

ON SIMPLICIAL COMMUTATIVE ALGEBRAS WITH FINITE ANDRÉ-QUILLEN HOMOLOGY

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ABSTRACT. In [22, 4] a conjecture was posed to the effect that if $R \rightarrow A$ is a homomorphism of Noetherian rings then the André-Quillen homology on the category of A -modules satisfies: $D_s(A|R; -) = 0$ for $s \gg 0$ implies $D_s(A|R; -) = 0$ for $s \geq 3$. In [28], an extended version of this conjecture was considered for which A is a simplicial commutative R -algebra with Noetherian homotopy such that $\text{char}(\pi_0 A) \neq 0$. In addition, a homotopy characterization of such algebras was described. The main goal of this paper is to develop a strategy for establishing this extended conjecture and provide a complete proof when R is Cohen-Macaulay of characteristic 2.

OVERVIEW

In [22], D. Quillen presented his viewpoint on the homology of algebras which extended, in the commutative case, the work of Lichtenbaum and Schlessinger and gave M. André's notion of homology. Furthermore, he observed that strong vanishing of this André-Quillen homology for finite type algebras held only when such algebras possessed the complete intersection property and conjectured that a weaker type of vanishing also characterized such algebras. In [4], L. Avramov clarified and extended Quillen's conjectures in the following manner. Let $f : R \rightarrow A$ be a homomorphism of Noetherian rings [Note: unless otherwise noted, all rings and algebras from this point on are commutative with unit]. Then f is a *locally complete intersection* provided for each $\mathfrak{q} \in \text{Spec } S$ the semi-completion $R_{\mathfrak{q} \cap R} \rightarrow \hat{A}_{\mathfrak{q}}$ suitably factors through a surjection with kernel being generated by a regular sequence (see below for more details).

Quillen's Conjecture: (see [4, 22]) *Let $R \rightarrow A$ be a homomorphism of Noetherian rings such that the André-Quillen homology satisfies $D_s(A|R; -) = 0$ (as functors of A -modules) for $s \gg 0$. Then*

- (1) $D_s(A|R; -) = 0$ for $s \geq 3$;
- (2) if $\text{fd}_R A < \infty$ then $R \rightarrow A$ is a *locally complete intersection* (and, hence, $D_s(A|R; -) = 0$ for $s \geq 2$).

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Part 2 of this conjecture was proved by Avramov in [4]. Part 1 was proved by Avramov and S. Iyengar for algebra retracts in [6].

Following ideas of Haynes Miller, an alternate approach to proving this conjecture was taken in [26, 27] when R is a field by viewing it as a special case of an algebraic version of a theorem of J.P. Serre [25]. Following this line of thinking, in [27, 28] the more general consideration of Noetherian algebras was extended to *simplicial commutative algebras with Noetherian homotopy*, that is, simplicial commutative algebras A such that $\pi_0 A$ is Noetherian and $\pi_* A$ is a finite graded $\pi_0 A$ -module. In using André-Quillen homology to analyse such, we can use the type of tools first clarified by André and Quillen: flat base change, transitivity sequence, localization etc. Cf. [1, 22, 28]. A particularly useful method for analysing simplicial commutative algebras in our present context through homology is the following generalization of the main result in [5], proved in [28]. For each $\wp \in \text{Spec}(\pi_0 A)$ the simplicial commutative algebra $A' = A \otimes_{\pi_0 A}^{\mathbf{L}} (\widehat{\pi_0 A})_{\wp}$ there is a complete local ring R' and a homotopy commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\eta} & A \\ \phi \downarrow & & \downarrow \psi \\ R' & \xrightarrow{\eta'} & A' \end{array}$$

with the following properties:

- (1) ϕ is a flat map and its closed fibre $R'/\wp R'$ is weakly regular;
- (2) ψ is a $D_*(-|R; k(\wp))$ -isomorphism;
- (3) η' induces a surjection $\eta'_* : R' \rightarrow (\pi_0 A', k(\wp))$ of local rings;
- (4) $\text{fd}_R(\pi_* A)$ finite implies that $\text{fd}_{R'}(\pi_* A')$ is finite

We call such a diagram a *homotopy factorization* of A . We can use such factorizations to extend the notion of locally complete intersection to simplicial commutative R -algebras with Noetherian homotopy. Specifically, we call such A a *locally homotopy n -intersection*, n a natural number, provided for each $\wp \in \text{Spec}(\pi_0 A)$ there is a factorization such that the *connected component at \wp* satisfies

$$A(\wp) := A' \otimes_{R'}^{\mathbf{L}} k(\wp) \simeq S_{k(\wp)}(W)$$

with W a connected simplicial $k(\wp)$ -module satisfying $\pi_s W = 0$ for $s > n$. Here and throughout $S_{k(\wp)}(-)$ denotes the free commutative $k(\wp)$ -algebra functor.

We can now state, inspired by Serre's theorem [25], our simplicial version of Quillen's conjecture:

Vanishing Conjecture: *Let R be a Noetherian ring and let A be a simplicial commutative R -algebra with finite Noetherian homotopy and $\text{char}(\pi_0 A) \neq 0$ such that the André-Quillen homology satisfies $D_s(A|R; -) = 0$ (as functors of $\pi_0 A$ -modules) for $s \gg 0$. Then*

- (1) A is a locally homotopy 2-intersection;
- (2) if $\text{fd}_R \pi_* A < \infty$ then A is a locally homotopy 1-intersection.

Part 2 of the Vanishing Conjecture was proved in [27, 28]. The goal of the first part of this paper is to outline a strategy for giving a proof for the whole Vanishing Conjecture. The strategy involves formulating a more local version of the Algebraic Serre Theorem proved in [27] used to prove part 2 of the Vanishing Conjecture. Specifically, we will analyse the behavior of homotopy operations on each $A(\wp)$, particular the divided p^{th} -powers and the *André operation* (so named because of the role it played in [3] which motivated the direction of this paper). In the first section, we will formulate a Nilpotence and Non-nilpotence Conjecture regarding the action of these operations which, when coupled together, imply the Vanishing Conjecture. The second section will then focus on proving the Nilpotence Conjecture at the prime 2. Finally, in the third section we will establish what will hopefully be our first step toward proving the Vanishing Conjecture when $\text{char}(\pi_0 A) = 2$. Specifically, we will establish our:

Main Theorem: *Let A be a simplicial commutative R -algebra with finite Noetherian homotopy such that R is Cohen-Macaulay of characteristic 2. Then $D_s(A|R; -) = 0$ (as a functor of $\pi_0 A$ -modules) for $s \gg 0$ if and only if A is a locally homotopy 2-intersection.*

As an immediate consequence, we obtain:

Corollary. *Let $R \rightarrow A$ be a homomorphism of Noetherian rings of characteristic 2 such that R is Cohen-Macaulay. Then $D_s(A|R; -) = 0$ for $s \gg 0$ implies $D_s(A|R; -) = 0$ for $s \geq 3$.*

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1. NILPOTENCE CONJECTURES

In this section we reformulate the Vanishing Conjecture in terms of a two part Nilpotence Conjecture which shifts the burden for global vanishing of André-Quillen homology to local vanishing of operations acting on the homotopy of components. We will first need a weaker notion of homotopy factorization in order to tighten our grip on how on how information from the homotopy of our simplicial algebra is transferred to the homotopy of its components. We will, throughout this section, be assuming basic properties of André-Quillen homology, referring the reader to [28] for details.

1.1. Weak homotopy factorizations. In the next subsection, we will recall that the conclusions of the Vanishing Conjectures are equivalent to certain strong global vanishing properties of André-Quillen homology. Our goal at present is to modify the notion of homotopy factorizations which suitably preserves the André-Quillen homology but puts a tighter control on the local ring R' .

Let A be a simplicial commutative R -algebra and denote $\pi_0 A$ by Λ . We may assume that A is a simplicial commutative Λ -algebra. Cf. [28, Theorem A]. Fix $\wp \in \text{Spec } \Lambda$

and let $\widehat{(-)}$ denote the completion functor on R -modules at \wp . Define the homotopy connected simplicial supplemented $\widehat{\Lambda}$ -algebra A' by

$$A' = A \otimes_{\Lambda}^{\mathbf{L}} \widehat{\Lambda}.$$

Proposition 1.1. *Suppose A is a simplicial commutative R -algebra with R a Noetherian ring. Then there exists a (complete) local ring (R'', \mathfrak{m}) , a simplicial commutative R'' -algebra A'' , and a homotopy commutative diagram*

$$(1.1) \quad \begin{array}{ccc} R & \xrightarrow{\eta} & A \\ \phi \downarrow & & \downarrow \psi \\ R'' & \xrightarrow{\eta''} & A'' \end{array}$$

with the following properties:

- (1) ϕ is a complete intersection at \mathfrak{m} ;
- (2) $\text{depth}(\mathfrak{m}) = 0$;
- (3) $D_{\geq 2}(A|R; k(\wp)) \cong D_{\geq 2}(A''|R''; k(\wp))$;
- (4) η'' induces a surjection of local rings $\eta''_*: R'' \rightarrow \pi_0 A''$;
- (5) If A has finite Noetherian homotopy then A'' has finite Noetherian homotopy.

Proof: Choose a homotopy factorization of A over \wp

$$\begin{array}{ccc} R & \xrightarrow{\eta} & A \\ \phi \downarrow & & \downarrow \psi \\ R' & \xrightarrow{\eta'} & A' \end{array}$$

which exists by [28, (2.8)].

Next, let \mathfrak{q} be the maximal ideal of R' . Let x_1, \dots, x_r be a maximal R' -subsequence of a minimal generating set for \mathfrak{q} . We define

$$R'' = R'/(x_1, \dots, x_r).$$

Then $\mathfrak{m} = \mathfrak{q}/(x_1, \dots, x_r)\mathfrak{q}$ has depth 0 since it contains only zero divisors. Furthermore, the composite $R_{\wp} \rightarrow R' \rightarrow R''$ is a complete intersection at \mathfrak{m} by definition. Cf. [4].

Now, let $A'' = A' \otimes_{R'}^{\mathbf{L}} R''$. Then

$$D_{\geq 2}(A|R; k(\wp)) \cong D_{\geq 2}(A'|R; k(\wp)) \cong D_{\geq 2}(A'|R'; k(\wp)) \cong D_{\geq 2}(A' \otimes_{R'}^{\mathbf{L}} R''|R''; k(\wp))$$

which follows from the properties of homotopy factorizations, the transitivity sequence, and flat base change [28, (2.4)]. Applying π_0 to the map $R'' \rightarrow A' \otimes_{R'}^{\mathbf{L}} R''$ gives the map $R'' \cong R' \otimes_{R'} R'' \rightarrow \pi_0(A') \otimes_{R'} R''$ which is a surjection. Thus $R'' \rightarrow \pi_0 A''$ is a surjection.

Finally, if A has finite Noetherian homotopy then so does A' (since $\pi_* A' \cong \widehat{\pi_* A}$). By [21, §II.6], there is a Kunnetth spectral sequence

$$E_{s,t}^2 = \text{Tor}_s^{R'}(\pi_t A', R'') \implies \pi_{s+t} A''.$$

Since $R' \rightarrow R''$ is a complete intersection, $\text{fd}_{R'} R'' < \infty$. Thus $\pi_* A''$ will be a finite module over $\pi_0 A'' \cong \widehat{\Lambda} \otimes_{R'} R''$. \square

We will call a diagram (1.1) satisfying the conditions (1) - (5) above a *weak homotopy factorization* for A .

1.2. Brief review of the homotopy of simplicial commutative algebras over a field. Let A be a simplicial commutative ℓ -algebra where ℓ is a field. In this section we review some basic facts about the homotopy groups of such objects, computed as the homotopy groups of simplicial ℓ -modules.

Let \mathcal{A}_ℓ be the category of *supplemented ℓ -algebras*, i.e. commutative ℓ -algebras augmented over ℓ . Let $s\mathcal{A}_\ell$ be the category of simplicial objects over \mathcal{A}_ℓ . Then for $A \in s\mathcal{A}_\ell$ and $n \geq 0$ we have a natural isomorphism

$$\pi_n A \cong [S_\ell(n), A]_{\text{Ho}(s\mathcal{A}_\ell)}$$

where $S_\ell(n) = S_\ell(K(n))$, $K(n)$ the simplicial ℓ -module satisfying $\pi_* K(n) \cong \ell$ concentrated in degree n . We will use this relation to determine the natural primary algebra structure on $\pi_* A$.

Given integers $r_1, \dots, r_m, t_1, \dots, t_n \neq 0$ an *multioperation* of degree $(r_1, \dots, r_m; t_1, \dots, t_n)$ is a natural map

$$\theta : \pi_{r_1} \times \dots \times \pi_{r_m} \rightarrow \pi_{t_1} \times \dots \times \pi_{t_n}$$

of functors on $s\mathcal{A}_\ell$. Let $\text{Nat}_{r_1, \dots, r_m; t_1, \dots, t_n}$ be the set of multioperations of degree $(r_1, \dots, r_m; t_1, \dots, t_n)$. It is straightforward to show that

$$\text{Nat}_{r_1, \dots, r_m; t_1, \dots, t_n} \cong \text{Nat}_{r_1, \dots, r_m; t_1} \times \dots \times \text{Nat}_{r_1, \dots, r_m; t_n}.$$

Now, we define

$$(1.2) \quad f : \text{Nat}_{r_1, \dots, r_m; t} \rightarrow \pi_t(S_\ell(r_1) \otimes_\ell \dots \otimes_\ell S_\ell(r_m))$$

as follows. Let $\mathcal{N} = \text{Nat}_{r_1, \dots, r_m; t}$ and let $X = S_\ell(r_1) \otimes_\ell \dots \otimes_\ell S_\ell(r_m)$. For each $1 \leq j \leq m$, let $\iota_j \in \pi_{r_j} X$ be the homotopy class of the inclusion $S_\ell(r_j) \rightarrow X$. Given $\theta \in \mathcal{N}$ there is an induced map

$$\theta_X : \pi_{r_1} X \times \dots \times \pi_{r_m} X \rightarrow \pi_t X.$$

Thus we can define $f : \mathcal{N} \rightarrow \pi_t X$ by

$$(1.3) \quad f(\theta) = \theta_X(\iota_1, \dots, \iota_m).$$

Proposition 1.2. $\text{Nat}_{r_1, \dots, r_m; t} \cong \pi_t(S_\ell(r_1) \otimes_\ell \dots \otimes_\ell S_\ell(r_m))$.

Proof: Since we have

$$\pi_{r_1} \times \dots \times \pi_{r_m} \cong [S_\ell(r_1) \otimes_\ell \dots \otimes_\ell S_\ell(r_m), -]_{\text{Ho}(s\mathcal{A}_\ell)}$$

the result follows from Yoneda's lemma [18]. □

Note:

(1) There is an obvious map

$$\text{Nat}_{r_1, \dots, r_m; t} \times \text{Nat}_{t; q} \rightarrow \text{Nat}_{r_1, \dots, r_m; q}$$

induced by composition.

(2) Nat is naturally an ℓ -module and f is naturally a linear map.

We now can address the issue of understanding possible relations among multioperations.

Corollary 1.3. *For $\theta \in \text{Nat}_{r_1, \dots, r_m; t}$ then any expression for θ in $\text{Nat}_{r_1, \dots, r_m; q}$ is determined by $f(\theta) \in \pi_t(S_\ell(r_1) \otimes_\ell \dots \otimes_\ell S_\ell(r_m))$. Furthermore, if $\psi \in \text{Nat}_{t; q}$ then $f(\psi \circ \theta) = f(\psi) \circ f(\theta)$, as composites of their homotopy representatives, in $\pi_t(S_\ell(r_1) \otimes_\ell \dots \otimes_\ell S_\ell(r_m))$.*

Proof: This again follows from Yoneda's lemma [18]. \square

Now, we are in a position to determine the full natural primary structure for homotopy in $s\mathcal{A}_\ell$. First, recall that for any field \mathbb{F} we have

$$(1.4) \quad S_{\mathbb{F}}(V \oplus W) \cong S_{\mathbb{F}}(V) \otimes S_{\mathbb{F}}(W).$$

Next, let $k = \mathbb{Q}$ if $\text{char}(\ell) = 0$ and let $k = \mathbb{F}_p$ if $\text{char}(\ell) = p \neq 0$. We seek a natural map of ℓ -algebras

$$\phi_V : S_\ell(V \otimes_k \ell) \rightarrow S_k(V) \otimes_k \ell$$

where V is a k -module. This can be defined as the adjunction of the inclusion $V \otimes_k \ell \rightarrow I(S_k(V) \otimes_k \ell)$ (here $I : \mathcal{A}_\ell \rightarrow \mathcal{V}_\ell$ is the augmentation ideal functor).

Proposition 1.4. *The natural map $\phi : S_\ell((-) \otimes_k \ell) \rightarrow S_k(-) \otimes_k \ell$ is an isomorphism of functors from k -modules to \mathcal{A}_ℓ .*

Proof: By the identity (1.4) and naturality, it is enough to provide a proof for one dimensional V , i.e. for $V \cong k\langle x \rangle$. Then $\phi_V : \ell[x] \rightarrow k[x] \otimes_k \ell$ is determined algebraically by the value $\phi_V(x) = x \otimes_k 1$. This is clearly an isomorphism. \square

Note: This and other similar results can also be shown to follow from the faithful flatness of the functor $(-) \otimes_k \ell$.

Corollary 1.5. *For $V \in s\mathcal{V}_k$ there is a natural isomorphism*

$$\pi_*(S_\ell(V \otimes_k \ell)) \cong \pi_*(S_k(V)) \otimes_k \ell.$$

As a consequence all natural primary homotopy operations for simplicial supplemented ℓ -algebras and their relations are determined by $\pi_ S_k(n)$ for all $n \in \mathbb{N}$.*

Proof: The first statement follows from Proposition 1.4 and the faithful flatness of $(-) \otimes_k \ell$. The second statement follows additionally from Corollary 1.3 and the Kunnet theorem. Recall that $S_\ell(n) \cong S_\ell(K_\ell(n))$ and we can take $K_\ell(n) = \ell\langle S^n \rangle \cong k\langle S^n \rangle \otimes_k \ell$, where S^n is a choice of simplicial set model for the n -sphere. \square

Note: The computation of $\pi_* S_{\mathbb{Q}}(n)$ can be traced back to at least as early as [9]. The computation of $\pi_* S_{\mathbb{F}_p}(n)$ can be found in [8, 9], for general p , and in [11] for $p = 2$. We will review the results of [11] in the next section.

For non-zero characteristics, we will be interested in two particular operations. Specifically, for $A \in s\mathcal{A}_\ell$, $\pi_* A$ is naturally a divided power algebra. Therefore, there is a *divided p^{th} -power operation*

$$\gamma_p : \pi_n A \rightarrow \pi_{pn} A.$$

Cartan, Bousfield, and Dwyer also construct an operation

$$\vartheta : \pi_n A \rightarrow \pi_{(p-1)n+1} A$$

which we call the *André operation* because of the role it played in [3] where M. André's showed that Gulliksen's result about the equality of deviations with simplicial dimensions for rational local rings [15] cannot be extended to the primary case. In the notion of [8, 8.8] and [11],

$$(1.5) \quad \vartheta = \begin{cases} \delta_{n-1} & p = 2, \\ \nu_{(n-1)/2} & p > 2. \end{cases}$$

A useful basic relation between the two operations is

$$(1.6) \quad \vartheta \gamma_p = 0$$

1.3. Nilpotency conjectures and consequences. We now are in a position to address the Vanishing Conjecture and reformulate it in terms of local conditions on homotopy groups. To begin, we need the following

Lemma 1.6. *Let $W \in s\mathcal{V}_\ell$, with $\text{char}(\ell) > 0$, and let $n \in \mathbb{N}$ be so that $\pi_j W \neq 0$ implies $n \geq j \geq 1$. Then*

- (1) $\gamma_p = 0$ on $\pi_* S_\ell(W)$ provided $n = 1$;
- (2) $\vartheta = 0$ on $\pi_* S_\ell(W)$ provided $n = 2$.

Proof: By Corollary 1.5, it is enough to provide a proof for $\ell = \mathbb{F}_p$. For $n = 1$, $\pi_* S_\ell(W)$ is a free exterior algebra generated by $\pi_1 W$, which has trivial γ_p -action. For $n = 2$, $\pi_* S_\ell(W)$ is a free divided power algebra generated by $\pi_* W$. Cf. [9]. Thus $\pi_* S_\ell(W)$ has trivial ϑ -action by relation (1.6). \square

Given $A \in s\mathcal{A}_\ell$ with $\text{char}(\ell) > 0$, we call A

- (1) Γ -nilpotent provided $\gamma_p^s(x) = 0$ for $s \gg 0$ for all $x \in \pi_* A$ and
- (2) André nilpotent provided $\vartheta^s(x) = 0$ for $s \gg 0$ for all $x \in \pi_* A$.

Next, given a Noetherian ring R and a simplicial commutative R -algebra A with Noetherian homotopy. For $\wp \in \text{Spec}(\pi_0 A)$ with $\text{char}(k(\wp)) > 0$, we call A

- (1) Γ -nilpotent at \wp provided there is a homotopy factorization at \wp such that $A(\wp)$ is Γ -nilpotent over $k(\wp)$, and
- (2) André nilpotent at \wp provided there is a weak homotopy factorization at \wp such that $A'' \otimes_{R''} k(\wp)$ is André nilpotent over $k(\wp)$.

Proposition 1.7. *Let A be a simplicial commutative R -algebra with Noetherian homotopy and $\wp \in \text{Spec}(\pi_0 A)$ such that $\text{char}(k(\wp)) > 0$. Then*

- (1) A is Γ -nilpotent at \wp provided A is a homotopy 1-intersection at \wp ;
- (2) A is André-nilpotent at \wp provided A is a homotopy 2-intersection at \wp .

Proof: Both follow from the definitions and Lemma 1.6. \square

We now can state our two nilpotence-type conjectures.

Nilpotence Conjecture: *Let A be a simplicial commutative R -algebra with finite Noetherian homotopy. Let $\wp \in \mathrm{Spec}(\pi_0 A)$ be such that $\mathrm{char}(k(\wp)) > 0$ and $D_s(A|R; k(\wp)) = 0$ for $s \gg 0$. Then:*

- (1) *A is Γ -nilpotent at \wp if and only if A is a homotopy 1-intersection at \wp ;*
- (2) *A is André nilpotent at \wp if and only if A is a homotopy 2-intersection at \wp .*

Non-Nilpotence Conjecture: *Let A be a simplicial commutative R -algebra with finite Noetherian homotopy. Let $\wp \in \mathrm{Spec}(\pi_0 A)$ be such that $\mathrm{char}(k(\wp)) > 0$. Then $D_s(A|R; k(\wp)) \neq 0$ for infinitely many $s \in \mathbb{N}$ provided A fails to be André nilpotent at \wp .*

Remark: A motivation for the Nilpotence Conjecture came from dual topological results centered around conjectures of Serre and Sullivan as addressed in [20, 16, 14]. See also [26] for further speculations on formulating other dual results.

Assuming both of these conjectures, we can now provide a:

Proof of the Vanishing Conjecture: First, we have $D_s(A|R; k(\wp)) = 0$ for $s \gg 0$. Since $\mathrm{char}(\pi_0 A) > 0$ then $\mathrm{char}(k(\wp)) > 0$. Thus, by the Non-Nilpotence Conjecture, A is André nilpotent at \wp . By the Nilpotence Conjecture, A is a homotopy 2-intersection. If, additionally, $\mathrm{fd}_R(\pi_* A) < \infty$ it follows that $\pi_*(A' \otimes_{R'} k(\wp))$ is finite and, hence, Γ -nilpotent. It follows from the Nilpotence Conjecture that A is a homotopy 1-intersection at \wp . \square

2. PROOF OF THE NILPOTENCE CONJECTURE AT THE PRIME 2

The goal of this section will be to provide a proof of the Nilpotence Conjecture when the base field has characteristic 2. This will involve a careful study of a certain map, the character map, defined on the homotopy of simplicial supplemented algebras with finite André-Quillen homology, whose non-triviality implies the Nilpotence Conjecture. In fact, in the process of analysing this character map, we will be able to establish an upper bound on the top non-trivial degree of the André-Quillen homology in terms of the non-nilpotence of certain operations acting on homotopy. Along the way we will review some results from [11] and [12] and generalize them to arbitrary fields of characteristic 2.

2.1. Connected envelopes and the character map. We close this section by providing a strategy for proving the Nilpotence Conjecture. This will involve first reviewing the concept of connected envelopes from [27]. We then construct the notion of a character map for connected simplicial supplemented algebras with finite André-Quillen homology and state a conjecture regarding this map whose validity implies the Nilpotence Conjecture.

Given A in $s\mathcal{A}_\ell$, which is connected, we define its **connected envelopes** to be a sequence of cofibrations

$$A = A(1) \xrightarrow{j_1} A(2) \xrightarrow{j_2} \dots \xrightarrow{j_{n-1}} A(n) \xrightarrow{j_n} \dots$$

with the following properties:

- (1) For each $n \geq 1$, $A(n)$ is a $(n - 1)$ -connected.
- (2) For $s \geq n$,

$$H_s^Q A(n) \cong H_s^Q A.$$

- (3) There is a cofibration sequence

$$S_\ell(H_n^Q A, n) \xrightarrow{f_n} A(n) \xrightarrow{j_n} A(n+1).$$

Here we write, for $B \in s\mathcal{A}_\ell$, $H_s^Q(A) := D_s(A|\ell; \ell)$ and, for $V \in \mathcal{V}_\ell$, $S_\ell(V, m) := S_\ell(K(V, m))$. Existence of connected envelopes is proved in [27, §2].

Note: Paul Goerss has pointed out that connected envelopes can also be constructed through a “reverse” decomposition via collapsing skeleta on the canonical CW approximation.

Now, for $A \in s\mathcal{A}_\ell$ connected, define the *André-Quillen dimension* of A to be

$$\text{AQ-dim}(A) = \max\{m \in \mathbb{N} \mid H_m^Q(A) \neq 0\}.$$

Assume that $n = \text{AQ-dim } A < \infty$. Then

$$A(n) \simeq S_\ell(H_n^Q(A), n).$$

Cf. [27, (2.1.3)]. Summarizing, we have

Proposition 2.1. *For $A \in s\mathcal{A}_\ell$ connected and $\text{AQ-dim } A < \infty$ there is a natural map*

$$\phi_A : A \rightarrow S_\ell(H_n^Q(A), n),$$

where $n = \text{AQ-dim } A$, with the property that $H_n^Q(\phi_A)$ is an isomorphism.

Now, assuming $\text{char}(\ell) > 0$, we noted that $\pi_* B$ is naturally a divided power algebra. Given a divided power algebra Λ in characteristic p , let $J \subset \Lambda$ to be the divided power ideal generated by all decomposables $w_1 w_2 \dots w_r$ and $\gamma_p(z)$ with $w_1, w_2, \dots, w_r, z \in \Lambda_{\geq 1}$. Define the Γ -indecposables to be

$$Q_\Gamma \Lambda = \Lambda/J.$$

Given $A \in s\mathcal{A}_\ell$ connected and $n = \text{AQ-dim } A$ finite, we define the *character map* of A to be

$$\Phi_A = Q_\Gamma \pi_*(\phi_A) : Q_\Gamma \pi_* A \rightarrow Q_\Gamma \pi_*(S_\ell(H_n^Q(A), n)).$$

Now, for $B \in s\mathcal{A}_\ell$, the action of the André operation ϑ on $\pi_* B$ induces an action on $Q_\Gamma \pi_* B$ by the relation (1.6) and the fact that ϑ kills decomposables of elements of positive degree. Cf. [8, (8.9)].

Theorem 2.2. *Let $A \in s\mathcal{A}_\ell$ be connected with $\text{char}(\ell) = 2$ and $H_*^Q(A)$ a non-trivial finite graded ℓ -module. Then Φ_A is non-trivial.*

Proof of Nilpotence Conjecture at the prime 2: Let $n = \text{AQ-dim } B$ where $B = A'' \otimes_{R''} \ell$ with $\ell = k(\wp)$. By Corollary 1.5 and [12, (3.5)], ϑ acts non-nilpotently on every non-trivial element of $Q_\Gamma \pi_*(S_\ell(H_n^Q(B), n))$ if $n \geq 3$. Therefore if $\pi_* B$ is André nilpotent then Theorem 2.2 implies that $n \leq 2$. Thus B is a homotopy 2-intersection by [27, (2.2)].

Since $H_*^Q(B) \cong H_*^Q(A(\wp))$, if A is additionally Γ -nilpotent at \wp then A is a homotopy 1-intersection at \wp , as $\pi_* A(\wp)$ is free as a divided power algebra. \square

The goal of the rest of this section will be to provide a proof of Theorem 2.2.

Remark: If A is a simplicial commutative R -algebra with finite Noetherian homotopy such that $\text{char}(\pi_0 A) = 2$, then, for each $\wp \in \text{Spec}(\pi_0 A)$, $\text{char}(k(\wp)) = 2$. Thus Theorem 2.2 coupled with the Non-Nilpotence Conjecture implies (1) of the Vanishing Conjecture when $\text{char}(\pi_0 A) = 2$. Furthermore, as an inspection of the proof of the Vanishing Conjecture above shows, (2) of the Vanishing Conjecture follows directly from the Nilpotence Conjecture. Thus we have an alternative proof of [28, Theorem B] when $\text{char}(\pi_0 A) = 2$.

2.2. Review of homotopy operations in characteristic 2. Let A be a simplicial commutative algebra of characteristic 2 (and, therefore, a simplicial \mathbb{F}_2 -algebra). Associated to A is a chain complex, $(C(A), \partial)$, where, for each $n \in \mathbb{N}$, we have

$$C(A)_n = A_n, \quad \partial = \sum_{i=0}^n d_i : C(A)_n \rightarrow C(A)_{n-1}.$$

It is standard that we have the identity [17]

$$\pi_n A \cong H_n(C(A)).$$

In [11], W. Dwyer showed the existence of natural chain maps

$$\Delta^k : (C(V) \otimes C(W))_{i+k} \rightarrow C(V \otimes W)_i \quad 0 \leq k \leq i,$$

where V and W are simplicial \mathbb{F}_2 -modules, having the following properties:

- (1) $\Delta^0 + T\Delta^0T = \Delta + \phi_0$;
- (2) $\Delta^k + T\Delta^kT = \partial\Delta^{k-1} + \Delta^{k-1}\partial$.

Here $T : C(V) \otimes C(W) \rightarrow C(W) \otimes C(V)$ is the twist map, $\Delta : C(V) \otimes C(W) \rightarrow C(V \otimes W)$ is the shuffle map [17, p. 243], and $\phi_k : C(V) \otimes C(W) \rightarrow C(V \otimes W)$ is the degree $(-k)$ map defined by

$$\phi_k(v \otimes w) = \begin{cases} 0 & \text{deg } v \neq k \text{ or } \text{deg } w \neq k; \\ v \otimes w & \text{otherwise.} \end{cases}$$

Note: Tensor product of chain complexes is graded tensor product and tensor product of simplicial modules is levelwise tensor product.

Now, for $x \in C(A)_n$ and $1 \leq i \leq n$, define $\Theta_i(x) \in C(A)_{n+i}$ by $\Theta_i(x) = \alpha_{n-i}(x)$ where

$$\alpha_t(x) = \mu\Delta^t(x \otimes x) + \mu\Delta^{t-1}(x \otimes \partial x),$$

and

$$\alpha_0(x) = \mu\Delta^0(x \otimes x),$$

where μ is the map $C(A \otimes A) \rightarrow C(A)$ induced by the product on A . As shown in [12, §3], these natural maps have the following properties:

- (1) $\partial\Theta_i(x) = \Theta_i(\partial x)$ for $2 \leq i \leq n$;
- (2) $\partial\Theta_n(x) = \mu\Delta(x \otimes \partial x)$;
- (3) $\partial\Theta_1(x) = \Theta_1(\partial x) + x^2$;
- (4) $\Theta_i(x + y) = \Theta_i(x) + \Theta_i(y) + \begin{cases} \partial\mu\Delta^{n-i-1}(x \otimes y) & 2 \leq i < n; \\ \mu\Delta(x \otimes y) & i = n. \end{cases}$

From these chain properties for the Θ_i , there are induced homotopy operations

$$\delta_i : \pi_n A \rightarrow \pi_{n+i} A \quad 2 \leq i \leq n,$$

or, upon letting $\alpha_t = \delta_{n-t}$, we have

$$\alpha_t : \pi_n A \rightarrow \pi_{2n-t} A \quad 0 \leq t \leq n - 2.$$

Note, in particular, that

$$(2.7) \quad \vartheta = \alpha_1.$$

The following is proved in [11, 13]:

Theorem 2.3. *The homotopy operations δ_i have the following properties:*

- (1) δ_i is a homomorphism for $2 \leq i \leq n - 1$ and $\delta_n = \gamma_2$ - the divided square;
- (2) δ_i acts on products as follows:

$$\delta_i(xy) = \begin{cases} \delta_i(x)y^2 & \deg y = 0; \\ x^2\delta_i(y) & \deg x = 0; \\ 0 & \text{otherwise}; \end{cases}$$

- (3) if $i < 2j$, then

$$\delta_i\delta_j = \sum_{\frac{i+1}{2} \leq k \leq \frac{i+j}{3}} \binom{j-i+k-1}{j-s} \delta_{i+j-k}\delta_k.$$

Corollary 2.4. *The homotopy operations α_t have the following properties:*

- (1) α_t is a homomorphism for $1 \leq t \leq n - 2$ and $\alpha_0 = \gamma_2$ - the divided square;
- (2) α_t acts on products as follows:

$$\alpha_t(xy) = \begin{cases} \alpha_t(x)y^2 & \deg y = 0; \\ x^2\alpha_t(y) & \deg x = 0; \\ 0 & \text{otherwise}; \end{cases}$$

- (3) if $s > t$, then

$$\alpha_s\alpha_t = \sum_{\frac{s+2t}{3} \leq q \leq \frac{s+t-1}{2}} \binom{s-q-1}{q-t} \alpha_{s+2t-2q}\alpha_q.$$

Proof of Corollary 2.4: The first two items follow immediately from Theorem 2.3 using the identity $\alpha_t(x) = \delta_{n-t}(x)$ where $\deg x = n$. The last relation follows from (3) of Theorem 2.3 upon letting $j = n - t$, $i = 2n - s - t$, and $k = n - q$. \square

Our goal at present is to describe homotopy operations for simplicial algebras over general fields ℓ of characteristic 2. Specifically, we will prove:

Theorem 2.5. *Let A be a simplicial supplemented ℓ -algebra with $\text{char}(\ell) = 2$. Then, for $2 \leq i \leq n$, the natural operation $\delta_i : \pi_n A \rightarrow \pi_{n+i} A$ satisfies properties (1) - (3) of Theorem 2.3. In particular, for $a, b \in \ell$ and $x, y \in \pi_n A$ we have*

$$\delta_i(ax + by) = a^2\delta_i(x) + b^2\delta_i(y) + \begin{cases} (ab)(xy) & i = n; \\ 0 & \text{otherwise.} \end{cases}$$

and for $u, v \in \pi_* A$

$$\delta_i((au)(bv)) = \begin{cases} (ab)^2(\delta_i(u)v^2) & \deg v = 0; \\ (ab)^2(u^2\delta_i(v)) & \deg u = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, homotopy operations $\pi_n A \rightarrow \pi_{n+k} A$, as natural maps of functors of simplicial supplemented ℓ -algebras, are determined algebraically over ℓ by the operations $\delta_{i_1}\delta_{i_2}\dots\delta_{i_r}$ with (i_1, \dots, i_r) an admissible sequence of degree k and excess $\leq n$.

Recall that the degree of $I = (i_1, \dots, i_r)$ is $i_1 + \dots + i_r$ and the excess of I is $i_1 - i_2 - \dots - i_r$. We will write throughout $\delta_I = \delta_{i_1}\delta_{i_2}\dots\delta_{i_r}$. Finally, we call I admissible provided $i_{q-1} \geq 2i_q$ for all $2 \leq q \leq r$.

To prove Theorem 2.5, we need two lemmas.

Lemma 2.6. *For $n \geq 1$, we have*

$$\pi_* S_{\mathbb{F}_2}(n) \cong \Gamma[\delta_I(\iota_n) \mid \text{excess}(I) < n].$$

It follows that for any field ℓ of characteristic 2

$$\pi_* S_\ell(n) \cong \Gamma_\ell[\delta_I(\iota_n) \mid \text{excess}(I) < n].$$

Proof: For the first statement, see [8, §7] or [11, Remark 2.3]. The second statement follows from the first and Proposition 1.4. \square

For the following, see [12, 12.4.2].

Lemma 2.7. *Let A and B be simplicial commutative \mathbb{F}_2 -algebras. Then the induced action of δ_i on $\pi_*(A) \otimes_{\mathbb{F}_2} \pi_* B$ is determined by*

$$\delta_i(x \otimes y) = \begin{cases} \delta_i(x) \otimes y^2 & \deg y = 0; \\ x^2 \otimes \delta_i(y) & \deg x = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Theorem 2.5: Since ℓ has characteristic 2, the operations δ_i are defined on $\pi_n A$ and satisfy (1) through (3) of Theorem 2.3. In particular, to compute $\delta_i(ax + by)$ it is enough, by Corollary 1.5, to compute

$$\delta_i(a\iota_n \otimes_\ell 1 + 1 \otimes_\ell b\iota_n) \in \pi_*(S_\ell(n)) \otimes_\ell \pi_*(S_\ell(n)).$$

Under the isomorphism (using Proposition 1.4 and Kunnetth Theorem)

$$\pi_*(S_\ell(n)) \otimes_\ell \pi_*(S_\ell(n)) \cong (\pi_*(S_{\mathbb{F}_2}(n)) \otimes_{\mathbb{F}_2} \pi_*(S_{\mathbb{F}_2}(n))) \otimes_{\mathbb{F}_2} \ell,$$

$\delta_i(a\iota_n \otimes_\ell 1 + 1 \otimes_\ell b\iota_n)$ corresponds to $\delta_i((\iota_n \otimes_{\mathbb{F}_2} 1) \otimes_{\mathbb{F}_2} a + (1 \otimes_{\mathbb{F}_2} \iota_n) \otimes_{\mathbb{F}_2} b)$. Thus the desired result follows from Lemma 2.7. Similarly, to compute $\delta_i((au)(bv))$ it is enough to compute $\delta_i((a\iota_m) \otimes_\ell (b\iota_n)) \in \pi_*(S_\ell(m)) \otimes_\ell \pi_*(S_\ell(n))$, or, equivalently, $\delta_i((\iota_m \otimes_{\mathbb{F}_2} \iota_n) \otimes_{\mathbb{F}_2} (ab)) \in (\pi_*(S_{\mathbb{F}_2}(m)) \otimes_{\mathbb{F}_2} \pi_*(S_{\mathbb{F}_2}(n))) \otimes_{\mathbb{F}_2} \ell$. This again can be computed using Lemma 2.7.

Finally, the last statement follows from Corollary 1.5 and Lemma 2.6. \square

Note: Theorem 2.5 shows that the operations δ_i and the relations (1) - (3) of Theorem 2.3 completely determine the homotopy operations for simplicial supplemented algebras over general fields of characteristic 2. Thus the Galois group of ℓ over \mathbb{F}_2 produces no new homotopy operation of positive degree nor alters the relations between them. This should not be surprising as the same considerations is known to hold rationally. See [24, §4].

2.3. Quillen's spectral sequence. We now modify the results of [12, §6] to enable to use Quillen's fundamental spectral sequence [22, 23] over general fields of characteristic 2.

To begin, we need to be more explicit about the functors $S_\ell(-)$. Let V be an ℓ -module. For $n \in \mathbb{N}$, define $S_{\ell,0}(V) = \ell$ and

$$S_{\ell,n}(V) = \ell \langle v_1 v_2 \dots v_n \mid v_i \in V \rangle.$$

Then

$$S_\ell(V) \cong \bigoplus_{n \in \mathbb{N}} S_{\ell,n}(V).$$

Next, let W be a non-negatively graded ℓ -module and define

$$(2.8) \quad \mathcal{S}_\ell(W) = \Gamma_\ell[\delta_I(w) \mid w \in W, I \text{ admissible, excess}(I) < \deg w]$$

which, by Corollary 2.4, can be expressed as

$$(2.9) \quad \mathcal{S}_\ell(W) \cong \Gamma_\ell[\alpha_1^{i_1} \alpha_2^{i_2} \dots \alpha_{n-2}^{i_{n-2}}(w) \mid w \in W, n = \deg w, i_1, \dots, i_{n-2} \in \mathbb{Z}_+].$$

For $u \in \mathcal{S}_\ell(W)$, we define the *weight* of u , $\text{wt}(u)$, as follows:

$$\text{wt}(u) = \begin{cases} 0 & \text{if } u \in \ell; \\ 1 & \text{if } u \in W; \\ \text{wt}(x) + \text{wt}(y) & \text{if } u = xy; \\ 2 \text{wt}(x) & \text{if } u = \delta_i(x). \end{cases}$$

We then define, for $n \in \mathbb{N}$,

$$\mathcal{S}_{\ell,n}(W) = \ell \langle u \in \mathcal{S}_\ell(W) \mid \text{wt}(u) = n \rangle.$$

Proposition 2.8. *For a simplicial \mathbb{F}_2 -module V and $n \in \mathbb{N}$ there are a natural isomorphisms*

$$S_{\ell,n}(V \otimes_{\mathbb{F}_2} \ell) \cong S_{\mathbb{F}_2,n}(V) \otimes_{\mathbb{F}_2} \ell$$

and

$$\mathcal{S}_{\ell,n}((\pi_* V) \otimes_{\mathbb{F}_2} \ell) \cong \mathcal{S}_{\mathbb{F}_2,n}(\pi_* V) \otimes_{\mathbb{F}_2} \ell.$$

As a consequence, if W is a simplicial ℓ -module then

$$\pi_* S_{\ell,n}(W) \cong \mathcal{S}_{\ell,n}(\pi_* W).$$

Proof: The first two statements can be proved just as for Proposition 1.4. For the last statement, note that [12, §3] shows that the isomorphism holds when $\ell = \mathbb{F}_2$. Note also that a standard argument (e.g. via Postnikov towers) shows that there is a simplicial set X and a homotopy equivalence $W \simeq \ell\langle X \rangle$. Thus

$$\pi_* S_{\ell,n}(W) \cong \mathcal{S}_{\mathbb{F}_2,n}(\pi_* V) \otimes_{\mathbb{F}_2} \ell$$

where $V = \mathbb{F}_2\langle X \rangle$. Since $\pi_* W \cong (\pi_* V) \otimes_{\mathbb{F}_2} \ell$ it follows that

$$\mathcal{S}_{\ell,n}(\pi_* W) \cong \mathcal{S}_{\mathbb{F}_2,n}(\pi_* V) \otimes_{\mathbb{F}_2} \ell.$$

□

We now follow [12, §6]. Let A be a simplicial supplemented ℓ -algebra and let IA be its augmentation ideal. We may assume, using the standard model category structure [21, §II.3], that A is almost free, i.e. $A_t \cong S_{\ell}(V_t)$ for all $t \geq 1$. Furthermore, the composite $V_t \subseteq IA_t \rightarrow QA_t$ to the indecomposables module is an isomorphism. We now form a decreasing filtration of A :

$$F_s = (IA)^s.$$

For A almost free,

$$E_s^0 A = F_s / F_{s+1} = (IA)^s / (IA)^{s+1} \cong S_{\ell,s}(QA).$$

Applying homotopy gives a spectral sequence

$$(2.10) \quad E_{s,t}^1 A = \pi_t E_t^0 A \cong \pi_t S_{\ell,s}(QA) \implies \pi_t A$$

with differentials

$$(2.11) \quad d_r : E_{s,t}^r A \rightarrow E_{s+r,t-1}^r.$$

This is called *Quillen's spectral sequence*.

Theorem 2.9. *For a simplicial supplemented ℓ -algebra A there is a spectral sequence of algebras*

$$E^1 A_{s,t} = \mathcal{S}_{\ell,s}(H_*^Q(A))_t \implies \pi_t A$$

with the following properties:

- (1) *The spectral sequence converges if $\pi_0 A \cong \ell$. In particular, $E_{s,t}^r A = 0$ for $t < s$ for all $r \geq 1$.*
- (2) *For $1 \leq r \leq \infty$ there are operations*

$$\delta_i : E_{s,t}^r A \rightarrow E_{2s,t+i}^r A \quad 2 \leq i \leq t$$

of indeterminacy $2r-1$ with the following properties:

- (a) If $r = 1$, then δ_i coincides with the induced operation $\mathcal{S}_{\ell,s}(H_*^Q(A))_t \rightarrow \mathcal{S}_{\ell,2s}(H_*^Q(A))_{t+i}$.
- (b) If $x \in E^r A$ and $2 \leq i < t$ then $\delta_i(x)$ survives to $E^{2r} A$ and

$$\begin{aligned} d_{2r}\delta_i(x) &= \delta_i(d_r x) \\ d_r\delta_t(x) &= x d_r x \end{aligned}$$

modulo indeterminacy.

- (c) The operations on $E^r A$ are induced by the operations on $E^{r-1} A$ and the operations on $E^\infty A$ are induced by the operations on $E^r A$ for $r < \infty$.
- (d) The operations on $E^\infty A$ are induced by the operations on $\pi_* A$.
- (e) Up to indeterminacy, the operations on $E^r A$ satisfy the properties of Theorem 2.5.

Before we indicate a proof of this omnibus result, a word of explanation is needed. First, an element $y \in E_{s,t}^r A$ is said to be defined up to *indeterminacy* q provided y is a coset representative for a particular element of $E_{s,t}^q A / B_{s,t}^q A$ where

$$B_{s,t}^q A \subseteq E_{s,t}^r A \quad q \geq r$$

is the ℓ -module of elements of $E_{s,t}^r A$ which survive to $E_{s,t}^q A$ but have zero residue class.

Also, if A is almost free, and hence cofibrant as a simplicial supplemented ℓ -algebra, then

$$\pi_*(QA) \cong H_*^Q(A).$$

Cf. [27, §1].

Proof: First, if A is almost free, we have a pairing

$$\pi_t(S_{\ell,s}(QA)) \otimes \pi_{t'}(S_{\ell,s'}(QA)) \xrightarrow{(\mu\Delta)^*} \pi_{t+t'}(S_{\ell,s+s'}(QA))$$

which gives a pairing

$$E_{s,t}^1 A \otimes E_{s',t'}^1 A \rightarrow E_{s+s',t+t'}^1 A$$

and induces an algebra structure on the spectral sequence.

For (1), we simply note that if A is connected then $\mathcal{S}_{\ell,s}(H_*^Q(A)) = 0$ for $t > s$. Convergence now follows from standard convergence theorems. Cf. [17].

For (2), we have a commutative diagram

$$\begin{array}{ccccc} C((IA)^s) \otimes_{\mathbb{F}_2} C((IA)^s) & \xrightarrow{\bar{\alpha}_t} & C((IA)^s \otimes_{\mathbb{F}_2} (IA)^s) & \longrightarrow & C((IA)^s \otimes_{\ell} (IA)^s) \\ \sigma \uparrow & & & & \downarrow \mu \\ C((IA)^s) & \xrightarrow{\alpha_t} & C((IA)^{2s}) & = & C((IA)^{2s}) \end{array}$$

where $\sigma(u) = u \otimes u$ and $\bar{\alpha}_t(a \otimes b) = \Delta^t(a \otimes b) + \Delta^{t-1}(a \otimes \partial b)$. This induces a map

$$\Theta_i : (IA)^s \rightarrow (IA)^{2s},$$

by again setting $\Theta_i(u) = \alpha_{n-i}(u)$ where $n = \deg u$.

Let $x \in E_{s,t}^r A$. Then, modulo $(IA)^{s+1}$, x is represented by $u \in (IA)^s$ with the property that $\partial u \in (IA)^{s+r}$. The class of u is not unique, but may be altered by adding elements $\partial b \in (IA)^s$ with $b \in (IA)^{s-r+1}$.

Define $\delta_i(x) \in E_{2s, t+i}^r A$ to be the residue class of $\Theta_i(u)$. Since

$$\partial\Theta_i(u) = \Theta_i(\partial u) \in (IA)^{2s+2r} \quad 2 \leq i < t,$$

and

$$\partial\Theta_t(u) = \mu\Delta(u \otimes \partial u) \in (IA)^{2s+r}.$$

Thus $\delta_i(x)$ is defined in $E_{2s, t+i}^r A$ and survives to $E^{2r} A$ with $d_{2r}\delta_i(x) = \delta_i(d_r x)$ for $2 \leq i < t$. Also $d_r\delta_t(x) = xd_r x$. This gives us (b).

Now we have a commuting diagram

$$\begin{array}{ccc} \pi_t((IA)^s) & \xrightarrow{(\Theta_i)_*} & \pi_{t+i}((IA)^{2s}) \\ \downarrow & & \downarrow \\ \pi_t A & \xrightarrow{\delta_i} & \pi_{t+i} A \end{array}$$

and an induced diagram

$$\begin{array}{ccc} \pi_t((IA)^s) & \xrightarrow{(\Theta_i)_*} & \pi_{t+i}((IA)^{2s}) \\ \downarrow & & \downarrow \\ \pi_t((IA)^s/(IA)^{s+1}) & \longrightarrow & \pi_{t+i}((IA)^{2s}/(IA)^{2s+1}) \\ \downarrow \cong & & \cong \downarrow \\ \mathcal{S}_{\ell, s}(H_*^Q(A))_t & \xrightarrow{\delta_i} & \mathcal{S}_{\ell, 2s}(H_*^Q(A))_{t+i} \end{array}$$

It is now straightforward to check (a), (c), (d), and (e). \square

2.4. Non-triviality of the character map. We now proceed to prove Theorem 2.2. We will in fact prove a more general theorem. Specifically:

Theorem 2.10. *Let A be a simplicial supplemented ℓ -algebra ($\text{char}(\ell) = 2$) such that $H_*^Q(A)$ is finite graded as an ℓ -module. Let $n = \text{AQ-dim } A$ and assume $n \geq 2$. Then there exists $x \in \pi_* A$ and $y \neq 0 \in H_n^Q(A)$ such that under the map*

$$\pi_* \phi_A : \pi_* A \rightarrow \pi_* \mathcal{S}_\ell(H_n^Q(A), n)$$

we have

$$(\pi_* \phi_A)(x) = \alpha_{n-2}^t(y)$$

for some $t \geq 1$.

Proof of Theorem 2.2: Assume $n = \text{AQ-dim } A \geq 3$. Let $y \in H_n^Q(A)$, and $x \in \pi_* A$ satisfy the properties of Theorem 2.10. By Equation 2.9, $\alpha_{n-2}^t(y) \neq 0$ in $Q_\Gamma \pi_*(\mathcal{S}_\ell(H_n^Q(A), n))$ for all $t \geq 1$. We conclude that $\Phi_A(x) \neq 0$.

If $n \leq 2$, then Φ_A is a surjection and, hence, non-trivial. \square

Now, in order to prove Theorem 2.10 we will need to know something about the annihilation properties of homotopy operations. Specifically, we will focus on composite operations of the form

$$\theta(s, t) = \delta_{2s} \delta_{2s-1} \dots \delta_{2t+1} \quad s > t$$

(where we set $\theta(t+1, t) = \delta_{2t+1}$).

Lemma 2.11. *Let $i \geq 2$ and $t \geq 1$ be such that $2^t < i$. Then $\theta(s, t)\delta_i = 0$ for $s \gg t$.*

Proof: Write $i = 2^{t-1} + n$ with $n \geq 1$. Note first that an application of the relation Theorem 2.3 (3) shows that for any $t \geq 1$,

$$\delta_{2^{t+1}}\delta_{2^{t+1}} = 0 = \delta_{2^{t+1}}\delta_{2^{t+2}}.$$

We thus assume, by induction, that for any t and $0 < j < n$, there exists $s \gg t$ such that

$$\theta(s, t)\delta_{2^t+j} = 0.$$

By another application of the relation Theorem 2.3 (3), we have

$$\delta_{2^{t+1}}\delta_{2^t+n} = \sum_{1 \leq r \leq \frac{n}{3}} \binom{n+r-1}{n-r} \delta_{2^{t+1}+n-r}\delta_{2^t+r}.$$

Notice that, for each such r , $2^{t+1} < 2^{t+1} + n - r < 2^{t+1} + n$. Thus, by induction, we can find $s \gg t + 1$ so that

$$\theta(s, t+1) \left(\sum_{1 \leq r \leq \frac{n}{3}} \binom{n+r-1}{n-r} \delta_{2^{t+1}+n-r}\delta_{2^t+r} \right) = 0.$$

We conclude that

$$\theta(s, t)\delta_{2^t+n} = \theta(s, t+1)\delta_{2^{t+1}}\delta_{2^t+n} = 0.$$

□

Corollary 2.12. *Let $I = (i_1, \dots, i_k)$ be an admissible sequence and let $t < k$. Then $\theta(s, t)\delta_I = 0$ for $s \gg t$.*

Proof: Since I is admissible, then

$$i_1 \geq 2i_2 \geq \dots \geq 2^{k-1}i_k \geq 2^k > 2^t.$$

Thus, by Lemma 2.11,

$$\theta(s, t)\delta_I = (\theta(s, t)\delta_{i_1})\delta_{i_2} \dots \delta_{i_k} = 0$$

for $s \gg t$.

□

Proposition 2.13. *Let A be a connected simplicial supplemented ℓ -algebra, $\text{char}(\ell) = 2$. Let $y \neq 0$ in $E_{1,n}^1 A \cong H_n^Q(A)$, $n \geq 2$. Then there exists $s \geq 1$ such that $\alpha_{n-2}^s(y) \in E_{2^s, n+2^s+1-2}^1 A$ survives to $E^\infty A$ (though possibly trivially).*

Proof: Choose $m \geq 1$ and suppose $\alpha_{n-2}^m(y)$ survives to $E^r A$, $r \geq 1$. By Theorem 2.9 (2) (a), we may assume that $r \geq 2^m$. Let $w = d_r([\alpha_{n-2}^m(y)]) \in E_{2^m+r, n+2^m+1-3}^r A$ by (2.11). By Theorem 2.9 (1), $w = 0$ provided $n + 2^m - 2 \leq r$. Thus if $r \geq n + 2^m - 2$ then the class of $\alpha_{n-2}^m(y)$ survives to $E^\infty A$ as all subsequent differentials will satisfy the same criterion.

Suppose next that $r < n + 2^m - 2$. Write $n + 2^m - q = r$ with $n \geq q > 2$. Assume, by induction, that if for some m the class of $\alpha_{n-2}^m(y)$ survives to $E_{2^m, n+2^m+1-2}^{n+2^m-j} A$ for $q > j$ then there exists $s \gg m$ such that the class of $\alpha_{n-2}^s(y)$ survives to $E^\infty A$. Again, let

$w = d_r([\alpha_{n-2}^m(y)]) \in E_{2^m+r, n+2^{m+1}-3}^r A$. Choose $u \in E_{2^m+r, n+2^{m+1}-3}^1 A$ to represent the class w . By Theorem 2.9 and Proposition 1.4, we have

$$u = \begin{cases} \sum_{I,l} a_{I,l} \delta_I(x_l) + \sum_J b_J z_J & r = 2^k - 2^m, k > m; \\ \sum_J b_J z_J & \text{otherwise} \end{cases}$$

where $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_r)$ are sequences with I admissible, $a_{I,k}, b_J \in \ell$, and $z_J = z_{j_1} z_{j_2} \dots z_{j_{r+2^m}}$ with $x_k, z_{j_1}, \dots, z_{j_{r+2^m}} \in H_*^Q(A)$.

First assume that $r \neq 2^k - 2^m$. Then $d_r([\alpha_{n-2}^m(y)]) = [u] \in E_{2^m+r, n+2^{m+1}-3}^r A$ with $u \in E_{2^m+r, n+2^{m+1}-3}^1 A$ decomposable. Note again that $\deg u > 2^m$. Thus, by Theorem 2.9 (2) (b), (c), and (e), $d_{2^r}(\delta_{2^{m+1}}[\alpha_{n-2}^m(y)]) = \delta_{2^{m+1}} d_r([\alpha_{n-2}^m(y)]) = \delta_{2^{m+1}}[u] = 0$. Thus $[\alpha_{n-2}^{m+1}y]$ survives to $E_{2^{m+1}, n+2^{m+2}-2}^{2r+1} A$. Now, let $2r+1 = n+2^{m+1}-j$ and recall that $r = n+2^m - q \geq 2^m$. Then

$$j = (n+2^{m+1}) - (2n+2^{m+1}-2q) - 1 = 2q - n - 1 = q - (n-q) - 1 < q.$$

Thus, by induction, there exists $s \gg m$ such that $[\alpha_{n-2}^s(y)]$ survives to E^∞ .

Now assume that $r = 2^k - 2^m$ with $k > m$. By definitions of α_{n-2} and $\theta(m, t)$,

$$\alpha_{n-2}^m(y) = \theta(m, 0)(y).$$

By Theorem 2.9 (2) (b) and (c), for $e > m$

$$d_{2^e-m_r}([\theta(e, 0)y]) = \theta(e, m) d_r([\theta(m, 0)(y)]) = \theta(e, m)w.$$

By Theorem 2.9 (2) (c) and (e) and Theorem 2.5, $\theta(e, m)w$ is represented by

$$\sum_{I,l} a_{I,l}^{2^e-m} \theta(e, m) \delta_I(z_l) \quad \text{modulo indeterminacy.}$$

Note that $2^m < \deg u$ so we can assume there are no decomposables in our choice of representative for $\theta(e, m)w$. As indicated above, we have for each $I = (i_1, \dots, i_k)$ that $k > m$. Thus, by Corollary 2.12, since the sum is finite, there exists $e \gg m$ such that

$$\theta(e, m) \delta_I = 0 \quad \text{for all } I.$$

Thus $d_{2^e-m_r}([\theta(e, 0)y]) = 0$ modulo indeterminacy. Therefore $[\theta(e, 0)y]$ survives to $E_{2^e, n+2^{e+1}-2}^{2^e-m_r+1} A$, so, by the previous case, there exists $s \gg e$ such that $[\alpha_{n-2}^s(y)] = [\theta(s, 0)y]$ survives to E^∞ . \square

Proof of Theorem 2.10: Choose $y \in H_n^Q(A) \cong E_{1,n}^1 A$ and choose $s \geq 1$ such that $\alpha_{n-2}^s(y) \in E^1 A$ survives to $E^\infty A$, which exists by Proposition 2.13. Under the induced map

$$E^r(\phi_A) : E^r A \rightarrow E^r S_\ell(H_n^Q(A), n)$$

we have

$$E^r(\phi_A)([\alpha_{n-2}^s(y)]) = [\alpha_{n-2}^s(y)]$$

for all $\infty \geq r \geq 1$. But, since

$$E^1 S_\ell(H_n^Q(A), n) \cong E^\infty S_\ell(H_n^Q(A), n),$$

we can conclude that $E^\infty(\phi_A)([\alpha_{n-2}^s(y)]) \neq 0$. Thus we can find a nontrivial $x \in \pi_*A$ which is represented by $\alpha_{n-2}^s(y)$ in $E^\infty A$ such that $(\pi_*\phi_A)(x) = \alpha_{n-2}^s(y)$. \square

Remarks:

- (1) From the proof of Proposition 2.13, an algorithm can be made to determine an s such that $\alpha_{n-2}^s(x)$ survives to $E^\infty A$ for $x \in H_n^Q(A)$. Choose $m \geq 1$ such that $\alpha_{n-2}^m(x)$ survives to $E^{2^m+1}A$, guaranteed by Theorem 2.9.2 (b) and Corollary 2.12 (see also [12, (6.9)]). Then, using the procedure in the proof, it can be shown that $\alpha_{n-2}^{m+n-2}(x)$ survives to $E^\infty A$.
- (2) Following the philosophy of [26], the reader can conjecture a dual topological version of Theorem 2.10 for nilpotent finite Postnikov towers, using connected covers, which would further generalize results of Serre from [25] in the spirit of [16, 14].

3. PROOF OF THE MAIN THEOREM

We now seek to establish a special case of the Vanishing Conjecture as described in the overview. The proof will utilize the validity of the Nilpotence Conjecture at the prime 2 while avoiding the need to evoke the Non-Nilpotence Conjecture.

Let A be a simplicial commutative \mathbb{F}_2 -algebra and let $(C(A), \partial)$ be the associated chain complex. The following is proved in [2, 11].

Proposition 3.1. *The shuffle map $\Delta : C(A) \otimes_{\mathbb{F}_2} C(A) \rightarrow C(A \otimes_{\mathbb{F}_2} A)$ induces a divided power algebra structure on $C(A)$. Specifically, for each $k \in \mathbb{Z}_+$, there is a function $\gamma_k : C(A)_n \rightarrow C(A)_{kn}$ satisfying:*

- (1) $\gamma_0(x) = 1$ and $\gamma_1(x) = x$
- (2) $\gamma_h(x)\gamma_k(x) = \binom{h+k}{h}\gamma_{h+k}(x)$
- (3) $\gamma_k(x+y) = \sum_{r+s=k} \gamma_r(x)\gamma_s(x)$
- (4) $\gamma_k(xy) = 0$ for $k \geq 2$ and $x, y \in C(A)_{\geq 1}$
- (5) $\gamma_k(xy) = x^k\gamma_k(y)$ for $x \in C(A)_0$ and $y \in C(A)_{\geq 2}$
- (6) $\gamma_k(\gamma_2(x)) = \gamma_{2k}(x)$
- (7) $\partial\gamma_k(x) = (\partial x)\gamma_{k-1}(x)$
- (8) $u \in C(A)_n$ a cycle then, for $[u] \in \pi_n A$, $\delta_n([u]) = [\gamma_2(u)]$.

Let $A \rightarrow B$ be a map of simplicial commutative \mathbb{F}_2 -algebras and $\rho : C(A) \rightarrow C(B)$ the induced map of chain complexes. Then for $u \in C(A)_n$ and all $n > i \geq 0$

$$\rho(\alpha_i(u)) = \alpha_i(\rho(u))$$

where $\alpha_i = \Theta_{n-i}$. Recall (2.7) that $\vartheta = \alpha_1$.

Lemma 3.2. *Let $A \rightarrow B$ be a map of simplicial commutative \mathbb{F}_2 -algebras and suppose $\pi_s A = 0$ for $s \gg 0$. Let $u \in C(A)_n$, $n \geq 3$, such that $\rho(\partial u) = 0$. Then $\rho(u)$ is a cycle in $C(B)$ and $\vartheta^r([\rho(u)]) = 0$ in $\pi_*(B)$ for $r \gg 0$ provided $\gamma_2^r(\partial u) = 0$ in $C(A)$ for $r \gg 0$.*

Proof: First, in $C(A)$, we have, by an induction using the formulas for Θ_i from §2.2, that

$$\partial\vartheta^r(u) = \gamma_2^r(\partial u).$$

Since $\gamma_2^r(\partial u) = 0$ for $r \gg 0$ and $H_s(C(A)) = 0$ for $s \gg 0$, it follows that $\vartheta^r(u)$ is a boundary in $C(A)$ for $r \gg 0$. We conclude that $\vartheta^r([\rho(u)]) = [\rho(\vartheta^r(u))] = 0$ in $\pi_*(B)$. \square

Corollary 3.3. *Let $A \rightarrow B$ be a level-wise surjection of simplicial commutative \mathbb{F}_2 -algebras such that γ_2 acts locally nilpotently on $(\partial C(A)) \cap \ker \rho$ and $\pi_s A = 0$ for $s \gg 0$. Then B is André nilpotent.*

Proof: Given $x \in \pi_n B$ with $n \geq 3$, let $w \in C(B)_n$ be a cycle representative for x and choose $u \in C(A)$ such that $\rho(u) = w$. Then $\rho(\partial u) = 0$ and $\gamma_2^r(\partial u) = 0$ for $r \gg 0$ by assumption. Thus $\vartheta^r(x) = 0$ for $r \gg 0$ by Lemma 3.2. \square

Proof of the Main Theorem: Let A be a simplicial commutative R -algebra with finite Noetherian homotopy, R a Cohen-Macaulay ring of characteristic 2, such that $D_s(A|R; -) = 0$ for $s \gg 0$, as a functor of $\pi_0 A$ -modules. Note that A has an induced simplicial \mathbb{F}_2 -algebra structure. Choose $\wp \in \text{Spec}(\pi_0 A)$. Choosing a weak homotopy factorization, $(R'', \mathfrak{m}) \rightarrow A''$, of A at \wp , which exists by Proposition 1.1, then, by Proposition 1.1, [5, (3.8.3) & (3.10)], and [19, §5], (R'', \mathfrak{m}) is a Cohen-Macaulay ring of depth zero and, hence, locally Artin. Thus, by Proposition 1.1, we may simply assume that R is locally Artinian, that A is a cofibrant simplicial commutative R -algebra, and that the unit map $R \rightarrow \pi_0 A$ is a surjective local homomorphism. We will now show that such A is André nilpotent at \wp . Note that if $\ell = k(\wp)$ then $\text{char}(\ell) = 2$ since R and A have characteristic 2.

Let $B = A \otimes_R^{\mathbf{L}} \ell = A \otimes_R \ell$. Then

$$(3.12) \quad C(B) \cong C(A) \otimes_R \ell \cong C(A)/\mathfrak{m}C(A).$$

Thus $\rho : C(A) \rightarrow C(B)$ is a surjection and $\ker \rho = \mathfrak{m}C(A)$. Since R is locally Artinian,

$$(3.13) \quad \mathfrak{m}^s = 0 \quad s \gg 0.$$

Cf. [19, 2.3]. Let $a, b \in \mathfrak{m}$ and let $x, y \in C(A)$ of degrees ≥ 2 . By Proposition 3.1 (3) and a straightforward induction,

$$\gamma_2^r(ax + by) = a^{2^r} \gamma_2^r(x) + b^{2^r} \gamma_2^r(y) \quad \text{modulo decomposables.}$$

Thus, by (3.13) and Proposition 3.1 (4), $\gamma_2^r(ax + by) = 0$ for $r \gg 0$. Hence, by a further induction, γ_2 acts locally nilpotently on $\mathfrak{m}C(A)$. Therefore, by Corollary 3.3, B is André nilpotent. We conclude, by the validity of the Nilpotence Conjecture at the prime 2, that A is a homotopy 2-intersection at \wp . \square

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