, for each integer $n \in \mathbb{N}$, of an object $\mathcal{O}(n) \in \mathcal{C}$, together with an action of the symmetric group Σ_n, a unit $\mathbf{1} \longrightarrow \mathcal{O}(1)$ and structural morphisms

$$\mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \longrightarrow \mathcal{O}(\sum_i n_i),$$

for all integers $k \geq 1$ and $n_1, \ldots, n_k$. These structural morphisms are required to satisfy suitable associativity, commutativity and unity rules that the reader may find in [Kr-Ma, I.1.1]. A *morphism* between two operads $\mathcal{O}$ and $\mathcal{O}'$ in $\mathcal{C}$ is the data of morphisms $\mathcal{O}(n) \longrightarrow \mathcal{O}'(n)$ commuting with the unit and the structural morphisms. With these definitions, operads in $\mathcal{C}$ form a well defined category that will be denoted by $Op(\mathcal{C})$.

Following [Hin], [Sp] and [Ber-Moe], a morphism $f : \mathcal{O} \longrightarrow \mathcal{O}'$ of operads in $\mathcal{C}$ is a fibration (resp. an equivalence) if for all $n \in \mathbb{N}$, the induced morphism $f_n : \mathcal{O}(n) \longrightarrow \mathcal{O}'(n)$ is a fibration (resp. an equivalence) in $\mathcal{C}$.

Theorem 3.1.1 *([Sp, Thm. 3.2]) The category $Op(\mathcal{C})$ of operads in $\mathcal{C}$, together with the class of fibrations and equivalences defined above, is a cofibrantly generated semi-model category.*

It is important to remark that $Op(\mathcal{C}_{\mathbb{U}})$ is a sub-model category of $\mathcal{C}$, which is $\mathbb{U}$-cofibrantly generated and $\mathbb{V}$-small.

The fundamental operad we are interested in, is the operad COM, classifying commutative and unital monoids in $\mathcal{C}$. Explicitly, it is defined by $COM(n) = \mathbf{1}$ for any $n \geq 0$, with the trivial action of Σ_n.

Definition 3.1.2 *([Sp, Def. 8.1]) A* unital E_∞-operad *in $\mathcal{C}$ is an operad $\mathcal{O} \in Op(\mathcal{C})$ satisfying the following conditions:*

- *There exists an equivalence $u : \mathcal{O} \longrightarrow COM$;*
- *The induced morphism*

  $$u : \mathcal{O}(0) \longrightarrow COM(0) = \mathbf{1}$$

  is an isomorphism;

- *For any $n \geq 0$, the object $\mathcal{O}(n)$ is cofibrant in the semi-model category $\mathcal{C}^{\Sigma_n}$ of Σ_n-equivariant objects in $\mathcal{C}$.*

It is important to remark that unital E_∞-operads in $\mathcal{C}$ always exist. This follows from [Sp, Lem. 8.2] and our assumptions on $\mathcal{C}$ which include that $\mathbf{1}$ is cofibrant and that $\mathcal{C}$ is left proper.

For any operad $\mathcal{O} \in Op(\mathcal{C})$, one can define the category of *$\mathcal{O}$-algebras* in $\mathcal{C}$. By definition, an $\mathcal{O}$-algebra is the data of an object $A \in \mathcal{C}$ and structural morphisms $\mathcal{O}(n) \otimes A^{\otimes n} \longrightarrow A$, for all $n \geq 0$. These structural morphisms are required to satisfy certain associativity, commutativity and unit rules that the reader may find in [Kr-Ma, *I*.2.1]. A morphism of $\mathcal{O}$-algebras is the data of a morphism $A \longrightarrow A'$ in $\mathcal{C}$, commuting with the structural morphisms. These definitions allow to define the category of algebras over a fixed operad $\mathcal{O}$ in $\mathcal{C}$, that will be denoted by $Alg(\mathcal{O})$.

Let $f : \mathcal{O} \longrightarrow \mathcal{O}'$ be a morphism in $Op(\mathcal{C})$. Then, there exists a natural restriction functor

$$f_* : Alg(\mathcal{O}') \longrightarrow Alg(\mathcal{O}).$$

This functor has a left adjoint

$$f^* : Alg(\mathcal{O}) \longrightarrow Alg(\mathcal{O}').$$

As for the case of operads, a morphism $f : A \longrightarrow A'$ of $\mathcal{O}$-algebras in $\mathcal{C}$, is a fibration (resp. an equivalence) if it is a fibration (resp. an equivalence) in when considered as a morphism in $\mathcal{C}$.

Theorem 3.1.3 *([Sp, Thm. 4.7] and [Sp, Cor. 6.7])*

1. *Let $\mathcal{O} \in Op(\mathcal{C})$ be a unital E_∞-operad in $\mathcal{C}$. Then the category $Alg(\mathcal{O})$ of $\mathcal{O}$-algebras in $\mathcal{C}$, together with the class of fibrations and equivalences defined above, is a cofibrantly generated semi-model category.*

2. *Let $f : \mathcal{O} \longrightarrow \mathcal{O}'$ be an equivalence between two operads in $\mathcal{C}$. If for every $n \geq 0$, $\mathcal{O}(n)$ and $\mathcal{O}'(n)$ are cofibrant in M^{Σ_n}, then the induced Quillen adjunction*

$$f^* : Alg(\mathcal{O}) \longrightarrow Alg(\mathcal{O}') \qquad Alg(\mathcal{O}) \longleftarrow Alg(\mathcal{O}') : f_*$$

is a Quillen equivalence.

Corollary 3.1.4 *The semi-model category $Alg(\mathcal{O})$ of algebras over a unital E_∞-algebra in $\mathcal{C}$, is independent, up to a Quillen equivalence, of the choice of $\mathcal{O} \in Op(\mathcal{C})$.*

Proof: This follows from part (2) of Theorem 3.1.3 and the fact that two unitals E_∞-operads in $\mathcal{C}$ are isomorphic in $Ho(Op(\mathcal{C}))$ (because they are both isomorphic to COM). □

The previous corollary justifies the following definition

Definition 3.1.5 *The semi-model category of* E_∞-algebras *in* $\mathcal{C}$ *is defined to be* $Alg(\mathcal{O})$*, where* $\mathcal{O}$ *is a unital* E_∞*-operad in* $\mathcal{C}$*. It will be denoted by* $E_\infty - Alg(\mathcal{C})$*.*

The opposite semi-model category $(E_\infty - Alg(\mathcal{C}))^{op}$ *will be called the semi-model category of* affine stacks *over* $\mathcal{C}$ *and will be denoted by* $\mathcal{C} - Aff$*.*

An E_∞*-algebra* A *considered as an object in* $\mathcal{C} - Aff$ *will be symbolically denoted by* $Spec\, A$*.*

The same notations and terminology will be used for the category $\mathcal{C}_{\mathbb{U}} - Aff := (E_\infty - Alg(\mathcal{C}_{\mathbb{U}}))^{op}$*.*

Remarks:

- By conventions, the model category $\mathcal{C}$ is an algebra over the monoidal model category $SSet$ of simplicial sets (i.e. is a simplicial monoidal model category) or over the category $C(\mathbb{Z})$ of complexes of abelian groups (i.e. is a complicial monoidal model category). There are well known and famous E_∞-operads in $SSet$ and $C(\mathbb{Z})$, for example the singular realizations of the *little* n*-cubes operad* and of the *linear isometries operad*, as well as their homology complexes (see [Kr-Ma, I.5]). These operads can be transported to $\mathcal{C}$ via the unit of the algebra structure $SSet \longrightarrow \mathcal{C}$ or $C(\mathbb{Z}) \longrightarrow \mathcal{C}$ and give rise to unital E_∞-operads in $\mathcal{C}$. This implies that in practice, there exist natural choices for the unital E_∞-operad in $\mathcal{C}$.

- It is important to note that $E_\infty - Alg(\mathcal{C}_{\mathbb{U}})$ is a sub-model category of $E_\infty - Alg(\mathcal{C})$, which is $\mathbb{U}$-cofibrantly generated and $\mathbb{V}$-small, as soon as the E_∞-operad has been chosen in $\mathcal{C}_{\mathbb{U}}$. The category $Aff(\mathcal{C}_{\mathbb{U}})$, opposite to the category $E_\infty - Alg(\mathcal{C}_{\mathbb{U}})$ is really the category we will be mostly interested in.

- If the model structure on $\mathcal{C}$ is trivial, then so is the model structure on $E_\infty - Alg(\mathcal{C})$. Actually, a unital E_∞-operad is then automatically isomorphic to the operad COM. Therefore in this case, $E_\infty - Alg(\mathcal{C})$ is just the trivial model category of commutative and unital monoids in $\mathcal{C}$.

Let $\mathcal{O}$ be an operad in $\mathcal{C}$ and A be an $\mathcal{O}$-algebra in $\mathcal{C}$. An A*-module* in $\mathcal{C}$ is the data of an object $M \in \mathcal{C}$ and structural morphisms $\mathcal{O}(n) \otimes A^{n-1} \otimes M \longrightarrow M$. These structural morphisms are required to satisfy certain associativity, commutativity and unit rules that the reader may find in [Kr-Ma, I.4.1]. A *morphism* of A-modules is the data of a morphism $M \longrightarrow M'$ in $\mathcal{C}$, commuting with the structural morphisms. These definitions allow to define the category of modules over a fixed operad algebra A over a fixed operad $\mathcal{O}$ in $\mathcal{C}$, which will be simply denoted by $Mod(A)$.

Let $\mathcal{O} \in Op(\mathcal{C})$ be an operad in $\mathcal{C}$ and $f : A \longrightarrow A'$ be a morphism in $Alg(\mathcal{O})$. Then, there exists a natural restriction functor

$$f_* : Mod(A') \longrightarrow Mod(A).$$

This functor has a left adjoint

$$f^* : Mod(A) \longrightarrow Mod(A').$$

As for the case of operads and algebras, a morphism $f : M \longrightarrow M'$ of A-modules in $\mathcal{C}$ is a fibration (resp. an equivalence) if it is a fibration (resp. an equivalence) when considered as a morphism in $\mathcal{C}$.

Theorem 3.1.6 *([Sp, Thm. 6.1] and [Sp, Cor. 6.7]).*
Let $\mathcal{O} \in Op(\mathcal{C})$ be a unital E_∞-operad in $\mathcal{C}$ and $A \in Alg(\mathcal{O})$ a cofibrant E_∞-algebra in $\mathcal{C}$. Then

1. *The category $Mod(A)$ of A-modules in $\mathcal{C}$, together with the classes of fibrations and equivalences defined above, is a cofibrantly generated model category;*

2. *If $f : A \longrightarrow A'$ is an equivalence between two cofibrant E_∞-algebras in $\mathcal{C}$, the adjunction*
$$f^* : Mod(A) \longrightarrow Mod(A') \qquad Mod(A) \longleftarrow Mod(A') : f_*$$
is a Quillen equivalence.

Again, if $\mathcal{O} \in Op(\mathcal{C}_{\mathbb{U}})$ is a unital E_∞-algebra and $A \in Alg(\mathcal{O})$ is an E_∞-algebra in $\mathcal{C}_{\mathbb{U}}$, then the category of A-modules in $\mathcal{C}_{\mathbb{U}}$ is a sub-model category of $Mod(A)$. It is furthermore $\mathbb{U}$-cofibrantly generated and $\mathbb{V}$-small.

The compatibility between the pushforward and pullback functors above, on the model categories of modules, is expressed through the following base change formula.

Proposition 3.1.7 *([Sp, Prop. 9.12]). Let*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ A' & \xrightarrow[f']{} & B' \end{array}$$

be a homotopy co-cartesian diagram of cofibrant E_∞-algebras in $\mathcal{C}$. Then, for any $M \in Ho(Mod(B))$, the natural base change morphism

$$\mathbb{L}(g)^* \mathbb{R}f_*(M) \longrightarrow \mathbb{R}f'_* \mathbb{L}(g')^*(M)$$

is an isomorphism in $Ho(Mod(A'))$.

3.2 Geometric stacks over a monoidal model category

For this paragraph, recall our basic setting of this section: an inclusion $\mathcal{C}_{\mathbb{U}} \subset \mathcal{C}$ of monoidal model categories, satisfying the conditions explained at the beginning of this section. We will assume that the semi-model category $\mathcal{C}_{\mathbb{U}} - Aff$ of affine stacks over $\mathcal{C}_{\mathbb{U}}$ is endowed with a topology τ. This semi-model site will be denoted by $(\mathcal{C}_{\mathbb{U}} - Aff, \tau)$ and the corresponding model category of stacks will simply be denoted by $\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau}$. For the sake of simplicity we assume the topology τ is sub-canonical (see Def. 2.4.3).

Recall from Section 1 the existence of a Quillen adjunction

$$\mathrm{Re} : \mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau} \longrightarrow \mathcal{C} - Aff \qquad \mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau} \longleftarrow \mathcal{C} - Aff : \mathtt{Spec},$$

where we denote by $\mathtt{Spec}$ the functor we have called $\underline{h}$ in Section 1, because this seems more natural when dealing with E_∞-algebras. This Quillen adjunction, as we saw in Section 1, induces a fully faithful functor (the Yoneda embedding)

$$\mathbb{R}\underline{Spec} : Ho(\mathcal{C}_{\mathbb{U}} - Aff) = Ho(E_\infty - Alg(\mathcal{C}_{\mathbb{U}}))^{op} \longrightarrow Ho(\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau}).$$

In order to give the definition of *n-geometric stacks*, we will need to add the following hypothesis on our topology τ:

Hypothesis 3.2.1 *Let $\{Spec\, B_i \longrightarrow Spec\, A\}_{i\in I}$ be a $\mathbb{U}$-small family of morphisms in $Ho(\mathcal{C}_{\mathbb{U}} - Aff)$ and, for each $i \in I$, let $\{Spec\, C_j \longrightarrow Spec\, B_i\}_{j\in J_i}$ be a τ-covering family. If the induced family in $Ho(\mathcal{C}_{\mathbb{U}} - Aff)$, $\{Spec\, C_j \longrightarrow Spec\, A\}_{i\in I, j\in J_i}$ is a τ-covering, then so is $\{Spec\, B_i \longrightarrow Spec\, A\}_{i\in I}$.*

The definition of an n-geometric stack over $\mathcal{C}_{\mathbb{U}}$ is given by induction on n. We define simultaneously the notion of n-geometric stack and the notion of n-covering family by induction on n. Note that the both definitions depend on the topology τ, and one should probably use the expression n_τ-geometric stacks. However, as we will not consider different topologies at the same time, we will omit the reference to τ.

Definition 3.2.2

- *The category of* 0-geometric stacks *over $\mathcal{C}_{\mathbb{U}}$ is the essential image of the functor $\mathbb{R}\mathtt{Spec}$. It will be denoted by $0-GeSt(\mathcal{C}_{\mathbb{U}})$ and is equivalent to $\mathcal{C}_{\mathbb{U}}-Aff$ via the Yoneda embedding. Note also that it does not depend on τ.*

  The category $0 - GeSt$ will also be called the category of affine stacks *over $\mathcal{C}_{\mathbb{U}}$.*

- *A morphism $f : F \longrightarrow F'$ in $Ho(\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})$ is* 0-representable *if for any 0-geometric stack H and any morphism $H \longrightarrow F'$, the homotopy pull-back $F \times^h_{F'} H$ is a 0-geometric stack (this is again independent of τ).*

- *A $\mathbb{U}$-small family of morphisms $\{f_i : F_i \longrightarrow F'\}_{i\in I}$, $I \in \mathbb{U}$, in $Ho(\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})$, is a* 0-covering *if it satisfies the two following conditions:*

  - *For any $i \in I$, the morphism f_i is 0-representable;*
  - *For any 0-geometric stack H, any morphism $H \longrightarrow F'$ and any $i \in I$, the homotopy pull-back family $\{F_i \times^h_{F'} H \longrightarrow H\}_{i\in I}$ (which is a $\mathbb{U}$-small family of morphisms of 0-geometric stacks by the first condition), corresponds to a τ-covering family in $Ho(\mathcal{C}_{\mathbb{U}} - Aff)$.*

Let us suppose that the full sub-category $(n-1) - GeSt(\mathcal{C}_{\mathbb{U}}) \subset Ho(\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})$ of $(n-1)$-geometric stacks has been defined, as well as the notion of a $(n-1)$-covering family in $Ho(\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})$.

- *A morphism $f : F \longrightarrow F'$ in $Ho(\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})$ is* $(n-1)$-representable *if, for every 0-geometric stack H and any morphism $H \longrightarrow F'$, the homotopy fibred product $F \times^h_{F'} H \in Ho(\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})$ is an $(n-1)$-geometric stack.*

- *A stack* $F \in Ho(\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})$ *is* n-geometric *if it satisfies the following two conditions:*

  - *The diagonal morphism* $F \longrightarrow F \times F$ *is* $(n-1)$*-representable.*
  - *There exists a* $\mathbb{U}$*-small* $(n-1)$*-covering family* $\{f_i : F_i \longrightarrow F\}_{i \in I}$, $I \in \mathbb{U}$, *such that each* F_i *is a* 0*-geometric stack. Such a family will be called a* $(n-1)$-atlas *for* F*.*

  The full sub-category of $Ho(\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})$ *consisting of* n*-geometric stacks will be denoted by* $n - GeSt(\mathcal{C}_{\mathbb{U}})$*.*

- *Let* F *be a* 0*-geometric stack. A* $\mathbb{U}$*-small family of morphisms* $\{f_i : F_i \longrightarrow F\}_{i \in I}$ *in* $n - GeSt(\mathcal{C}_{\mathbb{U}})$ *is a* special n-covering *if, for any* $i \in I$*, there exists a* $(n-1)$*-atlas* $\{H_{i,j} \longrightarrow F_i\}_{j \in J_i}$*, such that the induced family* $\{H_{i,j} \longrightarrow F\}_{i \in I, j \in J_i}$ *is a* 0*-covering in* $Ho(\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})$*.*

- *A* $\mathbb{U}$*-small family of morphisms* $\{f_i : F_i \longrightarrow F\}_{i \in I}$ *in* $Ho(\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})$ *is a* n-covering *if it satisfies the following two conditions¿*

  - *For any* $i \in I$*, the morphism* f_i *is* n*-representable;*
  - *For any* 0*-geometric stack* H *and any morphism* $H \longrightarrow F'$*, the homotopy pullback family* $\{F_i \times^h_{F'} H \longrightarrow H\}_{i \in I}$ *(which is a family of morphisms from* n*-geometric stacks to a* 0*-geometric stack), is a special* n*-covering family (as defined before).*

A stack will be simply called geometric if it is n*-geometric for some integer* n*.*

Remarks:

- For F an n-geometric stack, the integer n refers to the complexity of the geometry of F and not to its homotopical complexity as the usual expression *n-stack* refers to. In general, the notion of n-geometric stack has nothing to do with the notion of n-stack i.e. of n-truncated simplicial presheaf. These two notions relate each others only when the model structure on $\mathcal{C}$ is trivial and the reason is that in this case affine stacks are 0-truncated (i.e. are presheaves of constant simplicial sets).

- The reader should be warned that when $\mathcal{C}_{\mathbb{U}}$ is the monoidal trivial model category of $\mathbb{U}$-abelian groups, then our notion of n-geometric stacks for $n = 0, 1$ is *not* equivalent to the notion of algebraic spaces and algebraic stacks as commonly used (e.g., in [La-Mo]), say with τ the ffqc-topology. For example, a non-affine scheme is not a 0-geometric stack in our sense but it is a 1-geometric stack if it is separated. To get non-separated schemes one needs to consider 2-geometric stacks. In the same way, an Artin stack with a non-affine diagonal is not a 1-geometric stack in our sense. It is a 2-geometric stack if it is quasi-separated but only a 3-geometric stack in general.

Using a big induction argument on n like it is done in [S1], one proves the following basic proposition. We will not rewrite the argument in this version of the paper. Note however that the proof uses in an essential way the hypothesis 3.2.1 on our topology. This hypothesis is precisely used to prove independence of the choice of atlases.

Proposition 3.2.3 *With the notations as above:*

1. *There are natural inclusions $n - GeSt(\mathcal{C}_{\mathbb{U}}) \subset (n+1) - GeSt(\mathcal{C}_{\mathbb{U}})$;*
2. *The set of n-representable morphisms is stable by composition and base change. Any isomorphism is n-representable for any n;*
3. *The set of n-covering families is stable by compositions and base changes. Any isomorphism is a n-covering family for any n;*
4. *If $f : F \longrightarrow F'$ is a n-representable morphism and F' is a n-geometric stack, then so is F;*
5. *The sub-category $n - GeSt(\mathcal{C}_{\mathbb{U}}) \subset Ho(\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})$ is stable under homotopy fibred products.*

Proof: See [S1]. □

3.3 An example: Quotient stacks

The model category of stacks $\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau}$ is a $\mathbb{U}$-cofibrantly generated model category and therefore, for any $\mathbb{U}$-small category I, the category of I-diagrams $(\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})^I$ is a model category with the so-called projective model structure (see [Hi, Thm. 13.8.1]). Let us recall that fibrations and equivalences in $(\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})^I$ are defined levelwise. We will be interested in the case $I = \Delta^{op}$ i.e. in the category of simplicial objects in $\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau}$. As usual we will denote $s\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau} := (\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})^{\Delta^{op}}$ and, for $X_* \in s\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau}$, $X_n := X([n])$.

Recall that for any integer $n > 0$ and any $X_* \in s\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau}$, there exists a Segal morphism

$$S_n : X_n \longrightarrow \underbrace{X_1 \times^h_{X_0} X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1}_{n \text{ times}}$$

induced by the morphisms in Δ,

$$\alpha_i : [1] \longrightarrow [n] \qquad d_o : [0] \longrightarrow [1] \qquad d_1 : [0] \longrightarrow [1],$$

which are defined by

$$\alpha_i(0) = i \qquad \alpha_i(1) = i+1, \qquad 0 \leq i < n.$$

The following definition is a generalization of Segal's Δ^{op}-spaces to the case where the *space of objects* is not contractible.

Definition 3.3.1 *An object $X_* \in s\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau}$ is called a* Segal groupoid object *if it satisfies the following two conditions:*

1. *For every integer $n > 0$, the Segal morphism*

$$S_n : X_n \longrightarrow \underbrace{X_1 \times^h_{X_0} X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1}_{n\ times},$$

is an equivalence in $\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau}$;

2. *The natural morphism*

$$d_2 \times d_1 : X_2 \longrightarrow X_1 \times^h_{d_1, X_0, d_1} X_1$$

is an equivalence in $\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau}$.

Remark. Condition (2) implies that the induced simplicial object in $Ho(\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})$ is a groupoid object.

The colimit functor $colim : s\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau} \longrightarrow \mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau}$ is clearly a left Quillen functor and can then be left derived to a functor at the level of homotopy categories

$$| - | := hocolim_{\Delta^{op}} : Ho(s\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau}) \longrightarrow Ho(\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau}).$$

Definition 3.3.2 *If $X_* \in Ho(s\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})$ is a Segal groupoid, then $|X_*| \in Ho(\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})$ is called the* quotient stack *of X_*.*

The fundamental theorem of this section is the following generalization of the criterion of [S1, Prop. 4.1]. We will only provide a sketch of its proof in this version of the paper.

Theorem 3.3.3 *A stack $F \in Ho(\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})$ is n-geometric if and only if there exists a Segal groupoid $X_* \in Ho(s\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})$ such that $F \simeq |X_*|$ and satisfying the following two conditions:*

- *The stack X_0 is a $\mathbb{U}$-small disjoint union of $(n-1)$-geometric stacks;*
- *Each of the two natural morphisms $d_0, d_1 : X_1 \longrightarrow X_0$ is a $(n-1)$-covering family (with one element).*

Sketch of proof: Suppose that X_* is a Segal groupoid satisfying the two conditions of the theorem. Then, using [Se, Prop. 1.6] and corollary 2.5.6, one checks that the natural morphism $X_0 \longrightarrow |X_*|$ is such that $X_0 \times^h_{|X_*|} X_0 \simeq X_1$. From this one deduces easily that if $\{H_i \longrightarrow X_0\}_{i \in I}$ is an $(n-1)$-atlas for X_0, then $\{H_i \longrightarrow X_0 \longrightarrow |X_*|\}_{i \in I}$ is again an $(n-1)$-atlas for $|X_*|$. One can also check that the following diagram is homotopy cartesian

$$\begin{array}{ccc} |X_*| & \longrightarrow & |X_*| \times |X_*| \\ \uparrow & & \uparrow \\ X_1 & \longrightarrow & X_0 \times X_0. \end{array}$$

This implies that the diagonal of $|X_*|$ is $(n-1)$-representable and finally that $|X_*|$ is n-geometric.

For the other implication, let F be an n-geometric stack and $\{H_i \longrightarrow F\}_{i\in I}$ an $(n-1)$-atlas. Let $X_0 := \coprod_{i\in I} H_i \longrightarrow F$ be the induced morphism. The homotopy nerve of $X_0 \longrightarrow F$ is a simplicial object $X_* \in Ho(s\mathcal{C}_{\mathbb{U}} - Aff^{\sim,\tau})$ such that $|X_*| \simeq F$. Furthermore, the fact that F is n-geometric implies that X_* satisfies the two conditions of the theorem. □

4 Applications and perspectives

In this last section, we present two applications of our theory. The first one is an approach to *DG-schemes* in which we interpret them as *geometric stacks over the model category of complexes*. The second application is a definition of étale K-theory of E_∞-ring spectra. Several other applications will appear in a forthcoming version.

4.1 An approach to DG-schemes

For this paragraph let k be a commutative ring with unit, $C(k)$ the symmetric monoidal model category of complexes of k-modules in $\mathbb{V}$ and $C(k)_{\mathbb{U}}$ the full sub-category of $\mathbb{C}$ of objects belonging to $\mathbb{U}$. We adopt the convention that complexes are $\mathbb{Z}$-graded co-chain complexes (i.e differentials increase degrees). As explained in the previous section, we will work with a fixed unital E_∞-operad $\mathcal{O}$ in $C(k)_{\mathbb{U}}$. For the sake of simplicity, we will assume that for each n, one has $\mathcal{O}(n)^i = 0$ for $i > 0$ (i.e. the operad $\mathcal{O}$ is concentrated in non-positive degrees).

Applying definition 3.1.5, we can consider the semi-model categories of affine stacks in $C(k)$ and in $C(k)_{\mathbb{U}}$

$$C(k) - Aff \qquad C(k)_{\mathbb{U}} - Aff.$$

In this special case, it is known that $C(k) - Aff$ and $C(k)_{\mathbb{U}} - Aff$ are actually *model* categories (see [Hin]).

Let us fix one of the standard Grothendieck topologies τ_0 on the category of k-schemes (e.g. Zariski, Nisnevich, étale, faithfully flat, ...). Starting from τ_0, we construct a topology τ on the model category $C(k)_{\mathbb{U}} - Aff$ (Def. 2.2.1) in the following way. Recall that for $Spec\,A \in C(k) - Aff$, one can consider its cohomology algebra $H^*(A) = \oplus H^i(A)$ which is in a natural way a graded commutative k-algebra. The construction $A \mapsto H^*(A)$ is of course functorial and therefore defines a functor from $Ho(C(k) - Aff)^{op}$ to graded commutative k-algebras.

The following definition was inspired by the work of K. Behrend [Be], where an étale topology on differential graded algebras is used.

Definition 4.1.1 *A $\mathbb{U}$-small family of morphisms in $C(k)_{\mathbb{U}} - Aff$*

$$\{f_i : Spec\,A_i \longrightarrow Spec\,B\}_{i\in I}$$

is a τ-covering if it satisfies the following two conditions:

- *The induced family of morphisms of (usual) affine schemes*

$$\{f_i : Spec\, H^0(A_i) \longrightarrow Spec\, H^0(B)\}_{i\in I}$$

  is a τ_0*-covering;*

- *For any* $i \in I$*, the induced morphism*

$$H^*(B) \otimes_{H^0(B)} H^0(A_i) \longrightarrow H^*(A_i)$$

  is an isomorphism.

The topology τ *on* $C(k)_{\mathbb{U}} - Aff$*, associated to the Grothendieck topology* τ_0 *on* k*-schemes, will be called the* strong τ_0-topology. *Covering families in* $(C(k)_{\mathbb{U}} - Aff, \tau)$ *will be called* strongly τ_0-covering families.

The above definition allows one to introduce the strong Zariski (resp. Nisnevich, étale, faithfully flat and quasi-compact, ...) topology on $C(k)_{\mathbb{U}} - Aff$. The corresponding model site will be denoted by $(C(k)_{\mathbb{U}} - Aff, \mathrm{Zar})$ (resp. $(C(k)_{\mathbb{U}} - Aff, \mathrm{Nis})$, resp. $(C(k)_{\mathbb{U}} - Aff, \text{ét})$, resp. $(C(k)_{\mathbb{U}} - Aff, \mathrm{ffqc})$, ...). The associated model categories of stacks will be naturally denoted by

$$C(k)_{\mathbb{U}} - Aff^{\sim,\mathrm{Zar}} \qquad C(k)_{\mathbb{U}} - Aff^{\sim,\mathrm{Nis}} \qquad C(k)_{\mathbb{U}} - Aff^{\sim,\text{ét}} \qquad C(k)_{\mathbb{U}} - Aff^{\sim,\mathrm{ffqc}}$$

and so on

Proposition 4.1.2 *For any Grothendieck topology* τ_0 *on* $k - Sch$ *which is coarser than the faithfully flat and quasi-compact topology, the induced strong* τ_0*-topology* τ *on* $C(k)_{\mathbb{U}} - Aff$ *is sub-canonical.*

Proof: Using lemma 2.4.2, it is enough to show that for any τ-hypercover $Spec\, B_* \longrightarrow Spec\, A$ in $C(k)_{\mathbb{U}} - Aff$, the natural morphism

$$A \longrightarrow holim_{[n]\in\Delta} B_n$$

is an equivalence of E_∞-algebras. As the forgetful functor from the category of E_∞-algebras to the category of complexes commutes with homotopy limits, it is enough to show that $A \longrightarrow holim_{[n]\in\Delta} B_n$ is a quasi-isomorphism of complexes of k-modules. Furthermore, in the model category $C(k)$ the homotopy limits along Δ can be computed using total complexes and therefore it is enough to show that the natural morphism of complexes

$$A \longrightarrow Tot(B_*)$$

is a quasi-isomorphism. To prove this, we use the spectral sequence computing the cohomology of a total complex as described e.g. in [We, 5.6],

$$E_2^{p,q} = H^p(H^q(B_*)) \Rightarrow H^{p+q}(Tot(B_*)),$$

where $H^q(B_*)$ is the normalized complex associated to the co-simplicial k-module $([n] \mapsto H^q(B_n))$. Now, by definition of a τ-hypercover and by the hypothesis on τ_0, one can use the Tor spectral sequence (see [Kr-Ma, thm. V.7.3]) to prove that the co-simplicial algebra $([n] \mapsto H^*(B_n))$ corresponds to a faithfully flat hypercover of affine schemes

$$Spec\, H^*(B_*) \longrightarrow Spec\, H^*(A).$$

By the usual faithfully flat descent (see [Mi, §I]), the above spectral sequence degenerates and satisfies

$$E_2^{p,q} = 0 \text{ for } p \neq 0, \qquad E_2^{0,q} = H^q(A).$$

This in turns implies that $A \longrightarrow Tot(B_*)$ is a quasi-isomorphism. □

We recall from [Ci-Ka1] the notion of DG-scheme. We will actually adopt a slightly different definition which is adapted to the case of an arbitrary base ring k. In the case k is a field of characteristic zero, our notion and that of [Ci-Ka1] are homotopically equivalent (see below).

Let X be a k-scheme (all schemes will be separated and quasi-compact) and $CQCoh(\mathcal{O}_X)$ its category of complexes of quasi-coherent $\mathcal{O}_X$-modules. This category is an algebra over the symmetric monoidal category $C(k)$, therefore it makes sense to talk about E_∞-algebras in $CQCoh(\mathcal{O}_X)$ (see [Sp]).

Definition 4.1.3 *A (separated and quasi-compact) DG-*scheme *is a pair (X, A_X) where X is a (separated and quasi-compact) k-scheme and A_X is a E_∞-algebra in $CQCoh(\mathcal{O}_X)$ satisfying the following two conditions:*

- *A_X is concentrated in non-positive degrees (i.e. $A_X^i = o$ for $i > 0$);*
- *The unit morphism $\mathcal{O}_X \longrightarrow A_X^0$ is an isomorphism.*

A morphism *between DG-schemes $f : (X, A_X) \longrightarrow (Y, A_Y)$ is the data of a morphism of schemes $f : X \longrightarrow Y$ together with a morphism of E_∞-algebras in $CQCoh(\mathcal{O}_X)$, $f^*(A_Y) \longrightarrow A_X$.*

For a DG-scheme (X, A_X), the cohomology sheaf $H^0(A_X)$ is a quasi-coherent $\mathcal{O}_X$-algebra whose associated X-affine scheme will be denoted by

$$\mathrm{H}^0(X, A_X) := Spec\, H^0(A_X) \longrightarrow X.$$

Actually, as $A_X^0 \simeq \mathcal{O}_X$ and $A_X^1 = 0$, the scheme $\mathrm{H}^0(X, A_X)$ is a closed sub-scheme of X. The cohomology sheaves $H^*(A_X)$ are naturally quasi-coherent $H^0(A_X)$-modules and therefore correspond to quasi-coherent sheaves on the sub-scheme $\mathrm{H}^0(X, A_X)$. They will still be denoted by $H^*(A_X)$.

Definition 4.1.4 *A morphism of DG-schemes $f : (X, A_X) \longrightarrow (Y, A_Y)$ is a* quasi-isomorphism *if it satisfies the following two conditions:*

- *The induced morphism of schemes $\mathrm{H}^0(f) : \mathrm{H}^0(X, A_X) \longrightarrow \mathrm{H}^0(Y, A_Y)$ is an isomorphism;*

- *The natural morphism of quasi-coherent sheaves on* $\mathrm{H}^0(X, A_X) \simeq \mathrm{H}^0(Y, A_Y)$

$$H^*(A_Y) \longrightarrow H^*(A_X)$$

is an isomorphism.

The homotopy category of DG-schemes is the category obtained from the category of DG-schemes belongings to $\mathbb{U}$ *by formally inverting the quasi-isomorphisms. It will be denoted by* $Ho(DG - Sch)$.

Remarks:

- The category of DG-schemes in $\mathbb{U}$ is a $\mathbb{V}$-small category. Therefore, $Ho(DG - Sch)$ is also a $\mathbb{V}$-small category but it is not clear a priory that it is a $\mathbb{U}$-small category.
- When k is a field of characteristic zero, the definition of DG-scheme given in [Ci-Ka1] is not strictly equivalent to 4.1.3. However, it is well known that in this case the homotopy theory of commutative differential graded algebras is equivalent to the homotopy theory of E_∞-algebras. This fact implies easily that the homotopy category of DG-schemes as defined in [Ci-Ka1] (and called by the authors, the right derived category of k-schemes) is equivalent to our $Ho(DG - Sch)$.
- Let A be a E_∞-algebra in $\mathbb{U}$ such that $A^i = 0$ for $i > 0$. As the operad $\mathcal{O}$ is concentrated in non-positive degrees, the k-module A^0 carries an induced E_∞-algebra structure. As it is a complex concentrated in degree zero, this is then nothing else than a commutative and unital algebra structure. Moreover, it is clear that the natural morphism of complexes $A^0 \longrightarrow A$ is a morphism of E_∞-algebras. In particular, A is naturally a complex of A^0-modules. This implies that for any E_∞-algebra A such that $A^i = 0$ for $i > 0$, one can define a DG-scheme $X := \underline{Spec}\, A$, whose underlying scheme is $Spec\, A^0$ and with $A_X := \widetilde{A} \in QCoh(X)$. It is clear that any DG-scheme (X, A_X) such that X is an affine scheme is of the form $\underline{Spec}\, A$ for some E_∞-algebra in non-positive degrees A (in fact, one has $A \simeq \Gamma(X, A_X)$).

Proposition 4.1.5 *There exists a functor*

$$\Theta : Ho(DG - Sch) \longrightarrow Ho(C(k)_{\mathbb{U}} - Aff^{\sim, ffqc})$$

such that, for any E_∞*-algebra in non-positive degrees* A*, one has*

$$\Theta(\underline{Spec}\, A) \simeq \mathbb{R}Spec\, A.$$

Moreover, for every DG-scheme (X, A_X)*, the stack* $\Theta(X, A_X)$ *is* 1*-geometric.*

Sketch of Proof: Let (X, A_X) be a DG-scheme and let $\{U_i\}_{i\in I}$ be a finite Zariski covering of X by affine schemes. Taking the nerve of this covering yields a simplicial diagram of affine schemes

$$[n] \mapsto \coprod_{i_0,\dots,i_n \in I^{n+1}} U_{i_0,\dots,i_n},$$

where $U_{i_0,\ldots,i_n} := U_{i_0} \cap \cdots \cap U_{i_n}$. By restricting A_X on each $U_{i_0,\ldots,i_n}$, one actually obtains a simplicial diagram of DG-schemes

$$[n] \mapsto (\coprod_{i_0,\ldots,i_n \in I^{n+1}} U_{i_0,\ldots,i_n}, \coprod A_{U_{i_0,\ldots,i_n}}).$$

Moreover, as each $\coprod_{i_0,\ldots,i_n \in I^{n+1}} U_{i_0,\ldots,i_n}$ is an affine scheme, this diagram is actually the image by $\underline{Spec}$ of a co-simplicial diagram of E_∞-algebras or, equivalently, of a simplicial diagram in $C(k)_{\mathbb{U}} - Aff$

$$F(U,X): \qquad \begin{array}{rcl} \Delta^{op} & \longrightarrow & C(k)_{\mathbb{U}} - Aff \\ {[n]} & \longmapsto & Spec\, A_n \end{array}$$

Considering its image by $\mathbb{R}Spec$, this diagram induces a well defined object in $Ho(s(C(k)_{\mathbb{U}} - Aff^{\sim,\mathrm{ffqc}}))$, the homotopy category of simplicial objects in $C(k)_{\mathbb{U}} - Aff^{\sim,\mathrm{ffqc}}$

$$F(U,X): \qquad \begin{array}{rcl} \Delta^{op} & \longrightarrow & C(k)_{\mathbb{U}} - Aff^{\sim,\mathrm{ffqc}} \\ {[n]} & \longmapsto & \mathbb{R}Spec\, A_n. \end{array}$$

We then define

$$\Theta(U,X) := hocolim_{[n]\in\Delta^{op}} \mathbb{R}Spec\, A_n$$

as an object in $Ho(C(k)_{\mathbb{U}} - Aff^{\sim,\mathrm{ffqc}})$. With some work, it is not difficult to verify that the stack $\Theta(U,X)$ does not depend on the choice of the affine covering $\{U_i\}_{i\in I}$ and that $(X, A_X) \mapsto \Theta(U,X)$ defines a functor

$$\Theta : Ho(DG - Sch) \longrightarrow Ho(C(k)_{\mathbb{U}} - Aff^{\sim,\mathrm{ffqc}}).$$

By construction, it is clear that $\Theta(\underline{Spec}\, A) \simeq \mathbb{R}Spec\, A$.

Finally, to prove that $\Theta(X, A_X)$ is a 1-geometric stack, one applies the criterion 3.3.3. The conditions of 3.3.3 are satisfied because by construction $\Theta(X, A_X)$ is the geometric realization of the Segal groupoid $[n] \mapsto \mathbb{R}Spec\, A_n$, for which the natural morphisms $\mathbb{R}Spec\, A_1 \longrightarrow \mathbb{R}Spec\, A_0$ are clearly strong Zariski coverings and a fortiori coverings in $C(k)_{\mathbb{U}} - Aff^{\sim,\mathrm{ffqc}}$. □

We make the following

Conjecture 4.1.6 *The functor* Θ *of proposition 4.1.5 is fully faithful.*

This conjecture says that the homotopy theory of DG-schemes can be embedded into the homotopy theory of geometric stacks over the model category of complexes. In other words, the theory of DG-schemes should be *a part of algebraic geometry over the model category of complexes*. We propose the model category of stacks $C(k)_{\mathbb{U}} - Aff^{\sim,\mathrm{ffqc}}$ as a natural setting for the theory of DG-schemes and more generally, for the theory of DG-stacks. One of the reasons why we believe this is a natural candidate is that in this way DG-schemes would appear naturally as a part of a fully-fledged homotopy theory, in the abstract modern sense of Quillen model categories. Instead, trying to obtain in a complete elementary way a homotopy structure out of usual DG-schemes (e.g., defining the weaker structure of a category with fibrations and equivalences, by declaring smooth maps to be fibrations and quasi-isomorphisms to be equivalences, as it seems to be suggested in [Ka]) seems to run into difficulties and moreover it is not a priori clear what kind of flexibility such a construction could have.

4.2 Étale K-theory

The problem of definition étale K-theory was raised by P.A. Ostvær and we give below a possible answer. We were very delighted by the question since it looked as a particularly good test of applicability of our theory.

Let Sp^{Σ} be the model category of symmetric spectra in $\mathbb{V}$ and $Sp^{\Sigma}_{\mathbb{U}}$ its sub-model category of objects in $\mathbb{U}$ (see [Ho-Sh-Sm]). The wedge product of symmetric spectra makes Sp^{Σ} and $Sp^{\Sigma}_{\mathbb{U}}$ into symmetric monoidal model categories. Applying definition 3.1.5, we may consider the semi-model categories $Sp^{\Sigma}_{\mathbb{U}} - Aff$ of affine stacks over $Sp^{\Sigma}_{\mathbb{U}}$. Again, it is know that $Sp^{\Sigma}_{\mathbb{U}} - Aff$ is actually a model category.

For each object $Spec\,A \in Sp^{\Sigma}_{\mathbb{U}} - Aff$, one can consider the category of A-modules in $Sp^{\Sigma}_{\mathbb{U}}$, $Mod(A)_{\mathbb{U}}$ as defined in the previous section. As Sp^{Σ} satisfies the monoid axiom, $Mod(A)_{\mathbb{U}}$ is actually a model category (with fibrations and equivalences defined on the underlying objects) which is moreover Quillen equivalent to $Mod(QA')_{\mathbb{U}}$, where QA' is a cofibrant replacement of A. Therefore, in theorem 3.1.6 one does not need to ask A to be a cofibrant object in order to get a good theory of modules.

Recall from [Sp, Prop. 9.10] that the homotopy category $Ho(Mod(A)_{\mathbb{U}})$ is a closed symmetric monoidal category. One can therefore define the notion of strongly dualizable objects in $Ho(Mod(A)_{\mathbb{U}})$ (following [E-K-M-M, $\S III.7$]). The full sub-category of $Mod(A)^{c}_{\mathbb{U}}$ consisting of strongly dualizable objects will be denoted by $Mod(A)^{sd}_{\mathbb{U}}$, and will be equipped with the induced notion of cofibrations and equivalences coming from $Mod(A)_{\mathbb{U}}$. It is not difficult to check that with this structure, $Mod(A)^{sd}_{\mathbb{U}}$ is then a Waldhausen category (see [E-K-M-M, $\S VI$]). Furthermore, if $A \longrightarrow B$ is a morphism of E_{∞}-algebras in Sp^{Σ}, then the base change functor

$$f^* : Mod(A)^{sd}_{\mathbb{U}} \longrightarrow Mod(B)^{sd}_{\mathbb{U}},$$

being the restriction of a left Quillen functor, preserves equivalences and cofibrations. This makes the lax functor

$$\begin{array}{cccc} Mod(-)^{sd}_{\mathbb{U}} : & Sp^{\Sigma}_{\mathbb{U}} & \longrightarrow & Cat \\ & Spec\,A & \mapsto & Mod(A)^{sd}_{\mathbb{U}} \\ & (f : A \to B) & \mapsto & f^* \end{array}$$

into a lax presheaf of Waldhausen $\mathbb{V}$-small categories. Applying standard strictification techniques we deduce a presheaf of $\mathbb{V}$-simplicial sets of K-theory

$$\begin{array}{cccc} K(-) : & Sp^{\Sigma}_{\mathbb{U}} & \longrightarrow & SSet \\ & Spec\,A & \mapsto & K(Mod(A)^{sd}_{\mathbb{U}}). \end{array}$$

Definition 4.2.1 *The previous presheaf will be considered as an object in* $Sp^{\Sigma}_{\mathbb{U}} - Aff^{\wedge}$ *and will be called the* presheaf of K-theory over the symmetric monoidal model category $Sp^{\Sigma}_{\mathbb{U}}$. *For any* $Spec\, A \in Sp^{\Sigma}_{\mathbb{U}} - Aff$, *we will write*

$$\mathbb{K}(A) := K(Spec\, A).$$

Remark. The same construction as above works if one replaces $Sp^{\Sigma}_{\mathbb{U}}$ by a general symmetric monoidal model category allowing therefore to define the spectrum $\mathbb{K}(A)$ for any E_{∞}-algebra A in a general symmetric monoidal model category. It could be interesting to look at this construction for the *motivic* categories considered in [Sp, 14.8].

Definition 4.2.2 *Let* τ *be a topology on the model category* $Sp^{\Sigma}_{\mathbb{U}} - Aff$ *and* $Sp^{\Sigma}_{\mathbb{U}} - Aff^{\sim,\tau}$ *the associated model category of stacks. Let* $K \longrightarrow K_{\tau}$ *be a fibrant replacement of* K *in* $Sp^{\Sigma}_{\mathbb{U}} - Aff^{\sim,\tau}$.

The K_{τ}-theory space *of an* E_{∞}*-algebra* A *in* $Sp^{\Sigma}_{\mathbb{U}}$ *is defined by*

$$\mathbb{K}_{\tau}(A) := K_{\tau}(Spec\, A).$$

The natural morphism $K \longrightarrow K_{\tau}$ *induces a natural augmentation (localization morphism)*

$$\mathbb{K}(A) \longrightarrow \mathbb{K}_{\tau}(A).$$

Remark. Note that we have

$$\mathbb{K}_{\tau}(A) \simeq \mathbb{R}\underline{Hom}_{w,\tau}(h_{Spec\, A}, K) \simeq \mathbb{R}\underline{Hom}_{w,\tau}(\mathbb{R}Spec\, A, K).$$

An application: étale K-theory of E_{∞}-ring spectra.
One defines an étale topology on $Sp^{\Sigma}_{\mathbb{U}} - Aff$ by stating that a family $\{f_i : Spec\, B_i \longrightarrow Spec\, A\}_{i \in I}$ is an étale covering if it satisfies the following three conditions:

1. For all $i \in I$, the morphism $A \longrightarrow B_i$ is a formally étale morphism of E_{∞}-ring spectra (in the sense that the corresponding co-tangent complex $L_{B_i/A}$ of [Hin, 7] vanishes);

2. For all $i \in I$, the A-algebra B_i is finitely presented (in any reasonable sense, see e.g. [Ma-Re] or [Ro, p. 7] in the "absolute" case, for connective, p-complete spectra)[4];

3. The family of base change functors

$$\{\mathbb{L}f_i^* : Ho(Mod(A)_{\mathbb{U}}) \longrightarrow Ho(Mod(B_i)_{\mathbb{U}})\}_{i \in I}$$

   is conservative i.e. a morphism in $Ho(Mod(A)_{\mathbb{U}})$ is an isomorphism if and only if, for any $i \in I$, its image in $Ho(Mod(B_i)_{\mathbb{U}})$ is an isomorphism.

One can check that these conditions actually define a topology *ét* on $Sp^{\Sigma}_{\mathbb{U}} - Aff$ Therefore, using definition 4.2.2, one can associate to any E_{∞}-ring spectrum A in $Sp^{\Sigma}_{\mathbb{U}}$ its *étale K-theory space* $\mathbb{K}_{et}(A)$.

[4] A precise definition would need more polishing and insight in the general case; we expect to give all details in a forthcoming version of this paper. However we are convinced that any topologically natural definition should work well to finally give a topology.

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