ON THE HOMOTOPY TYPE OF THE CLASSIFYING SPACE OF THE EXCEPTIONAL LIE GROUP OF RANK 4

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Abstract. Previous work of several authors shows that the exceptional Lie group of rank 4, F_4, as a p-compact group, is determined up to isomorphism by the isomorphism type of its maximal torus normalizer for $p > 2$. This paper considers the case $p = 2$ proving that F_4 as 2-compact group is also determined up to isomorphism by the isomorphism type of its maximal torus normalizer. This allow the authors to determine the integral homotopy type of F_4 among connected finite loop spaces with maximal tori.

1. Introduction

One of the major problems in Homotopy Theory is the understanding and classification of finite loop spaces. A loop space $L := (L, BL, e)$ consists of a pair of spaces L and BL, BL pointed, and a homotopy equivalence $e : \Omega BL \simeq L$ defining a loop structure on L. The space BL is called the classifying space of L. A loop space $L := (L, BL, e)$ is called finite (resp. $\mathbb{F}_p$-finite) if $H^*(L; \mathbb{Z})$ (resp. $H^*(L; \mathbb{F}_p)$) is finitely generated as graded abelian group. Examples of finite loop spaces are given by compact Lie groups; for every compact Lie group G, being BG its honest classifying space, there exists a canonical equivalence $e : G \simeq \Omega BG$ which establishes a finite loop space structure (G, BG, e) on G.

A great inroad in the subject was the advent of p-compact groups. In the celebrated paper [14], Dwyer and Wilkerson introduced the concept of p-compact group, a homotopy theoretic generalization of compact Lie group. Given a prime number p, a loop space $X := (X, BX, e)$ is said to be a p-compact group if X is $\mathbb{F}_p$-finite, and BX is p-complete in the sense of Bousfield-Kan [7]. Again examples of p-compact groups are given by the p-completion of compact Lie groups, the triple $G^\wedge_p := (G^\wedge_p, BG^\wedge_p, e)$, is a p-compact group when $\pi_0 G$ is a finite p-group. In this way p-compact tori appear: a p-compact torus of rank n is a triple (T, BT, e) where $BT \simeq K\big((\mathbb{Z}^\wedge_p)^{\oplus n}, 2\big)$ is an Eilenberg-MacLane space of degree 2, that is, a p-compact torus of rank n is the p-completion of a rank n torus. Further examples are given by the realization of polynomial algebras, i.e., by pairs $(\Omega BX, BX)$, where BX has polynomial mod p cohomology (see [1], [8], [13], [28], [31] and [39]).

These purely homotopy theoretic objects, possess much of the rich internal structure of compact Lie groups so it is possible to set up a dictionary translating constructions and arguments from the algebraic theory of groups to the homotopical setting of p-compact groups (see the original [14], or the reviews [11], [20] and [27]). In particular, it is possible to define:

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- Homomorphisms [14, §3.1]: A homomorphism $X \xrightarrow{f} Y$ of p-compact groups is a pointed map $BX \xrightarrow{Bf} BY$. The homomorphism f is an isomorphism if Bf is a homotopy equivalence. It is a monomorphism if the homotopy fiber Y/X of Bf is $\mathbb{F}_p$-finite or equivalently if $H^*(BX, \mathbb{F}_p)$ is a finitely generated module over $H^*(BY, \mathbb{F}_p)$ via Bf^*.
- Centralizers [14, §3.4]: For a homomorphism $Y \xrightarrow{f} X$ of p-compact groups, the centralizer $C_X\big(f(Y)\big)$ is defined by the equation $BC_X\big(f(Y)\big) := \mathrm{map}(BY, BX)_{Bf}$.
- Maximal tori [14, Definition 8.9]: A monomorphism $T \to X$ of a p-compact torus into a p-compact group X is a maximal torus if $C_X(T)$ is a p-compact toral group and if $C_X(T)/T$ is homotopically discrete. Every p compact group admits maximal tori [14, Theorem 8.13]).
- Weyl group [14, Definition 9.2]: Let $BT_X \xrightarrow{Bf_T} BX$ be a maximal torus of a p-compact group X. Assume that Bf_T is already a fibration and treat $\mathbb{W}_X$ as the space of self-maps of BT_X over BX. Composition gives $\mathbb{W}_X$ the structure of an associative topological monoid. It is shown [14, Proposition 9.5]) that $\mathbb{W}_X$ is homotopically discrete and therefore $W_X := \pi_0\mathbb{W}_X$ is a (finite) group. Moreover, the action of W_X on BT_X induces a faithful representation
$$W_X \rightarrowtail \mathrm{GL}(H^*_{\mathbb{Q}_p} BT_X) \cong \mathrm{GL}_n(\mathbb{Q}_p^\wedge)$$
whose image is generated by pseudo reflections, i.e. W_X is a pseudo reflection group [14, Theorem 9.7].
- Maximal torus normalizers [14, Definition 9.8]): Let $BT_X \xrightarrow{Bf_T} BX$ be a maximal torus of a p-compact group X. The normalizer of T_X denoted by NT_X, or simply by N_X, is the loop space such that BNT_X is the Borel construction of the action of $\mathbb{W}_X$ on BT_X.

It has been conjectured (see [23]) that two p-compact groups are isomorphic if and only if their maximal torus normalizers are isomorphic, that is, if X and Y are p-compact groups such that N_X and N_Y are homotopy equivalent loop spaces, then $X \cong Y$ as p-compact groups. Those p-compact groups that verify the conjecture are called N-determined.

It has been shown that for $p > 2$, all p-compact groups are N-determined (see [2]), but for $p = 2$, the conjecture fails as maximal torus normalizer cannot control connectivity. So $SO(3)^\wedge_2$ is not N-determined since its maximal torus normalizer is isomorphic to $O(2)^\wedge_2$, which is again a 2-compact group, but $SO(3)^\wedge_2$ and $O(2)^\wedge_2$ are not isomorphic 2-compact groups, although they have isomorphic maximal torus normalizers.

So one is forced to consider a weaker version of the normalizer conjecture for $p = 2$ (see [29]): we shall say that a 2-compact group X is weakly N-determined if for any 2-compact group Y, $X \cong Y$ as 2-compact groups if and only if there exists an homotopy equivalence of loop space $N_X \simeq N_Y$ inducing an isomorphism $\pi_0 X \cong \pi_0 Y$. Of course, N-determined 2-compact groups are weakly N-determined.

It has been shown that the 2-compact groups $(G_2)^\wedge_2$ ([34]) and $O(n)^\wedge_2$ ([29]) are weakly N-determined 2-compact groups (although they are not N-determined), and that unitary groups ([25]) and $DI(4)$ ([30]) are N-determined .

All those previous cases cited above share a common property: the classifying spaces realize polynomial algebras which are the invariants of a precise action on the cohomology of an elementary abelian 2-group (with a suitable Frobenius twist). There are two more 2-compact group families with that nice property: the 2-completion of simplectic groups $Sp(n)$, and the 2-completion of the exceptional Lie group of rank 4, F_4 (see Proposition 2.5). This fact makes centralizer calculations easier, and a tricky use of the 2-stubborn subgroup decomposition ([18]) of F_4 allows us to prove that the 2-compact group $(F_4)^\wedge_2$ is N-determined, that is,

Theorem 1.1. *Let X be a 2-compact group with maximal torus normalizer $N \xrightarrow{j} X$, if N is homotopy equivalent to $N_{(F_4)^\wedge_2}$ as a loop space, then X and $(F_4)^\wedge_2$ are isomorphic 2-compact groups.*

The case of the 2-compact groups obtained by the 2-completion of $Sp(n)$ is considered in [33].

The normalizer conjecture can also be stated for finite loop spaces as a weak version of Wilkerson's conjecture on finite loop spaces with maximal tori (see [36]): Let L be a connected finite loop space with maximal torus normalizer isomorphic to that of a compact connected Lie group G, then it is conjecture that BL is homotopy equivalent to BG.

If L is a connected finite loop space with maximal torus normalizer N, isomorphic to that of a compact connected Lie group G, proving that $BL \simeq BG$ is equivalent to prove that BL and BG lie in the same adic genus (see [26]). The rational case is trivial so we fix our attention to the p-completion of BL. As L is finite and connected, its p-completion gives rise to a p-compact group $L^\wedge_p$ such that

- its classifying space is $BL^\wedge_p$
- its maximal torus normalizer is just the fiberwise completion of N by the fibration $BT \longrightarrow BN \longrightarrow BW_L = BW_G$.

In other words, p-completion of L, or BL, gives rise to a connected p-compact group whose maximal torus normalizer is isomorphic to that of the p-compact group $G^\wedge_p$. Therefore BL and BG lie in the same genus if and only if the p-compact group $G^\wedge_p$ is weakly N-determined for all primes p.

Previous works show that the group F_4 considered as a p-compact group is N-determined for $p > 2$ (see [25] for $p > 3$, [35] for $p = 3$ and [2] for the general case). Those results combined with Theorem 1.1 allow us to prove:

Corollary 1.2. *Let L be a connected finite loop space with maximal torus normalizer isomorphic to that of F_4. Then BL is homotopy equivalent to BF_4.*

Organization of the paper: In Section §2 we describe the mod 2 cohomology of BF_4. This description allow us to easily calculate the Quillen category of F_4 at $p = 2$ in Section §3. In Section §4, we exhibit an interesting connection between the p-stubborn category and the Quilllen category of a Lie group G. Finally, in Section §5 we prove Theorem 1.1

Notation: Here $\mathcal{A}_2$ is the mod 2 Steenrod algebra, all spaces are assumed to have the homotopy type of CW-complexes. Given a space Y, we write H^*Y for $H^*(Y;\mathbb{F}_2)$, $H^*_{\mathbb{Q}^\wedge_2}(Y)$ for $H^*(Y;\mathbb{Z}^\wedge_2) \otimes \mathbb{Q}$ and $Y^\wedge_p$ for the Bousfield-Kan $(\mathbb{Z}/p)_\infty$-completion, or p-completion, of the space Y ([7]). Given a p-compact group X, $\mathcal{Q}_p(X)$ denotes the Quillen category of X at the prime p (see Section 3). For a compact Lie group G, and a prime p, we write $\mathcal{R}_p(G)$ for the p-stubborn category (see Section 4). For a group G and an element $g \in G$, c_g denotes the inner group automorphism induced by conjugation by g. Given two groups K and G, we denote by $\mathrm{Mono}(K, G)$ the set of G-conjugacy classes of monomorphisms $K \overset{f}{\rightarrowtail} G$.

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2. The mod 2 cohomology of BF_4

The algebra structure of H^*BF_4 has been known since Borel: in [4], the cohomology ring of H^*F_4 is calculated, and in view of [3, Théorèmes 17.3 and 19.2], the generators can be taken universally transgressive so it follows,

Theorem 2.1. *The mod 2 cohomology of BF_4 is isomorphic to the polynomial algebra*

$$H^*BF_4 = \mathbb{F}_2[y_4, y_6, y_7, y_{16}, y_{24}].$$

Moreover, $Sq^2 y_4 = y_6$, $Sq^1 y_6 = y_7$, and $Sq^8 y_{16} = y_{24}$.

Nevertheless, the aim of this section is giving a more appropriate description of H^*BF_4 as the ring of invariants on 5 variables under the action of a certain group of block matrices. That description involves some understanding of the elementary abelian 2-subgroups of F_4. Recall that:

- Given a group G and a subgroup H, it is said that H is pure if all its elements, but the neutrum, are in the same G-conjugacy class.
- Given an abelian group G, ${}_nG$ denotes the subgroup of elements of order n.

Then the basic data about elementary abelian subgroups in F_4 can be found in [16, Tables I and VI, and Theorem 7.3]:

Proposition 2.2. *In F_4 there are exactly two conjugacy classes of elements of order two listed below*

Class	*Centralizer*	*no. in T*
$2A$	$SU(2) \times_{\mathbb{Z}/2} Sp(3)$	12
$2B$	$Spin(9)$	3

Moreover:

i) *Any elementary abelian 2-subgroup of F_4 is toral if and only if it does not contain a $2A$-pure eights group.*
ii) *There is, up to conjugacy, a unique maximal elementary abelian 2-subgroup represented, say, by $E_{32} := \langle {}_2T, \theta \rangle$, where θ is an involution in $N_{F_4}T$ inducing -1 on T. Then, E_{32} has rank 5 and there is a $2B$-pure subgroup $\hat{E}_4$ of order 4 such that $E_{32} \backslash \hat{E}_4 = E_{32} \cap 2A$. Also, $E_{32} = C_{F_4}(E_{32})$ and $N_{F_4}(E_{32})/E_{32} \cong 2^{2 \cdot 3} : (\mathrm{GL}_3(2) \times \Sigma_3)$.*

Indeed, the subgroup $E_{32} \subset F_4$ can be easily identified up to conjugation:

Proposition 2.3. *Consider the standard chain of inclusions*

$$G_2 \subset Spin(8) \subset Spin(9) \subset F_4.$$

Then the maximal elementary abelian 2-subgroup in F_4 is (up to conjugation) $V_3 \oplus Z(Spin(8))$ where V_3 is the maximal (non toral) elementary abelian 2-subgroup in G_2, and $\big(V_3 \oplus Z(Spin(8))\big) \cap T_{F_4} = {}_2T_{F_4}$.

Proof. According to [34, Proposition 5.3], or [16, Table I and Theorem 6.1], there exists $V_3 = (\mathbb{Z}/2)^3$ maximal elementary abelian 2-subgroup of G_2. As G_2 is centerfree, then $G_2 \cap Z(Spin(8)) = \{1\}$, and $Z(Spin(8)) \subset C_{F_4}(G_2)$. Hence $V_3 \oplus Z(Spin(8)) \cong (\mathbb{Z}/2)^5$ and this group must be the maximal elementary abelian 2-subgroup of F_4 (up to conjugation).

Finally, the chain of inclusions

$$G_2 \subset Spin(8) \subset Spin(9) \subset F_4$$

induces the chain

$$T_{G_2} \subset T_{Spin(8)} \subset T_{Spin(9)} \subset T_{F_4},$$

and as:

- $V_3 \cap T_{G_2} = {}_2T_{G_2}$ by [34, Propositions 5.3 and 5.4] and
- $Z(Spin(8)) \subset T_{Spin(8)}$

then $(\mathbb{Z}/2)^4 \cong \big(V_3 \oplus Z(Spin(8))\big) \cap T_{F_4} = {}_2T_{F_4}$. □

It is also possible to give the matrix representation of $N_{F_4}(E_{32})/E_{32} \subset \mathrm{Aut}(E_{32}) = \mathrm{GL}_5(2)$:

Proposition 2.4. *For an appropriate choice of basis, the matrix representation of $N_{F_4}(E_{32})/E_{32} \subset \mathrm{Aut}(E_{32}) = \mathrm{GL}_5(2)$ consists of matrices of the form (action on columns)*

$$\begin{pmatrix} \mathrm{GL}_2(2) & \begin{matrix} * & * & * \\ * & * & * \end{matrix} \\ 0 & \mathrm{GL}_3(2) \end{pmatrix}$$

where each $* = 0$ or 1.

Proof. The description of $W_{F_4}(E_{32}) := N_{F_4}(E_{32})/E_{32} \subset \mathrm{GL}_5(2)$ in Proposition 2.2 shows that $W_{F_4}(E_{32})$ contains a 2-Sylow subgroup of $\mathrm{GL}_5(2)$. Therefore, up to conjugacy in $\mathrm{GL}_5(2)$, or equivalently, up to a change of base in E_{32}, $W_{F_4}(E_{32})$ contains B, the subgroup of upper triangular matrices.

Following [10, §65B and §65C], B is the standard Borel subgroup of $\mathrm{GL}_5(2)$, and as $B \subset W_{F_4}(E_{32})$, then $W_{F_4}(E_{32})$ must be a standard parabolic subgroup of $\mathrm{GL}_5(2)$. There is 16 of such groups, and only two of them have the same order as $W_{F_4}(E_{32})$ (hence one of them is $W_{F_4}(E_{32})$). Those two standard parabolic subgroups, namely P_1 and P_2, consist of matrices of the form (action on columns)

$$\begin{pmatrix} \mathrm{GL}_2(2) & \begin{matrix} * & * & * \\ * & * & * \end{matrix} \\ 0 & \mathrm{GL}_3(2) \end{pmatrix} \text{ or } \begin{pmatrix} \mathrm{GL}_3(2) & \begin{matrix} * & * \\ * & * \\ * & * \end{matrix} \\ 0 & \mathrm{GL}_2(2) \end{pmatrix}$$

respectively. But the conjugacy class distribution described in Proposition 2.2.*ii)*, forces $W_{F_4}(E_{32}) = P_1$. □

The following theorem was communicated to the authors by C. Wilkerson:

Theorem 2.5. *Let $E_{32} = (\mathbb{Z}/2)^5$ be a maximal elementary abelian 2-subgroup of F_4, then $H^*BF_4 = (H^*BE_{32})^{W_{F_4}(E_{32})}$. Therefore, the Steenrod algebra action on H^*BF_4 is given by the following table*

x	Sq^1x	Sq^2x	Sq^4x	Sq^8x	$Sq^{16}x$
y_4	0	y_6	y_4^2	0	0
y_6	y_7	0	y_4y_6	0	0
y_7	0	0	y_4y_7	0	0
y_{16}	0	0	0	$y_{24}+y_4^2y_{16}$	y_{16}^2
y_{24}	0	0	y_4y_{24}	$y_4^2y_{24}$	$y_{16}y_{24}+y_4y_6^2y_{24}$

Proof. By Proposition 2.2 and Lemma 3.1 there exists $E_{32}\cong(\mathbb{Z}/2)^5\subset F_4$, maximal elementary abelian 2-subgroup, whose Weyl group in F_4, $W_{F_4}(E_{32})$, consists of matrices of type (action on columns)

$$\begin{pmatrix} \mathrm{GL}_2(2) & \begin{smallmatrix} * & * & * \\ * & * & * \end{smallmatrix} \\ 0 & \mathrm{GL}_3(2) \end{pmatrix}$$

where each $*=0$ or 1. Therefore the action on $H^*BE_{32}=\mathbb{F}_2[t_1,t_2,t_3,t_4,t_5]$, where $|t_i|=1$, is given by the transposition of those matrices. Now we will calculate the invariants $H^*(BE_{32})^{W_{F_4}(E_{32})}$.

Consider $P:=\mathbb{F}_2[z_4,z_6,z_7,z_{16},z_{24}]$ where subindex indicates degree and

- z_4, z_6 and z_7 are the Dickson invariants in the three variables t_3, t_4 and t_5,
- $z_{16}=a_1^2+a_2^2+a_1a_2$, $z_{24}=a_1a_2(a_1+a_2)$, being $a_i=t_i^8+z_4t_i^4+z_6t_i^2+z_7t_i$ for $i=1,\dots,5$. Notice that $a_i=0$ if $i\neq 1,2$.

It is clear that z_4, z_6, and z_7 are invariants by the action of $W_{F_4}(E_{32})$. Now we will show that z_{16} and z_{24} are invariants too.

If $M=[m_{i,j}]_{1\le i,j\le 5}\in W_{F_4}(E_{32})$ then (M^T means "M transposed"):

$$\begin{aligned} M^Ta_i &= (\sum_{n=1}^5 m_{i,n}t_n)^8+z_4(\sum_{n=1}^5 m_{i,n}t_n)^4+z_6(\sum_{n=1}^5 m_{i,n}t_n)^2+z_7(\sum_{n=1}^5 m_{i,n}t_n)= \\ &= \sum_{n=1}^5 m_{i,n}(t_n^8+z_4t_n^4+z_6t_n^2+z_7t_n)= \\ &= \sum_{n=1}^5 m_{i,n}a_i= \\ &= m_{i,1}a_1+m_{i,2}a_2, \end{aligned}$$

and therefore

$$\begin{aligned} M^Tz_{16} &= (m_{1,1}a_1+m_{1,2}a_2)^2+(m_{2,1}a_1+m_{2,2}a_2)^2+ \\ &\quad +(m_{1,1}a_1+m_{1,2}a_2)(m_{2,1}a_1+m_{2,2}a_2)= \\ &= (m_{1,1}+m_{1,1}m_{2,1}+m_{2,1})a_1^2+(m_{1,2}+m_{1,2}m_{2,2}+m_{2,2})a_2^2+ \\ &\quad +(m_{1,2}m_{2,1}+m_{1,1}m_{2,2})a_1a_2= \\ &= a_1^2+a_2^2+a_1a_2= \\ &= z_{16}. \end{aligned}$$

as the matrix $[m_{i,j}]_{1\le i,j\le 2}\in\mathrm{GL}_2(\mathbb{F}_2)$ which implies

$$\begin{aligned} m_{1,2}m_{2,1}+m_{1,1}m_{2,2}&=1, \\ m_{1,1}+m_{1,1}m_{2,1}+m_{2,1}&=1, \\ m_{1,2}+m_{1,2}m_{2,2}+m_{2,2}&=1. \end{aligned}$$

For any matrix $[m_{i,j}]_{1\leq i,j\leq 2} \in \mathrm{GL}_2(\mathbb{F}_2)$ it is also true that

$$\begin{aligned} m_{1,1}m_{2,1}(m_{1,2}+m_{2,2}) + m_{1,1} + m_{2,1} &= 1, \\ m_{1,2}m_{2,2}(m_{1,1}+m_{2,1}) + m_{1,2} + m_{2,2} &= 1, \\ m_{1,1}m_{2,1}(m_{1,1}+m_{2,1}) &= 0, \\ m_{1,2}m_{2,2}(m_{1,2}+m_{2,2}) &= 0, \end{aligned}$$

and therefore

$$\begin{aligned} M^T z_{24} &= (m_{1,1}a_1 + m_{1,2}a_2)(m_{2,1}a_1 + m_{2,2}a_2)(m_{1,1}a_1 + m_{1,2}a_2 + m_{2,1}a_1 + m_{2,2}a_2) = \\ &= m_{1,1}m_{2,1}(m_{1,1}+m_{2,1})a_1^3 + (m_{1,1}m_{2,1}(m_{1,2}+m_{2,2}) + m_{1,1} + m_{2,1})a_1^2a_2 + \\ &\quad + (m_{1,2}m_{2,2}(m_{1,1}+m_{2,1}) + m_{1,2} + m_{2,2})a_1a_2^2 + m_{1,2}m_{2,2}(m_{1,2}+m_{2,2})a_2^3 = \\ &= a_1^2a_2 + a_1a_2^2 = \\ &= z_{24}. \end{aligned}$$

So we show that $P \subset H^*(BE_{32})^{W_{F_4}(E_{32})}$. As the ring $\mathbb{F}_2[t_1,\dots,t_5] = H^*(BE_{32})$ is an integral extension over P, by [37, Lemma 3.2], the degree is

$$\begin{aligned} \left[\mathbb{F}_2[t_1,\dots,t_5]\colon P\right] &= \deg(z_4)\deg(z_6)\deg(z_7)\deg(z_{16})\deg(z_{24}) = \\ &= 2^{10}3^27. \end{aligned}$$

The degree of the extension

$$\left[\mathbb{F}_2[t_1,\dots,t_5]\colon \mathbb{F}_2[t_1,\dots,t_5]^{W_{F_4}(E_{32})}\right] = |W_{F_4}(E_{32})|$$

is $2^{10}3^27$ [32, Theorem 79]. By the degree formula,

$$\begin{aligned} &\left[\mathbb{F}_2[t_1,\dots,t_5]\colon P\right] = \\ &\qquad = \left[\mathbb{F}_2[t_1,\dots,t_5]\colon \mathbb{F}_2[t_1,\dots,t_5]^{W_{F_4}(E_{32})}\right]\left[\mathbb{F}_2[t_1,\dots,t_5]^{W_{F_4}(E_{32})}\colon P\right] \end{aligned}$$

and we see that

$$\left[\mathbb{F}_2[t_1,\dots,t_5]^{W_{F_4}(E_{32})}\colon \mathbb{F}_2[y_4,y_6,y_7,y_{16},y_{24}]\right] = 1.$$

Thefore

$$H^*(BE_{32})^{W_{F_4}(E_{32})} = \mathbb{F}_2[y_4,y_6,y_7,y_{16},y_{24}],$$

and the Steenrod algebra structure can be read from [37, Corollary 2.4].

Finally, consider T a maximal torus in F_4, then the diagram

$$\begin{array}{ccc} B(E_{32}\cap T) & \xrightarrow{Bi_{E_{32}\cap T}} & BT \\ \big\downarrow & & \big\downarrow{\scriptstyle Bi_T} \\ BE_{32} & \xrightarrow{Bi_{E_{32}}} & BF_4 \end{array}$$

is homotopy commutative, and

- $Bi^*_{E_{32}\cap T}$ is monomorphism, as by Proposition 2.3, $E_{32}\cap T = {}_2T$,
- $\ker Bi^*_T$ is the ideal of H^*BF_4 generated by y_7,

hence $\ker Bi^*_{E_{32}} \subset \ker Bi^*_T = \langle y_7\rangle$, which forces

$$Bi^*_{E_{32}}(y_4) = z_4, \quad Bi^*_{E_{32}}(y_6) = Sq^2Bi^*_{E_{32}}(y_4) = z_6 \quad \text{and} \quad Bi^*_{E_{32}}(y_7) = Sq^1Bi^*_{E_{32}}(y_6) = z_7.$$

Hence $Bi^*_{E_{32}}$ is injective and, as the Poincaré series of H^*BF_4 and $(H^*BE_{32})^{W_{F_4}(E_{32})}$ agree, $H^*BF_4 \cong (H^*BE_{32})^{W_{F_4}(E_{32})}$. $\square$

An easy consequence is

Corollary 2.6. *Let* $E_{32} = (\mathbb{Z}/2)^5 \xrightarrow{i} Spin(9)$ *be the maximal elementary abelian 2-subgroup of* $Spin(9)$ *described in Lemma 3.1. Then the cohomological morphism* Bi^* *is injective.*

Proof. The morphism $BE_{32} \xrightarrow{Bi} BSpin(9)$ is obtained from $BE_{32} \xrightarrow{Bi_{E_{32}}} BF_4$ by taking centralizers of a rank 1 elementary abelian subgroup. Now calculating the cohomology of centralizers of elementary abelian subgroups is just applying Lannes' $\mathcal{T}$ functor, which is exact. Therefore, as $H^*BF_4 \xrightarrow{Bi^*_{E_{32}}} H^*BE_{32}$ is injective by Theorem 2.5, then $H^*BSpin(9) = \mathcal{T}_\phi(H^*BF_4) \xrightarrow{Bi^* = \mathcal{T}_\phi(Bi^*_{E_{32}})} \mathcal{T}_\phi(H^*BE_{32}) = H^*BE_{32}$ is injective, where $\mathcal{T}_\phi$ is the right component of $\mathcal{T}$. □

3. The Quillen category of F_4

In this section we calculate the Quillen category of F_4. First we recall the definition of Quillen category of a group. The Quillen category of a group G at a prime p, namely $\mathcal{Q}_p(G)$, is defined as the category whose objects are pairs $(V,\alpha) \in \mathcal{A}b \times \mathrm{Mono}(V,G)$ such that V is a non-trivial elementary abelian p-group, and with morphisms $\mathrm{Mor}_{\mathcal{Q}_p(G)}\big((V_1,\alpha_1),(V_2,\alpha_2)\big)$, the set of group homomorphism $f : V_1 \to V_2$ such that $(V_1,\alpha_1) = (V_1,\alpha_2 f)$. The group of automorphisms in the Quillen category of an object (V,α) is what, in Section cohomology-section, we called the Weyl group of $\alpha(V)$ in G.

Another equivalent description of the Quillen category, that can be applied to the case of p-compact groups, can be found in [12]: given X a p-compact group, $\mathcal{Q}_p(X)$ is the category whose objects consist of pairs $(V,\alpha^*) \in \mathcal{A}b \times \mathrm{Mono}_{\mathcal{A}_p}(H^*BX, H^*BV)$ such that V is a non-trivial elementary abelian p-group, and with morphisms $\mathrm{Mor}_{\mathcal{Q}_p(G)}\big((V_1,\alpha_1^*),(V_2,\alpha_2^*)\big)$, the set of group homomorphism $f : V_1 \to V_2$ such that $(V_1,\alpha_1^*) = (V_1, Bf^*\alpha_2^*)$.

Actually, we choose, for simplicity, a skeletal subcategory of $\mathcal{Q}_2(F_4)$; that is, a full subcategory containing just one representative for each isomorphism class of objects in $\mathcal{Q}_2(F_4)$. This election is described in Proposition 3.2.

As the Weyl group index of $Spin(9)$ in F_4 is 3, any 2-subgroup of F_4 lives (up to conjugation) in $Spin(9)$. Therefore our calculations will be done in $Spin(9)$.

We use the description of the groups $Spin(n)$ given in [9, Chapter 10]. Let $\{e_i\}$ denote a basis of the suitable Clifford algebra, then we fix some distinguished elements in $Spin(9)$. Define

$$b_1 = -1, \quad b_2 = e_1e_2e_3e_4e_5e_6e_7e_8, \quad a_1 = e_1e_2e_3e_4,$$
$$a_2 = e_1e_2e_5e_6, \quad \text{and} \quad a_3 = e_1e_3e_5e_7.$$

Let $T \subset Spin(9)$ be the preimage of the standard maximal torus in $SO(9)$ with cover map $Spin(9) \xrightarrow{\rho} SO(9)$, that is

$$\begin{aligned} T &= \left\{\Pi_{i=1}^4 e_{2i-1}(\cos(t_i)e_{2i-1} + \sin(t_i)e_{2i}) \in Spin(9)\right\} \\ &= \left\{\Pi_{i=1}^4 (-\cos(t_i) + \sin(t_i)e_{2i-1}e_{2i}) \in Spin(9)\right\}. \end{aligned}$$

This allow us to fix a representative of the maximal elementary abelian 2-subgroup of F_4, as well as its Weyl group. From the proof of [16, Theorem 7.3] we get

Lemma 3.1. *The subgroup*

$$(\mathbb{Z}/2)^5 \cong V_{32} = \langle b_1, b_2, a_1, a_2, a_3 \rangle \subset Spin(9) \subset F_4$$

is a representative of the maximal elementary abelian 2-subgroup of $Spin(9)$ and F_4, such that $V_{32} \cap T = \langle b_1, b_2, a_1, a_2 \rangle$. Moreover, this choice of base makes its Weyl group consists of matrices of the form (action on columns)

$$\begin{pmatrix} \mathrm{GL}_2(2) & \begin{smallmatrix} * & * & * \\ * & * & * \end{smallmatrix} \\ 0 & \mathrm{GL}_3(2) \end{pmatrix}$$

where each $* = 0$ or 1.

Now, we can describe the Quillen category of F_4.

Proposition 3.2. *The Quillen category of F_4 at $p = 2$, $\mathcal{Q}_2(F_4)$, contains exactly 11 isomorphism classes of object with representatives listed below. All the representatives are presented as subgroups of $Spin(9) \subset F_4$*

Class	*Representative*	*Centralizer*	*Weyl*	*Toral*
$2A$	$\langle a_1 \rangle$	$SU(2) \times_{\mathbb{Z}/2} Sp(3)$	$\mathrm{GL}_1(2)$	*Yes*
$2B$	$\langle b_1 \rangle$	$Spin(9)$	$\mathrm{GL}_1(2)$	*Yes*
$4A^3$	$\langle a_1, a_2 \rangle$	$(S^1 \times_{\mathbb{Z}/2} U(3)) \rtimes \mathbb{Z}/2$	$\mathrm{GL}_2(2)$	*Yes*
$4A^2B$	$\langle b_1, a_1 \rangle$	$Spin(4) \times_{\mathbb{Z}/2} Spin(5)$	$\mathbb{Z}/2$	*Yes*
$4B^3$	$\langle b_1, b_2 \rangle$	$Spin(8)$	$\mathrm{GL}_2(2)$	*Yes*
$8A^7$	$\langle a_1, a_2, a_3 \rangle$	$(\mathbb{Z}/2)^2 \times O(3)$	$\mathrm{GL}_3(2)$	*No*
$8A^6B$	$\langle b_1, a_1, a_2 \rangle$	$(S^1 \times_{\mathbb{Z}/2} S^1 \times_{\mathbb{Z}/2} U(2)) \rtimes \mathbb{Z}/2$	$2^2.\,\mathrm{GL}_2(2)$	*Yes*
$8A^4B^3$	$\langle b_1, b_2, a_1 \rangle$	$Spin(4) \times_{\mathbb{Z}/2} Spin(4)$	$2^2.\,\mathrm{GL}_2(2)$	*Yes*
$16A^{14}B$	$\langle b_1, a_1, a_2, a_3 \rangle$	$(\mathbb{Z}/2)^3 \times O(2)$	$2^3.\,\mathrm{GL}_3(2)$	*No*
$16A^{12}B^3$	$V_{32} \cap T$	$T : \langle a_3 \rangle$	$W_{F_4}/Z(W_{F_4})$	*Yes*
$32A^{28}B^3$	V_{32}	V_{32}	$W_{F_4}(V_{32})$	*No*

where the notation XA^nB^m means a group of order X such that n (resp. m) elements are in the conjugacy class A (resp. B).

Proof. Here we use the Dwyer-Wilkerson's approach to the Quillen category of G at the prime p, which means that $\mathcal{Q}_p(G)$ can be read from the mod p cohomology of its classifying space.

The group V_{32} is a maximal elementary abelian 2-subgroup and in Proposition 2.5 we have prove that $H^*BF_4 = (H^*BV_{32})^{W_{F_4}(V_{32})}$. Therefore, given any $\alpha^* \in \mathrm{Mono}_{\mathcal{A}_2}(H^*BF_4, H^*BV)$ where V is an elementary abelian 2-subgroup, there exists $f \in \mathrm{Mono}(V, V_{32})$ such that $\alpha^* = Bf^*Bi^*_{V_{32}}$. This implies that the objects in $\mathcal{Q}_2(F_4)$ are just the orbits of the action of $W_{F_4}(V_{32})$ on V_{32} and the Weyl groups are the isotropy subgroups of $W_{F_4}(V_{32})$. Centralizers are calculated by direct computation in $Spin(9)$ and in $SU(2) \times_{\mathbb{Z}/2} Sp(3)$. □

Remark 3.3. *Notice that the representatives shown in Proposition 3.2 live in $N_{Spin(9)} \subset N$, and verify that the centralizer in N_{F_4} is the normalizer of the centralizer in F_4, that is, the representatives are shown as preferred lifts (see [21]) of their inclusions in F_4.*

To finish this section, we define an interesting subcategory of $\mathcal{Q}_2(F_4)$. We call $\mathcal{Q}_2^{Stub}(F_4)$ the full subcategory of $\mathcal{Q}_2(F_4)$ whose objects are the representatives of the classes $2B$, $4A^2B$, $4B^3$, $8A^6B$, $8A^4B^3$, $16A^{12}B^3$ and $32A^{28}B^3$ listed in Proposition 3.2. Then

Remark 3.4. *Notice that the objects of $\mathcal{Q}_2^{Stub}(F_4)$ are all toral but V_{32}, as it is shown in Proposition 3.2. Therefore, these toral objects are shown as the unique preferred lifts of their inclusions in F_4, and any $\mathcal{Q}_2^{Stub}(F_4)$-automorphism of E, $E \neq V_{32}$, can be realized as conjugation by an element in N_{F_4}* [25, Proposition 4.1].

We also prove:

Proposition 3.5. *For any object (V, α) in $\mathcal{Q}_2^{Stub}(F_4)$, the self equivalences of the classifying space centralizer $BC_{F_4}(\alpha(V))$ are determined by restriction to its maximal torus normalizer.*

Proof. Let (V, α) be an object in $\mathcal{Q}_2^{Stub}(F_4)$, and let $C_{F_4}(\alpha(V))$ and $C_{F_4}(\alpha(V))_0$ be the centralizer of $\alpha(V)$ in F_4 and its connected component respectively. As it is shown in the table in Proposition 3.2, $C_{F_4}(\alpha(V))_0$ is not of type $SO(2n+1)\times Sp(n)$, hence by [19, Corollary 3.5], its self equivalences are determined by the restriction to its maximal torus normalizer.

Finally, as the center of $C_{F_4}(\alpha(V))_0$ equals the center of its maximal torus normalizer, by [22, Proposition 4.2] the self equivalences of $C_{F_4}(\alpha(V))$ are also determined by the restriction to its maximal torus normalizer. □

4. The p-stubborn category versus Quillen category

In this section we show that there exists an interesting connection between $\mathcal{R}_p(G)$, the p-stubborn category of G, and $\mathcal{Q}_p(G)$. Finally we discuss this relation in the particular case of F_4 at $p = 2$.

First we recall the definition of p-stubborn subgroup of a Lie group G. Let p be a fixed prime. A p-toral group is a compact Lie group P whose component of the unit, P_0, is a torus and whose group of components P/P_0, is a finite p-group. Given a compact Lie group G, a p-toral subgroup P of G is said to be p-stubborn if the quotient $N_G(P)/P$, where $N_G(P)$ is the normalizer of P in G, is finite and does not contain any nontrivial normal p-subgroup. Therefore, the category $\mathcal{R}_p(G)$ is defined as the full subcategory of the orbit category whose objects are those orbits G/P for which $P \subset G$ is a p-stubborn subgroup. Notice that $\mathcal{R}_p(G)$ can also be thought as the category whose objects are $P \subset G$, p-stubborn subgroups, and morphisms $\mathrm{Mor}_{\mathcal{R}_p(G)}(P_1, P_2) = P_2/N_G(P_1, P_2)$, where $N_G(P_1, P_2) = \{g \in G | gP_1g^{-1} \subset P_2\}$. Therefore morphisms in $\mathcal{R}_p(G)$ can be thought as conjugations modulo action of the target group and we will denote them as $c_{P_2 g}$.

Given a Lie group G, the natural way of associating an elementary abelian p-subgroup of to a p-stubborn subgroup is via the center. In what follows, given a group P, ${}_pZ(P)$ denotes the elements of order p in the center of P (thus ${}_pZ(P)$ is an elementary abelian p-group).

Theorem 4.1. *Fix p a prime, and let G be a compact Lie group such that $\pi_0 G$ is a p-group. Then there exists a contravariant functor $\mathbf{F}_G\colon \mathcal{R}_p(G) \rightsquigarrow \mathcal{Q}_p(G)$ given by $\mathbf{F}_G(P) = {}_pZ(P)$ and $\mathbf{F}_G(c_{P_2 g}) = c_{g^{-1}}$.*

Proof. Notice that for a p-stubborn subgroup $P \subset G$, it holds that $Z(P) = C_G(P)$, since $\pi_0 G$ is a p-group [18, Lemma 1.5.(*ii*)]. Therefore $c_g(P_1) \subset P_2$ gives rise to $Z(P_2) = C_G(P_2) \subset C_G(c_g(P_1)) = Z(c_g(P_1))$. Because c_g is isomorphism $c_g(Z(P_1)) = Z(c_g(P_1))$, and hence $c_{g^{-1}}(Z(P_2)) \subset Z(P_1)$. From this we get $c_{g^{-1}}({}_pZ(P_2)) \subset {}_pZ(P_1)$. Finally, notice that if $P_2g' = P_2g$, then $g' = qg$ for some $q \in P_2$, and therefore $c_{g^{-1}} = c_{g'^{-1}} : {}_pZ(P_2) \longrightarrow {}_pZ(P_1)$ as group morphisms. Thus $\mathbf{F}_G$ is a contravariant functor between $\mathcal{R}_p(G)$ and $\mathcal{Q}_p(G)$. □

Remark 4.2. *The importance of Theorem 4.1 is that allow us to relate the two classical homology decompositions: via centralizers* [17] *and via p-stubborns* [18]. *Indeed, given a Lie group G such that $\pi_0 G$ is a p-group, we can consider*

$$P \xrightarrow{i_P} C_G({}_pZ(P))$$

the natural inclusion and Theorem 4.1 shows that the diagram

$$\{BP\}_{\mathcal{R}_p(G)} \xrightarrow{\{Bi_P\}} \{C_G(E)\}_{\mathcal{Q}_p(G)}$$

is homotopy commutative.

To finish this section, we consider the case of F_4. Consider the skeletal subcategory of $\mathcal{R}_2(F_4)$ given by those 2-stubborn subgroups in N_{F_4} such that ${}_2Z(P)$ is one of the representatives in Proposition 3.2, and denote it again by $\mathcal{R}_2(F_4)$, then

Lemma 4.3. *The image of $\mathcal{R}_2(F_4)$ by the functor $\mathbf{F}_{F_4}$ is a subcategory of $\mathcal{Q}_2^{Stub}(F_4)$.*

Proof. As F_4 is connected, for any $P \in \mathcal{R}_2(F_4)$, it holds that $Z(P) = C_{F_4}(P)$. Notice that $Spin(9) \subset F_4$, and by transfer arguments $P \subset Spin(9)$. Hence

$$Z(Spin(9)) \subset C_{Spin(9)}(P) \subset C_{F_4}(P) = Z(P),$$

and therefore $Z(P)$, and ${}_2Z(P)$, contains an element of class $2B$.

Finally, notice that there is no 2-stubborn subgroup $P \subset F_4$, whose center belongs to the class $16A^{14}B$, as in that case $P \subset (\mathbb{Z}/2)^3 \times O(2)$, which implies $P = (\mathbb{Z}/2)^3 \times O(2)$ and $N_{F_4}(P)/P = 2^3.\mathrm{GL}_3(2)$, which is not 2-reduced.

Therefore, the 2-center of a 2-stubborn subgroup in F_4 is either in the class $2B$, $4A^2B$, $4B^3$, $8A^6B$, $8A^4B^3$, $16A^{12}B^3$ or $32A^{28}B^3$. □

5. The proof of Theorem 1.1

Throughout this section we prove Theorem 1.1. In what follows,

- X is a 2-compact group whose maximal torus normalizer $N \xrightarrow{j} X$ is isomorphic to that of F_4, and
- F_4, $Spin(9)$, $C_{Spin(9)}(E)$ and $C_{F_4}(E)$ will denote the 2-compact group obtained from the 2-completion of the respective Lie groups.

First we show that X is a connected 2-compact group as F_4 is so. This is a consequence of the order of the normal subgroups of the Weyl group of X, W_X, which is isomorphic to the Weyl group of F_4. The order of these normal subgroups of W_{F_4} are described in the following lemma:

Lemma 5.1. *The group W_{F_4} has exactly twelve normal subgroups, and they have orders* 1, 2, 32, 96, 96, 192, 192, 288, 576, 576, 576 *and* 1152.

Proof. The proof was done by direct calculation by means of MAGMA [5], using the generators obtained from the Cartan matrix in [6, Planche VIII]. □

Proposition 5.2. *X is connected.*

Proof. Assume that X is not connected and call X_0 the connected component of the unit. Then there exists a short exact sequence:

$$0 \longrightarrow W_{X_0} \longrightarrow W_X \cong W_{F_4} \longrightarrow \pi_0 X \longrightarrow 0.$$

As $\pi_0 X$ is a 2-group, W_{X_0} must be a normal subgroup of $W_X \cong W_{F_4}$ of index a power of 2. Lemma 5.1 shows that $\#|W_{X_0}| = 288$, 576 or 1152. But according to [8], no 2-adic pseudoreflection group of rank 4 has order 288 or 576, hence $\#|W_{X_0}| = 1152$ and $\pi_0 X = 0$. Thus X is connected. $\square$

As X is connected, previous work by Dwyer-Wilkerson [15] allow us to determine the group component of centralizers in X so obtaining the isomorphism type of some centralizers

Proposition 5.3. *There exists a map* $f_{\langle b_1 \rangle} : B\mathbb{Z}/2 \longrightarrow BN \xrightarrow{Bj} BX$ *such that*

$$\begin{array}{ccc}
BN_{Spin(9)} \overset{\psi}{\simeq} \mathrm{map}(B\mathbb{Z}/2, BN)_{f_{\langle b_1\rangle}} & \xrightarrow{ev} & BN \\
\downarrow \qquad\qquad \downarrow{\scriptstyle Bj_*} & & \downarrow{\scriptstyle Bj} \\
BSpin(9) \overset{\phi}{\simeq} \mathrm{map}(B\mathbb{Z}/2, BX)_{f_{\langle b_1\rangle}} & \xrightarrow{ev} & BX
\end{array}$$

Proof. As $N \cong N_{F_4}$, we can identify $\mathbb{Z}/2 \cong \langle b_1 \rangle \subset N$ such that by Remark 3.3 $C_N(\mathbb{Z}/2) = N_{Spin(9)}$. Moreover, as X is connected, by [15, Theorem 7.6] we get that the Weyl group of $C_X(\mathbb{Z}/2)$ agress with the Weyl group of its connected component, and then $\pi_0 C_X(\mathbb{Z}/2) = 0$. Therefore $C_X(\mathbb{Z}/2)$ is a connected 2-compact group with normalizer isomorphic to that of $Spin(9)$. According to [29], $C_X(\mathbb{Z}/2) \cong Spin(9)$ as 2-compact groups. $\square$

Now, as the Weyl group index of $Spin(9)$ in X is 3, by transfer arguments we know that the cohomology of BX injects into the cohomology of $BSpin(9)$ via $f_{Spin(9)} = ev \circ \phi$, this allow us to determine H^*BX.

Proposition 5.4. *There is an isomorphism of unstable algebras* $H^*BX \cong_{\mathcal{A}_2} H^*BF_4$ *induced by* $f^*_{Spin(9)}$, *that is,* $Bi^*_{Spin(9)}(H^*BF_4) = f^*_{Spin(9)}(H^*BX)$.

Proof. Recall the situation: X is a 2-compact group whose maximal torus normalizer $N \xrightarrow{j} X$ is isomorphic to that of F_4, we shall prove that $H^*BX \cong H^*BF_4$ as unstable algebras over $\mathcal{A}_2$.

According to Proposition 5.2, X is a connected 2-compact group, and it has the same Weyl group as F_4. Therefore

$$H^*_{\mathbb{Q}_2^\wedge} BX = (H^*_{\mathbb{Q}_2^\wedge} BT)^{W_X} = (H^*_{\mathbb{Q}_2^\wedge} BT)^{W_{F_4}} = \mathbb{Q}_2^\wedge[q_4, q_{12}, q_{16}, q_{24}],$$

and the Bockstein spectral sequence associated to H^*BX converges to $\mathbb{F}_2[q_4, q_{12}, q_{16}, q_{24}]$.

Now by Proposition 5.3, we know that there exists a commutative diagram:

$$\begin{array}{ccc}
BN_{Spin(9)} & \longrightarrow & BN \\
\downarrow & & \downarrow{\scriptstyle Bj} \\
BSpin(9) & \xrightarrow{f_{Spin(9)}} & BX
\end{array}$$

As the Weyl group index of $Spin(9)$ in X is 3, by transfer arguments we know that the cohomology of BX injects into the cohomology of $BSpin(9)$. Let V_{32} be as defined in Lemma 3.1. According to Corollary 2.6, $H^*BSpin(9)$ injects in H^*BV_{32}, thus H^*BX does so. Therefore $H^*BX \subset (H^*BV_{32})^{W_X(V_{32})}$ where $W_X(V_{32})$ is the Weyl group of V_{32} in X.

Our next step is to determine $W_X(V_{32})$, or at least a big enough subgroup. The Weyl group of V_{32} in $Spin(9)$ can be read from Proposition 3.2, and consists of matrices of type (action on columns)

$$\begin{pmatrix} \begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix} & \begin{smallmatrix} * & * & * \\ * & * & * \end{smallmatrix} \\ 0 & \mathrm{GL}_3(2) \end{pmatrix}$$

where each $* = 0$ or 1. From Proposition 3.2, we can also calculate the Weyl group of V_{32} in N, and it consists of matrices of type (action on columns)

$$\begin{pmatrix} \mathrm{GL}_2(2) & \begin{smallmatrix} * & * \\ * & * \end{smallmatrix} & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ 0 & \mathrm{GL}_2(2) & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ 0 & 0 & 1 \end{pmatrix}$$

where again each $* = 0$ or 1. Therefore $W_X(V_{32})$ does contain the group generated by those two matrix groups, which is exactly $W_{F_4}(V_{32})$.

Hence $H^*BX \subset (H^*BV_{32})^{W_X(V_{32})} \subset (H^*BV_{32})^{W_{F_4}(V_{32})} = H^*BF_4$, that is H^*BX is a subalgebra of H^*BF_4 and we showed its Bockstein spectral sequence converges to $\mathbb{F}_2[q_4, q_{12}, q_{16}, q_{24}]$. As the element q_4 comes from something in dimension 4, and $H^4BF_4 = \langle y_4 \rangle$, then $q_4 = [y_4]$. Thus $y_4 \in H^*BX$, $y_6 = Sq^2 y_4 \in H^*BX$, and $y_7 = Sq^1 y_6 \in H^*BX$. As y_6^2 survives in the Bockstein spectral sequence, then $q_{12} = [y_6^2]$. The element q_{16} must be a class coming from $H^{16}BX \subset \langle y_{16}, y_4 y_6^2 \rangle = H^{16}BF_4$. As $[y_4 y_6^2] = [y_4][y_6^2] = q_4 q_{12}$ is independent of q_{16}, then $q_{16} = [y_{16} + \lambda y_4 y_6^2]$. We already saw that $y_4 y_6^2$ is in H^*BX, therefore $y_{16} \in H^*BX$. Because $Sq^8 y_{16} = y_4^2 y_{16} + y_{24} \in H^*BX$ and $y_4^2 y_{16} \in H^*BX$, also the element y_{24} is in H^*BX, which shows that $H^*BX = \mathbb{F}_2[y_4, y_6, y_7, y_{16}, y_{24}] = H^*BF_4$.

Finally, notice that the isomorphism, namely φ, is constructed such that the following diagram is commutative:

$$\begin{array}{ccccc} H^*BX & \xrightarrow{f^*_{Spin(9)}} & H^*BSpin(9) & \xrightarrow{Bi^*_{V_{32}}} & H^*BV_{32} \\ \Big\downarrow \varphi & \nearrow_{Bi^*_{Spin(9)}} & & & \\ H^*BF_4 & & & & \end{array}$$

As $Bi^*_{V_{32}}$ is injective by Corollary 2.6, then $Bi^*_{Spin(9)}(H^*BF_4) = f^*_{Spin(9)}(H^*BX)$. $\square$

A direct consequence of Proposition 5.4 and Lannes' theory is

Corollary 5.5. *$\mathcal{Q}_2(X)$ is isomorphic to $\mathcal{Q}_2(F_4)$ as categories.*

The Quillen category of F_4 at the prime 2 is calculated in Section 3 (actually a skeletal subcategory of $\mathcal{Q}_2(F_4)$ is calculated). We are interested in a full subcategory of $\mathcal{Q}_2(X)$, namely $\mathcal{Q}_2^{Stub}(X) \cong \mathcal{Q}_2^{Stub}(F_4)$, where the isomorphism is the restriction of the isomorphism given in Corollary 5.5.

By Proposition 5.4, the diagram

$$\{BE\}_{\mathcal{Q}_2^{Stub}(F_4)} \xrightarrow{\{f_E\}} BX$$

is commutative, where $f_E : BE \longrightarrow BN_{Spin(9)} \longrightarrow BSpin(9) \xrightarrow{f_{Spin(9)}} BX$. Taking centralizers in X over the category $\mathcal{Q}_2^{Stub}(X) = \mathcal{Q}_2^{Stub}(F_4)$ we obtain the commutative diagram

$$\{\mathrm{map}(BE, BX)_{f_E}\}_{\mathcal{Q}_2^{Stub}(F_4)} \xrightarrow{\{ev\}} BX. \tag{5.1}$$

Noticing that if $E \in \mathcal{Q}_2^{Stub}(X) = \mathcal{Q}_2^{Stub}(F_4)$, then $E = E \cdot \langle b_1 \rangle$, by Lemma 7.10 in [15] we have that

$$\operatorname{map}(BE, BX)_{f_E} \simeq \operatorname{map}(BE, \operatorname{map}(B\langle b_1 \rangle, BX)_{f_{\langle b_1 \rangle}})_{f_E}$$

and using Proposition 5.3 we obtain

$$\begin{aligned} (5.2) \quad & \operatorname{map}(BE, BX)_{f_E} \simeq \\ & \simeq \operatorname{map}(BE, \operatorname{map}(B\langle b_1 \rangle, BX)_{f_{\langle b_1 \rangle}})_{f_E} \overset{\phi_*}{\simeq} \operatorname{map}(BE, BSpin(9))_{Bi_E} \simeq \\ & \simeq BC_{Spin(9)}(E) = BC_{F_4}(E) \end{aligned}$$

and

$$\begin{aligned} (5.3) \quad & \operatorname{map}(BE, BN)_{Bi_E} \simeq \\ & \simeq \operatorname{map}(BE, \operatorname{map}(B\langle b_1 \rangle, BN)_{Bi_{\langle b_1 \rangle}})_{Bi_E} \overset{\psi_*}{\simeq} \operatorname{map}(BE, BN_{Spin(9)})_{Bi_E} \simeq \\ & \simeq BC_{N_{Spin(9)}}(E) = BN_{C_{F_4}(E)} \end{aligned}$$

where the last equality comes from the fact that $BE \xrightarrow{Bi_E} BN$ is a preferred lift as showed in Remark 3.4.

Now, the idea is to replace $\operatorname{map}(BE, BX)_{f_E}$ by $BC_{F_4}(E)$ in diagram (5.1). Before doing so, we check the replacement will keep the diagram commutative (up to homotopy). As any $\mathcal{Q}_2^{Stub}(F_4)$-morphism $E \longrightarrow E'$ is composition of the fixed inclusion $E \rightarrowtail E'$ with a $\mathcal{Q}_2^{Stub}(F_4)$-automorphism $E' \longrightarrow E'$, we have to worry only about $\mathcal{Q}_2^{Stub}(F_4)$-automorphism.

If $E = V_{32}$, then by Lannes' $\mathcal{T}$ functor $\operatorname{map}(BE, BX)_{f_E} \simeq BV_{32} = BC_{F_4}(V_{32})$, and by Lannes' theory and Proposition 5.4 the diagram commutes up to homotopy.

Now assume $E \in \mathcal{Q}_2^{Stub}(X) = \mathcal{Q}_2^{Stub}(F_4)$ such that $E \neq V_{32}$. Then by Remark 3.4 the inclusion $E \subset N$ is the only preferred lift of E in F_4 (and therefore in X), and any $\mathcal{Q}_2^{Stub}(F_4)$-automorphism $E \xrightarrow{h} E$ is induced by conjugation by $g \in N$. So we have the homotopy commutative diagram

$$\begin{array}{ccc} BE & \xrightarrow{Bi_E} & BN \\ {\scriptstyle h}\downarrow & & \| \\ BE & \xrightarrow{Bi_E} & BN \end{array}$$

which together with diagrams (5.2) and (5.3) induces the homotopy commutative diagram

$$\begin{array}{ccccccc}
\mathrm{map}(BE,BX)_{f_E} & & & \xrightarrow{\phi_*} & & & BC_{F_4}(E)\\
 & \nwarrow^{j_*} & & & & \nearrow & \\
\Big\downarrow h^* & & \mathrm{map}(BE,BN)_{Bi_E} & \xrightarrow{\psi_*} & BN_{C_{F_4}(E)} & & \Big\downarrow \phi_*(h^*)\\
 & & \Big\downarrow h^* & & \Big\downarrow \psi_*(h^*) & & \\
 & & \mathrm{map}(BE,BN)_{Bi_E} & \xrightarrow{\psi_*} & BN_{C_{F_4}(E)} & & \\
 & \swarrow^{j_*} & & & & \searrow & \\
\mathrm{map}(BE,BX)_{f_E} & & & \xrightarrow{\phi_*} & & & BC_{F_4}(E)
\end{array}$$

Now, notice that h is just c_g in N, hence we can replace $\psi_*(h^*)$ by $Bc_{g^{-1}}$ in the diagram above and, as self equivalences of $BC_{F_4}(E)$ are determined by restriction to $BN_{C_{F_4}(E)}$ by Proposition 3.5, we can also replace $\phi_*(h^*)$ by $Bc_{g^{-1}}$ above. So gluing together this information with diagram (5.1) we obtain the homotopy commutative diagram

$$\{BC_{F_4}(E)\}_{\mathcal{Q}_2^{Stub}(F_4)} \xrightarrow{\{\phi_*\}} \{\mathrm{map}(BE,BX)_{f_E}\}_{\mathcal{Q}_2^{Stub}(F_4)} \xrightarrow{\{ev\}} BX,$$

that composed with the commutative diagram constructed in Remark 4.2 gives rise to the (homotopy) commutative diagram

$$\{BP\}_{\mathcal{R}_2(F_4)} \xrightarrow{\{ev\circ\phi_*\circ Bi_P\}} BX. \tag{5.4}$$

Now, as one cannot take the hocolim of a diagram in the homotopy category, the diagram above does not ensure the existence of a map from $\mathrm{hocolim}\{BP\}_{\mathcal{R}_2(F_4)}$ (which is BF_4 up to 2-completion) to BX. Such a map exists if some obstruction classes living in

$$\varprojlim_{\mathcal{R}_2(F_4)}{}^{i+1}\pi_i(\mathrm{map}(BP,BX)_{ev\circ\phi_*\circ Bi_P})$$

vanish [38, Proposition 3]. Here $\lim^i$ is the i-th derived functor of the inverse limit functor [7, Chapter XI,§6].

Proposition 5.6. *For any i, and j, it holds $\lim^i \pi_j(\mathrm{map}(BP,BX)_{f_P}) = 0$, where $f_P = ev\circ\phi_*\circ Bi_P$.*

Proof. Since every P contains $\langle b_1\rangle = Z(Spin(9))$, and $\langle b_1\rangle \subset Z(P)$ then

$$\mathrm{map}(BP,BX)_{f_P} \simeq \mathrm{map}(BP,BC_X(\langle b_1\rangle))_{\phi_*\circ Bi_P} \overset{\phi_*}{\simeq}$$
$$\overset{\phi_*}{\simeq} \mathrm{map}(BP,BSpin(9))_{Bi_P} \simeq BZ(P),$$

and the induced map by Bc_g behaves as $Bc_{g^{-1}}$ by means of Theorem 4.1. Exactly the same description holds for $\mathrm{map}(BP, BF_4)_{Bi_P}$. Hence

$$\begin{aligned}\lim_{\overleftarrow{\mathcal{R}_2(F_4)}}{}^{i} \pi_j(\mathrm{map}(BP, BX)_{f_P}) &= \lim_{\overleftarrow{\mathcal{R}_2(F_4)}}{}^{i} \pi_j(\mathrm{map}(BP, BF_4)_{Bi_P}) \\ &= 0 \text{ by [18].}\end{aligned}$$

□

So the obstruction classes vanish and the diagram (5.4) induces a map $BF_4 \xrightarrow{f} BX$. To finish the proof of Theorem 1.1, we have to show that f induces an isomorphism of 2-compact groups. Notice that by construction the diagram

$$\begin{CD} BN_2 \cong (BN_{Spin(9)})_2 @>>> BN \\ @VVV @VV{Bj}V \\ BF_4 @>{f}>> BX \end{CD}$$

commutes, where N_2 is the 2-normalizer of the maximal torus in F_4 as well as the maximal 2-stubborn subgroup of F_4. All the induced cohomological maps (but f^*) are known to be injective, hence f^* is so. As the Poincaré series of H^*BF_4 and H^*BX agree, f^* is a cohomological isomorphism, thus a 2-compact group isomorphism.

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