Polynomials Maps and Even Dimensional Spheres

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Abstract. We construct, for every even dimensional sphere S^n, $n \geq 2$, and every odd integer k, a homogeneous polynomial map $f : S^n \to S^n$ of Brouwer degree k and algebraic degree $2 \mid k \mid -1$.

A *polynomial map* from $X \subset \mathbb{R}^m$ to $Y \subset \mathbb{R}^r$ is the restriction to X of a polynomial map $F : \mathbb{R}^m \to \mathbb{R}^r$ such that $F(X) \subset Y$. When each component of F is homogeneous of degree k, we will say that the polynomial map from $X \subset \mathbb{R}^m$ to $Y \subset \mathbb{R}^r$ is *homogeneous* of degree k. As usual S^n is the sphere on $\mathbb{R}^{n+1}$ defined by the equation $x_1^2 + ... + x_{n+1}^2 = 1$, in short $\| x \|^2 = 1$, whereas S_r^n, $1 \leq r \leq n$, will be the differentiable manifold, diffeomorphic to S^n , defined by the equation $(x_1^2 + ... + x_r^2)^2 + x_{r+1}^2 + ... + x_{n+1}^2 = 1$. In this work we show:

Theorem 1. *Suppose n even and ≥ 2. Let k be an integer. Then:*
(a) If k is odd, there exists a homogeneous polynomial map from S^n to S^n of Brouwer degree k and algebraic degree $2 \mid k \mid -1$.
(b) If k is even there exists, for each $2 \leq 2r \leq n$, a polynomial map from S_{2r}^n to S^n of Brouwer degree k.

Representing elements of $\pi_n(S^n)$ by polynomial maps is an old question [1] which was affirmatively solved by Wood, in 1968, provided that n is odd (theorem 1 of [3], see[4] as well for the complex sphere). Nevertheless, as far as I know, this problem is still open for n even; our theorem settles it when the Brouwer degree is odd. In both cases the polynomial maps constructed are homogeneous; therefore the problem of representing elements of $\pi_n(S^n)$ by homogeneous polynomial maps is solved now, since only zero and the odd topological degrees may be represented in this way when n is even [2].

The proof of theorem 1 of [3] makes use of a natural polynomial map of topological degree 2 (lemmas 4 and 5). Nothing similar is known for n even; however the polynomial map $x \in \mathbb{R}^3 \to (x_1^2 - x_2^2, 2x_1x_2, x_3) \in \mathbb{R}^3$ send S_2^2 into S^2 with topological degree 2. Part (b) of our theorem generalizes this map.

For proving theorem 1 we start constructing a family of real polynomials in one variable. Let $\varphi_\ell = \sum_{j=0}^{\ell} a_j t^j$ be the Taylor expansion of $\varphi = (1-t)^{-1/2}$, at zero, up to order ℓ; that is to say $a_j = \dfrac{(2j-1)(2j-3)\cdots 1}{2^j \cdot j!}$. Since the radius of convergence of the power series $\sum_{j=0}^{\infty} a_j t^j$ is 1 and each $a_j > 0$, we have $a_0 = 1 \leq \varphi_\ell \leq (1-t)^{-1/2}$, $t \in [0,1)$, whence $(t-1)\varphi_\ell^2(t) + 1 \geq 0$ and $\varphi_\ell(t) \geq 1$ when $t \geq 0$ (both inequalities are obvious if $t \geq 1$).

On the other hand if we set $\varphi = t^{\ell+1}R + \varphi_\ell$ then $\sum_{j=0}^{\infty} t^j = (1-t)^{-1} = \varphi^2 = t^{\ell+1}\tilde{R} + \varphi_\ell^2$ on $(-1,1)$; Therefore $\varphi_\ell^2 = t^{\ell+1}S + \sum_{j=0}^{\ell} t^j$ where S is a polynomial in t. It follows, from that, the existence of a polynomial λ_ℓ of degree ℓ such that $(t-1)\varphi_\ell^2(t) + 1 = t^{\ell+1}\lambda_\ell$.

Lemma 1. *For every ℓ one has $(t-1)\varphi^2 + 1 = t^{\ell+1}\lambda_\ell$ where λ_ℓ is a polynomial of degree ℓ. Moreover $\lambda_\ell(t) \geq 0$ and $\varphi(t) > 0$ for each $t \in \mathbb{R}$ if ℓ is even, and for any $t \geq 0$ if ℓ is odd.*

Proof. It will suffice to show that $\lambda_\ell(t) \geq 0$ and $\varphi_\ell(t) > 0$ if $\ell \geq 2$ is even and $t < 0$. First we will prove, by induction on ℓ, the existence of a $\delta_\ell > 0$ such that φ_ℓ is strictly decreasing on $(-\infty, -1 + \delta_\ell)$. Note that $\varphi_\ell = a_\ell t^\ell + a_{\ell-1}t^{\ell-1} + \varphi_{\ell-2} = a((2\ell-1)t^\ell + 2\ell t^{\ell-1}) + \varphi_{\ell-2}$ where $a > 0$.

By induction hypothesis or because $\varphi_0 = 1$, the polynomial $\varphi_{\ell-2}$ is decreasing on

$(-\infty, -1+\delta_{\ell-2})$, or on $\mathbb{R}$ if $\ell = 2$. But the derivative $((2\ell-1)t^{\ell} + 2\ell t^{\ell-1})' = ((2\ell - 1)\ell t^{\ell-1} + 2\ell(\ell-1)t^{\ell-2}) < 0$ on $(-\infty, -1]$, so φ_ℓ is strictly decreasing on some interval $(-\infty, -1+\delta_\ell)$.

We show now that $\varphi_\ell(t) > (1-t)^{-1/2} > 0$ if $t < 0$. As $(1-t)^{-1/2}$ is strictly increasing, it is enough to prove our result on $(-1, 0)$. On this interval $lim_{\ell\to\infty}\{\varphi_\ell(t)\} = \sum_{j=0}^{\infty} a_j t^j = (1-t)^{-1/2}$. But the series $\sum_{j=0}^{\infty} a_j t^j$ is alternating and the sequence $\{a_j \mid t \mid^j\}_{j\in\mathbb{N}}$, whose limit is zero, strictly decreasing; then $\varphi_\ell(t) > (1-t)^{-1/2} > 0$ for ℓ even.

Finally, if $\varphi_\ell(t) > (1-t)^{-1/2} > 0$ for any $t < 0$, a straightforward calculation shows that $(t-1)\varphi_\ell^2(t) + 1 < 0$, whence $\lambda_\ell(t) \geq 0$ since $t^{\ell+1} < 0$. $\square$

Recall that any polynomial μ which do not takes negative values has even degree and can be write $\mu = \mu_1^2 + \mu_2^2$, where μ_1 and μ_2 are polynomials of degree $\leq$ half of degree of μ. Therefore by setting $k = \ell + 1$, $\alpha = \varphi_\ell$, $\lambda_\ell = \mu$, $\beta_1 = \mu_1$ and $\beta_2 = \mu_2$ one has:

Corollary 1. *For any odd natural number k there exist three polynomials α, β_1, β_2, the first one of degree $k-1$ and the other two with degree $\leq \frac{k-1}{2}$, such that $\alpha(t) > 0$ and $(1-t)\alpha^2(t) + t^k(\beta_1^2(t) + \beta_2^2(t)) = 1$ anywhere.*

Let us proof part (a) of theorem 1. Since topological degrees ± 1 may be represented by linear maps, we can assume $k \geq 1$. On $\mathbb{C} \times \mathbb{R}^{n-1} = \mathbb{R}^{n+1}$, endowed with coordinates $(z, y) = (z, y_1, ..., y_{n-1})$ for which $S^n = \{(z, y); \mid z \mid^2 + y_1^2 + ... + y_{n-1}^2 = 1\}$, we define

$$F(z, y) = ((\beta_1(\mid z \mid^2) + i\beta_2(\mid z \mid^2))z^k, \alpha(\mid z \mid^2)y)$$

where α, β_1 and β_2 are as in corollary 1. Then $F(S^n) \subset S^n$.

Set $S^1 = \{(z, 0); \mid z \mid^2 = 1\} \subset S^n$. As $\alpha(t) > 0$ for each $t \in \mathbb{R}$, $F^{-1}(S^1) = S^1$ and F preserves the orientation transversely to S^1. Hence the maps $F_{|S^1}$ and $F_{|S^n}$ have the same topological degree, that is to say k.

By construction all the monomials of F have odd degree $\leq 2k-1$. Multiplying everyone of them by a suitable power of $\mid z \mid^2 + y_1^2 + ... + y_{n-1}^2$ the map F becomes homogeneous of algebraic degree $2k-1$, whereas $F_{|S^n}$ do not change.

For proving (b), first we set $\tilde{\lambda}_\ell(t) = \lambda_\ell(t^2)$ and $\tilde{\varphi}_\ell(t) = \varphi_\ell(t^2)$. By lemma 1 we have $(t^2-1)\tilde{\varphi}_\ell^2(t) + 1 = t^{2\ell+2}\tilde{\lambda}_\ell(t)$, $\tilde{\varphi}_\ell(t) > 0$ and $\tilde{\lambda}_\ell(t) \geq 0$ for any $t \in \mathbb{R}$. This allows us to find out, for every natural number $\tilde{k} \geq 1$, three polynomials $\tilde{\alpha}, \tilde{\beta}_1, \tilde{\beta}_2$ such that $\tilde{\alpha}(t) > 0$ and $(1-t^2)\tilde{\alpha}^2(t) + t^{2\tilde{k}}(\tilde{\beta}_1^2(t) + \tilde{\beta}_2^2(t)) = 1$ anywhere.

Consider on $\mathbb{R}^{n+1} = \mathbb{R}^{2r} \times \mathbb{R}^{n-2r+1}$ coordinates $(x, y) = (x_1, ..., x_{2r}, y_1, ..., y_{n-2r+1})$. Let $f : \mathbb{R}^{2r} \to \mathbb{R}^{2r}$ be a homogeneous polynomial map of algebraic degree $2\tilde{k}$, sending S^{2r-1} into S^{2r-1} with topological degree $k = \pm 2\tilde{k}$, which always exists (see [3]) and $J : \mathbb{R}^{2r} \to \mathbb{R}^{2r}$ the isomorphism given by $Jx = (-x_2, x_1, ..., -x_{2r}, x_{2r-1})$, that is to say the canonical complex structure of $\mathbb{R}^{2r}$. One defines (if $\tilde{k} = 0$ just consider a constant map):

$$F(z, y) = (\tilde{\beta}_1(\| x \|^2)f(x) + \tilde{\beta}_2(\| x \|^2)Jf(x), \tilde{\alpha}(\| x \|^2)y)$$

Then $F(S^n_{2r}) \subset S^n$ and the same argument as in part (a), applied to $S^{2r-1} = \{(x, 0); \| x \|^2 = 1\} \subset S^n_{2r}$, shows that the topological degree of $F : S^n_{2r} \to S^n$ equals k.

References

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