

# ON THE MOD $p$ COHOMOLOGY OF $BPU(p)$

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ABSTRACT. We study the mod  $p$  cohomology of the classifying space of the projective unitary group  $PU(p)$ . We first prove that old conjectures due to J.F. Adams, and Kono and Yagita [16] about the structure of the mod  $p$  cohomology of classifying space of connected compact Lie groups held in the case of  $PU(p)$ . Finally, we prove that the classifying space of the projective unitary group  $PU(p)$  is determined by its mod  $p$  cohomology as an unstable algebra over the Steenrod algebra for  $p > 3$ , completing previous works [10] and [6] for the cases  $p = 2, 3$ .

## 1. INTRODUCTION

Compact Lie groups provide an example of one of the classical mathematical maxims: “the richer is the mathematical structure of an object, the more rigid the object is”. So for example all the rich mathematical structure associated to a connected compact Lie group is so intimately linked that it is completely recovered (perhaps, up to local isomorphism) from some small data like the Dynkin diagram or the maximal torus normalizer [8].

In homotopy theory, this rigidity in the structure of a compact Lie group  $G$  is expected to be inherited by the classifying space  $BG$  and related structures. So for example, in the appropriate homotopical setting of  $p$ -compact groups [12], maximal torus normalizers do characterize the isomorphic type of  $BG$ , at least at odd primes [3].

The aim of this work is to study the rigidity of the mod  $p$  cohomology of  $BG$ , namely  $H^*(BG; \mathbb{F}_p)$ , proving several conjectures in the particular case of  $G$  being the projective unitary group  $PU(p)$ , which is obtained as the quotient of the unitary group of rank  $p$ ,  $U(p)$ , by the subgroup  $\{\text{Diag}(\alpha, \dots, \alpha) \mid \alpha \in S^1\}$  of diagonal matrices.

In [12, Theorem 1.1], it is shown that  $H^*(BG; \mathbb{F}_p)$  is a Noetherian algebra for any compact connected Lie group  $G$ , so by [29, Theorem 1.4] (or directly [28, Theorem 6.2]) we know that the kernel of the natural map

$$(1) \quad H^*(BG; \mathbb{F}_p) \longrightarrow \varinjlim_{\mathcal{A}_p(G)} H^*(BE; \mathbb{F}_p),$$

where  $\mathcal{A}_p(G)$  stands for the Quillen category of elementary abelian  $p$ -subgroups of  $G$  [28, 29, 17, 11], contains only nilpotent elements. For  $p > 2$ , a more stronger conjecture shows up

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**Conjecture 1.1** (J.F. Adams). *Let  $G$  be a compact connected Lie group, and  $p$  be an odd prime. Then the mod  $p$  cohomology of  $BG$  is detected by elementary abelian  $p$ -subgroups [1, Definition 4.2], i.e. the natural map (1) is a monomorphism.*

Conjecture 1.1 trivially holds in the torsion free cases (see [3, Theorem 12.1]). In the case of torsion, only a few examples have been worked out, all of them at  $p = 3$ :  $F_4$  [5, Teorema 5],  $E_6$  [23] and  $PU(3)$  [16, Theorem 3.3]. Our first result generalizes the last reference proving,

**Theorem A.** The group  $PU(p)$  verifies Conjecture 1.1 at the odd prime  $p$ , i.e.  $H^*(BPU(p); \mathbb{F}_p)$  is detected by elementary abelian  $p$ -subgroups, i.e. the natural map (1) is a monomorphism for the case  $G = PU(p)$ .

*Proof.* See Theorem 2.4 □

The knowledge of the structure of  $H^*(BG; \mathbb{F}_p)$  plays an important role when trying to understand other generalized cohomologies of  $BG$  as it is shown in [16]. So for example, understanding Milnor primitive operations (see Section 3) is a crucial step in the use of the Atiyah-Hirzebruch spectral sequence [20, pag. 496]. A new conjecture arises [16, Conjecture 5]

**Conjecture 1.2** (Kono-Yagita). *If  $G$  is a connected compact Lie group, then for each odd dimensional element  $x \in H^*(BG; \mathbb{F}_p)$ , there is  $i$  such that  $Q_m x \neq 0$  for all  $m \geq i$ , where  $Q_m$  are the Milnor primitive operators.*

Then our result generalizes the study of  $PU(3)$  carried out in [16] proving,

**Theorem B.** The group  $PU(p)$  verifies Conjecture 1.2 for every odd prime  $p$ , i.e. for each odd dimensional element  $x \in H^*(BPU(p); \mathbb{F}_p)$ , there is  $i$  such that  $Q_m x \neq 0$  for all  $m \geq i$ , where  $Q_m$  are the Milnor primitive operators.

*Proof.* See Theorem 3.3. □

**Remark 1.3.** It is worth to remark that while the proof of Conjectures 1.1 and 1.2 in previous known cases is heavily based in a precise understanding of the cohomology rings involved, i.e. generators and relations, the proofs of Theorems A and B is done by geometrical methods and without using any information about the algebra structure of  $H^*BPU(p)$ .

So many restrictions on  $H^*(BG; \mathbb{F}_p)$  suggest that these algebras do not show up in nature very frequently. In other words, any space  $X$  whose mod  $p$  cohomology is isomorphic to that of  $BG$ , for a connected compact Lie group  $G$ , should be topologically related with  $BG$  in some way. This idea is captured in the next conjecture [26, Conjecture 4.4]

**Conjecture 1.4.** *Let  $G$  be a compact connected Lie group, and let  $X$  be a  $p$ -complete space such that  $H^*(X; \mathbb{F}_p) \cong H^*(BG; \mathbb{F}_p)$  as algebras over the mod  $p$  Steenrod algebra  $\mathcal{A}_p$ . Then  $X \simeq BG_p^\wedge$ .*

The first result of this kind appeared in [9], where Dwyer, Miller and Wilkerson proved Conjecture 1.4 for  $G = SU(2) = S^3$  at  $p = 2$ . In [10], the same authors considered the case when  $p$  does not divide the order of the Weyl group of  $G$ . Notbohm in [24] considered the case when  $p$  divides the order of the Weyl group of  $G$ , but  $BG$

has no torsion. For the case when torsion exists, there are only few known results [6, 30, 31, 32]. We prove,

**Theorem C.** Let  $X$  be a  $p$ -complete space such that  $H^*(X; \mathbb{F}_p) \cong H^*(BPU(p); \mathbb{F}_p)$  as an unstable algebra over the Steenrod algebra  $\mathcal{A}_p$ . Then  $X$  is homotopy equivalent to  $BPU(p)_p^\wedge$ .

*Proof.* If  $p = 2$ , then  $PU(2) = SO(3)$ , and the theorem is known [9]. If  $p = 3$  the theorem is proved in [6]. The case  $p \geq 5$  is consider in Section 4.  $\square$

**Notation:** Here all spaces are assumed to have the homotopy type of CW-complexes. Completion means Bousfield-Kan completion [4]. For a given space  $X$ , we write  $H^*X$  for the mod  $p$  cohomology  $H^*(X; \mathbb{F}_p)$  and  $X_p^\wedge$  for Bousfield-Kan  $(\mathbb{Z}_p)_\infty$ -completion or  $p$ -completion of the space  $X$ . Given a group  $G$  and a  $\mathbb{Z}G$ -module  $M$ , we write  $\mathcal{H}^*(G; M)$  for the cohomology of  $G$  with (twisted) coefficients in  $M$ . We assume that the reader is familiar with the Lannes' theory [18].

## 2. THE ADAMS' CONJECTURE

The aim of this section is to prove Adams' conjecture (Conjecture 1.1) for the group  $PU(p)$  at the prime  $p > 2$ . We start identifying some important subgroups of a compact connected Lie group  $G$ . Let  $T(G) \subset G$  be a maximal torus and  $N(G) \subset G$  its normalizer. Define  $N_p(G) \subset N(G)$ , the  $p$ -normalizer of the maximal torus  $T(G)$ , as the preimage of a  $p$ -Sylow subgroup in the Weyl group of  $G$ ,  $W_G = N(G)/T(G)$ .

**Lemma 2.1.** *The groups  $N_p(PU(p))$  and  $N_p(SU(p))$  are isomorphic.*

*Proof.* Notice first that  $N_p(PU(p)) = N_p(SU(p))/\{\text{Diag}(\alpha, \dots, \alpha) \mid \alpha \in S^1\}$ . Now, every element in  $N_p(SU(p))$  can be written in a unique way as  $\text{Diag}(z_1, \dots, z_p)P^i$ , where  $P$  is the permutation matrix corresponding to the cyclic permutation  $(1, 2, \dots, p)$ . Then  $\varphi: N_p(PU(p)) \longrightarrow N_p(SU(p))$  given by

$$\varphi([\text{Diag}(z_1, \dots, z_p)P^i]) = \text{Diag}\left(\frac{z_1}{z_2}, \dots, \frac{z_{p-1}}{z_p}, \frac{z_p}{z_1}\right)P^i$$

provides the desired isomorphism.  $\square$

The isomorphism constructed above is very convenient as we can prove Conjecture 1.1 for  $N_p(SU(p))$ ,

**Lemma 2.2.** *The cohomology  $H^*BN_p(SU(p))$  is detected by elementary abelian subgroups.*

*Proof.* Notice tha  $N_p(U(p)) \cong S^1 \wr \mathbb{Z}/p$ , hence by [1, Theorem 4.3] we know that  $H^*BN_p(U(p))$  is detected by elementary abelian subgroups. Moreover  $H^*BN_p(U(p))$  is detected by just two subgroups,  $V_t = \mathbb{Z}/p^{\oplus p} \subset T(U(p))$  the maximal elementary abelian toral subgroup and  $V_n = \mathbb{Z}/p \oplus \mathbb{Z}/p = (Z(U(p)) \times \mathbb{Z}/p) \cap SU(p)$  [1, Lemma 4.4], where  $Z(U(p))$  is the center of the group  $U(p)$

Now the fibration

$$S^1 \longrightarrow BSU(p) \longrightarrow BU(p)$$

gives rise to a fibration

$$S^1 \longrightarrow BN_p(SU(p)) \xrightarrow{Bj} BN_p(U(p)),$$

whose the Gysin sequence is

$$\cdots \longrightarrow H^*BN_p(U(p)) \xrightarrow{Bj^*} H^*BN_p(SU(p)) \xrightarrow{d} H^{*-1}BN_p(U(p)) \longrightarrow \cdots$$

Let  $x \in H^*BN_p(SU(p))$  and suppose  $d(x) \neq 0$ . Let  $V \longrightarrow BN_p(U(p))$  be an elementary abelian group detecting  $d(x)$ . Then  $V' = \langle V, Z(U(p)) \rangle \cap BN_p(SU(p))$  is an elementary abelian group, which appears in the fibration  $S^1 \longrightarrow BV' \longrightarrow B\langle V, Z(U(p)) \rangle$ , and detects the element  $x$ .

If  $d(x) = 0$ , then  $x = Bj^*(y)$  for some  $y \in H^*BN_p(U(p))$  and  $y$  is detected by  $V_t$  or  $V_n$  defined above, so the element  $x$  is detected by  $V_n$  or  $V_t \cap N_p(SU(p)) \cong (\mathbb{Z}/p)^{p-1}$ .  $\square$

An easy consequence of the previous lemmas is

**Lemma 2.3.** *The mod  $p$  cohomology of  $BN(PU(p))$  is detected by elementary abelian  $p$ -subgroups.*

*Proof.* Combining Lemmas 2.1 and 2.2 we obtain that  $H^*BN_p(PU(p))$  is detected by elementary abelian  $p$ -subgroups. Then, because the index  $[N(PU(p)) : N_p(PU(p))] = (p-1)!$  is nonzero in  $\mathbb{F}_p$ , the transfer argument [33, Lemma 6.7.17] shows that  $H^*BN(PU(p)) \longrightarrow H^*BN_p(PU(p))$  is a monomorphism. Therefore  $H^*BN(PU(p))$  is also detected by elementary abelian  $p$ -subgroups.  $\square$

Finally,

**Theorem 2.4.** *The mod  $p$  cohomology of  $BPU(p)$  is detected by elementary abelian  $p$ -subgroups.*

*Proof.* According to [21, Theorem 1.2 & Lemma 3.1],  $H^*(PU(p)/N(PU(p)))$  is finite and its Euler characteristic,  $\chi(PU(p)/N(PU(p)))$ , equals 1. Therefore, the transfer argument [12, Theorem 9.13] shows that  $H^*BPU(p) \longrightarrow H^*BN(PU(p))$  is a monomorphism. As  $H^*BN(PU(p))$  is detected by elementary abelian subgroups by previous lemma,  $H^*BPU(p)$  is so.  $\square$

### 3. THE KONO-YAGITA CONJECTURE

In this very short section we provide a proof of Theorem B (see Theorem 3.3) by means of Theorem A. Recall that for an odd prime  $p$ , the Milnor primitive operators are inductively defined as  $Q_0 = \beta$  and  $Q_{n+1} = \mathcal{P}^{p^l}Q_n - Q_n\mathcal{P}^{p^l}$  where  $\beta$  and  $\mathcal{P}^j$  are the Bockstein and the  $j$ -th Steenrod power respectively.

As quoted above, we use Theorem A to prove Theorem B, hence we need some information about elementary abelian subgroups in  $PU(p)$ . This information is collected in the following proposition ([7, Corollary 3.4] or [3, Theorem 9.1])

**Proposition 3.1.** *The group  $PU(p)$  contains two conjugacy classes of maximal elementary abelian subgroups corresponding to the conjugacy classes of the maximal toral elementary abelian and a rank two nontoral.*

In fact, those two subgroups of  $PU(p)$  already showed up along the proof of Lemma 2.2 after the identification in Lemma 2.1.

The first lemma in this section shows that Conjecture 1.2 holds for rank two elementary abelian groups,

**Lemma 3.2.** *Let  $x$  be an odd dimensional element of  $H^*B(\mathbb{Z}/p)^2 = E(x_1, x_2) \otimes \mathbb{F}_p[y_1, y_2]$ , then there exists an  $i$  such that  $Q_mx$  is not trivial for all  $m > i$ .*

*Proof.* First notice that  $Q_n x_i = y_i^{p^n}$  and  $Q_n y_i = 0$ . Now, if  $x$  is odd dimensional, then  $x = x_1 f + x_2 g$ , where  $f, g \in \mathbb{F}_p[y_1, y_2]$ . If  $Q_n x$  is nontrivial for all  $n$ , lemma holds. So, let  $r$  be an integer such that  $Q_r x = 0$ . Then  $Q_r x = y_1^{p^r} f + y_2^{p^r} g = 0$  and therefore there exists  $h \in \mathbb{F}_p[y_1, y_2]$  such that  $f = y_2^{p^r} h$  and  $g = -y_1^{p^r} h$ . For  $m > r$  we have that

$$Q_m x = y_1^{p^m} f + y_2^{p^m} g = y_1^{p^m} y_2^{p^r} h - y_2^{p^m} y_1^{p^r} h = (y_1^{p^m - p^r} - y_2^{p^m - p^r}) y_1^{p^r} y_2^{p^r} h$$

is nontrivial. □

Finally

**Theorem 3.3.** *For each odd dimensional element  $x \in H^*BPU(p)$ , there is  $i$  such that such that  $Q_mx \neq 0$  for all  $m \geq i$ .*

*Proof.* Let  $x$  be in  $H^*BPU(p)$  an odd dimensional element. By Theorem A,  $Bj^*(x)$  is nontrivial for some  $j: E \longrightarrow PU(p)$ , where  $E$  is an elementary abelian  $p$ -group. If  $E$  is toral, then  $j$  factors through maximal torus  $i_T: T \longrightarrow PU(p)$ . As  $H^*BT$  is concentrated in even degrees,  $Bj^*$  is trivial on elements of odd degree. Therefore  $Bj^*(x)$  is a non trivial odd dimensional element in  $H^*BV$  for  $j: V \longrightarrow PU(p)$  the non toral elementary abelian subgroup which is of rank two by Proposition 3.1. By the previous lemma, there exists  $i$  such that for all  $m > i$ ,  $Q_m Bj^*(x) = Bj^*(Q_m x)$  is nontrivial. Thus for all  $m > i$ ,  $Q_m x$  is nontrivial. □

#### 4. COHOMOLOGICAL UNIQUENESS

In this section we proceed to prove Theorem C in the case  $p > 3$ . So in what follows  $X$  is a  $p$ -complete space, such that there exists an isomorphism  $\phi: H^*BPU(p) \cong H^*X$  as an unstable algebra over the Steenrod algebra  $\mathcal{A}_p$ , for  $p > 3$ .

The idea is to construct a homotopy equivalence  $BPU(p)_p^\wedge \longrightarrow X$  by means of the cohomology decomposition of  $BPU(p)$  given by  $p$ -stubborn subgroups [14].

Recall that given a compact Lie group  $G$ , a subgroup  $P \subset G$  is called  $p$ -stubborn [14, pag. 186] if the following conditions hold:

- The connected component of  $P$  is a torus and  $\pi_0 P$  is a  $p$ -group.
- The quotient group  $N_G(P)/P$  is finite and possesses no nontrivial normal  $p$ -subgroups

Then if  $\mathcal{R}_p(G)$  denotes the full subcategory of the orbit category of  $G$  whose objects are the homogeneous spaces  $G/P$  where  $P \subset G$  is  $p$ -stubborn, the natural map

$$\text{hocolim}_{G/P \in \mathcal{R}_p(G)} EG/P \longrightarrow BG$$

induces an isomorphism of homology with  $\mathbb{Z}_{(p)}$ -coefficients [14, Theorem 4].

The  $p$ -stubborn subgroups of  $PU(p)$  are described in the next proposition.

**Proposition 4.1.** *The group  $PU(p)$  contains exactly three  $p$ -stubborn subgroups up to conjugation:*

- (1) *the maximal torus  $T$ ,*
- (2) *the  $p$ -normalizer  $N_p \stackrel{\text{def}}{=} N_p(PU(p))$  of the maximal torus, and*
- (3) *the group  $E_2 = (\mathbb{Z}/p)^2$  generated by the diagonal matrix  $\text{Diag}(1, \zeta, \dots, \zeta^{p-1})$ , where  $\zeta$  is a  $p^{\text{th}}$  root of the unit, and a permutation matrix which corresponds to the cyclic permutation  $(1, 2, \dots, p)$ .*

*Proof.* By [14, Proposition 1.6],  $P \subset SU(p)$  is a  $p$ -stubborn subgroup if and only if  $P/(P \cap Z)$  is a  $p$ -stubborn subgroup of  $PU(p)$ , where  $Z \cong \mathbb{Z}/p$  is the center of  $SU(p)$ . Finally, [27, Theorems 6, 8 & 10] describe all the conjugacy classes of  $p$ -stubborn groups in  $SU(p)$ , what leads to the desired result.  $\square$

Let  $\tilde{\mathcal{R}}_p(PU(p))$  be the full subcategory of  $\mathcal{R}_p(PU(p))$  with only the three objects:  $PU(p)/T$ ,  $PU(p)/N_p$ , and  $PU(p)/E_2$ . Then the strategy is to construct a homotopy commutative diagram (Lemma 4.3)

$$\{EG/P \simeq BP\}_{PU(p)/P \in \tilde{\mathcal{R}}_p(PU(p))} \xrightarrow{f_P} X$$

such that it can be lifted to the topological category (after Proposition 4.5) so then we can recover  $BPU(p)$  (up to  $p$ -completion) as a hocolim.

As every  $p$ -stubborn  $P \subset PU(p)$  such that  $PU(p)/P \in \tilde{\mathcal{R}}_p(PU(p))$  appears as a subgroup of  $N \stackrel{\text{def}}{=} N(PU(p))$ , we first construct a map  $BN \longrightarrow X$ .

**Theorem 4.2.** *There exists a map  $f_N: BN \longrightarrow X$  such that the diagram*

$$(2) \quad \begin{array}{ccc} & H^*BN & \\ Bi_N^* \nearrow & & \nwarrow f_N^* \\ H^*BPU(p) & \xrightarrow[\cong]{\phi} & H^*X \end{array}$$

*commutes.*

*Proof.* Let  $i_E: E = (\mathbb{Z}/p)^{p-1} \longrightarrow T \longrightarrow PU(p)$  be the maximal elementary abelian  $p$ -subgroup of  $PU(p)$ . By Lannes' theory [18, Théorème 3.1.1.], there exists a map  $f_E: BE \longrightarrow X$  such that  $f_E^* = Bi_E^* \phi^{-1}: H^*X \longrightarrow H^*BE$ . By [18, Proposition 3.4.6.],

$$T_{Bi_E^*}^E H^*BPU(p)_p^\wedge \cong H^* \text{Map}(BE, BPU(p)_p^\wedge)_{Bi_E^\wedge}.$$

Since

$$\text{Map}(BE, BPU(p)_p^\wedge)_{Bi_E^\wedge} \simeq BC_{PU(p)}(E)_p^\wedge \simeq BT_p^\wedge,$$

where  $C_{PU(p)}(E)$  denotes the centralizer ([13],[25]), it follows that

$$T_{f_E^*}^E H^*X \cong T_{Bi^*}^E H^*BPU(p) \cong H^*BT_p^\wedge.$$

Because  $T_{f_E^*}^E H^* X$  is zero in dimension 1, we can use [18, Théorème 3.2.1.] and obtain

$$T_{f_E^*}^E H^* X \cong H^* \text{Map}(BE, X)_{f_E}.$$

Therefore the mapping space  $\text{Map}(BE, X)_{f_E}$  has the same cohomology ring as  $BT_p^\wedge$ . The mapping space  $\text{Map}(BE, X)_{f_E}$  is  $p$ -complete [18, Proposition 3.4.4], hence  $BT_p^\wedge \simeq \text{Map}(BE, X)_{f_E}$ .

Now, the standard action of  $W_{PU(p)} = \Sigma_p$  on  $T$  restricts to an action on  $E$ , which induces an action of  $\Sigma_p$  on  $\text{Map}(BE, X)$ . If  $\sigma \in \Sigma_p$ , then  $Bi_E \simeq Bi_E \sigma$ , and therefore

$$f_E^* = Bi_E^* \phi^{-1} = \sigma^* Bi_E^* \phi^{-1} = \sigma^* f_E^*,$$

and by Lannes' theory [18, Théorème 3.1.1],  $f_E \simeq f_E \sigma$ . This means that  $\Sigma_p$  acts on  $\text{Map}(BE, X)_{f_E}$ .

Consider now the space  $Y = \text{Map}(BE, X)_{f_E} \times_{\Sigma_p} E\Sigma_p$  which fits in the fibration

$$\text{Map}(BE, X)_{f_E} \longrightarrow Y \longrightarrow B\Sigma_p.$$

Fibrations with fiber  $\text{Map}(BE, X)_{f_E}$  and base  $B\Sigma_p$  are classified by

$$\mathcal{H}^n(B\Sigma_p; \pi_n(\text{Map}(BE, X)_{f_E})) = H^2(B\Sigma_p; \pi_2(\text{Map}(BE, X)_{f_E})).$$

According to [2, Theorem 3.6], this group is trivial (recall that  $p \geq 5$ ) which shows that  $Y \simeq BN_p^\circ$ , the fiberwise  $p$ -completion of  $BN$ .

Let  $f_N: \text{Map}(BE, X)_{f_E} \times_{\Sigma_p} E\Sigma_p \longrightarrow X$  denote the evaluation map. We have to prove that the diagram (2) commutes, that is, that  $f_N^* \phi = Bi_N^*$ . Let us define  $a = f_N^* \phi$ .

Since  $H^*BN$  is detected by elementary abelian subgroups (Lemma 2.3), it is enough to prove that  $Bj_V^* a = Bj_V^* Bi_N^*$  for every elementary abelian  $p$ -group  $j_V: V \longrightarrow N$ . In fact, the proof of Lemma 2.3 and Lemma 2.2 shows that it is enough to consider  $E$ , the maximal toral elementary abelian subgroup, and  $V_n$ , the nontoral elementary abelian subgroup of rank two that coincides with  $E_2$  in Proposition 4.1.

By construction of the map  $f_N$ , the composition

$$H^*BPU(p) \xrightarrow{a} H^*BN \xrightarrow{Bi^*} H^*BT$$

is the same as  $Bi_T^*$ . Therefore  $Bj_E^* a = Bj_E^* Bi_N^*$  for  $V = E$ .

Let now consider the case  $V = E_2$ . As  $Bj_E$  factors through  $BT$ ,  $Bj_E$  can detect only even dimensional elements in  $H^*BN$ . Therefore,  $Bj_V$  detects  $H^{odd}BN$ . In particular  $Bj_V$  detects  $H^3BN \supset H^3BPU(p) \subset \beta H^2BPU(p)$  which is nontrivial (notice that  $H_2(BPU(p), \mathbb{Z}) \cong \pi_1 PU(p) = \mathbb{Z}/p$  so the Universal Coefficient Theorem for cohomology [19, Theorem 4.3 in pag. 163] and the description of the Bockstein morphism [20, pag. 455] imply the statement). As  $f_T^* H^2 X \neq 0$  by construction, then  $f_N^* H^3 X \subset \beta H^2 X \neq 0$  and  $f_N Bj_V$  detects  $H^3 X$  as well. Finally, by Lemma 3.1 the group  $PU(p)$  has only one nontoral elementary  $p$ -subgroup, hence by Lannes' theory there exists just one morphism of unstable algebras  $H^*BPU(p) \xrightarrow{\psi} H^*B\hat{V}$  such that  $\hat{V}$  is a nontrivial elementary abelian,  $H^*B\hat{V}$  is a finite module over  $H^*BPU(p)$  (via  $\psi$ ) and  $\psi H^{odd}BPU(p) \neq 0$ , thus  $\hat{V} = V = E_2$  and  $\psi = Bj_V^* Bi_N^*$ , as well as  $\psi = Bj_V^* \phi f_N^* = Bj_V^* a$ , thus  $Bj_V^* Bi_N^* = j_V'^* Bi_N^* = Bj_V^* a$  also in this case.  $\square$

Define maps  $f_P: EPU(p)/P \simeq BP \longrightarrow BN \xrightarrow{f_N} X$  for  $P = T, N_p$ , and  $E_2$ . This gives rise to a diagram

$$(3) \quad \{EG/P \simeq BP\}_{PU(p)/P \in \tilde{\mathcal{R}}_p(PU(p))} \xrightarrow{f_P} X$$

Next lemma shows diagram (3) commutes up to homotopy.

**Lemma 4.3.** *For every two objects  $P$  and  $Q$  in  $\tilde{\mathcal{R}}_p(PU(p))$  and morphism  $c_g \in \text{Mor}(P, Q)$  the diagram*

$$\begin{array}{ccc} BP & \xrightarrow{f_P} & X \\ Bc_g \downarrow & & \parallel \\ BQ & \xrightarrow{f_Q} & X \end{array}$$

commutes.

*Proof.* Because every morphism in  $\tilde{\mathcal{R}}_p(PU(p))$  is a composition of an automorphism and an inclusion, it is enough to prove that the diagram

$$\begin{array}{ccc} BP & \xrightarrow{f_P} & X \\ Bc_g \downarrow & & \parallel \\ BP & \xrightarrow{f_P} & X \end{array}$$

commutes for every object  $PU(p)/P$  in  $\tilde{\mathcal{R}}_p(PU(p))$ . If  $P = T$ , then the element  $g$  is in the normalizer  $N$ , hence the diagram

$$\begin{array}{ccccc} BP & \longrightarrow & BN & \xrightarrow{f_N} & X \\ Bc_g \downarrow & & \parallel & & \parallel \\ BP & \longrightarrow & BN & \xrightarrow{f_N} & X \end{array}$$

commutes.

Let  $P = N_p$ . Since  $c_g(N_p) = N_p$ , and  $T$  is the connected component of  $N_p$ , also  $c_g(T) = T$ , hence  $g \in N$ . Again we get a commutative diagram as in the previous case.

Let  $P = E_2$ . Then  $Bi_{E_2}^* = Bc_g^* Bi_{E_2}^*$ , since  $Bi_{E_2} \simeq Bi_{E_2} Bc_g$ , and therefore

$$f_{E_2}^* = Bi_{E_2}^* \phi^{-1} = Bc_g^* Bi_{E_2}^* \phi^{-1} = Bc_g^* f_{E_2}^*.$$

By Lannes' theory [18, Théorème 3.1.1.],  $f_{E_2} \simeq f_{E_2} Bc_g$ , which finishes the proof.  $\square$

The diagram (3) commutes only up to homotopy, hence we do not know if the collection of maps  $\{f_P\}_{PU(p)/P \in \tilde{\mathcal{R}}_p(PU(p))}$  induces a map

$$\text{hocolim}_{PU(p)/P \in \tilde{\mathcal{R}}_p(PU(p))} EPU(p)/P \longrightarrow X.$$

The obstructions lie in the groups

$$\varprojlim_{\mathcal{R}_p(PU(p))}^i \pi_j(\text{Map}(BP, X)_{f_P}),$$

where  $\text{lim}^i$  is the  $i$ -th derived functor of the inverse limit functor ([4] and [34]). Now we will prove that all obstruction groups are trivial.

Let

$$\Pi_j^X, \Pi_j^{PU(p)}: \tilde{\mathcal{R}}_p(PU(p)) \longrightarrow \mathcal{A}b$$

be functors defined by

$$\Pi_j^X(PU(p)/P) = \pi_j(\text{Map}(BP, X)_{f_P}),$$

$$\Pi_j^{PU(p)}(PU(p)/P) = \pi_j(\text{Map}(BP, BPU(p)_p^\wedge)_{(Bi_P)_p^\wedge}),$$

where  $\mathcal{A}b$  is the category of abelian groups. Note that  $\text{Map}(BP, BPU(p)_p^\wedge)_{(Bi_P)_p^\wedge} \cong BZ(P)_2^\wedge$  [14, Theorem 3.2] therefore  $\Pi_1^{PU(p)}(PU(p)/P)$  is well defined and, by the next lemma, also  $\Pi_1^X(PU(p)/P)$  is well defined.

**Lemma 4.4.** *There exists a natural transformation  $\mathcal{T}: \Pi_j^{PU(p)} \longrightarrow \Pi_j^X$  which is an equivalence.*

*Proof.* Let  $P$  be the maximal torus  $T$  of the  $p$ -normalizer  $N_p$ , and let  $E \cong (\mathbb{Z}/p)^{p-1}$  be the maximal toral elementary abelian subgroup in  $N$ . We apply Lannes'  $T$  functor to the diagram

$$(4) \quad \begin{array}{ccc} & H^*BN & \\ Bi_N^* \nearrow & & \nwarrow f_N^* \\ H^*BPU(p) & & H^*X \end{array}$$

and get

$$\begin{array}{ccc} & T_{Bi_E^*}^E H^*BN & \\ T_{Bi_E^*}^E H^*BPU(p) \nearrow & & \nwarrow T_{f_E^*}^E H^*X \end{array}$$

By [18, Théorème 3.4.5], it follows that

$$T_{Bi_E^*}^E H^*BN \cong H^*BC_T(E) = H^*BT,$$

$$T_{Bi_E^*}^E H^*BPU(p) \cong H^*BC_{PU(p)}(E) = H^*BT,$$

and the left map in the above diagram is an isomorphism. Because  $T_{f_E^*}^E H^*X \cong T_{Bi_E^*}^E H^*BPU(p) \cong H^*BT$ , it is zero in the degree 1, hence by [18, Théorème 3.2.1.],

$T_{f_E}^E H^* X \cong H^* \text{Map}(BE, X)_{f_E}$  and the right map in the diagram is an isomorphism. We conclude that in the diagram

$$\begin{array}{ccc} & \text{Map}(BE, BN_p^\circ)_{(Bi_E)_p^\wedge} & \\ & \swarrow \quad \searrow & \\ \text{Map}(BE, BPU(p)_p^\wedge)_{(Bi_E)_p^\wedge} & & \text{Map}(BE, X)_{f_E} \end{array}$$

both maps are equivariant mod  $p$  equivalences. Taking homotopy fixed points we obtain the following diagram

$$\begin{array}{ccc} & \text{Map}(BE, BN_p^\circ)^{h(P/E)}_{(Bi_E)_p^\wedge} & \\ & \swarrow \quad \searrow & \\ \text{Map}(BE, BPU(p)_p^\wedge)^{h(P/E)}_{(Bi_E)_p^\wedge} & & \text{Map}(BE, X)_{f_E}^{h(P/E)}. \end{array}$$

where both maps are mod  $p$  equivalences, since an equivariant mod  $p$  equivalence between 1-connected spaces induces a mod  $p$  equivalence between the homotopy fixed-point sets. Using  $\text{Map}(BP, \cdot) \simeq \text{Map}(BE, \cdot)^{h(P/E)}$ , we obtain mod  $p$  equivalences

$$\begin{array}{ccc} & \text{Map}(BP, BN_p^\circ)_{(Bi_P)_p^\wedge} & \\ & \swarrow \quad \searrow & \\ \text{Map}(BP, BPU(p)_p^\wedge)_{(Bi_P)_p^\wedge} & & \text{Map}(BP, X)_{f_P}. \end{array}$$

Let us consider the remaining case  $P = E_2 \cong (\mathbb{Z}/2)^2$ . Applying Lannes' functor to diagram (4) gives

$$\begin{array}{ccc} & T_{Bi_P^*}^P H^* BN & \\ & \swarrow \quad \searrow & \\ T_{Bi_P^*}^P H^* BPU(p) & & T_{f_P}^P H^* X \end{array}$$

By [18, Théorème 3.4.5], we get

$$T_{Bi_P^*}^P H^* BN \cong H^* BC_N(P) = H^* BP,$$

$$T_{Bi_P^*}^P H^* BPU(p) \cong H^* BC_{PU(p)}(P) = H^* BP,$$

and the left map is an isomorphism. Since  $T_{f_P}^P H^* X$  is free in dimension  $\leq 2$ , it follows by [18, Théorème 3.2.4] that  $T_{f_P}^P H^* X \cong H^* \text{Map}(BP, X)_{f_P}$  and also the right map is an isomorphism. So, in the diagram

$$\begin{array}{ccc} & \text{Map}(BP, BN_p^\circ)_{Bi_P} & \\ & \swarrow \quad \searrow & \\ \text{Map}(BP, BPU(p)_p^\wedge)_{Bi_P} & & \text{Map}(BP, X)_{f_P} \end{array}$$

both maps are mod  $p$  equivalences.

We have shown that in all cases ( $P = N_p, T$ , or  $E_2$ ) the map

$$\mathrm{Map}(BP, BPU(p)^\wedge_{(Bi_P)_p}) \longrightarrow \mathrm{Map}(BP, X)_{f_P}$$

is a mod  $p$  equivalence. Because  $\mathrm{Map}(BP, BPU(p)^\wedge_{(Bi_P)_p})$  and  $\mathrm{Map}(BP, X)_{f_P}$  are  $p$ -complete spaces [18, Proposition 3.4.4.], above map is a homotopy equivalence. To see that this homotopy equivalence is natural, we have to prove that it commutes with maps induced by conjugation, which means that we have to show that the diagram

$$\begin{array}{ccc} & \mathrm{Map}(BP, BN_p^\circ)_{(Bi_P)_p^{Bc_g}} & \\ & \swarrow \quad \searrow & \\ \mathrm{Map}(BP, BPU(p)^\wedge_{(Bi_P)_p}) & \longleftarrow \mathrm{Map}(BP, BN_p^\circ)_{(Bi_P)_p} \longrightarrow & \mathrm{Map}(BP, X)_{f_P} \end{array}$$

commutes. This follows from two commutative diagrams,

$$\begin{array}{ccc} BC_{PU(p)}(P)_p^\wedge & \xlongequal{\quad\quad\quad} & BC_N(P)_p^\wedge \\ \downarrow Bc_g & \searrow \simeq & \downarrow \simeq \\ & \mathrm{Map}(BP, BPU(p)^\wedge_{(Bi_P)_p}) \longleftarrow \mathrm{Map}(BP, BN_p^\circ)_{(Bi_P)_p} & \\ & \parallel & \\ & \mathrm{Map}(BP, BPU(p)^\wedge_{(Bi_P)_p^{Bc_g}}) \longleftarrow \mathrm{Map}(BP, BN_p^\circ)_{(Bi_P)_p^{Bc_g}} & \\ \simeq \nearrow & & \uparrow \simeq \\ BC_{PU(p)}(c_g(P))_p^\wedge & \xlongequal{\quad\quad\quad} & BC_N(c_g(P))_p^\wedge \end{array}$$

and

$$\begin{array}{ccc} BC_N(P)_p^\wedge & \xlongequal{\quad\quad\quad} & BC_{PU(p)}(P)_p^\wedge \\ \downarrow \simeq & & \swarrow \simeq \\ \mathrm{Map}(BP, BN_p^\circ)_{(Bi_P)_p} & \longrightarrow & \mathrm{Map}(BP, X)_{f_P} \\ & & \parallel \\ \mathrm{Map}(BP, BN_p^\circ)_{(Bi_P)_p^{Bc_g}} & \longrightarrow & \mathrm{Map}(BP, X)_{f_P^{Bc_g}} \\ \uparrow \simeq & & \swarrow \simeq \\ BC_N(c_g(P))_p^\wedge & \xlongequal{\quad\quad\quad} & BC_{PU(p)}(c_g(P))_p^\wedge \end{array}$$

which can be glued together.  $\square$

**Proposition 4.5.** *For all  $i, j \geq 1$ ,*

$$\varprojlim_{\mathcal{R}_p(PU(p))}^i \pi_j(\mathrm{Map}(BP, X)_{f_P}) = 0.$$

*Proof.* By the previous lemma,

$$\varprojlim_{\mathcal{R}_p(PU(p))}^i \pi_j(\text{Map}(BP, X)_{f_P}) = \varprojlim_{\mathcal{R}_p(PU(p))}^i \pi_j(\text{Map}(BP, BPU(p)_p^\wedge)_{(Bi_P)_p^\wedge})$$

and the right side is 0 [14, Theorem 4.8].  $\square$

Because all obstructions vanish, there exists a map  $f: BPU(p)_p^\wedge \longrightarrow X$ . By construction of the map  $f$ , the diagram

$$\begin{array}{ccc} & (BN_p)_p^\wedge & \\ Bi_N \swarrow & & \searrow f_N \\ BPU(p)_p^\wedge & \xrightarrow{f} & X \end{array}$$

commutes. The Euler characteristic  $\chi(PU(p)/N_p) \not\equiv 0 \pmod{p}$ , hence a transfer argument shows that  $Bi_N^*$  is a monomorphism. By Theorem 4.2, also  $f_N^*$  is a monomorphism. Therefore,  $f^*$  is a monomorphism and, because  $H^*BPU(p) \cong H^*X$  is a finite dimensional in each degree,  $f^*$  is an isomorphism. This shows that  $f$  is a homotopy equivalence.

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