# ON SIMPLICIAL COMMUTATIVE ALGEBRAS WITH NOETHERIAN HOMOTOPY

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Abstract. In this paper, we introduce a strategy for studying simplicial commutative algebras over general commutative rings $R$. Given such a simplicial algebra $A$, this strategy involves replacing $A$ with a connected simplicial commutative $k(\wp)$-algebra $A(\wp)$, for each $\wp \in \mathrm{Spec}(\pi_0 A)$, which we call the **connected component of** $A$ **at** $\wp$. These components retain most of the André-Quillen homology of $A$ when the coefficients are $k(\wp)$-modules ($k(\wp)$ = residue field of $\wp$ in $\pi_0 A$). Thus these components should carry quite a bit of the homotopy theoretic information for $A$. Our aim will be to apply this strategy to those simplicial algebras which possess **Noetherian homotopy**. This allows us to have sophisticated techniques from commutative algebra at our disposal. One consequence of our efforts will be to resolve a more general form of a conjecture of Quillen that was posed in [13].

## Overview

Our focus, in this paper, is to take the view that the study of Noetherian rings and algebras through homological methods is a special case of the study of simplicial commutative algebras having Noetherian homotopy type. Our goal is to show that such simplicial algebras can be given a suitably rigid structure in the homotopy category, which then allows us to bring in methods from commutative algebra. Such methods should enable more facile techniques from homological algebra to be ferried in for the purpose of elaborating the global structure of such simplicial algebras.

To begin, we define for a simplicial commutative algebra $A$ to have *Noetherian homotopy* provided:

1. $\pi_0 A$ is a Noetherian ring, and
2. each $\pi_m A$ is a finite $\pi_0 A$-module.

If, more strongly, $\pi_* A$ is a finite graded $\pi_0 A$-module, we that $A$ has *finite Noetherian homotopy.*

In order to achieve a more systematic study of simplicial algebras with Noetherian homotopy, particularly to allow us a straighter path to proving our main result, Theorem B below, we first seek to rigidify the action of $\pi_0$ from the homotopy groups to the simplicial algebra. This is accomplished by the following:

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**Theorem A:** *Any simplicial commutative algebra $A$ is weakly equivalent to a connected simplicial supplemented $\pi_0 A$-algebra.*

Theorem A provides the means to import in methods from commutative algebra, most notably localizations and completions. In particular, we use these methods as a means to provide a proof of a conjecture posed in [13] which generalizes a conjecture of Quillen regarding the vanishing of André-Quillen homology. Our larger interests lie in providing an understanding of the of the homotopy type of a simplicial commutative algebra $A$ with Noetherian homotopy over a Noetherian ring $R$ through its André-Quillen homology $D(A|R;-)$. Here we shall view this homology as a functor of $\pi_0 A$-modules. This enables us to be specific about the homology's rigidity properties.

Before stating our result, we first need a homotopy invariant notion of complete intersection. To obtain one, we first define a map $A \to B$ of simplicial commutative $R$-algebras, augmented over a field $\ell$, to be *virtually acyclic* provided $D_{\geq 1}(B|A;\ell) = 0$. Also, if $W$ is a graded $\ell$-module, define the simplicial $\ell$-algebra $S_\bullet(W)$ by

$$S_\bullet(W) = \bigotimes_n S(W_n, n)$$

where $S(V,n)$ is the free commutative $\ell$-algebra generated by the Eilenberg-MacLane space $K(V,n)$.

Define a simplicial commutative $R$-algebra $A$ over $\ell$ to be a *homotopy n-intersection,* for $n \geq 1$, provided there is a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \eta\downarrow & & \downarrow \eta' \\ A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ \ell & \xrightarrow{=} & \ell \end{array}$$

with the horizontal maps being virtually acyclic over $\ell$ and in the homotopy category there is an isomorphism

$$A' \otimes^{\mathbf{L}}_{R'} \ell \cong S_\bullet(W)$$

with $W$ a graded $\ell$-module satisfying $W_{>n} = 0$. We call a general simplicial commutative $R$-algebra $A$ a *locally homotopy n-intersection* if, for each $\wp \in \mathrm{Spec}(\pi_0 A)$, $A$ is a homotopy n-intersection over the residue field $k(\wp)$

Recall that the *flat dimension* of an $R$-module $M$ to be the positive integer $\mathrm{fd}_R M$ such that

$$\mathrm{fd}_R M \leq m \iff \mathrm{Tor}_i^R(M,-) = 0 \quad \text{for} \quad i > m. \tag{0.1}$$

**Theorem B:** *Let $A$ be a simplicial commutative $R$-algebra with finite Noetherian homotopy,* $\mathrm{char}(\pi_0 A) \neq 0$, *and* $\mathrm{fd}_R(\pi_* A)$ *is finite. Then $D_s(A|R;-) = 0$ for $s \gg 0$ if and only if $A$ is a locally homotopy 1-intersection.*

This resolves a conjecture posed in [13] generalizing a conjecture of Quillen [11, 5.7].

**Notes:**

1. Theorem B fails when $\operatorname{char}(\pi_0 A) = 0$, as shown in [13].
2. Theorem B fails for general simplicial algebras having Noetherian homotopy. The case of the simplicial algebras $S(V, n)$ over a field of non-zero characteristic provide counterexamples, by computations of Cartan [5].
3. A homomorphism between Noetherian rings is a locally complete intersection if and only if it is a locally homotopy 1-intersection, as shown in [2, 13].

Quillen further conjectured a more general result [11, 5.6] which drops the finite flat dimension condition. We would like to indicate a possible simplicial version of this conjecture of Quillen. To formulate it, we first indicate a special vanishing result for André-Quillen homology that we will prove.

**Theorem C:** *Let $A$ be a simplicial commutative $R$-algebra with Noetherian homotopy. Then $D_s(A|R; -) = 0$ for $s \geq 3$ if and only if $A$ is a locally homotopy 2-intersection.*

This now leads us to pose the following:

**Conjecture:** *Let $A$ have finite Noetherian homotopy with* $\operatorname{char}(\pi_0 A) \neq 0$. *Then $D_s(A|R; -) = 0$ for $s \gg 0$ implies that $A$ is a locally homotopy 2-intersection.*

The strategy for proving Theorem B is to show that $D_s(A|R; k(\wp)) = 0$ for $s \geq 2$ for each $\wp \in \operatorname{Spec}(\pi_0 A)$. This is sufficient by a result of André [1, S.30]. Following a strategy of Avramov [2], we use Theorem A coupled with commutative algebra techniques developed in [3] to replace $A$ with $A(\wp)$, its *connected component at* $\wp$, which has the following properties:

1. $A(\wp)$ is a connected simplicial supplemented $k(\wp)$-algebra;
2. $\operatorname{fd}_R(\pi_* A) < \infty$ implies that $A(\wp)$ has finite Noetherian homotopy;
3. $D_s(A|R; k(\wp)) \cong D_s(A(\wp)|k(\wp); k(\wp))$ for $s \geq 2$.

Theorem B now follows from the algebraic version of a theorem of Serre established in [13].

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## 1. Postnikov Systems and Theorem A

Throughout this paper, we fix a commutative ring with unit $\Lambda$ and let $\mathcal{A}lg_\Lambda$ be the category of (unitary) commutative rings augmented over $\Lambda$. Finally, we denote by ${}_\Lambda\mathcal{A}lg_\Lambda$ the category of $\Lambda$-algebras in $\mathcal{A}lg_\Lambda$.

We will also be assuming the reader has an acquaintance with closed (simplicial) model category theory. Our main resource is [10]. We will further need specific results on the model category structure for simplicial commutative rings and algebras. Our primary sources are [10, 12, 6].

1.1. **Postnikov Systems.** Let $A$ be an object in the category $s\mathcal{A}lg_\Lambda$ of simplicial commutative rings over $\Lambda$. We review the construction of a Postnikov tower for $A$ derived from [4, 7] which we will be use in the proof of Theorem A.

Following [7, §5], define the $n$th Postnikov section of $A$ as follows: for fixed $k$, let $I_{n,k} \to A_k$ be the kernel of the map

$$d : A_k \to \prod_{\phi:[m]\to[k]} A_n$$

where $\phi$ runs over all injections in the ordinal number category with $m \le n$, $d$ is induced by the maps $\phi^* : A_k \to A_m$, and $\prod$ denotes the product in the category of algebras augmented over $\Lambda$. Define

$$A(n)_k = A_k/I_{n,k} \tag{1.2}$$

Notice that there is a quotient map in $s\mathcal{A}lg_\Lambda$, $A \to A(n)$, and that if $k \le n$, $A(n)_k = A_k$. There are also quotient maps

$$q_n : A(n) \to A(n-1) \tag{1.3}$$

and $A \cong \lim A(n)$. Let $F(n)$ be the fibre of $q_n$, i.e.

$$F(n) = \ker(q_n : A(n) \to A(n-1)). \tag{1.4}$$

Note that $F(n) \to A(n) \overset{q_n}{\to} A(n-1)$ forgets to a fibration sequence as simplicial abelian groups. As such, the following can be proved just as in [7, 5.5].

**Lemma 1.1.** *The homotopy groups of $F(n)$ are computed as follows:*

$$\pi_k F(n) = \begin{cases} \pi_n A & k = n; \\ 0 & k \neq n. \end{cases}$$

1.2. **Eilenberg-MacLane objects.** Following [4, §5], define an object $A$ of $s\mathcal{A}lg_\Lambda$ to be *of type* $K_\Lambda$ if $\pi_0 A \cong \Lambda$ and the higher homotopy groups of $A$ are trivial. Suppose $M$ is a $\Lambda$-module. We say that a map $A \to B$ is *of type* $K_\Lambda(M,n)$ $n \ge 1$, if $A$ is of type $K_\Lambda$, $\pi_0 B \cong \Lambda$, $\pi_n B \cong M$ (as a $\Lambda$-module), all other homotopy groups of $B$ are trivial, and the map $A \to B$ is a $\pi_0$-isomorphism.

For a general map $f : A \to B$ in $s\mathcal{A}lg_\Lambda$, let $C$ be the pushout of the diagram $B' \leftarrow A' \to A(0)'$ obtained by using a functorial construction to replace $A$ by a cofibrant object and the two maps $A \to B$ and $A \to A(0)$ by cofibrations. There is then a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \sim\uparrow & & \uparrow\sim \\ A' & \xrightarrow{f'} & B' \\ \downarrow & & \downarrow \\ A(0)' & \xrightarrow{\Delta_n(f)} & C(n+1) \end{array} \tag{1.5}$$

The bottom map $\Delta_n(f)$ is called the *difference construction of f*. The following can be proved just as in [4, 6.3].

**Proposition 1.2.** *Suppose that $A \to B$ is a map of simplicial commutative algebras which is a $\pi_0$-isomorphism and whose homotopy fibre $F$ is (n-1)-connected. Let $M = \pi_n F$. Then $M$ is naturally a $\Lambda$-module for $\Lambda = \pi_0 B$ and $\Delta_n(f)$ is a map of type $K_\Lambda(M, n+1)$. If $\pi_k F$ vanishes except for $k = n$, then the right-hand square in 1.5 is a homotopy fibre square.*

1.3. **Differentials functor.** For an object $A$ in $\mathcal{A}lg_\Lambda$, define its $\Lambda$-*differentials* to be the $\Lambda$-module

$$D_\Lambda A = J/J^2 \otimes_A \Lambda$$

where $J$ is the kernel of the product $A \otimes A \to A$. As a functor to the category of $\Lambda$-modules, $D_\Lambda$ posseses a right adjoint - the functor

$$(-)_+ : Mod_\Lambda \to \mathcal{A}lg_\Lambda$$

defined by $M_+ = M \oplus \Lambda$ with the usual twisted product

$$(x, a) \cdot (y, b) = (bx + ay, ab).$$

An equivalent identification of the differentials functor

$$D_\Lambda \cong I/I^2 \otimes_A \Lambda, \tag{1.6}$$

where $I$ is the augmentation ideal of $A$, which can be seen to follow from Yoneda's lemma.

The next proposition is proved in [10, §II.5].

**Proposition 1.3.** *The prolonged adjoint pair of functors*

$$D_\Lambda : s\mathcal{A}lg_\Lambda \Longleftrightarrow sMod_\Lambda : (-)_+$$

*induces an adjoint pair on the homotopy categories*

$$\mathbf{L}D_\Lambda : Ho(s\mathcal{A}lg_\Lambda) \Longleftrightarrow Ho(sMod_\Lambda) : \mathbf{R}(-)_+.$$

Finally, the following useful property of the derived functor of differentials follows from [12, 7.3].

**Proposition 1.4.** *If $f : A \to B$ is a $\pi_{\leq n}$-isomorphism, then $\mathbf{L}D_\Lambda(f)$ is a $\pi_{\leq n}$-isomorphism.*

1.4. **Characterizing $K_\Lambda(M, n)$-type.** Fix a $\Lambda$-module $M$. In $sMod_\Lambda$, the fibration $p_n : E(M, n) \to K(M, n)$ is determined by the Dold-Kan correspondence by to correspond to the map of normalized chain complexes $\{M \xrightarrow{1} M\} \to \{M\}$ with the source concentrated in degrees $n$ and $n-1$, the target concentrated in degree $n$, and the map being the identity in degree $n$ and trivial otherwise.

Applying $(-)_+$ to $p_n$ gives a $K_\Lambda(M, n)$-type fibration in $s\mathcal{A}lg_\Lambda$

$$(p_n)_+ : E_\Lambda(M, n) \to K_\Lambda(M, n)$$

which we call the *canonical map of type* $K_\Lambda(M, n)$.

**Proposition 1.5.** *Let $A \to B$ be of type $K_\Lambda(M,n)$ between cofibrant objects in $s\mathcal{A}lg_\Lambda$. Then there is a commuting diagram in $s\mathcal{A}lg_\Lambda$*

$$\begin{array}{ccc} A & \xrightarrow{\sim} & E_\Lambda(M,n) \\ \downarrow & & \downarrow p_n \\ B & \xrightarrow{\sim} & K_\Lambda(M,n) \end{array}$$

*with the horizontal maps being weak equivalences.*

*Proof.* To begin, note that the canonical map $B \to \Lambda$ is (n-1)-connected. Thus the induced map $D_\Lambda B \to 0$ is (n-1)-connected by Proposition 1.4. Let $I = \ker(B \to \Lambda)$. Filtering $B$ by powers of $I$ we note that $B$ cofibrant implies that

$$I^q/I^{q+1} = S_q^\Lambda(I/I^2) \cong S_q^\Lambda(D_\Lambda B)$$

where the last identity always holds when the augmentation is surjective, by (1.6). Thus there is a convergent spectral sequence

$$E^1_{p,q} = H_{p+q}[S_q^\Lambda(D_\Lambda B)] \Longrightarrow \pi_{p+q}B.$$

From the connectivity indicated above and [12, 7.40], $E^1_{p,q} = 0$ for $0 < p+q \leq 2(q-2)+n$. Thus we obtain

$$M \cong \pi_n B \cong \pi_n D_\Lambda B.$$

Thus there is an n-connected map $D_\Lambda B \to K(M,n)$ and its adjoint $B \to K_\Lambda(M,n)$ will be a weak equivalence by the computations above and the assumption that $A \to B$ is of type $K_\Lambda(M,n)$.

Finally, $A \to \Lambda$ is a weak equivalence, hence $D_\Lambda A \to 0$ is a weak equivalence by Proposition 1.4. Since $A$, and hence $D_\Lambda A$, are cofibrant, the composite $D_\Lambda A \to D_\Lambda B \to K(M,n)$ lifts to a map $D_\Lambda A \to E(M,n)$, whose adjoint $A \to E_\Lambda(M,n)$ is necessarily a weak equivalence. □

1.5. **Proof of Theorem A.** Fix an object $A$ in $s\mathcal{A}lg_\Lambda$. We will show, by induction, that there is a map $X \to Y$ in $s_\Lambda\mathcal{A}lg_\Lambda$ and a commutative diagram in $\mathrm{Ho}(s\mathcal{A}lg_\Lambda)$

$$\begin{array}{ccc} A(n) & \xrightarrow{\sim} & X \\ q_n \downarrow & & \downarrow \\ A(n-1) & \xrightarrow{\sim} & Y \end{array} \tag{1.7}$$

with the horizontal maps being equivalences. It is clear for $n = 0$ as $A(0) \to \Lambda$ is a weak equivalence.

Using 1.5, some closed model category theory and induction, we may assume that there is a trivial fibration $\sigma : A(n-1)' \to Y$ with the target $Y$ a cofibrant object in $s_\Lambda\mathcal{A}lg_\Lambda$.

**Lemma 1.6.** *Let $M = \pi_n A$. Then there is a commuting diagram in $Ho(s\mathcal{A}lg_\Lambda)$ of the form*

$$\begin{array}{ccc} A(n-1)' & \longrightarrow & C(n+1) \\ \sim\downarrow \sigma & & \downarrow\sim \\ Y & \longrightarrow & K_\Lambda(M,n+1) \end{array}$$

*with the top arrow from 1.5.*

*Proof.* First, note that since $\sigma : A(n-1)' \to Y$ is a trivial fibration between suitably cofibrant objects (see above) it follows from that and from 1.6 that

$$D_\Lambda \sigma : D_\Lambda A(n-1)' \to D_\Lambda Y$$

is a trivial fibration between cofibrant objects in $sMod_\Lambda$. By [10, I.1.7], $D_\Lambda \sigma$ has a homotopy left inverse $i$ ($i \circ D_\Lambda \sigma \simeq \mathrm{Id}_{D_\Lambda A(n-1)}$).

Next, utilizing Lemma 1.5, let $t : A(n-1)' \to K_\Lambda(M, n+1)$ be the composite of $A(n-1)' \to C(n+1) \to K_\Lambda(M, n+1)$. Let $w : D_\Lambda Y \to K(M, n+1)$ be the composite $(D_\Lambda t) \circ i$. Then $w \circ D_\Lambda \sigma \simeq D_\Lambda t$ and the result now follows from Proposition 1.3. □

From the previous lemma, we may form the homotopy pullback diagram in $s_\Lambda \mathcal{A}lg_\Lambda$

$$\begin{array}{ccc} X & \longrightarrow & E_\Lambda(M, n+1) \\ \downarrow & & \downarrow (p_n)_+ \\ Y & \longrightarrow & K_\Lambda(M, n+1). \end{array} \tag{1.8}$$

By Proposition 1.2, the diagram below is also a homotopy pullback in $s\mathcal{A}lg_\Lambda$

$$\begin{array}{ccc} A(n)' & \longrightarrow & A(0)' \\ q_n' \downarrow & & \downarrow \Delta[q_n] \\ A(n-1)' & \longrightarrow & C(n+1). \end{array} \tag{1.9}$$

By Proposition 1.5 and Lemma 1.6, there is an induced map of diagrams 1.9 to 1.8 in the category $\mathrm{Ho}(s\mathcal{A}lg_\Lambda)$. Since fibrations and pullbacks in $s\mathcal{A}lg_\Lambda$ are fibrations and pullbacks as simplicial groups, a computation of homotopy groups can be performed utilizing Lemma 1.1 to show that the induced map $A(n)' \to X$ is a weak equivalence. This completes the induction step.

## 2. André-Quillen homology and Theorems B and C

2.1. **Base change property of André-Quillen homology.** Recall that the *cotangent complex* of a simplicial $R$-algebra $A$ is defined to be the object of $\mathrm{Ho}(Mod_A)$

$$\mathcal{L}(A|R) := \Omega_{P|R} \otimes_P A \tag{2.10}$$

where the $T$-module $\Omega_{T|S} = J/J^2$, $J = \ker(T \otimes_S T \to T)$, denotes the *Kahler differentials* of an $S$-algebra $T$, and $P \to A$ is a cofibrant replacement of $A$ as a simplicial $R$-algebra.

**Note:** As in §1.3, $\Omega_{T|S}$ is left adjoint to the functor $M \mapsto M \oplus T$ where the image has a $T$-algebra structure with $M^2 = 0$.

Also recall that given another simplicial $R$-algebra $B$, the *derived tensor product* of $A$ and $B$ to be the object of $\mathrm{Ho}(sMod_R)$

$$A \otimes_R^{\mathbf{L}} B := P \otimes_R Q$$

where $Q \to B$ is a cofibrant replacement of $B$.

We now derive a base change property for the cotangent complex following [12].

**Lemma 2.1.** *If* $\operatorname{Tor}_q^R(A_k, B_k) = 0$ *for all* $k \geq 0$ *and all* $q > 0$ *then* $A \otimes_R^{\mathbf{L}} B \simeq A \otimes_R B$.

*Proof.* This follows immediately from the spectral sequence [10, §II.6]

$$E^2_{p,q} = \pi_p \operatorname{Tor}_q^R(A, B) \Longrightarrow \pi_{p+q}(A \otimes_R^{\mathbf{L}} B).$$

□

**Lemma 2.2.** $\Omega_{A\otimes_R B|B} \cong \Omega_{A|R} \otimes_R B$

*Proof.* Let $A' = A \otimes_R B$ and fix an $A'$-module $M$. Then

$$\begin{aligned} \hom_{A'}(\Omega_{A'|B}, M) &\cong \hom_{{}_B Alg_{A'}}(A', M \oplus A') \\ &\cong \hom_{{}_R Alg_A}(A, M \oplus A) \\ &\cong \hom_A(\Omega_{A|R}, M) \\ &\cong \hom_{A'}(\Omega_{A|R} \otimes_R B, M). \end{aligned}$$

The result now follows from Yoneda's lemma. □

**Proposition 2.3.** $\mathcal{L}(A \otimes_R^{\mathbf{L}} B|B) \simeq \mathcal{L}(A|R) \otimes_R^{\mathbf{L}} B$

*Proof.* Fix cofibrant replacements $P$ and $Q$ for $A$ and $B$, respectively. Then

$$\mathcal{L}(A \otimes_R^{\mathbf{L}} B|B) = \Omega_{P\otimes_R Q|Q} \cong \Omega_{P|R} \otimes_R Q \tag{2.11}$$

by Lemma 2.2. Since $P$ is projective as a simplicial $R$-module then $\Omega_{P|R}$ is a projective $P$-module. Thus, by Lemma 2.1, the map $\Omega_{P|R} \xrightarrow{\sim} \Omega_{P|R} \otimes_P A$ is a weak equivalence. Since $Q$ is projective, Lemma 2.1 further tells us that

$$\Omega_{P|R} \otimes_R Q \xrightarrow{\sim} (\Omega_{P|R} \otimes_P A) \otimes_R Q \cong \mathcal{L}(A|R) \otimes_R^{\mathbf{L}} B \tag{2.12}$$

is a weak equivalence. The result now follows by combining 2.11 with 2.12. □

**Corollary 2.4.** *As a functor of* $A \otimes_R B$*-modules,* $D_*(A \otimes_R^{\mathbf{L}} B|B; -) \cong D_*(A|R; -)$.

*Proof.* This follows from Proposition 2.3 and the identity $D_*(T|S; M) := \pi_*[\mathcal{L}(T|S) \otimes_T M]$. □

2.2. **Proof of Theorem B.** We first recall the main result of [13].

**Theorem 2.5.** *Let* $A$ *be a homotopy connected simplicial supplemented commutative algebra over a field* $\ell$ *of non-zero characteristic. Then* $D_s(A|\ell; \ell) = 0$ *for* $s \gg 0$ *implies that there is an equivalence* $S_\ell(D_1(A|\ell; \ell), 1) \cong A$ *in the homotopy category.*

We now begin by establishing a special case of Theorem A. To that end let $A$ be a simplicial commutative $R$-algebra and assume that the unit $R \to \pi_0 A = \Lambda$ is a surjection. For $\wp \in \operatorname{Spec}\Lambda$, define the *connected component of* $A$ *at* $\wp$ to be the connected simplicial supplemented $k(\wp)$-algebra

$$A(\wp) = A \otimes_R^{\mathbf{L}} k(\wp).$$

**Lemma 2.6.** *Let* $A$ *be as above. Then*

1. $D_*(A|R; k(\wp)) \cong D_*(A(\wp)|k(\wp); k(\wp))$*, and*

2. *if $A$ also has finite Noetherian homotopy and* $\mathrm{fd}_R(\pi_* A) < \infty$ *it follows that* $A(\wp)$ *has finite Noetherian homotopy.*

*Proof.* 1. follows from Corollary 2.4. For 2., [10, §II.6] gives a spectral sequence

$$E^2_{s,t} = \mathrm{Tor}^R_s(\pi_t A, k(\wp)) \Longrightarrow \pi_{s+t}(A \otimes^{\mathbf{L}}_R k(\wp)).$$

From the finiteness conditions, each $E^2_{s,t}$ is a finite $k(\wp)$-module and vanishes for $s, t \gg 0$. Thus $A \otimes^{\mathbf{L}}_R k(\wp))$ has finite Noetherian homotopy. □

**Corollary 2.7.** *Let $A$ be as in Lemma 2.6.2 and further assume that* $\mathrm{char}(k(\wp)) \neq 0$. *Then $D_s(A|R; k(\wp)) = 0$ for $s \gg 0$ implies that $D_s(A|R; k(\wp)) = 0$ for $s \geq 2$.*

*Proof.* This follows from Lemma 2.6 and Theorem 2.5. □

Now assume that the simplicial algebra $A$ in question is a homotopy connected simplicial supplemented $\Lambda$-algebra, by Theorem A. We further assume that $A$ has Noetherian homotopy.

Fix $\wp \in \mathrm{Spec}\,\Lambda$ and let $\widehat{(-)}$ denote the completion functor on $R$-modules at $\wp$. Define the homotopy connected simplicial supplemented $\widehat{\Lambda}$-algebra $A'$ by

$$A' = A \otimes^{\mathbf{L}}_\Lambda \widehat{\Lambda}.$$

**Proposition 2.8.** *Suppose $A$ is a simplicial commutative $R$-algebra, with $R$ a Noetherian ring. Then $\pi_* A' \cong \widehat{\pi_* A}$ and there exists a (complete) Noetherian $R'$ that fits into the following commutative diagram in* $\mathrm{Ho}(s_R\mathcal{A}lg)$

$$\begin{array}{ccc} R & \xrightarrow{\eta} & A \\ \phi\downarrow & & \downarrow\psi \\ R' & \xrightarrow{\eta'} & A' \end{array}$$

*with the following properties:*

1. *$\phi$ is a flat map and its closed fibre $R'/\wp R'$ is weakly regular;*
2. *$\psi$ is a $D_*(-|R; k(\wp))$-isomomorphism;*
3. *$\eta'$ induces a surjection $\eta'_* : R' \to \pi_0 A'$;*
4. $\mathrm{fd}_R(\pi_* A)$ *finite implies that* $\mathrm{fd}_{R'}(\pi_* A')$ *is finite*

*Proof:* First, Quillen's spectral sequence [10, II.6] $\mathrm{Tor}^\Lambda_*(\pi_* A, \widehat{\Lambda}) \Longrightarrow \pi_* A'$ collapses to give the first result since $\widehat{\Lambda}$ is flat over $\Lambda$ and each $\pi_m A$ is finite over $\Lambda$ [9, 8.7 and 8.8].

Next, by [3, 1.1], the unit ring homomorphism $R \to \widehat{\Lambda}$ factors as $R \xrightarrow{\phi} R' \xrightarrow{\eta'_*} \widehat{\Lambda}$ with $\phi$ having the properties described in 1. and $\eta'_*$ is a surjection. Thus the induced map $\eta' : R' \to A'$ induces a surjection on $\pi_0$, giving 3., and the desired diagram commutes.

Now, by the transitivity sequence [12, 4.12] applied to $R \to A \to A'$, 2. follows from the isomorphism

$$D_*(A'|A; k(\wp)) \cong D_*(\widehat{\Lambda}|\Lambda; k(\wp)) \cong 0$$

which follows from Corollary 2.4.

Finally, 4. follows from [3, 3.2], as $A$ has Noetherian homotopy. □

Now, let $A$ have finite Noetherian homotopy with $D_s(A|R;-) = 0$ for $s \gg 0$. From Proposition 2.8, Theorem 2.5, Corollary 2.7, and [1, §S.30], if $\mathrm{fd}_R(\pi_* A) < \infty$ then $A(\wp) \cong S_{k(\wp)}(D_1(A|R;k(\wp),1)$, for each $\wp \in \mathrm{Spec}(\pi_0 A)$, if and only if $D(A|R;-) = 0$. Thus Theorem B follows from the definition of locally homotopy complete intersection (see introduction) and a transitivity sequence argument.

2.3. **Proof of Theorem C.** Let $A$ be a simplicial commutative $R$-algebra with Noetherian homotopy. It follows from Lemma 2.6.1, Proposition 2.8, and [1, §S.30], that $D_{\geq 3}(A|R;-) = 0$ if and only if $D_{\geq 3}(A(\wp)|k(\wp);k(\wp)) = 0$, for all $\wp \in \mathrm{Spec}(\pi_0 A)$. From the definition of locally virtual homotopy complete intersection (see introduction), Theorem C will follow if we can show that, for each prime ideal $\wp$, $A(\wp) \cong S_\bullet(D_{\leq 2}(A|R;k(\wp)))$ in the homotopy category. But this in turn follows from [13, (2.2)].

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