ON THE COHOMOLOGY AND THE CHOW RING OF THE CLASSIFYING SPACE OF PGL_p

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ABSTRACT. We investigate the integral cohomology ring and the Chow ring of the classifying space of the complex projective linear group PGL_p, when p is an odd prime. In particular, we determine their additive structures completely, and we reduce the problem of determining their multiplicative structures to a problem in invariant theory.

CONTENTS

1. INTRODUCTION

Let G be a complex linear group. One of the main invariants associated with G is the cohomology H^*_G of the classifying space $\mathrm{B}G$. B. Totaro (see [13]) has also introduced an algebraic version of the cohomology of the classifying space of an algebraic group G over a field k, the Chow ring A^*_G of the classifying space of G. When $k = \mathbb{C}$ there is a cycle ring homomorphism

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$\mathrm{A}^*_G \to \mathrm{H}^*_G$. Chow rings are normally infinitely harder to study than cohomology; it remarkable that, in contrast, A^*_G seems to be better behaved, and easier to study, than H^*_G. For example, when G is a finite abelian group, A^*_G is the symmetric algebra over $\mathbb{Z}$ of the dual group $\widehat{G}$; while the cohomology ring contains this symmetric algebra, but is much more complicated (for example, will contain elements of odd degree), unless G is cyclic.

This ring A^*_G has also been computed for $G = \mathrm{GL}_n$, SL_n, Sp_n by Totaro and R. Pandharipande, and for $G = \mathrm{SO}_n$ by R. Field. However, not much is known for the PGL_n series. Even the cohomology of $\mathrm{B}\,\mathrm{PGL}_n$ seems to be mysterious. Algebraic topologists tend to work with cohomology with coefficients in a field, the case in which their very impressive toolkits work the best. When p does not divide n, the comology ring $\mathrm{H}^*(\mathrm{B}\,\mathrm{PGL}_n, \mathbb{Z}/p\mathbb{Z})$ is a well understood polynomial ring. Also, since $\mathrm{PGL}_2 = \mathrm{SO}_3$, the ring $\mathrm{H}^*(\mathrm{B}\,\mathrm{PGL}_2, \mathbb{Z}/2\mathbb{Z})$ is also well understood. The other results that I am aware of on $\mathrm{H}^*(\mathrm{B}\,\mathrm{PGL}_n, \mathbb{Z}/p\mathbb{Z})$ are the following.

(1) In [9], the authors compute the cohomology ring $\mathrm{H}^*(\mathrm{B}\,\mathrm{PGL}_3, \mathbb{Z}/3\mathbb{Z})$.
(2) The ring $\mathrm{H}^*(\mathrm{B}\,\mathrm{PGL}_n, \mathbb{Z}/2\mathbb{Z})$ is known when $n \equiv 2 \pmod 4$ ([8] and [12]).
(3) Some interesting facts on $\mathrm{H}^*(\mathrm{B}\,\mathrm{PGL}_p, \mathbb{Z}/p\mathbb{Z})$ are proved in [14].

On the other hand, to my knowledge no one has studied the integral cohomology ring $\mathrm{H}^*_{\mathrm{PGL}_n}$.

In the algebraic case, the only known results about $\mathrm{A}^*_{\mathrm{PGL}_n}$, apart from the case of $\mathrm{PGL}_2 = \mathrm{SO}_3$, concern PGL_3 and were proved by Vezzosi in [15]. Here he determines almost completely the structure of $\mathrm{A}^*_{\mathrm{PGL}_3}$ by generators and relations; the only ambiguity is about one of the generators, denoted by χ and living in $\mathrm{A}^6_{\mathrm{PGL}_3}$, about which he knows that it is 3-torsion, but is not able to determine whether it is 0. This χ maps to 0 in the cohomology ring $\mathrm{H}^*_{\mathrm{PGL}_3}$; according to a conjecture of Totaro, the cycle map $\mathrm{A}^*_{\mathrm{PGL}_3} \to \mathrm{H}^*_{\mathrm{PGL}_3}$ should be injective; so, if the conjecture is correct, χ should be 0.

Despite this only partial success, the ideas of [15] are very important. The main one is to make use of the *stratification method* to get generators. This is how it works. Recall that Edidin and Graham ([2]) have generalized Totaro's ideas to give a full-fledged equivariant intersection theory. Let V be a representation of a group G; then we have $\mathrm{A}^*_G = \mathrm{A}^*_G(V)$. Suppose that we have a stratification $V_0, \dots, V_t$ of V by locally closed invariant subvarieties, such that each $V_{\le i} \overset{\text{def}}{=} \cap_{j \le i} V_j$ is open in V, each V_i is closed in $V_{\le i}$, and $V_t = V \setminus \{0\}$. If we can determine generators for $\mathrm{A}^*_G(V_i)$ for each i, then one can use the localization sequence

$$\mathrm{A}^*_G(V_i) \longrightarrow \mathrm{A}^*_G(V_{\le i}) \longrightarrow \mathrm{A}^*_G(V_{\le i-1}) \longrightarrow 0$$

and induction to get generators for $\mathrm{A}^*_G(V \setminus \{0\})$; and since $\mathrm{A}^*_G(V \setminus \{0\}) = \mathrm{A}^*_G / \bigl(\mathrm{c}_r(V)\bigr)$, where $\mathrm{c}_r(V) \in \mathrm{A}^r_G$ is the r^{th} Chern class of V, we obtain that A^*_G is generated by lifts to A^*_G of the generators for $\mathrm{A}^*_G(V \setminus \{0\})$, plus $\mathrm{c}_r(V)$.

The stratification methods gives a unified approach for all the known calculations of A^*_G for classical groups (see [10]).

Vezzosi applies the method to the adjoint representation space $V = \mathfrak{sl}_3$ consisting of matrices with trace 0. The open subscheme V_0 is the subscheme of matrices with distict eigenvalues; its Chow ring is related with the Chow ring of the normalizer N_3 of a maximal torus $\mathrm{T}_{\mathrm{PGL}_p}$ in PGL_3. In order to get relations, Vezzosi uses an unpublished result of Totaro, implying that the restriction homomorphism $\mathrm{A}^*_{\mathrm{PGL}_3} \to \mathrm{A}^*_{N_3}$ is injective. The reason why he is not able to determine whether χ is 0 or not is that he does not have a good description of the 3-torsion in $\mathrm{A}^*_{N_3}$.

In this paper we extend Vezzosi's approach to the case of PGL_p, where p is an odd prime; and we also show how this can give considerable information on the cohomology of a classifying space.

Let $\mathrm{T}_{\mathrm{PGL}_p}$ be the standard maximal torus in PGL_p, consisting of classes of diagonal matrices, S_p its Weyl group. Here are our main results (see Section 3 for details).

(1) The natural homomorphism $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}} \to \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$ is surjective, and has a natural splitting $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p} \to \mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$, which is a ring homomorphism.
(2) The ring $\mathrm{A}^*_{\mathrm{PGL}_p}$ is generated as an algebra over $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$ by a single p-torsion element $\rho \in \mathrm{A}^{p+1}_{\mathrm{PGL}_p}$; we also describe the relations.
(3) The ring $\mathrm{H}^*_{\mathrm{PGL}_p}$ is generated as an algebra over $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$ by two elements: the image $\rho \in \mathrm{H}^{2p+2}_{\mathrm{PGL}_p}$ of the class above and the Brauer class $\beta \in \mathrm{H}^3_{\mathrm{PGL}_p}$; we also describe the relations.
(4) Using (2) and (3) above, we describe completely the additive structures of $\mathrm{A}^*_{\mathrm{PGL}_p}$ and $\mathrm{H}^*_{\mathrm{PGL}_p}$.
(5) For $p = 3$ we give a presentation of $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_3}}\right)^{\mathrm{S}_3}$ by generators and relations (this is already in [15]); and this, together with (2) and (3) above, gives presentations of $\mathrm{A}^*_{\mathrm{PGL}_3}$ and $\mathrm{H}^*_{\mathrm{PGL}_3}$, completing the work of [15].
(6) The cycle homomorphism $\mathrm{A}^*_{\mathrm{PGL}_p} \to \mathrm{H}^{\mathrm{even}}_{\mathrm{PGL}_p}$ into the even-dimensional cohomology is an isomorphism.

The ring $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$ is complicated when $p > 3$; see the discussion in Section 14.

The class ρ in (2) seems interesting, and gives a new invariant for sheaves of Azumaya algebras of prime rank (Remark 11.3). In [11], Elisa Targa shows that ρ is not a polynomial in Chern classes of representations of PGL_p.

Many of the ideas in this paper come from [15]. The main new contributions here are the contents of Sections 6 and 7 (the heart of these results are Proposition 6.1, and the proof of Lemma 6.5), which substantially improve our understanding of the cohomology and Chow ring of the classifying space of N_p, and Proposition 10.1, which gives a way of showing that in the stratification method no new generators come from the strata corresponding to non-zero matrices with multiple eigenvalues, thus avoiding the painful case-by-case analysis that was necessary in [15].

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2. Notations and conventions

All algebraic groups and schemes will be of finite type over a fixed field k of charateristic 0. Furthermore, we will fix an odd prime p, and assume that k contains a fixed p^{th} root of 1, denoted by ω. When $k = \mathbb{C}$, we take $\omega = \mathrm{e}^{2\pi\mathrm{i}/p}$.

The hypothesis that the characteristic be 0 is only used in the proof Theorem 9.3, which should however hold over an arbitrary field. If so, it would be enough to assume here that the characteristic of k be different from p.

Our main tool is Edidin and Graham's equivariant intersection theory (see [2]), which works over an arbitrary field; when we discuss cohomology, instead, we will always assume that $k = \mathbb{C}$. All finite groups will be considered as algebraic groups over k, in the usual fashion. We denote by $\mathbb{G}_{\mathrm{m}}$ the multiplicative group of non-zero scalars over k, $\boldsymbol{\mu}_n$ the algebraic group of n^{th} roots of 1 over k.

Whenever V is a vector space over k, we also consider it as a scheme over k, as the spectrum of the symmetric algebra of the dual vector space $V^\vee$. If V is a representation of an algebraic group G, then there is an action of G on V as a scheme over k.

We denote by $\mathrm{T}_{\mathrm{GL}_p}$, $\mathrm{T}_{\mathrm{SL}_p}$ and $\mathrm{T}_{\mathrm{PGL}_p}$ the standard maximal tori in the respective groups, those consisting of diagonal matrices. We identify the Weyl groups of these three groups with the symmetric group S_p. We also denote the normalizer of $\mathrm{T}_{\mathrm{PGL}_p}$ in PGL_p by $\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}$.

If $a_1, \ldots, a_p$ are elements of k^*, we will denote by $[a_1, \ldots, a_p]$ the diagonal matrix in GL_p with entries $a_1, \ldots, a_p$, and also its class in PGL_p. In general, we will ofter use the same symbol for a matrix in GL_p and its class in PGL_p; this should not give rise to confusion.

It is well known that the arrows

$$\mathrm{A}^*_{\mathrm{GL}_p} \longrightarrow (\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}})^{\mathrm{S}_p}, \quad \mathrm{H}^*_{\mathrm{GL}_p} \longrightarrow (\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}})^{\mathrm{S}_p}$$

and

$$\mathrm{A}^*_{\mathrm{SL}_p} \longrightarrow (\mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}})^{\mathrm{S}_p}, \quad \mathrm{H}^*_{\mathrm{GL}_p} \longrightarrow (\mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}})^{\mathrm{S}_p}$$

induced by the embeddings $\mathrm{T}_{\mathrm{GL}_p} \hookrightarrow \mathrm{GL}_p$ and $\mathrm{T}_{\mathrm{SL}_p} \hookrightarrow \mathrm{SL}_p$ are isomorphisms. If we denote by $x_i \in \mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}} = \mathrm{H}^*_{\mathrm{T}_{\mathrm{GL}_p}}$ the first Chern class of the i^{th} projection $\mathrm{T}_{\mathrm{GL}_p} \to \mathbb{G}_{\mathrm{m}}$, or its restriction to $\mathrm{T}_{\mathrm{SL}_p}$, then $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}} = \mathrm{H}^*_{\mathrm{T}_{\mathrm{GL}_p}}$ is the polynomial ring $\mathbb{Z}[x_1, \dots, x_p]$, while $\mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}} = \mathrm{H}^*_{\mathrm{T}_{\mathrm{SL}_p}}$ equals $\mathbb{Z}[x_1, \dots, x_p]/(x_1 + \cdots + x_p)$. If we denote by $\sigma_1, \dots, \sigma_p$ the elementary symmetric functions in the x_i, then we conclude that

$$\mathrm{A}^*_{\mathrm{GL}_p} = \mathrm{H}^*_{\mathrm{GL}_p} = \mathbb{Z}[\sigma_1, \dots, \sigma_p]$$

while

$$\mathrm{A}^*_{\mathrm{SL}_p} = \mathrm{H}^*_{\mathrm{SL}_p} = \mathbb{Z}[\sigma_1, \dots, \sigma_p]/(\sigma_1) = \mathbb{Z}[\sigma_2, \dots, \sigma_p].$$

The ring $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}} = \mathrm{H}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$ is the subring of $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}$ generated by the differences $x_i - x_j$. In particular it contains the element $\delta = \prod_{i \neq j}(x_i - x_j)$, which we call the *discriminant* (up to sign, it is the classical discriminant); it will play an important role in what follows.

We will use the following notation: if R is a ring, $t_1, \dots, t_n$ are elements of R, $f_1, \dots, f_r$ are polynomials in $\mathbb{Z}[x_1, \dots, x_n]$, we write

$$R = \mathbb{Z}[t_1, \dots, t_n]/\bigl(f_1(t_1, \dots, t_n), \dots, f_r(t_1, \dots, t_n)\bigr)$$

to indicate the the ring R is generated by $t_1, \dots, t_n$, and the kernel of the evaluation map $\mathbb{Z}[x_1, \dots, x_n] \to R$ sending x_i to t_i is generated by $f_1, \dots, f_r$. When there are no f_i this means that R is a polynomial ring in the t_i.

3. The main results

Consider the embedding $\boldsymbol{\mu}_p \hookrightarrow \mathrm{T}_{\mathrm{PGL}_p}$ defined by $\zeta \mapsto [\zeta, \zeta^2, \dots, \zeta^{p-1}, 1]$. This induces a restriction homomorphism

$$\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}} \to \mathrm{A}^*_{\boldsymbol{\mu}_p} = \mathbb{Z}[\eta]/(p\eta),$$

where η is the first Chern class of the embedding $\boldsymbol{\mu}_p \subseteq \mathbb{G}_{\mathrm{m}}$.

The restriction of the discriminant $\delta \in (\mathrm{A}^{p^2-p}_{\mathrm{T}_{\mathrm{PGL}_p}})^{\mathrm{S}_p}$ to $\boldsymbol{\mu}_p$ is the element

$$\prod_{i \neq j}(i\eta - j\eta) = \Bigl(\prod_{i \neq j}(i - j)\Bigr)\eta^{p^2-p}$$

of $\mathbb{Z}[\eta]/(p\eta)$; this is non-zero multiple of η^{p^2-p} (in fact, it is easy to check that it equals $-\eta^{p^2-p}$).

Proposition 3.1. *The image of the restriction homomorphism*

$$\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p} \longrightarrow \mathbb{Z}[\eta]/(p\eta)$$

is the subring generated by η^{p^2-p}.

This is proved at the end of Section 7.

Theorem 3.2. *There exists a canonical ring homomorphism* $\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p} \to \mathrm{A}^*_{\mathrm{PGL}_p}$ *whose composite with the restriction homomorphism* $\mathrm{A}^*_{\mathrm{PGL}_p} \to \bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p}$ *is the identity.*

This is proved in Section 12.

As a consequence, $\mathrm{A}^*_{\mathrm{PGL}_p}$ and $\mathrm{H}^*_{\mathrm{PGL}_p}$ can be regarded as $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$-algebras.

Theorem 3.3. *The $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$-algebra $\mathrm{A}^*_{\mathrm{PGL}_p}$ is generated by an element $\rho \in \mathrm{A}^{p+1}_{pgl}$, and the ideal of relations is generated by the following:*
(a) *$p\rho = 0$, and*
(b) *$\rho u = 0$ for all u in the kernel of the homomorphism $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p} \to \mathrm{A}^*_{\boldsymbol{\mu}_p}$.*

There is a similar description of the cohomology: besides the element ρ, now considered as living in $\mathrm{H}^{2p+2}_{\mathrm{PGL}_p}$, we need a single class β in degree 3. This class is essentially the tautological *Brauer class*. That is, if we call $\mathcal{C}$ the sheaf of complex valued continuous functions and $\mathcal{C}^*$ the sheaf of complex valued nowhere vanishing continuous functions on the classifying space $\mathrm{B}\,\mathrm{PGL}_p$, the tautological PGL_p principal bundle on $\mathrm{B}\,\mathrm{PGL}_p$ has a class in the topological Brauer group $\mathrm{H}^2(\mathrm{B}\,\mathrm{PGL}_p, \mathcal{C}^*)_{\mathrm{tors}}$ (see [6]). On the other hand, the exponential sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi \mathrm{i}} \mathcal{C} \longrightarrow \mathcal{C}^* \longrightarrow 1$$

induces a boundary homomorphism

$$\mathrm{H}^2(\mathrm{B}\,\mathrm{PGL}_p, \mathcal{C}^*) \longrightarrow \mathrm{H}^3(\mathrm{B}\,\mathrm{PGL}_p, \mathbb{Z}) = \mathrm{H}^3_{\mathrm{PGL}_p}$$

which is an isomorphism, since $\mathrm{B}\,\mathrm{PGL}_p$ is paracompact, hence $\mathrm{H}^i(\mathrm{B}\,\mathrm{PGL}_p, \mathcal{C}) = 0$ for all $i > 0$. Our class β is, up to sign, the image under this boundary homomorphism of the Brauer class of the tautological bundle.

Theorem 3.4. *The ring $\mathrm{H}^*_{\mathrm{PGL}_p}$ is the commutative $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$-algebra generated by an element β of degree 3 and the element ρ of degree $2p+2$. The ideal of relations is generated by the following:*
(a) *$\beta^2 = 0$,*
(b) *$p\rho = p\beta = 0$, and*
(c) *$\rho u = \beta u = 0$ for all u in the kernel of the homomorphism $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p} \to \mathrm{A}^*_{\boldsymbol{\mu}_p}$.*

Corollary 3.5. *The cycle homomorphism induces an isomorphism of $\mathrm{A}^*_{\mathrm{PGL}_p}$ with $\mathrm{H}^{\mathrm{even}}_{\mathrm{PGL}_p}$.*

From here it is not hard to get the additive structure of $\mathrm{A}^*_{\mathrm{PGL}_p}$ and $\mathrm{H}^*_{\mathrm{PGL}_p}$. For each integer m, denote by $r(m,p)$ the number of partitions of m into numbers between 2 and p. If we denote by $\pi(m,p)$ the number of partitions of m with numbers at most equal to p (a more usual notation for this is $p(m,p)$, which does not look very good), then $r(m,p) = \pi(m,p) - \pi(m-1,p)$.

We will also denote by $s(m,p)$ the number of ways of writing m as a linear combination $(p^2-p)i + (p+1)j$, with $i \geq 0$ and $j > 0$; and by $s'(m,p)$ the number of ways of writing m as a the same linear combination, with $i \geq 0$

and $j \geq 0$. Obviously we have $s'(m,p) = s(m,p)$, unless m is divisible by $p^2 - p$, in which case $s'(m,p) = s(m,p) + 1$.

Theorem 3.6.

(a) *The groups* $\mathrm{A}^m_{\mathrm{PGL}_p}$ *is isomorphic to*

$$\mathbb{Z}^{r(m,p)} \oplus (\mathbb{Z}/p\mathbb{Z})^{s(m,p)}.$$

(b) *The group* $\mathrm{H}^m_{\mathrm{PGL}_p}$ *is isomorphic to* $\mathrm{A}^{m/2}_{\mathrm{PGL}_p}$ *when* m *is even, and is isomorphic to*

$$(\mathbb{Z}/p\mathbb{Z})^{s'\left(\frac{m-3}{2},p\right)}$$

when m *is odd.*

When $p = 3$ we are able to get a description of $\mathrm{A}^*_{\mathrm{PGL}_p}$ and $\mathrm{H}^*_{\mathrm{PGL}_p}$ by generator and relations, completing the work of [15].

Theorem 3.7.

(a) $\mathrm{A}^*_{\mathrm{PGL}_3}$ *is the commutative* $\mathbb{Z}$*-algebra generated by elements* γ_2, γ_3, δ, ρ, *of degrees* 2, 3, 6 *and* 4 *respectively, with relations*

$$27\delta - (4\gamma_2^3 + \gamma_3^2), \quad 3\rho, \quad \gamma_2\rho, \quad \gamma_3\rho.$$

(b) $\mathrm{H}^*_{\mathrm{PGL}_3}$ γ_2, γ_3, δ, ρ *and* β *of degrees* 4, 6, 12, 8*a and* 3 *respectively, with relations*

$$27\delta - (4\gamma_2^3 + \gamma_3^2), \quad 3\rho, \quad 3\beta, \quad \beta^2, \quad \gamma_2\rho, \quad \gamma_3\rho, \quad \gamma_2\beta, \quad \gamma_3\beta.$$

The rest of the paper is dedicated to the proofs of these results. We start by recalling some basic facts on equivariant intersection theory.

4. Preliminaries on equivariant intersection theory

In this section the base field k will be arbitrary.

We refer to [13], [2] and [15] for the definitions and the basic properites of the Chow ring A^*_G of the classifying space of an algebraic group G over a field k, and of the Chow group $\mathrm{A}^*_G(X)$ when X is a scheme, or algebraic space, over k on which G acts, and their main properties. Almost all X that appear in this paper will be smooth, in which case $\mathrm{A}^*_G(X)$ is a commutative ring; the single exception will be in the proof of Lemma 6.5.

The connection between these two notions is that $\mathrm{A}^*_G = \mathrm{A}^*_G(\operatorname{Spec} k)$.

Recall that $\mathrm{A}^*_G(X)$ is contravariant for equivariant morphism of smooth varieties; that is, if $f\colon X \to Y$ is a G-equivariant morphism of smooth G-schemes, there in an induced ring homomorphism $f^*\colon\ \mathrm{A}^*_G(X) \to \mathrm{A}^*_G(Y)$.

If $k = \mathbb{C}$, and X is a smooth algebraic variety on which G acts, there is a cycle ring homomorphism $\mathrm{A}^*_G(X) \to \mathrm{H}^*_G(X)$ from the equivariant Chow ring to the equivariant cohomology ring; this is compatible with pullbacks.

Furthermore, if f is proper there is a pushforward $f_*\colon\ \mathrm{A}^*_G(Y) \to \mathrm{A}^*_G(X)$; this is not a ring homomorphism, but it satisfies the projection formula

$$f_*(\xi \cdot f^*\eta) = f_*\xi \cdot \eta$$

for any $\xi \in \mathrm{A}^*_G(X)$ and $\eta \in \mathrm{A}^*_G(Y)$.

$$\mathrm{A}^*_G(Y) \xrightarrow{\iota_*} \mathrm{A}^*_G(X) \longrightarrow \mathrm{A}^*_G(X \setminus Y) \longrightarrow 0.$$

The analogous statement for cohomology is different: here the restriction homomorphism $\mathrm{H}^*_G(X) \to \mathrm{H}^*_G(X \setminus Y)$ is not necessarily surjective. However, when X and Y are smooth we have a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^{i-1}_G(X \setminus Y) \xrightarrow{\partial} \mathrm{H}^{i-2r}_G(Y) \xrightarrow{\iota_*} \mathrm{H}^i_G(X) \longrightarrow \mathrm{H}^i_G(X \setminus Y) \xrightarrow{\partial} \mathrm{H}^{i-2r+1}_G(Y) \longrightarrow \cdots$$

where r is the codimension of Y in X.

Furthermore, if $H \to G$ is a homomorphism of algebraic groups, and G acts on a smooth scheme X, we can define an action of H on X by composing with the given homomorphism $H \to G$. Then we have a restriction homomorphism

$$\operatorname{res}^G_H \colon \mathrm{A}^*_G(X) \longrightarrow \mathrm{A}^*_H(X).$$

Here is another property that will be used often. Suppose that H is an algebraic subgroup of G. We can define a ring homomorphism $\mathrm{A}^*_G(G/H) \to \mathrm{A}^*_H$ by composing the restriction homomorphism $\mathrm{A}^*_G(G/H) \to \mathrm{A}^*_H(G/H)$ with the pullback $\mathrm{A}^*_H(G/H) \to \mathrm{A}^*_H(\operatorname{Spec} k) = \mathrm{A}^*_H$ obtained by the homomorphism $\operatorname{Spec} k \to G/H$ whose image is the image of the identity in $G(k)$. Then this ring homomorphism is an isomorphism.

More generally, suppose that H acts on a scheme X. We define the induced space $G \times^H X$ as usual, as the quotient $(G \times X)/H$ by the free right action given by the formula $(g, x)h = (gh, h^{-1}x)$. This carries a natural left action of G defined by the formula $g'(g, x) = (g'g, x)$. There is also an embedding $X \simeq H \times^H X \hookrightarrow G \times^H X$ that is H-equivariant: and the composite of the restriction homomorphism $\mathrm{A}^*_G(G \times^H X) \to \mathrm{A}^*_H(G \times^H X)$ with the pullback $\mathrm{A}^*_H(G \times^H X) \to \mathrm{A}^*_H(X)$ is an isomorphism.

Furthermore, if V is a representation of G, then there are Chern classes $\mathrm{c}_i(V) \in \mathrm{A}^i_G$, satisfying the usual properties. More generally, if X is a smooth scheme over k with an action of G, every G-equivariant vector bundles $E \to X$ has Chern classes $\mathrm{c}_i(E) \in \mathrm{A}^i_G(X)$.

The following fact will be used often.

Lemma 4.1. *Let $E \to X$ be an equivariant vector bundle of constant rank r, $s \colon X \to E$ the 0-section, $E_0 \subseteq E$ the complement of the 0-section. Then the sequence*

$$\mathrm{A}^*_G(X) \xrightarrow{\mathrm{c}_r(E)} \mathrm{A}^*_G(X) \longrightarrow \mathrm{A}^*_G(E_0) \longrightarrow 0,$$

where the second arrow is the pullback along $E_0 \to \operatorname{Spec} k$, is exact.

Furthermore, when $k = \mathbb{C}$ we also have a long exact sequence

$$\cdots \longrightarrow \mathrm{H}_G^{i-1}(E_0) \xrightarrow{\ \partial\ } \mathrm{H}_G^{i-2r}(X) \xrightarrow{\ \mathrm{c}_r(E)\ } \mathrm{H}_G^{i}(X) \longrightarrow \mathrm{H}_G^{i}(E_0) \xrightarrow{\ \partial\ } \mathrm{H}_G^{i-2r+1}(X) \longrightarrow \cdots$$

Proof. This follows from the following facts:

(1) the pullbacks $\mathrm{A}_G^*(X) \to \mathrm{A}_G^*(E)$ and $\mathrm{H}_G^*(X) \to \mathrm{H}_G^*(E)$ are isomorphisms,

(2) the self-intersection formula, that says that the homomorphisms $s^*s_*\colon \mathrm{A}_G^*(X) \to \mathrm{A}_G^*(X)$ and $s^*s_*\colon \mathrm{H}_G^*(X) \to \mathrm{H}_G^*(X)$ are multiplication by $\mathrm{c}_r(E)$, and

(3) the localization sequences for Chow rings and cohomology. ♠

Let us recall the following results from [13].

(1) If $T = \mathbb{G}_{\mathrm{m}}^n$ is a torus, and we denote by $x_i \in \mathrm{A}_T^1$ the first Chern class of the i^{th} projection $T \to \mathbb{G}_{\mathrm{m}}$, considered as a representation, then

$$\mathrm{A}_T^* = \mathbb{Z}[x_1, \dots, x_n].$$

(2) If $\mathrm{T}_{\mathrm{GL}_n}$ is the standard maximal torus in GL_n consisting of diagonal matrices, then the restriction homomorphism $\mathrm{A}_{\mathrm{GL}_n}^* \to \mathrm{A}_{\mathrm{T}_{\mathrm{GL}_n}}^*$ induces an isomorphism

$$\begin{aligned} \mathrm{A}_{\mathrm{GL}_n}^* &\simeq \mathbb{Z}[x_1, \dots, x_n]^{\mathrm{S}_n} \\ &= \mathbb{Z}[\sigma_1, \dots, \sigma_n] \end{aligned}$$

where the σ_i are the elementary symmetric functions of the x_i.

(3) If $\mathrm{T}_{\mathrm{SL}_n}$ is the standard maximal torus in SL_n consisting of diagonal matrices, and we denote by x_i the restriction to $\mathrm{A}_{\mathrm{SL}_n}^*$ of $x_i \in \mathrm{A}_{\mathrm{GL}_n}^*$, then we have

$$\mathrm{A}_{\mathrm{T}_{\mathrm{SL}_n}}^* = \mathbb{Z}[x_1, \dots, x_n]/(\sigma_1);$$

furthermore the restriction homomorphism $\mathrm{A}_{\mathrm{SL}_n}^* \to \mathrm{A}_{\mathrm{T}_{\mathrm{SL}_n}}^*$ induces an isomorphism

$$\begin{aligned} \mathrm{A}_{\mathrm{SL}_n}^* &\simeq \bigl(\mathbb{Z}[x_1, \dots, x_n]/(\sigma_1)\bigr)^{\mathrm{S}_n} \\ &= \mathbb{Z}[\sigma_1, \sigma_2, \dots, \sigma_n]/(\sigma_1). \end{aligned}$$

(4) If $t \in \mathrm{A}_{\boldsymbol{\mu}_n}^*$ is the first Chern class of the embedding $\boldsymbol{\mu}_n \hookrightarrow \mathbb{G}_{\mathrm{m}}$, considered as a 1-dimensional representation, then we have

$$\mathrm{A}_{\boldsymbol{\mu}_n}^* = \mathbb{Z}[t](nt).$$

Furthemore, if G is any of the groups above and $k = \mathbb{C}$, then the cycle homomorphism $\mathrm{A}^*_G \to \mathrm{H}^*_G$ is an isomorphism.

The following result is implicit in [13]. Let G be a finite algebraic group that is a product of copies of $\boldsymbol{\mu}_n$, for various n. This is equivalent to saying that G is a finite diagonalizable group scheme, or that G is the Cartier dual of a finite abelian group Γ, considered as a group scheme over k. By Cartier duality, we have that Γ is the character group $\widehat{G} \overset{\mathrm{def}}{=} \mathrm{Hom}(G, \mathbb{G}_{\mathrm{m}})$.

Proposition 4.2. *Consider the group homomorphism $\widehat{G} \to \mathrm{A}^1_G$ that sends each character $\chi\colon G \to \mathbb{G}_{\mathrm{m}}$ into $\mathrm{c}_1(\chi)$. The induced ring homomorphism $\mathrm{Sym}_{\mathbb{Z}} \widehat{G} \to \mathrm{A}^*_G$ is an isomorphism.*

A more concrete way of stating this is the following. Set

$$G = \boldsymbol{\mu}_{n_1} \times \cdots \times \boldsymbol{\mu}_{n_r}.$$

For each $i = 1, \ldots, n$ call χ_i the character obtained by composing the i^{th} projection $G \to \boldsymbol{\mu}_{n_i}$ with the embedding $\boldsymbol{\mu}_{n_i} \hookrightarrow \mathbb{G}_{\mathrm{m}}$, and set $\xi_i = \mathrm{c}_1(\chi_i) \in \mathrm{A}^1_G$. Then

$$\mathrm{A}^*_G = \mathbb{Z}[\xi_1, \ldots, \xi_r]/(n_1\xi_1, \ldots, n_r\xi_r).$$

Proof. When $G = \boldsymbol{\mu}_n$, this follows from Totaro's calculation cited above. The general case follows by induction on r from the following Lemma.

Lemma 4.3. *If H is an algebraic group over k, the ring homomorphism*

$$\mathrm{A}^*_H \otimes_{\mathbb{Z}} \mathrm{A}^*_{\boldsymbol{\mu}_n} \longrightarrow \mathrm{A}^*_{H\times\boldsymbol{\mu}_n}$$

induced by the pullbacks $\mathrm{A}^*_H \to \mathrm{A}^*_{H\times\boldsymbol{\mu}_n}$ and $\mathrm{A}^*_{\boldsymbol{\mu}_n} \to \mathrm{A}^*_{H\times\boldsymbol{\mu}_n}$ along the two projections $H \times \boldsymbol{\mu}_n \to H$ and $H \times \boldsymbol{\mu}_n \to \boldsymbol{\mu}_n$ is an isomorphism.

Proof. This follows easily, for example, from the Chow–Künneth formula in [13, Section 6], because $\boldsymbol{\mu}_p$ has a representation $V = k^n$ on which it acts by multiplication, with an open subscheme $U \overset{\mathrm{def}}{=} V \setminus \{0\}$ on which it acts trivially; and the quotient $U/\boldsymbol{\mu}_p$ is the total space of a $\mathbb{G}_{\mathrm{m}}$-torsor on $\mathbb{P}^{n-1}$, and, as such, it is a union of open subschemes of affine spaces.

It is also not hard to prove directly. ♠

There is also a very important *transfer* operation (sometimes called *induction*). Suppose that H is an algebraic subgroup of G of finite index. The transfer homomorphism

$$\mathrm{tsf}^H_G\colon \mathrm{A}^*_H \longrightarrow \mathrm{A}^*_G$$

(see [15]) is the proper pushforward from $\mathrm{A}^*_H \simeq \mathrm{A}^*_G(G/H)$ to $\mathrm{A}^*_G(\mathrm{Spec}\, k) = \mathrm{A}^*_G$.

This is not a ring homomorphism; however, the projection formula holds, that is, if $\xi \in \mathrm{A}^*_G(X)$ and $\eta \in \mathrm{A}^*_H(X)$, we have

$$\mathrm{tsf}^H_G\bigl(\xi \cdot \mathrm{res}^G_H \eta\bigr) = \xi \cdot \mathrm{tsf}^H_G \eta$$

(in other words, tsf^H_G is a homomorphism of $\mathrm{A}^*_G(X)$-modules).

We are going to need the analogue of Mackey's formula in this context. Let H and K be algebraic subgroups of G, and assume that H has finite index in G. We will also assume that the quotient G/H is reduced, and a disjoint union of copies of $\operatorname{Spec} k$ (this is automatically verified when k is algebraically closed of characteristic 0). Then it is easy to see that the double quotient $K\backslash G/H$ is also the disjoint union of copies of $\operatorname{Spec} k$. Furthermore, we assume that every element of $(K\backslash G/H)(k)$ is in the image of some element of $G(k)$. Of course this will always happen if k is algebraically closed; with some work, this hypothesis can be removed, but it is going to be verified in all the cases to which we will apply the formula).

Denote by $\mathcal{C}$ a set of representatives in $G(k)$ for classes in $(K\backslash G/H)(k)$. For each $s \in \mathcal{C}$, set

$$K_s \stackrel{\mathrm{def}}{=} K \cap sHs^{-1} \subseteq G.$$

Obviously K_s is a subgroup of finite index of K; there is also an embedding $K_s \hookrightarrow H$ defined by $k \mapsto s^{-1}ks$.

Proposition 4.4 (Mackey's formula)**.**

$$\operatorname{res}^G_K \operatorname{tsf}^H_G = \sum_{s\in\mathcal{C}} \operatorname{tsf}^{K_s}_K \operatorname{res}^H_{K_s} \colon \ \mathrm{A}^*_H \longrightarrow \mathrm{A}^*_K\,.$$

Proof. This is standard. We have that the equivariant cohomology rings $\mathrm{A}^*_G(G/H)$ and $\mathrm{A}^*_G(G/K)$ are canonically isomorphic to A^*_H and A^*_K, respectively. The retriction homomorphism $\mathrm{A}^*_G \to \mathrm{A}^*_K$ corresponds to the pullback $\mathrm{A}^*_G(\operatorname{Spec} k) \to \mathrm{A}^*_G(G/K)$, and the tranfer homomorphism corresponds to the proper pushforward $\mathrm{A}^*_G(G/H) \to \mathrm{A}^*_G(\operatorname{Spec} k)$.

Since proper pushforwards and flat pullbacks commute, from the cartesian diagram

$$\begin{array}{ccc} G/K \times G/H & \xrightarrow{\ \mathrm{pr}_2\ } & G/H \\ \big\downarrow{\scriptstyle \mathrm{pr}_1} & & \big\downarrow{\scriptstyle \pi} \\ G/K & \xrightarrow{\ \ \rho\ \ } & \operatorname{Spec} k \end{array}$$

we get the equality

$$\operatorname{res}^G_K \operatorname{tsf}^H_G = \rho^*\pi_* = \mathrm{pr}_{1*}\,\mathrm{pr}_2^* \colon \ \mathrm{A}^*_H \longrightarrow \mathrm{A}^*_K\,.$$

Now we need to express $G/K \times G/H$ as a disjoint union of orbits by the diagonal action of G. There is a G-invariant morphism $G \times G \to G$, defined by the rule $(a, b) \mapsto a^{-1}b$, that induces a morphism $G/K \times G/H \to K\backslash G/H$. For each $s \in \mathcal{C}$, call Ω_s the inverse image of $s \in (K\backslash G/H)(k)$, so that $G/K \times G/H$ is a disjoint union $\coprod_{s\in\mathcal{C}} \Omega_s$. It is easy to verify that Ω_s is the orbit of the class $[1, s] \in (G/K \times G/H)(k)$ of the element $(1, s) \in (G \times G)(k)$, and that the stabilizer of $[1, s]$ is precisely K_s. From this we get an isomorphism

$$G/K \times G/H \simeq \coprod_{s\in\mathcal{C}} G/K_s$$

from which the statement follows easily. ♠

Proposition 4.5. *Assume that G is smooth. Let $f\colon X \to Y$ a proper G-equivariant morphism of G-schemes. Assume that for every G-invariant closed subvariety $W \subseteq Y$ there exists a G-invariant closed subvariety of X mapping birationally onto W. Then the pushforward $f_*\colon \mathrm{A}^*_G X \to \mathrm{A}^*_G Y$ is surjective.*

Here by G-invariant closed subvariety of X we mean a closed subscheme V of X that is reduced, and such that G permutes the irreducible components of V transitively (one sometimes says that V is *primitive*).

This property can be expressed by saying that X is an *equivariant Chow envelope of* Y (see [3, Definition 18.3]).

Proof. In the non-equivariant setting the result follows from the definition of proper pushforward.

In our setting, let us notice first of all that if $Y' \to Y$ is a G-equivariant morphism and $X' \overset{\text{def}}{=} Y' \times_Y X$, the projection $X' \to Y'$ is also a Chow envelope (this is easy, and left to the reader). Therefore, if U is an open subscheme of a representation of G on which G acts freely, the morphism $f \times \mathrm{id}_U\colon X \times U \to Y \times U$ is an equivariant Chow envelope. But since G is smooth, it is easily seen that pullback from $(X \times U)/G$ to $X \times U$ defines a bijective correspondence between closed subvarieties of $(X \times U)/G$ and closed invariant subvarieties of $X \times U$; hence the $(X \times U)/G$ is a Chow envelope of $(Y \times U)/G$. So the proper pushforward $\mathrm{A}^*\bigl((X \times U)/G\bigr) \to \mathrm{A}^*\bigl((Y \times U)/G\bigr)$ is surjective, and this completes the proof. ♠

5. On $\mathrm{C}_p \times \boldsymbol{\mu}_p$

A key role in our proof is played by a finite subgroups $\mathrm{C}_p \times \boldsymbol{\mu}_p \subseteq \mathrm{PGL}_p$.

We denote by $\mathrm{C}_p \subseteq \mathrm{S}_p$ the subgroup generated by the cycle $\sigma \overset{\text{def}}{=} (1\,2\,\dots\,p)$. We embed S_p into PGL_p as usual by identifying a permutation $\alpha \in \mathrm{S}_p$ with the corresponding permutation matrix, obtained by applying α to the indices of the canonical basis $\mathrm{e}_1, \dots, \mathrm{e}_p$ of V (so that $\sigma \mathrm{e}_i = \mathrm{e}_{i+1}$, where addition is modulo p).

If we denote by τ the generator

$$[\omega, \dots, \omega^{p-1}, 1]$$

of $\boldsymbol{\mu}_p \subseteq \mathrm{PGL}_p$, we have that

$$\tau\sigma = \omega\sigma\tau \quad \text{in } \mathrm{GL}_p;$$

so σ and τ commute in PGL_p, and they generate a subgroup

$$\mathrm{C}_p \times \boldsymbol{\mu}_p \subseteq \mathrm{PGL}_p.$$

We denote by α and β the characters $\mathrm{C}_p \times \boldsymbol{\mu}_p \to \mathbb{G}_{\mathrm{m}}$ defined as

$$\alpha(\sigma) = \omega \quad \text{and} \quad \alpha(\tau) = 1$$

and

$$\beta(\sigma) = 1 \quad \text{and} \quad \beta(\tau) = \omega.$$

The following fact will be useful later.

Lemma 5.1. *If i and j are integers between 1 and p, consider the matrix $\sigma^i\tau^j$ in the algebra $\mathfrak{gl}_p$ of $p\times p$ matrices. Then if $(i,j)\neq(p,p)$, the matrix $\sigma^i\tau^j$ has trace 0, and its eigenvalues are precisely the p-roots of 1.*

Each $\sigma^i\tau^j$ is a semi-invariant for the action of $\mathrm{C}_p\times\boldsymbol{\mu}_p$, with character $\alpha^{-j}\beta^i$. Furthermore the $\sigma^i\tau^j$ form a basis of $\mathfrak{gl}_p$, and those with $(i,j)\neq(p,p)$ form a bases of $\mathfrak{sl}_p$.

Proof. The fact that $\mathrm{C}_p\times\boldsymbol{\mu}_p$ acts on $\sigma^i\tau^j$ via the character $\alpha^{-j}\beta^i$ is an elementary calculation, using the relation $\tau\sigma=\omega\sigma\tau$. From this it follows that the $\sigma^i\tau^j$ are linearly independent, and therefore form a basis of $\mathfrak{gl}_p$. The statement about the trace is also easy.

Let us check that the $\sigma^i\tau^j$ with $(i,j)\neq(p,p)$ have the elements of $\boldsymbol{\mu}_p$ as eigenvalues. When $i=p$ we get a diagonal matrix with eigenvalues are $\omega^j,\ldots,\omega^{pj}$, which are all the elements of $\boldsymbol{\mu}_p$, because p is a prime and j is not divisible by p. Assume that $i\neq p$. The numbers i, $2i$, …, pi, reduced modulo p, coincide with 1, …, p. If λ is a p^{th} root of 1, and $\mathrm{e}_1,\ldots,\mathrm{e}_p$ is the canonical basis of k^n, then the vector

$$\sum_{t=1}^{p}\lambda^{-t}\omega^{ij\binom{t}{2}}\mathrm{e}_{ti}$$

is easily seen to be an eigenvector of $\sigma^i\tau^j$ with eigenvalue λ (using the fact that

$$\binom{t_1}{2}\equiv\binom{t_2}{2}\pmod p$$

when $t_1\equiv t_2\pmod p$, which holds because p is odd, and the relations $\sigma\mathrm{e}_i=\mathrm{e}_{i+1}$ and $\tau\mathrm{e}_i=\omega^i\mathrm{e}_i$). This concludes the proof of the Lemma. ♠

Corollary 5.2. *Any two elements in $\mathrm{C}_p\times\boldsymbol{\mu}_p$ different from the identity are conjugate in PGL_p.*

Remark 5.3. It is interesting to observe that the Proposition, and its Corollary, are false for $p=2$; then the matrix $\sigma\tau$ has eigenvalues $\pm\sqrt{-1}$, which are not square roots of 1.

We will denote by ξ and η the first Chern classes in $\mathrm{A}^1_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$ of the characters α and β. Then we have

$$\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}=\mathbb{Z}[\xi,\eta]/(p\xi,p\eta).$$

We will identify $\mathrm{C}_p\times\boldsymbol{\mu}_p$ with $\mathbb{F}_p\times\mathbb{F}_p$, by sending σ to $(1,0)$ and τ to $(0,1)$; this identifies the automorphism group of $\mathrm{C}_p\times\boldsymbol{\mu}_p$ with $\mathrm{GL}_2(\mathbb{F}_p)$.

We are interested in the action of the normalizer $\mathrm{N}_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\mathrm{PGL}_p$ of $\mathrm{C}_p\times\boldsymbol{\mu}_p$ in PGL_p on $\mathrm{C}_p\times\boldsymbol{\mu}_p$ and on the Chow ring $\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$.

Proposition 5.4. *Consider the homomorphism*

$$\mathrm{N}_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\mathrm{PGL}_p\longrightarrow\mathrm{GL}_2(\mathbb{F}_p)$$

defined by the action of $\mathrm{N}_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\mathrm{PGL}_p$ *on* $\mathrm{C}_p\times\boldsymbol{\mu}_p$*. Its kernel is* $\mathrm{C}_p\times\boldsymbol{\mu}_p$*, while its image is* $\mathrm{SL}_2(\mathbb{F}_p)$*.*

Furthermore, the ring of invariants

$$\big(\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\big)^{\mathrm{SL}_2(\mathbb{F}_p)}$$

is the subring of $\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$ *generated by the two homogeneous polynomials*

$$\begin{aligned} q &\stackrel{\text{def}}{=} \eta^{p^2-p} + \xi^{p-1}(\xi^{p-1}-\eta^{p-1})^{p-1} \\ &= \xi^{p^2-p} + \eta^{p-1}(\xi^{p-1}-\eta^{p-1})^{p-1} \end{aligned}$$

and

$$r \stackrel{\text{def}}{=} \xi\eta(\xi^{p-1}-\eta^{p-1})$$

The equality of the two polynomials that appear in the definition of q is not immediately obvious, but is easy to prove, by subtracting them and using the identity

$$(\xi^{p-1}-\eta^{p-1})^p = \xi^{p^2-p} - \eta^{p^2-p}.$$

Proof. First of all, let us show that the image of the homomorphism above is contained in $\mathrm{SL}_2(\mathbb{F}_p)$. There is canonical symplectic form

$$\bigwedge\nolimits^2(\mathrm{C}_p\times\boldsymbol{\mu}_p)\longrightarrow\boldsymbol{\mu}_p$$

defined as follows: if a and b are in $\mathrm{C}_p\times\boldsymbol{\mu}_p\subseteq\mathrm{PGL}_p$, lift them to matrices $\overline{a}$ and $\overline{b}$ in GL_p. Then the commutator $\overline{a}\overline{b}\overline{a}^{-1}\overline{b}^{-1}$ is a scalar multiple of the identity matrix I_p; it is easy to see that the scalar factor, which we denote by $\langle a,b\rangle$, is in $\boldsymbol{\mu}_p$, and that it only depends on a and b, that is, it is independent of the liftings. The resulting function

$$\langle -,-\rangle\colon(\mathrm{C}_p\times\boldsymbol{\mu}_p)\times(\mathrm{C}_p\times\boldsymbol{\mu}_p)\longrightarrow\boldsymbol{\mu}_p$$

is the desired symplectic form.

Now, $\mathrm{SL}_2(\mathbb{F}_p)$ has $p(p^2-1)$ elements. According to Corollary 5.2, the action of $\mathrm{N}_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\mathrm{PGL}_p$ is transitive on the non-zero vectors in $\mathbb{F}_p^2$; so the order of the image of $\mathrm{N}_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\mathrm{PGL}_p$ in $\mathrm{SL}_2(\mathbb{F}_p)$ has order divisible by p^2-1. It is easy to check that the diagonal matrix

$$[1,\omega,\omega^3,\dots,\underbrace{\omega^{\binom{i}{2}}}_{i^{\text{th}}\text{ place}},\dots,\omega,1]$$

is also in $\mathrm{N}_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\mathrm{PGL}_p$, acts non-trivially on $\mathrm{C}_p\times\boldsymbol{\mu}_p$, and has order p. So the order of the image of $\mathrm{N}_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\mathrm{PGL}_p$ is divisible by p; it follows that it is equal to all of $\mathrm{SL}_2(\mathbb{F}_p)$.

It is not hard to check that the centralizer of $\mathrm{C}_p\times\boldsymbol{\mu}_p$ equals $\mathrm{C}_p\times\boldsymbol{\mu}_p$; and this completes the proof of the first part of the statement.

To study the invariant subring $\left(\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}\right)^{\mathrm{SL}_2(\mathbb{F}_p)}$, we use the natural surjective homomorphism

$$\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p} = \mathbb{Z}[\xi, \eta]/(p\xi, p\eta) \longrightarrow \mathbb{F}_p[\xi, \eta],$$

which is an isomorphism in all degrees except 1; it is enough to show that the ring of invariants $\mathbb{F}_p[\xi, \eta]^{\mathrm{SL}_2(\mathbb{F}_p)}$ is the polynomial subring $\mathbb{F}_p[q, r]$.

To look for invariants in $\mathbb{F}_p[\xi, \eta]$, we compute the symmetric functions of the vectors in the dual vector space $\left(\mathbb{F}_p^2\right)^\vee$; these are the homogeneous components of the polynomial

$$\prod_{i,j \in \mathbb{F}_p} (1 + i\xi + j\eta),$$

which are evidently invariant under $\mathrm{GL}_2(\mathbb{F}_p)$.

Lemma 5.5.

$$\prod_{0 \leq i,j \leq p-1} (1 + i\xi + j\eta) = 1 - q + r^{p-1}.$$

Proof. Using the formula

$$\prod_{i \in \mathbb{F}_p} (a + ib) = a^p - ab^{p-1},$$

which holds for any two elements a and b of a commmutative $\mathbb{F}_p$-algebra, we obtain

$$\begin{aligned}
\prod_{i,j \in \mathbb{F}_p} (1 + i\xi + j\eta) &= \prod_{i \in \mathbb{F}_p} \left((1 + i\xi)^p - (1 + i\xi)\eta^{p-1}\right) \\
&= \prod_{i \in \mathbb{F}_p} \left((1 - \eta^{p-1}) + i(\xi^p - \xi\eta^{p-1})\right) \\
&= (1 - \eta^{p-1})^p - (1 - \eta^{p-1})(\xi^p - \xi\eta^{p-1})^{p-1} \\
&= 1 - (\eta^{p^2-p} + (\xi^p - \xi\eta^{p-1})^{p-1}) \\
&\qquad + \xi^{p-1}\eta^{p-1}(\xi^{p-1} - \eta^{p-1})^{p-1} \\
&= 1 - q + r^{p-1}.
\end{aligned}$$

♠

This shows that q and r^{p-1} are invariant under $\mathrm{GL}_2(\mathbb{F}_p)$. The polynomial r is not invariant under $\mathrm{GL}_2(\mathbb{F}_p)$, but it is invariant under $\mathrm{SL}_2(\mathbb{F}_p)$. The simplest way to verify this is to observe that r must be a semi-invariant of $\mathrm{GL}_2(\mathbb{F}_p)$ (if $g \in \mathrm{GL}_2(\mathbb{F}_p)$, then $(gr)^{p-1} = r^{p-1}$, and this means that gr and r differ by a constant in $\mathbb{F}_p^*$). But the commutator subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$ is well known to be $\mathrm{SL}_2(\mathbb{F}_p)$; so any character $\mathrm{GL}_2(\mathbb{F}_p) \to \mathbb{F}_p^*$ is trivial on $\mathrm{SL}_2(\mathbb{F}_p)$, and r is invariant under $\mathrm{SL}_2(\mathbb{F}_p)$.

We have left to check that q and r generate the ring of invariants $\mathbb{F}_p[\xi, \eta]^{\mathrm{SL}_2(\mathbb{F}_p)}$. The equalities

$$\xi^{p^2-1} - q\xi^{p-1} + r^{p-1} = \eta^{p^2-1} - q\eta^{p-1} + r^{p-1} = 0,$$

which are easily checked by homogenizing the equality of Lemma 5.5, that is, by adding an indeterminate t and obtaining

$$\prod_{\substack{0\leq i,j\leq p-1\\(i,j)\neq(0,0)}}(t+i\xi+j\eta)=t^{p^2-1}-qt^{p-1}+r^{p-1},$$

ensure that the extension $\mathbb{F}_p[q,r]\subseteq\mathbb{F}_p[\xi,\eta]$ is finite. Hence it is flat, and its degree equals

$$\begin{aligned}\dim_{\mathbb{F}_p}\mathbb{F}_p[\xi,\eta]/(q,r)&=\dim_{\mathbb{F}_p}\mathbb{F}[\xi,\eta]/(q,\xi)\\&\qquad+\dim_{\mathbb{F}_p}\mathbb{F}[\xi,\eta]/(q,\eta)\\&\qquad+\dim_{\mathbb{F}_p}\mathbb{F}[\xi,\eta]/\bigl(q,\xi^{p-1}-\eta^{p-1}\bigr)\\&=\dim_{\mathbb{F}_p}\mathbb{F}_p[\xi,\eta]/(\eta^{p^2-p},\xi)\\&\qquad+\dim_{\mathbb{F}_p}\mathbb{F}_p[\xi,\eta]/(\xi^{p^2-p},\eta)\\&\qquad+\dim_{\mathbb{F}_p}\mathbb{F}_p[\xi,\eta]/\bigl(\xi^{p^2-p},\xi^{p-1}-\eta^{p-1}\bigr)\\&=(p^2-p)+(p^2-p)+(p^2-p)(p-1)\\&=p(p^2-1),\end{aligned}$$

which is the order of $\mathrm{SL}_2(\mathbb{F}_p)$. So the degrees of the field extensions

$$\mathbb{F}_p(q,r)\subseteq\mathbb{F}_p(\xi,\eta)\quad\text{and}\quad\mathbb{F}_p(\xi,\eta)^{\mathrm{SL}_2(\mathbb{F}_p)}\subseteq\mathbb{F}_p(\xi,\eta)$$

both equal $p(p^2-1)$, so $\mathbb{F}_p(q,r)=\mathbb{F}_p(\xi,\eta)^{\mathrm{SL}_2(\mathbb{F}_p)}$; and the result follows, because $\mathbb{F}_p[q,r]$ is integrally closed. ♠

For later use, let us record the following fact. The image the restriction homomorphism $\mathrm{A}^*_{\mathrm{PGL}_p}\to\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$ is contained in $\bigl(\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\bigr)^{\mathrm{SL}_2(\mathbb{F}_p)}$. We are going to need formulas for the restriction of the Chern classes $\mathrm{c}_i(\mathfrak{sl}_p)$ to $\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$.

Lemma 5.6. *Let i be a positive integer. Then the restriction of $\mathrm{c}_i(\mathfrak{sl}_p)$ to $\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$ is $-q$ if $i=p^2-p$, is r^{p-1} if $i=p^2-1$, and is 0 in all other cases.*

Proof. The total Chern class of $\mathfrak{gl}_p$ coincides with the total Chern class of $\mathfrak{sl}_p$, because $\mathfrak{gl}_p$ is the direct some of $\mathfrak{sl}_p$ and a trivial representation. From Lemma 5.1 we see that this total Chern class, when restricted to $\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$, equals

$$\sum_{i,j=1}^{p}(1+i\xi+j\eta);$$

and then the result follows from Lemma 5.5. ♠

We will also need to know about the cohomology ring $\mathrm{H}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$. For any cyclic group $\mathrm{C}_n\simeq\mu_n$, the homomorphism $\mathrm{A}^*_{\mathrm{C}_n}\to\mathrm{H}^*_{\mathrm{C}_n}$ is an isomorphism.

This does not extend to $\mathrm{C}_p \times \boldsymbol{\mu}_p$; however, from the universal coefficients theorem for cohomology, for each index k we have a split exact sequence

$$0 \longrightarrow \bigoplus_{i+j=k} \mathrm{H}^i_{\mathrm{C}_p} \otimes \mathrm{H}^j_{\boldsymbol{\mu}_p} \longrightarrow \mathrm{H}^k_{\mathrm{C}_p \times \boldsymbol{\mu}_p} \longrightarrow \bigoplus_{i+j=k+1} \mathrm{Tor}_1^{\mathbb{Z}}\bigl(\mathrm{H}^i_{\mathrm{C}_p}, \mathrm{H}^j_{\boldsymbol{\mu}_p}\bigr) \longrightarrow 0;$$

furthermore, since the exterior product homomorphism $\mathrm{A}^*_{\mathrm{C}_p} \otimes \mathrm{A}^*_{\boldsymbol{\mu}_p} \to \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ is an isomorphism, the image of the term $\bigoplus_{i+j=k} \mathrm{H}^i_{\mathrm{C}_p} \otimes \mathrm{H}^j_{\boldsymbol{\mu}_p}$ into $\mathrm{H}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ is the image of the cycle homomorphism $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p} \to \mathrm{H}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$. From this it is easy to deduce that the cycle homomorphism induces an isomorphism of $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ with the even dimensional part $\mathrm{H}^{\mathrm{even}}_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ of the cohomology.

We have isomorphisms

$$\mathrm{H}^3_{\mathrm{C}_p \times \boldsymbol{\mu}_p} \simeq \mathrm{Tor}_1^{\mathbb{Z}}\bigl(\mathrm{H}^2_{\mathrm{C}_p}, \mathrm{H}^2_{\boldsymbol{\mu}_p}\bigr) \simeq \mathbb{Z}/p\mathbb{Z};$$

chose a generator s of $\mathrm{H}^3_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ (later we will make a canonical choice). We have that $s^2 = 0$, because p is odd, and s has odd degree.

The odd-dimensional part $\mathrm{H}^{\mathrm{odd}}_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ of the cohomology is isomorphic to the direct sum $\bigoplus_{i,j} \mathrm{Tor}_1^{\mathbb{Z}}\bigl(\mathrm{H}^i_{\mathrm{C}_p}, \mathrm{H}^j_{\boldsymbol{\mu}_p}\bigr)$, with a shift by 1 in degree. Both $\mathrm{H}^{\mathrm{odd}}_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ and $\bigoplus_{i,j} \mathrm{Tor}_1^{\mathbb{Z}}\bigl(\mathrm{H}^i_{\mathrm{C}_p}, \mathrm{H}^j_{\boldsymbol{\mu}_p}\bigr)$ have natural structures of modules over $\mathrm{H}^*_{\mathrm{C}_p} \otimes \mathrm{H}^*_{\boldsymbol{\mu}_p} = \mathrm{H}^{\mathrm{even}}_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$, and the isomorphism above is an isomorphism of modules. But $\bigoplus_{i,j} \mathrm{Tor}_1^{\mathbb{Z}}\bigl(\mathrm{H}^i_{\mathrm{C}_p}, \mathrm{H}^j_{\boldsymbol{\mu}_p}\bigr)$ is easily seen to be a cyclic $\mathrm{H}^{\mathrm{even}}_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$-module generated by s. From this we obtain the following result.

Proposition 5.7.

$$\mathrm{H}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p} = \mathbb{Z}[\xi, \eta, s]/(p\xi, p\eta, ps, s^2).$$

We are also interested in the action of $\mathrm{SL}_2(\mathbb{F}_p)$ on $\mathrm{H}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$. I claim that the class s is invariant: this is equivalent to the following.

Lemma 5.8. *The action of* $\mathrm{SL}_2(\mathbb{F}_p)$ *on* $\mathrm{H}^3_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ *is trivial.*

This follows, for example, from the construction of Section 11, where we construct a class $\beta \in \mathrm{H}^3_{\mathrm{PGL}_p}$ that maps to a non-zero element of $\mathrm{H}^3_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$. It would be logically correct to postpone the proof to Section 11, as this fact is not used before then; but this would not be very satisfactory, so we prove it now directly.

Proof. Consider the exact sequence

$$\mathrm{H}^2(\mathrm{C}_p \times \boldsymbol{\mu}_p, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\beta} \mathrm{H}^3(\mathrm{C}_p \times \boldsymbol{\mu}_p, \mathbb{Z}) \xrightarrow{p} \mathrm{H}^3(\mathrm{C}_p \times \boldsymbol{\mu}_p, \mathbb{Z})$$

coming from the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0;$$

since $\mathrm{H}^3(\mathrm{C}_p \times \boldsymbol{\mu}_p, \mathbb{Z}) = \mathrm{H}^3_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ is $\mathbb{Z}/p\mathbb{Z}$, we see that the Bockstein homomorphism $\beta\colon\ \mathrm{H}^2(\mathrm{C}_p \times \boldsymbol{\mu}_p, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{H}^3(\mathrm{C}_p \times \boldsymbol{\mu}_p, \mathbb{Z})$ is surjective. It is also

$\mathrm{SL}_2(\mathbb{F}_p)$-equivariant. By Künneth's formula, the exterior product induces an isomorphism of the direct sum

$$\mathrm{H}^2(\mathrm{C}_p, \mathbb{Z}/p\mathbb{Z}) \oplus \bigl(\mathrm{H}^1(\mathrm{C}_p, \mathbb{Z}/p\mathbb{Z}) \otimes \mathrm{H}^1(\boldsymbol{\mu}_p, \mathbb{Z}/p\mathbb{Z})\bigr) \oplus \mathrm{H}^2(\boldsymbol{\mu}_p, \mathbb{Z}/p\mathbb{Z})$$

with $\mathrm{H}^2(\mathrm{C}_p \times \boldsymbol{\mu}_p, \mathbb{Z}/p\mathbb{Z})$. Now, from the commutativity of the diagram

$$\begin{array}{ccc}
\mathrm{H}^2(\mathrm{C}_p, \mathbb{Z}/p\mathbb{Z}) & \longrightarrow & \mathrm{H}^3(\mathrm{C}_p, \mathbb{Z}) = 0 \\
\downarrow & & \downarrow \\
\mathrm{H}^2(\mathrm{C}_p \times \boldsymbol{\mu}_p, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\beta} & \mathrm{H}^3(\mathrm{C}_p \times \boldsymbol{\mu}_p, \mathbb{Z})
\end{array}$$

where the rows are Bockstein homomorphisms and the columns are induced by projection $\mathrm{C}_p \times \boldsymbol{\mu}_p \to \mathrm{C}_p$, we see that the Bockstein homomorphism β sends $\mathrm{H}^2(\mathrm{C}_p, \mathbb{Z}/p\mathbb{Z})$, and $\mathrm{H}^2(\boldsymbol{\mu}_p, \mathbb{Z}/p\mathbb{Z})$ for analogous reasons, to 0. Hence the composite of the exterior product map

$$\mathrm{H}^1(\mathrm{C}_p, \mathbb{Z}/p\mathbb{Z}) \otimes \mathrm{H}^1(\boldsymbol{\mu}_p, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathrm{H}^2(\mathrm{C}_p \times \boldsymbol{\mu}_p, \mathbb{Z}/p\mathbb{Z})$$

with β is surjective. But we have an isomorphism

$$\mathrm{H}^1(\mathrm{C}_p \times \boldsymbol{\mu}_p, \mathbb{Z}/p\mathbb{Z}) \simeq \mathrm{H}^1(\mathrm{C}_p, \mathbb{Z}/p\mathbb{Z}) \oplus \mathrm{H}^1(\boldsymbol{\mu}_p, \mathbb{Z}/p\mathbb{Z})$$

which induces an isomorphism

$$\bigwedge^2 \mathrm{H}^1(\mathrm{C}_p \times \boldsymbol{\mu}_p, \mathbb{Z}/p\mathbb{Z}) \simeq \mathrm{H}^1(\mathrm{C}_p, \mathbb{Z}/p\mathbb{Z}) \otimes \mathrm{H}^1(\boldsymbol{\mu}_p, \mathbb{Z}/p\mathbb{Z}).$$

This shows that the composite of the map

$$\bigwedge^2 \mathrm{H}^1(\mathrm{C}_p \times \boldsymbol{\mu}_p, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathrm{H}^2(\mathrm{C}_p \times \boldsymbol{\mu}_p, \mathbb{Z}/p\mathbb{Z})$$

with the Bockstein homomorphism β is surjective, hence an isomorphism, because both groups are isomorphic to $\mathbb{Z}/p\mathbb{Z}$. It is also evidently $\mathrm{GL}_2(\mathbb{F}_2)$-equivariant. The action of $\mathrm{GL}_2(\mathbb{F}_p)$ on the exterior power $\bigwedge^2 \mathrm{H}^1(\mathrm{C}_p \times \boldsymbol{\mu}_p, \mathbb{Z}/p\mathbb{Z})$ is by multiplication by the inverse of the determinant; hence $\mathrm{SL}_2(\mathbb{F}_p)$ acts trivially, and this completes the proof. ♠

From this we deduce the following fact.

Proposition 5.9. *The ring of invariants* $\bigl(\mathrm{H}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}\bigr)^{\mathrm{SL}_2(\mathbb{F}_p)}$ *is generated by* q, r *and* s.

Remark 5.10. The group $\mathrm{C}_p \times \boldsymbol{\mu}_p$ is important in the theory of division algebras. Suppose that K is a field containing k, and $E \to \operatorname{Spec} K$ is a non-trivial PGL_p principal bundle. This corresponds to a central division algebra D over K of degree p. Recall that D is *cyclic* when there are elements a and b of K^*, such that D is generated by two elements x and y, satisfying the relations $x^p = a$, $y^p = b$, $yx = \omega xy$. It is not hard to show that D is cyclic if and only if E has a reduction of structure group to $\mathrm{C}_p \times \boldsymbol{\mu}_p$.

One of the main open problems in the theory of division algebra is whether all division algebras of prime degree is cyclic. Let V be a representation of PGL_p over k with a non-empty open invariant subset U on which PGL_p acts freely. Let K be the fraction field of U/PGL_p, E the pullback to $\operatorname{Spec} K$ of the PGL_p-torsor $U \to U/G$ and D the corresponding division algebra; it is well known that D cyclic if and only if every division algebra of degree p over a field containing k is cyclic.

The obvious way to show that D is *not* cyclic is to show that there is an invariant for division algebras that is 0 for cyclic algebras, but not 0 for D. However, the result proved here implies that there is no such invariant in the cohomology ring $\mathrm{H}^*_{\mathrm{PGL}_p}$. In fact, consider a non-zero invariant $\xi \in \mathrm{H}^*_{\mathrm{PGL}_p}$. Then either ξ has even degree, so it comes from $\mathrm{A}^*_{\mathrm{PGL}_p}$, hence it restricts to 0 in V/PGL_p for some open invariant subset $V \subseteq U$, or it has odd degree, and then it maps to 0 in $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$, and it does not map to 0 in $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$.

This is related with the fact that one can not find such an invariant in étale cohomology with $\mathbb{Z}/p\mathbb{Z}$ coefficients (see [4, §22.10]).

6. On $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$

Proposition 6.1. *Assume that $k = \mathbb{C}$. Then the cycle homomorphism $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} \to \mathrm{H}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$ is an isomorphism.*

Proof. This the first illustration of the stratification method: we take a geometrically meaningful representation of $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$ and we stratify it.

Denote by $V \overset{\text{def}}{=} \mathbb{A}^p$ the standard representation of GL_p, restricted to $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$. We denote by $V_{\leq i}$ the Zariski open $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$-invariant subset consisting of p-uples of complex numbers such that at most i of them are 0, and by $V_i \overset{\text{def}}{=} V_{\leq i} \setminus V_{\leq i-1}$ the smooth locally closed subvariety of p-uples consisting of vectors with exactly i coordinates that are 0. Obviously $V_{\leq p-1} = V \setminus \{0\}$ and $V_p = 0$.

Lemma 6.2. *For each $0 \leq i \leq p-1$, the cycle homomorphism $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} V_i \to \mathrm{H}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} V_i$ is an isomorphism.*

Proof. First of all, assume that $i = 0$. Then the action of $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$ on V_0 is transitive, and the stabilizer of $(1, \ldots, 1) \in V_0(k)$ is C_p; hence we have a commutative diagram

$$\begin{array}{ccc} \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}(V_0) & \longrightarrow & \mathrm{H}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}(V_0) \\ \downarrow & & \downarrow \\ \mathrm{A}^*_{\mathrm{C}_p} & \longrightarrow & \mathrm{H}^*_{\mathrm{C}_p} \end{array}$$

where the rows are cycle homomorphisms and the columns are isomorphisms. Since the bottom row is also an isomorphism, the thesis follows.

When $i > 0$ the argument is similar. The action of $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$ on V_i expresses V_i as a disjoint union of open orbits $\Omega_1, \ldots, \Omega_r$, where $r \overset{\text{def}}{=} \frac{1}{p}\binom{p}{i}$, and the stabilizer of a point of each Ω_j is an i-dimensional torus T_j; hence we get a commutative diagram

$$\begin{array}{ccc}
\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}(V_i) & \longrightarrow & \mathrm{H}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}(V_i) \\
\big\downarrow & & \big\downarrow \\
\bigoplus_{h=1}^r \mathrm{A}^*_{T_j} & \longrightarrow & \bigoplus_{h=1}^r \mathrm{H}^*_{T_j}
\end{array}$$

where the columns and the bottom row are isomorphisms. ♠

Lemma 6.3. *For each $0 \leq i \leq p-1$, the cycle homomorphism $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} V_{\leq i} \to \mathrm{H}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} V_{\leq i}$ is an isomorphism.*

Proof. We proceed by induction on i. When $i = 0$ we have $V_{\leq 0} = V_0$, and the thesis follows from the previous lemma. For the inductive step, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}(V_i) & \longrightarrow & \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}(V_{\leq i}) & \longrightarrow & \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}(V_{\leq i-1}) & \longrightarrow & 0 \\
\big\downarrow & & \big\downarrow & & \big\downarrow \textcircled{1} & & \\
\mathrm{H}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}(V_i) & \xrightarrow{\textcircled{3}} & \mathrm{H}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}(V_{\leq i}) & \xrightarrow{\textcircled{2}} & \mathrm{H}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}(V_{\leq i-1}); & &
\end{array}$$

by inductive hypothesis, the arrow marked with ① is an isomorphism, hence the arrow marked with ② is surjective. However, the bottom row of the diagram extends to a Gysin exact sequence

$$\mathrm{H}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}(V_i) \to \mathrm{H}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}(V_{\leq i}) \to \mathrm{H}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}(V_{\leq i-1}) \to \mathrm{H}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}(V_i) \to \cdots$$

showing that the arrow marked with ③ is injective. From this, and the fact that the left hand column is an isomorphism, it follows that the middle column is also an isomorphism, as desidered. ♠

Let us proceed with the proof of the Theorem. For each i we have an commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} & \xrightarrow{c_p(V)} & \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} & \longrightarrow & \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}(V \setminus \{0\}) & \longrightarrow & 0 \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\mathrm{H}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} & \xrightarrow{c_p(V)} & \mathrm{H}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} & \longrightarrow & \mathrm{H}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}(V \setminus \{0\}) & &
\end{array}$$

Now, by Lemma 6.3 the right hand column is an isomorphism, hence, arguing as in the proof of Lemma 6.3, we conclude that the bottom row of the diagram is a short exact sequence.

If i is odd, we have $\mathrm{H}^i_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}(V \setminus \{0\}) = 0$, hence the multiplication homomorphism

$$\mathrm{H}^{i-2p}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} \xrightarrow{c_p(V)} \mathrm{H}^i_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$$

is an isomorphism. From this we deduce, by induction on i, that $\mathrm{H}^i_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} = 0$ for all odd i.

When i is even, one proceeds similarly by induction on i, with a straightforward diagram chasing in the diagram above. ♠

Let us compute the Chow ring of the classifying space of $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$. The Weyl group S_p acts on $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}} = \mathbb{Z}[x_1, \dots, x_p]$ by permuting the x_i's. Consider the action of C_p on $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}$: the group permutes the monomials, and the only monomials that are left fixed are the ones of the form $\sigma_p^r = x_1^r \dots x_p^r$, while on the others the action of C_p is free. We will call the monomials that are not powers of σ_p *free monomials*. Then $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}$ splits as a direct sum $\mathbb{Z}[\sigma_p] \oplus M$, where M is the free $\mathbb{Z}\mathrm{C}_p$-module generated by the free monomials. Hence the ring of invariants $(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}})^{\mathrm{C}_p}$ is a direct sum $\mathbb{Z}[\sigma_p] \oplus M^{\mathrm{C}_p}$, and M^{C_p} is a free abelian group on the generators $\sum_{s \in \mathrm{C}_p} sm$, where m is a free monomial.

We will denote by $\xi \in \mathrm{A}^1_{\mathrm{C}_p}$ the first Chern class of the character $\mathrm{C}_p \to \mathbb{G}_{\mathrm{m}}$ obtained by sending the generator $(1, \dots, p)$ of C_p to the fixed generator ω of $\boldsymbol{\mu}_p$, and also its pullback to $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$ through the projection $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p} \to \mathrm{C}_p$.

We will also use the subgroup $\boldsymbol{\mu}_p \subseteq \mathrm{T}_{\mathrm{GL}_p}$ of matrices of the form $\zeta \mathrm{I}_p$, where $\zeta \in \boldsymbol{\mu}_p$. The Chow ring $\mathrm{A}^*_{\boldsymbol{\mu}_p}$ is of the form $\mathbb{Z}[\eta]/(p\eta)$, where η is the first Chern class of the 1-dimensional representation given by the embedding $\boldsymbol{\mu}_p \hookrightarrow \mathbb{G}_{\mathrm{m}}$. The action of C_p on $\boldsymbol{\mu}_p$ is trivial, so there a copy of $\mathrm{C}_p \times \boldsymbol{\mu}_p$ in $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$; the Chow ring $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ is $\mathbb{Z}[\xi, \eta]/(p\xi, p\eta)$.

Here are the facts about $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$ that we are going to need.

Proposition 6.4.

(a) *The image of the restriction homomorphism*

$$(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}})^{\mathrm{C}_p} \longrightarrow \mathrm{A}^*_{\boldsymbol{\mu}_p} = \mathbb{Z}[\eta]/(p\eta)$$

is the subring generated by η^p, which is the image of σ_p. The kernel is the subgroup of $(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}})^{\mathrm{C}_p}$ generated by the $\sum_{s \in \mathrm{C}_p} sm$, where m is a free monomial, and by $p\sigma_p$.

(b) *The ring homomorphism $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} \to \bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\bigr)^{\mathrm{C}_p}$ induced by the embedding $\mathrm{T}_{\mathrm{GL}_p} \hookrightarrow \mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$ is surjective, and admits a canonical splitting $\phi \colon \bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\bigr)^{\mathrm{C}_p} \to \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$, which is a ring homomorphism.*

(c) *As an algebra over $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$, the ring $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$ is generated by the element ξ, while the ideal of relations is generated by the following: $p\xi = 0$, and $\phi(u)\xi = 0$ for all u in the kernel of the ring homomorphism $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}} \to \mathrm{A}^*_{\boldsymbol{\mu}_p}$ induced by the embedding $\boldsymbol{\mu}_p \hookrightarrow \mathrm{T}_{\mathrm{GL}_p}$.*

(d) *The ring homomorphism $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} \to \mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}} \times \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ induced by the embeddings $\mathrm{T}_{\mathrm{GL}_p} \hookrightarrow \mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$ and $\mathrm{C}_p \times \boldsymbol{\mu}_p \hookrightarrow \mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$ is injective.*

(e) *The restriction homomorphism* $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} \to (\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}})^{\mathrm{C}_p}$ *sends the kernel of* $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} \to \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ *bijectively onto the kernel of* $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}} \to \mathrm{A}^*_{\boldsymbol{\mu}_p}$.

Proof. Let us prove part (a). All the x_i in $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}$ map to η in $\mathrm{A}^*_{\boldsymbol{\mu}_p}$, so σ_p maps to η^p, and all the $\sum_{s \in \mathrm{C}_p} sm$ map to $p\eta^{\deg m} = 0$.

Let us prove (b). First of all let us construct the splitting $\phi\colon \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\right)^{\mathrm{C}_p} \to \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$ as a homomorphism of abelian groups. The group $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\right)^{\mathrm{C}_p}$ is free over the powers of σ_p and the $\sum_{s \in \mathrm{C}_p} sm$.

The restriction of the canonical representation V of $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$ to the maximal torus $\mathrm{T}_{\mathrm{GL}_p}$ splits are a direct sum of 1-dimensional representations with first Chern characters $x_1, \ldots, x_p$; hence the i^{th} Chern class $\mathrm{c}_i(V) \in \mathrm{A}^i_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$ restricts to $\sigma_i \in (\mathrm{A}^i_{\mathrm{T}_{\mathrm{GL}_p}})$. We define the splitting by the rules

(a) $\phi(\sigma_p^r) = \mathrm{c}_p(V)^r \in \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$ for each $r > 0$, and

(b) $\phi(\sum_{s \in \mathrm{C}_p} sm) = \mathrm{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} m \in \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$ for each free monomial m.

Notice that the transfer in the second part of the definition only depends on the orbit of m; hence ϕ is well defined.

We need to check that ϕ is a ring homomorphism, by taking two basis element u and v and showing that $\phi(uv)$ equals $\phi(u)\phi(v)$. This is clear when both u and v are powers of σ_p.

Consider the product $\sigma_p^r \sum_{s \in \mathrm{C}_p} sm = \sum_{s \in \mathrm{C}_p} s(\sigma_p^r m)$; we have

$$\begin{aligned}
\phi\Big(\sigma_p^r \sum_{s \in \mathrm{C}_p} sm\Big) &= \phi\Big(\sum_{s \in \mathrm{C}_p} s(\sigma_p^r m)\Big) \\
&= \mathrm{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} (\sigma_p^r m) \quad \text{(because } \sigma_p^r m \text{ is still a free monomial)} \\
&= \mathrm{c}_p(V)^r \, \mathrm{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} (m) \quad \text{(by the projection formula)} \\
&= \phi(\sigma_p^r)\phi\Big(\sum_{s \in \mathrm{C}_p} sm\Big).
\end{aligned}$$

Now the hardest case. Notice that if m is any monomial, not necessarily free, we have the equality

$$\phi\Big(\sum_{s \in \mathrm{C}_p} sm\Big) = \mathrm{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} m.$$

When m is free this holds by definition, whereas when $m = \sigma_p^r$ we have

$$\begin{aligned}
\phi\Big(\sum_{s \in \mathrm{C}_p} \sigma_p^r\Big) &= p\phi(\sigma_p^r) \\
&= p\, \mathrm{c}_p(V)^r \\
&= \mathrm{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} \mathrm{res}^{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{T}_{\mathrm{GL}_p}} \mathrm{c}_p(V)^r \\
&= \mathrm{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} \sigma_p^r.
\end{aligned}$$

Take two free monomials m and n. We have

$$\begin{aligned}
\phi\Big(\sum_{s\in \mathrm{C}_p} sm\cdot \sum_{s\in\mathrm{C}_p} sn\Big) &= \phi\Big(\sum_{s,t\in\mathrm{C}_p} sm\cdot tn\Big)\\
&= \phi\Big(\sum_{s,t\in\mathrm{C}_p} sm\cdot stn\Big)\\
&= \sum_{t\in\mathrm{C}_p}\phi\Big(\sum_{s\in\mathrm{C}_p} s(m\cdot tn)\Big)\\
&= \sum_{t\in\mathrm{C}_p}\mathrm{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(m\cdot tn)\\
&= \mathrm{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}\Big(m\cdot\sum_{t\in\mathrm{C}_p} tn\Big)\\
&= \mathrm{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}\Big(m\cdot \mathrm{res}^{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{T}_{\mathrm{GL}_p}}\,\mathrm{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}\, n\Big)\\
&= \mathrm{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(m)\,\mathrm{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(n)\\
&= \phi\Big(\sum_{s\in\mathrm{C}_p} sm\Big)\phi\Big(\sum_{s\in\mathrm{C}_p} sn\Big)
\end{aligned}$$

as claimed. This ends the proof of part (b).

For parts (c) and (d), notice the following fact: since the restriction of ξ to $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}$ is 0, from the projection formula it follows that $\xi\,\mathrm{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(m) = 0 \in \mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}$ for any $m\in \mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}$; hence we get that $\phi(\sum_{s\in\mathrm{C}_p} m)\xi = 0 \in \mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}$, as claimed. Thus, the relations of the statement of the Proposition hold true.

Denote by $\mathrm{A}^+_{\mathrm{T}_{\mathrm{GL}_p}}$ the ideal of $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}$ generated by homogeneous elements of positive degree. Then the image of $\big(\mathrm{A}^+_{\mathrm{T}_{\mathrm{GL}_p}}\big)^{\mathrm{C}_p}$ in $\mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}$ via ϕ maps to 0 under the restriction homomorphism $\mathrm{res}^{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p}\colon \mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}} \to \mathrm{A}^*_{\mathrm{C}_p}$. In fact, the image of $\mathrm{A}^+_{\mathrm{T}_{\mathrm{GL}_p}}$ is generated by elements of the form $\mathrm{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}} m$, where $m\in \mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}$ is a monomial of positive degree, and by positive powers $\mathrm{c}_p(V)^r$ of the top Chern class of V. The fact that the restriction of $\mathrm{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}} m$ is 0 follows from Mackey's formula. On the other hand, the restriction of V to C_p is a direct sum of 1-dimensional representations with first Chern classes $0, \xi, 2\xi, \ldots, (p-1)\xi$, so the restriction of V has trivial top Chern class.

Lemma 6.5. *The kernel of the restriction homomorphism*

$$\mathrm{res}^{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p}\colon \mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}} \longrightarrow \mathrm{A}^*_{\mathrm{C}_p}$$

consists of the sum the image of $(\mathrm{A}^{+}_{\mathrm{T}_{\mathrm{GL}_p}})^{\mathrm{C}_p}$ *in* $\mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}$ *via* ϕ, *and of the ideal* $\bigl(\mathrm{c}_p(V)\bigr) \subseteq \mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}$.

Proof. Consider the hyperplane H_i in the canonical representation $V = \mathbb{A}^p$ defined by the vanishing of the i^{th} coordinate. Denote by $H = \cup_{i=1}^p H_i \subseteq V$ the union. If $V_0 = V \setminus H$ we have an exact sequence

$$\mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(H) \longrightarrow \mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(V) \longrightarrow \mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(V_0) \longrightarrow 0.$$

We identify $\mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(V)$ with $\mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}$ via the pullback $\mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}} \to \mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(V)$, which is an isomorphism. The action of $\mathrm{T}_{\mathrm{GL}_p}$ on V_0 is free and transitive, and the stabilizer of the point $(1, 1, \ldots, 1)$ is $\mathrm{C}_p \subseteq \mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$. Hence we have an isomorphism of $\mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(V_0)$ with $\mathrm{A}^*_{\mathrm{C}_p}$, and the pullback $\mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(V) \to \mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(V_0)$ is identified with the restriction homomorphism $\mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}} \to \mathrm{A}^*_{\mathrm{C}_p}$. So the kernel of this restriction is the image of $\mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(H)$.

Denote by $\widetilde{H}$ the disjoint union $\coprod_{i=1}^p H_i \sqcup \{0\}$ of the H_i with the origin $\{0\} \subseteq V$. I claim that the proper pushforward $\mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(\widetilde{H}) \to \mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(H)$ is surjective. This follows from Proposition 4.5: we need to check that every $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$-invariant closed subvariety of H is the birational image of a $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$-invariant subvariety of $\widetilde{H}$. Denote by W a $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$-invariant closed subvariety of H. If $W = \{0\}$ we are done. Otherwise it is easy to see that W will be the union of p $\mathrm{T}_{\mathrm{GL}_p}$-invariant irreducible components $W_1, \ldots, W_p$, such that each W_i is contained in H_i. Then the disjoint union $\coprod_{i=1}^p W_i \subseteq \coprod_{i=1}^p H_i \subseteq \widetilde{H}$ is $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$-invariant and maps birationally onto W. Hence we conclude that the kernel of the restriction homomorphism is the sum of the images of the proper pushforwards

$$\mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(\{0\}) \longrightarrow \mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(V)$$

and

$$\mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(\coprod_{i=1}^p H_i) \longrightarrow \mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(V).$$

After identifying $\mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(V)$ with $\mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}$, the first pushforward is just multiplication by $\mathrm{c}_p(V)$, so its image is the ideal $\bigl(\mathrm{c}_p(V)\bigr) \subseteq \mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}$.

Notice that the disjoint union $\coprod_{i=1}^p H_i$ is canonically isomorphic, as a $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$-scheme, to $(\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}) \times^{\mathrm{T}_{\mathrm{GL}_p}} H_1$; hence there is a canonical isomorphism

$$\mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}\left(\coprod_{i=1}^p H_i\right) \simeq \mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}(H_1).$$

The pushforward $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}(H_1) \to \mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(V)$ is the composite of the proper pushforward $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}(H_1) \to \mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}(V)$, followed by the transfer homomorphism $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}(V) \to \mathrm{A}^*_{\mathrm{C}_p\ltimes \mathrm{T}_{\mathrm{GL}_p}}(V)$. After identifying $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}(H_1)$ and $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}(V)$

with $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}$, $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}(V)$ with $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$, we see that this implies that the image of $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}\bigl(\coprod_{i=1}^p H_i\bigr)$ in $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}(V) = \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$ is the image of the ideal $(x_1) \subseteq \mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}$ under the transfer map $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}} \to \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$. So each element of the image of $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}\bigl(\coprod_{i=1}^p H_i\bigr)$ can be written as a linear combination with integer coefficients of transfers of monomials of positive degree: and this completes the proof of the Lemma. ♠

Now we show that $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$ is generated, as an algebra over $\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\bigr)^{\mathrm{C}_p}$, by the single element ξ. Take an element α of $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$ of degree d, and write its image in $\mathrm{A}^*_{\mathrm{C}_p} = \mathbb{Z}[\xi]/(p\xi)$ in the form $m\xi^d$, where m is an integer. Then $\alpha - m\xi^d \in \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$ maps to 0 in $\mathrm{A}^*_{\mathrm{C}_p}$, so according to Lemma 6.5 it is of the form $\beta + \sigma_p\gamma$, where β is in $\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\bigr)^{\mathrm{C}_p}$ and $\gamma \in \mathrm{A}^{d-p}_{\mathrm{C}_p}$. The proof is concluded by induction on d.

Now we prove that the relations indicated generate the ideal of relations, and, simultaneously, part (d).

Take an element $\alpha \in \mathrm{A}^d_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$; using the given relations, we can write α in the form $\alpha_0 + \alpha_1\xi + \alpha_2\xi^2 + \cdots$, where $\alpha_0 \in \bigl(\mathrm{A}^d_{\mathrm{T}_{\mathrm{GL}_p}}\bigr)^{\mathrm{C}_p}$, while for each $i > 0$ the element α_i is of the form $d_i\sigma_p^r$, where $0 \le d_i \le p-1$, and $rp = d-i$, when p divides $d-i$, and 0 otherwise.

Assume that the image of α in $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}} \times \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ is 0. The image of α in $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}$ is α_0, hence $\alpha_0 = 0$.

Lemma 6.6. *The restriction of $\phi(\sigma_p) = \mathrm{c}_p(V)$ to $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p} = \mathbb{Z}[\xi, \eta](p\xi, p\eta)$ equals $\eta^p - \eta\xi^{p-1}$.*

Proof. The restriction of V to $\mathrm{C}_p \times \boldsymbol{\mu}_p$ decomposes as a direct sum of 1-dimensional representations with first Chern classes η, $\eta - \xi$, $\eta - 2\xi$, ..., $\eta - (p-1)\xi$, and

$$\eta(\eta - \xi)(\eta - 2\xi)\dots\bigl(\eta - (p-1)\xi\bigr) = \eta^p - \eta\xi^{p-1}.$$ ♠

Since ξ and $\eta^p - \eta\xi^{p-1}$ are algebraically independent in the polynomial ring $\mathbb{F}_p[\xi, \eta]$, it follows that all the α_i are all 0. This finishes the proof of (c) and (d).

Finally, let us prove part (e).

Injectivity follows immediately from part (d). To show that the restriction homomorphism is surjective, it is sufficient to show that if u is in the kernel of the homomorphism $\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\bigr)^{\mathrm{C}_p} \to \mathrm{A}^*_{\boldsymbol{\mu}_p}$, then $\phi(u)$ is in the kernel of $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} \to \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$. Each element of $\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\bigr)^{\mathrm{C}_p}$ of the form $\sum_{s \in \mathrm{C}_p} sm$ goes to 0 in $\mathrm{A}^*_{\mathrm{C}_p}$, while σ_p goes to η^p; hence u is a linear combination of elements of the form $\sum_{s \in \mathrm{C}_p} m$ and $p\sigma_p^r$. So $\phi(u)$ is a linear combination of element of $\mathrm{A}^*_{\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{GL}_p}}$ of the form $p\,\mathrm{c}_p(V)^r$ and $\operatorname{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{GL}_p}} m$; from the following Lemma we see that all these elements to $\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$ is 0.

Lemma 6.7. *If u is an element of positive degree in $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}$, the restriction of $\operatorname{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{GL}_p}} u$ to $\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$ is 0.*

Proof. The double coset space $(\mathrm{C}_p\times\boldsymbol{\mu}_p)\backslash(\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{GL}_p})/\mathrm{T}_{\mathrm{GL}_p}$ consists of a single point and $(\mathrm{C}_p\times\boldsymbol{\mu}_p)\cap\mathrm{T}_{\mathrm{GL}_p}=\boldsymbol{\mu}_p$, so we have

$$\operatorname{res}^{\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\operatorname{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{GL}_p}} u=\operatorname{tsf}^{\boldsymbol{\mu}_p}_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\operatorname{res}^{\mathrm{T}_{\mathrm{GL}_p}}_{\boldsymbol{\mu}_p} u.$$

However, I claim that the transfer homomorphism

$$\operatorname{tsf}^{\boldsymbol{\mu}_p}_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\colon \mathrm{A}^*_{\boldsymbol{\mu}_p}\longrightarrow\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$$

is 0 in positive degree. In fact, the restriction homomorphism

$$\operatorname{res}^{\mathrm{C}_p\times\boldsymbol{\mu}_p}_{\boldsymbol{\mu}_p}\colon \mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\longrightarrow\mathrm{A}^*_{\boldsymbol{\mu}_p}$$

is surjective, because the embedding $\boldsymbol{\mu}_p\hookrightarrow\mathrm{C}_p\times\boldsymbol{\mu}_p$ is split by the projection $\mathrm{C}_p\times\boldsymbol{\mu}_p\to\boldsymbol{\mu}_p$. It follows immediately, again from Mackey's formula, that the composition $\operatorname{tsf}^{\boldsymbol{\mu}_p}_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\operatorname{res}^{\mathrm{C}_p\times\boldsymbol{\mu}_p}_{\boldsymbol{\mu}_p}$ is multiplication by p; and all classes in $\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$ in positive degree are p-torsion. ♠

This concludes the proof of Proposition 6.4. ♠

Remark 6.8. When $k=\mathbb{C}$, Propositions 6.1 and 6.4 give a description of the cohomology $\mathrm{H}^*_{\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{GL}_p}}$. This can be proved directly, by studying the Hochschild–Serre spectral sequence

$$E_2^{ij}=\mathrm{H}^i\bigl(\mathrm{C}_p,\mathrm{H}^j_{\mathrm{T}_{\mathrm{GL}_p}}\bigr)\Longrightarrow\mathrm{H}^{i+j}_{\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{GL}_p}}.$$

7. On $\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{PGL}_p}$

In this section we study the Chow ring of the classifying space of the group $\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{PGL}_p}$. Here is our main result. Consider the subgroup $\boldsymbol{\mu}_p\subseteq\mathrm{T}_{\mathrm{PGL}_p}$ defined, as in the Introduction, by the formula $\zeta\mapsto[\zeta,\zeta^2,\dots,\zeta^{p-1},1]$. This defines a homomorphism of rings $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\to\mathrm{A}^*_{\boldsymbol{\mu}_p}$.

Proposition 7.1.

(a) *The image of the restriction homomorphism*

$$(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}})^{\mathrm{C}_p}\longrightarrow\mathrm{A}^*_{\boldsymbol{\mu}_p}=\mathbb{Z}[\eta]/(p\eta)$$

is the subring generated by η^p.

(b) *The ring homomorphism $\mathrm{A}^*_{\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{PGL}_p}}\to\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{C}_p}$ induced by the embedding $\mathrm{T}_{\mathrm{PGL}_p}\hookrightarrow\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{PGL}_p}$ is surjective, and admits a canonical splitting $\phi\colon\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{C}_p}\to\mathrm{A}^*_{\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{PGL}_p}}$, which is a ring homomorphism.*

(c) *As an algebra over* $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{C}_p}$, *the ring* $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ *is generated by the element* ξ, *while the ideal of relations is generated by the following:* $p\xi = 0$, *and* $\phi(u)\xi = 0$ *for all* u *in the kernel of the ring homomorphism* $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}} \to \mathrm{A}^*_{\boldsymbol{\mu}_p}$ *induced by the embedding* $\boldsymbol{\mu}_p \hookrightarrow \mathrm{T}_{\mathrm{PGL}_p}$.

(d) *The ring homomorphisms*

$$\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \longrightarrow \mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}} \times \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$$

and

$$\mathrm{H}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \longrightarrow \mathrm{H}^*_{\mathrm{T}_{\mathrm{PGL}_p}} \times \mathrm{H}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$$

induced by the embeddings $\mathrm{T}_{\mathrm{PGL}_p} \hookrightarrow \mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}$ *and* $\mathrm{C}_p \times \boldsymbol{\mu}_p \hookrightarrow \mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}$ *is injective.*

(e) *The restriction homomorphism* $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \to (\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}})^{\mathrm{C}_p}$ *sends the kernel of* $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \to \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ *bijectively onto the kernel of* $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}} \to \mathrm{A}^*_{\boldsymbol{\mu}_p}$.

Proof. One of the main ideas in the paper is to exploit the fact, already used in [15] and rediscovered in [14], that there is an isomorphism of tori

$$\Phi\colon \mathrm{T}_{\mathrm{PGL}_p} \simeq \mathrm{T}_{\mathrm{SL}_p}$$

defined by

$$\Phi(t_1, \dots, t_p) = [t_1/t_p, t_2/t_1, t_2/t_2, \dots, t_{p-1}/t_{p-2}, t_p/t_{p-1}].$$

This isomorphism is not S_p-equivariant, but it is C_p-equivariant; therefore it induces an isomorphism

$$\Phi\colon \mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p} \simeq \mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{SL}_p}.$$

The composite of the embedding $\boldsymbol{\mu}_p \hookrightarrow \mathrm{T}_{\mathrm{PGL}_p}$ with the isomorphism Φ is the embedding $\boldsymbol{\mu}_p \hookrightarrow \mathrm{T}_{\mathrm{SL}_p}$ defined by $\zeta \mapsto [\zeta, \zeta, \dots, \zeta]$.

Now, take an open subset U of a representation of $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$ on which $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$ acts freely. The projection $U/\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{SL}_p} \to U/\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}$ is a $\mathbb{G}_{\mathrm{m}}$-torsor, coming from the determinant $\det\colon \mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p} \to \mathbb{G}_{\mathrm{m}}$ of the canonical representation V of $\mathrm{T}_{\mathrm{GL}_p}$. Lemma 4.1 implies that there is an exact sequence

$$\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} \xrightarrow{\mathrm{c}_1(V)} \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} \longrightarrow \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{SL}_p}} \longrightarrow 0$$

and a ring isomorphism $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{SL}_p}} \simeq \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} / \big(\mathrm{c}_1(V)\big)$.

Consider the splitting $\phi\colon \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\right)^{\mathrm{C}_p} \to \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$ constructed in the previous section. I claim that $\mathrm{c}_1(V)$ coincides with $\phi(\sigma_1) = \mathrm{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} x_1$. To prove this it is enough, according to Proposition 6.4 (d), to show that these two classes coincide after restriction to $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}$ and to $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$. The restrictions of both classes to $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}$ coincide with $x_1 + \cdots + x_p$.

The action of $\mathrm{C}_p \times \boldsymbol{\mu}_p$ on V splits as a direct sum of 1-dimensional representations with characters $\eta+\xi, \eta+2\xi, \dots, \eta+(p-1)\xi, \eta$, so the restriction of $\mathrm{c}_1(V)$ to $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ is

$$\eta + \xi + \eta + 2\xi + \cdots + \eta + (p-1)\xi + \eta = p\eta + \frac{p(p-1)}{2}\xi = 0.$$

So we need to show that the restriction of $\operatorname{tsf}^{\mathrm{T}_{\mathrm{GL}_p}}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} x_1$ to $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ is also 0. This is a particular case of Lemma 6.7.

There is also an exact sequence

$$0 \longrightarrow \mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}} \xrightarrow{\sigma_1} \mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}} \longrightarrow \mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}} \longrightarrow 0,$$

so $\mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}}$ is the quotient $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}/(\sigma_1)$.

Lemma 7.2. *If G is a subgroup of S_p, the projection $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\right)^G \to \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}}\right)^G$ induces an isomorphism*

$$\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}}\right)^G/(\sigma_1) \simeq \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\right)^G.$$

Proof. This is equivalent to saying that the exact sequence above stays exact after taking G-invariants; but we have that $\mathrm{H}^1\left(G, \mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\right) = 0$, because $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}$ is a torsion-free permutation module under G. ♠

Part (a) comes from the surjectivity of the restriction homomorphism $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\right)^{\mathrm{C}_p} \to \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}}\right)^{\mathrm{C}_p}$ and Proposition 6.4 (a).

We construct the splitting $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}}\right)^{\mathrm{C}_p} \to \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{SL}_p}}$ by taking the splitting $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\right)^{\mathrm{C}_p} \to \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$ constructed in the previous section, tensoring it with $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\right)^{\mathrm{C}_p}/(\sigma_1)$ over $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\right)^{\mathrm{C}_p}$, to get a ring homomorphism

$$\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\right)^{\mathrm{C}_p}/(\sigma_1) \longrightarrow \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}/(\sigma_1)$$

and using the isomorphisms

$$\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}}\right)^{\mathrm{C}_p} \simeq \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\right)^{\mathrm{C}_p}/(\sigma_1)$$

and

$$\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}/(\sigma_1) \simeq \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{SL}_p}}$$

constructed above. This proves part (b). Part (c) follows from Proposition 6.4 (c).

To prove part (e) consider the diagram of restriction homomorphisms

$$\begin{array}{ccccc}
\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} & \longrightarrow & \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{SL}_p}} & \longrightarrow & \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p} \\
\downarrow & & \downarrow & & \downarrow \\
\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\right)^{\mathrm{C}_p} & \longrightarrow & \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}}\right)^{\mathrm{C}_p} & \longrightarrow & \mathrm{A}^*_{\boldsymbol{\mu}_p}.
\end{array}$$

The surjectivity of the map in the statement follows from Proposition 6.4 (e) and from the fact that the first arrow in the bottom row is surjective.

To prove injectivity take an element u of $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{SL}_p}}$ that maps to 0 in $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ and in $\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}}\bigr)^{\mathrm{C}_p}$. Let v be an element of $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$ mapping to u. Since the kernel of the homomorphism $\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\bigr)^{\mathrm{C}_p} \to \bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}}\bigr)^{\mathrm{C}_p}$ is generated by σ_1, we can write the image of v in $\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\bigr)^{\mathrm{C}_p}$ as $\sigma_1 w$ for some $w \in \bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\bigr)^{\mathrm{C}_p}$. Then the element $v - \phi(\sigma_1 w)$ maps to 0 in $\mathrm{A}^*_{\boldsymbol{\mu}_p}$ and in $\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\bigr)^{\mathrm{C}_p}$; hence, by Proposition 6.4 (d), it is 0. So $v = \phi(\sigma_1)\phi(w)$ maps to 0 in $\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}}\bigr)^{\mathrm{C}_p}$, as claimed.

Let us prove part (d). The statement on Chow rings is an immediate consequence of part (e).

For the cohomology, we will argue as follows. We have a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^{i-1}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{SL}_p}} \xrightarrow{\partial} \mathrm{H}^{i-2}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} \xrightarrow{\mathrm{c}_1(V)} \mathrm{H}^{i}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} \longrightarrow \mathrm{H}^{i}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{SL}_p}} \xrightarrow{\partial} \mathrm{H}^{i-1}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} \longrightarrow \cdots$$

By Proposition 6.1, the cycle homomorphism $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} \to \mathrm{H}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$ is an isomorphism. Hence, for each i we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{A}^{i-1}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} & \longrightarrow & \mathrm{A}^{i}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} & \longrightarrow & \mathrm{A}^{i}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{SL}_p}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{H}^{2i-2}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} & \longrightarrow & \mathrm{H}^{2i}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} & \longrightarrow & \mathrm{H}^{2i}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{SL}_p}} & \longrightarrow & \mathrm{H}^{2i-1}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} = 0
\end{array}$$

in which the first two columns are isomorphisms. This implies that the third column is also an isomorphism: so the cycle homomorphism $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{SL}_p}} \to \mathrm{H}^{\mathrm{even}}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{SL}_p}}$ is an isomorphism. Therefore the homomorphism $\mathrm{H}^{\mathrm{even}}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{SL}_p}} \to \mathrm{H}^{\mathrm{even}}_{\mathrm{T}_{\mathrm{SL}_p}} \times \mathrm{H}^{\mathrm{even}}_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ is injective.

When i is odd, we have an exact sequence

$$0 = \mathrm{H}^{i}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} \longrightarrow \mathrm{H}^{i}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{SL}_p}} \xrightarrow{\partial} \mathrm{H}^{i}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} \xrightarrow{\mathrm{c}_1(V)} \mathrm{H}^{i+2}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}};$$

hence the boundary homomorphism $\partial\colon \mathrm{H}^{\mathrm{odd}}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{SL}_p}}\ \mathrm{H}^{\mathrm{even}}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}}$ yields an isomorphism of $\mathrm{H}^{\mathrm{odd}}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{SL}_p}}$ with the annihilator of the element $\mathrm{c}_1(V)$ of $\mathrm{H}^{\mathrm{even}}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{GL}_p}} =$

$\mathrm{A}^*_{\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{GL}_p}}$. From the description of the ring $\mathrm{A}^*_{\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{SL}_p}}$ in (c), it is easy to conclude that this annihilator is the ideal generated by ξ.

Consider a free action of $\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{GL}_p}$ on an open subscheme U of a representation. The diagram of embeddings

$$\begin{array}{ccc} \mathrm{C}_p\times\boldsymbol{\mu}_p & \longrightarrow & \mathrm{C}_p\times\mathbb{G}_{\mathrm{m}} \\ \downarrow & & \downarrow \\ \mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{SL}_p} & \longrightarrow & \mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{GL}_p} \end{array}$$

induces a cartesian diagram

$$\begin{array}{ccc} U/\mathrm{C}_p\times\boldsymbol{\mu}_p & \longrightarrow & U/\mathrm{C}_p\ltimes\mathbb{G}_{\mathrm{m}} \\ \downarrow & & \downarrow \\ U/\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{SL}_p} & \longrightarrow & U/\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{GL}_p} \end{array}$$

in which the rows are principal $\mathbb{G}_{\mathrm{m}}$-bundles, and the columns are $\mathbb{G}_{\mathrm{m}}$-equivariants. This in turn induces a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^{\mathrm{odd}}_{\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{SL}_p}} & \xrightarrow{\partial} & \mathrm{H}^{\mathrm{even}}_{\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{GL}_p}} \\ \downarrow & & \downarrow \\ \mathrm{H}^{\mathrm{odd}}_{\mathrm{C}_p\times\boldsymbol{\mu}_p} & \longrightarrow & \mathrm{H}^{\mathrm{even}}_{\mathrm{C}_p\times\mathbb{G}_{\mathrm{m}}} \end{array}$$

in which the top row is injective, and has as its image the ideal $(\xi)\subseteq\mathrm{H}^{\mathrm{even}}_{\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{GL}_p}}$ as we have just seen. Furthermore, every element of $(\xi)\subseteq\mathrm{H}^{\mathrm{even}}_{\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{GL}_p}}$ maps to 0 in $\mathrm{H}^*_{\mathrm{T}_{\mathrm{GL}_p}}$, because it is torsion: hence (ξ) injects into $\mathrm{H}^{\mathrm{even}}_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$, by Proposition 6.4 (d). Since $\mathrm{C}_p\times\boldsymbol{\mu}_p$ is contained into $\mathrm{C}_p\times\mathbb{G}_{\mathrm{m}}$, it follows that (ξ) also injects into $\mathrm{H}^{\mathrm{even}}_{\mathrm{C}_p\times\mathbb{G}_{\mathrm{m}}}$. So the composite arrow $\mathrm{H}^{\mathrm{odd}}_{\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{SL}_p}}\to\mathrm{H}^{\mathrm{even}}_{\mathrm{C}_p\times\mathbb{G}_{\mathrm{m}}}$ in the commutative diagram above is injective. It follows that the left hand column is injective.

This ends the proof of Proposition 7.1. ♠

Proof of Proposition 3.1. We need to analyze the action of the normalizer N_p of C_p in S_p on the Chow ring $\mathrm{A}^*_{\mathrm{C}_p\ltimes\mathrm{T}_{\mathrm{PGL}_p}}$. If we identify $\{1,\dots,p\}$ with the field $\mathbb{F}_p$ with p elements, by sending each i into its class modulo p, then C_p can be identified with the additive group $\mathbb{F}_p$ itself, acting by translations. There is also the multiplicative subgroup $\mathbb{F}_p^*$ of S_p, acting via multiplication. This is contained in the normalizer of $\mathrm{C}_p=\mathbb{F}_p$, and, since p is a prime, it is easy to show that the normalizer of C_p inside S_p is in fact the subgroup generated by $\mathbb{F}_p$ and $\mathbb{F}_p^*$, which is the semi-direct product $\mathbb{F}_p^*\ltimes\mathbb{F}_p$.

The subgroup $\mathrm{C}_p=\mathbb{F}_p$ acts trivially, so in fact the action is through $\mathbb{F}_p^*$. The action of $\mathbb{F}_p^*$ leaves $\boldsymbol{\mu}_p$ invariant, and the result of the action of $a\in\mathbb{F}_p^*$ on $\zeta\in\boldsymbol{\mu}_p$ is ζ^a: hence a acts on $\mathrm{A}^*_{\boldsymbol{\mu}_p}=\mathbb{Z}[\eta]/(p\eta)$ by sending η to $a\eta$, and the ring of invariants is the subring generated by η^{p-1}. The image of $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$ into $\mathrm{A}^*_{\boldsymbol{\mu}_p}$ is the subring generated by η^p, by Proposition 7.1, and its intersection with the ring of invariants in $\mathrm{A}^*_{\boldsymbol{\mu}_p}$ is the subring generated by $\eta^{p(p-1)}$. This shows that the image of $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$ into $\mathrm{A}^*_{\boldsymbol{\mu}_p}$ is contained in the subring generated by $\eta^{p(p-1)}$. The opposite inclusion is ensured by the fact that the discriminant $\delta \in \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$ maps to $-\eta^{p(p-1)}$. ♠

8. On $\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}$

The group S_p does not act on $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}$, only the normalizer $\mathbb{F}_p^* \ltimes \mathbb{F}_p$ of C_p does. Nevertheless, we define the subring $\left(\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$ of $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ consisting of all the elements that are invariant under $\mathbb{F}_p^* \ltimes \mathbb{F}_p$, and whose images in $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$ are S_p-invariant. The restriction homomorphism $\mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \to \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ has its image in $\left(\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$.

The result we need about $\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}$ is the following.

Proposition 8.1. *The localized restriction homomorphism*

$$\mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \otimes \mathbb{Z}[1/(p-1)!] \longrightarrow \left(\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p} \otimes \mathbb{Z}[1/(p-1)!]$$

is an isomorphism.

Of course the statement can not be correct without inverting $(p-1)!$, because the torsion part of $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ is all p-torsion, while $\mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ contains a lot of $(p-1)!$-torsion coming from $\mathrm{A}^*_{\mathrm{S}_p}$. This is complicated, but fortunately we do not need to worry about it.

Proof. Injectivity is clear: because of the projection formula, the composite

$$\mathrm{tsf}^{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \mathrm{res}^{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \colon \mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \longrightarrow \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$$

is multiplication by $\mathrm{tsf}^{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} = (p-1)!$.

To show surjectivity, take a class $u \in \left(\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$, and set

$$v \overset{\mathrm{def}}{=} \mathrm{tsf}^{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} u \in \mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} .$$

We apply Mackey's formula (Proposition 4.4). The double quotient

$$\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p} \backslash \mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p} / \mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p} = \mathrm{C}_p \backslash \mathrm{S}_p / \mathrm{C}_p$$

consists of $p-1$ elements coming from the normalizer $\mathbb{F}_p^* \ltimes \mathbb{F}_p$, and $(p-1)\frac{(p-2)!-1}{p}$ elements with the property that, if we call s a representative in $\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}$, we have

$$s(\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p})s^{-1} \cap \mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p} = \mathrm{T}_{\mathrm{PGL}_p}.$$

Therefore

$$\operatorname{res}^{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} v = (p-1)u + (p-1)\frac{(p-2)!-1}{p} \operatorname{tsf}^{\mathrm{T}_{\mathrm{PGL}_p}}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \operatorname{res}^{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}_{\mathrm{T}_{\mathrm{PGL}_p}} u;$$

hence it is enough to show that an element in the image of the trasfer map

$$\operatorname{tsf}^{\mathrm{T}_{\mathrm{PGL}_p}}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \colon \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p} \longrightarrow \left(\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$$

is in the image of $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$, up to a multiple of $(p-1)!$. But again an easy application of Mackey's formula reveals that

$$\operatorname{res}^{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}_{\mathrm{T}_{\mathrm{PGL}_p}} \operatorname{tsf}^{\mathrm{T}_{\mathrm{PGL}_p}}_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} w = (p-1)!w$$

for all $w \in \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$, and this finishes the proof. ♠

9. Some results on $\mathrm{A}^*_{\mathrm{PGL}_p}$

In this section we prove some auxilliary results, which play an important role in the proof of the main theorems.

The following observation is in [15, Corollary 2.4].

Proposition 9.1. *If ξ is a torsion element of $\mathrm{A}^*_{\mathrm{PGL}_p}$, or $\mathrm{H}^*_{\mathrm{PGL}_p}$, then $p\xi = 0$.*

Proof. Suppose that $\xi \in \mathrm{A}^m_{\mathrm{PGL}_p}$. Take a representation V of PGL_p with an open subset U on which PGL_p acts freely, such that the codimension of $V \setminus U$ has codimension larger than m, so that $\mathrm{A}^m_{\mathrm{PGL}_p} = \mathrm{A}^m(B)$, where we have set $B \overset{\mathrm{def}}{=} U/\mathrm{PGL}_p$. Let $\pi \colon E \to B$ be the Brauer–Severi scheme associated with the PGL_p-torsor $U \to B$: this is the projection $U/H \to U/\mathrm{PGL}_p$, where H is the parabolic subgroup of PGL_p consisting of classes of matrices (a_{ij}) with $a_{i1} = 0$ when $i > 1$. The embedding $H \hookrightarrow \mathrm{PGL}_p$ lift to an embedding $H \hookrightarrow \mathrm{GL}_p$, as the subgroup of matrices (a_{ij}) with $a_{i1} = 0$ when $i > 1$, and $a_{11} = 1$; hence the pullback $\mathrm{A}^m(B) \to \mathrm{A}^m(E)$ factors through $\mathrm{A}^m_{\mathrm{GL}_p}$, which is torsion-free. It follows that ξ maps to 0 in $\mathrm{A}^m(E)$.

Now consider the Chern class $\mathrm{c}_{p-1}(\mathrm{T}_{E/B}) \in \mathrm{A}^{p-1}(E)$ of the relative tangent bundle. This has the property that $\pi_* \mathrm{c}_{p-1}(\mathrm{T}_{E/B}) = p[B] \in \mathrm{A}^0(B)$; hence, by the projection formula we have

$$\begin{aligned} p\xi &= \xi \cdot \pi_*\bigl(\mathrm{c}_{p-1}(\mathrm{T}_{E/B})\bigr) \\ &= \pi_*\bigl(\pi^*\xi \cdot \mathrm{c}_{p-1}(\mathrm{T}_{E/B})\bigr) \\ &= 0. \end{aligned}$$

The proof for cohomology is identical, except for notation. ♠

Proposition 9.2. *The restriction homomorphisms $\mathrm{A}^*_{\mathrm{PGL}_p} \to \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ and $\mathrm{H}^*_{\mathrm{PGL}_p} \to \mathrm{H}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ are injective.*

Proof. By a classical result of Gottlieb ([5]) the homomorphism $\mathrm{H}^*_{\mathrm{PGL}_p} \to \mathrm{H}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ is injective; while the injectivity of $\mathrm{A}^*_{\mathrm{PGL}_p} \to \mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ is a recent result of Totaro. This is unpublished: a sketch of proof is presented in [15].

Theorem 9.3 (Totaro)**.** *If G is a linear algebraic group over a field k of characteristic* 0 *acting on a scheme X of finite type over k, and N is the normalizer of maximal torus, then the restriction homomorphism $\mathrm{A}^*_G(X) \to \mathrm{A}^*_N(X)$ is injective.*

Now, the kernels of the homomorphisms in the statement are p-torsion, by Proposition 9.1, while the kernels of $\mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \to \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ and $\mathrm{H}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \to \mathrm{H}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ are $(p-1)!$-torsion, by the projection formula, so the statement follows. ♠

Here is the basic result that we are going to use in order to verify that a given relation holds in $\mathrm{A}^*_{\mathrm{PGL}_p}$ and $\mathrm{H}^*_{\mathrm{PGL}_p}$.

Proposition 9.4. *The homomorphisms*

$$\mathrm{A}^*_{\mathrm{PGL}_p} \longrightarrow \mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}} \times \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$$

and

$$\mathrm{H}^*_{\mathrm{PGL}_p} \longrightarrow \mathrm{H}^*_{\mathrm{T}_{\mathrm{PGL}_p}} \times \mathrm{H}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$$

obtained from the embeddings $\mathrm{T}_{\mathrm{PGL}_p} \hookrightarrow \mathrm{PGL}_p$ and $\mathrm{C}_p \times \boldsymbol{\mu}_p \hookrightarrow \mathrm{PGL}_p$ are injective.

Proof. This follows from Propositions 9.4 and 7.1 (d). ♠

Here is another fundamental fact, which is one of the cornerstones of the treatment of PGL_3 in [15]. In the Lie algebra $\mathfrak{sl}_p$ of matrices of trace 0 consider the Zariski open subset $\mathfrak{sl}_p^0$ consisting of matrices with distinct eigenvalues; this is invariant by the action of PGL_p. Furthermore, we will consider the subspace $D_p \subseteq \mathfrak{sl}_p$ of diagonal matrices with trace equal to zero, and $D_p^0 = D_p \cap \mathfrak{sl}_p^0$. The subspaces D_p and D_p^0 are invariant under the action of $\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p} \subseteq \mathrm{PGL}_p$.

Proposition 9.5 (see [15], Proposition 3.1)**.** *The composites of restriction homomorphisms*

$$\mathrm{A}^*_{\mathrm{PGL}_p}(\mathfrak{sl}_p^0) \longrightarrow \mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(\mathfrak{sl}_p^0) \longrightarrow \mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(D_p^0)$$

and

$$\mathrm{H}^*_{\mathrm{PGL}_p}(\mathfrak{sl}_p^0) \longrightarrow \mathrm{H}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(\mathfrak{sl}_p^0) \longrightarrow \mathrm{H}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(D_p^0)$$

are isomorphisms.

Proof. The $\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}$-equivariant embedding $D_p^0 \subseteq \mathfrak{sl}_p^0$ induces a PGL_p-equivariant morphism $\mathrm{PGL}_p \times^{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} D_p^0 \to \mathfrak{sl}_p^0$, which sends the class of a pair (A, X) into AXA^{-1}. This morphism is easily seen to be an isomorphism, and the proof follows. ♠

Corollary 9.6. *The restriction homomorphisms*

$$\mathrm{A}^*_{\mathrm{PGL}_p} \to \mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}} \quad \textit{and} \quad \mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \to \mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$$

have the same image.

Proof. In the commutative diagram of restriction homomorphisms

$$\begin{array}{ccc} \mathrm{A}^*_{\mathrm{PGL}_p} & \longrightarrow & \mathrm{A}^*_{\mathrm{PGL}_p}(\mathfrak{sl}_p^0) \\ \downarrow & & \downarrow \\ \mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}} & \longrightarrow & \mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}(\mathfrak{sl}_p^0) \end{array}$$

the top row is surjective. On the other hand, the action on $\mathrm{T}_{\mathrm{PGL}_p}$ on $\mathfrak{sl}_p^0$ is trivial and $\mathfrak{sl}_p^0$ is an open subscheme of an affine space, so the bottom row is an isomorphism. It follows that the image of $\mathrm{A}^*_{\mathrm{PGL}_p}$ in $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$ maps isomorphically onto the image of $\mathrm{A}^*_{\mathrm{PGL}_p}(\mathfrak{sl}_p^0)$ in $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}(\mathfrak{sl}_p^0)$. A similar argument shows that the image of $\mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ in $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$ maps isomorphically onto the image of $\mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(D_p^0)$ in $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}(D_p^0)$. By we also have a commutative diagram

$$\begin{array}{ccc} \mathrm{A}^*_{\mathrm{PGL}_p}(\mathfrak{sl}_p^0) & \longrightarrow & \mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(D_p^0) \\ \downarrow & & \downarrow \\ \mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}(\mathfrak{sl}_p^0) & \longrightarrow & \mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}(D_p^0) \end{array}$$

where the top row is an isomorphism, by Proposition 9.5, and this concludes the proof. ♠

10. Localization

Consider the top Chern classes

$$\mathrm{c}_{p^2-1}(\mathfrak{sl}_p) \in \mathrm{A}^{p^2-1}_{\mathrm{PGL}_p} \quad \text{and} \quad \mathrm{c}_{p-1}(D_p) \in \mathrm{A}^{p-1}_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}.$$

We have the following fact.

Proposition 10.1. *The restriction homomorphism* $\mathrm{A}^*_{\mathrm{PGL}_p} \to \mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ *carries* $\mathrm{c}_{p^2-1}(\mathfrak{sl}_p)$ *into the ideal* $\bigl(\mathrm{c}_{p-1}(D_p)\bigr) \subseteq \mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$. *The induced homomorphism*

$$\mathrm{A}^*_{\mathrm{PGL}_p} / \bigl(\mathrm{c}_{p^2-1}(\mathfrak{sl}_p)\bigr) \longrightarrow \mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} / \bigl(\mathrm{c}_{p-1}(D_p)\bigr)$$

becomes an isomorphism when tensored with $\mathbb{Z}[1/(p-1)!]$.

Proof. The representation D_p of $\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}$ is naturally embedded in $\mathfrak{sl}_p$, so we have that

$$\mathrm{c}_{p^2-1}(\mathfrak{sl}_p) = \mathrm{c}_{p-1}(D_p)\, \mathrm{c}_{p+1}(\mathfrak{sl}_p/D_p) \in \mathrm{A}^{p^2-1}_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}},$$

and this proves the first statement.

The pullbacks

$$\mathrm{A}^*_{\mathrm{PGL}_p} \longrightarrow \mathrm{A}^*_{\mathrm{PGL}_p}(\mathfrak{sl}_p \setminus \{0\}) \quad \text{and} \quad \mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \longrightarrow \mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(D_p \setminus \{0\})$$

are surjective, and their kernels are the ideals generated by $\mathrm{c}_{p^2-1}(\mathfrak{sl}_p)$ and $\mathrm{c}_{p-1}(D_p)$ respectively: so it enough to show that the homomorphism

$$\mathrm{A}^*_{\mathrm{PGL}_p}(\mathfrak{sl}_p \setminus \{0\}) \longrightarrow \mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(D_p \setminus \{0\})$$

obtained by restricting the groups, and then pulling back along the embedding $D_p \setminus \{0\} \hookrightarrow \mathfrak{sl}_p \setminus \{0\}$ becomes an isomorphism after inverting $(p-1)!$.

Now, consider the diagram

$$\begin{array}{ccc} \mathrm{A}^*_{\mathrm{PGL}_p}(\mathfrak{sl}_p \setminus \{0\}) & \longrightarrow & \mathrm{A}^*_{\mathrm{PGL}_p}(\mathfrak{sl}^*_p) \\ \downarrow & & \downarrow \\ \mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(D_p \setminus \{0\}) & \longrightarrow & \mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(D^*_p) \end{array}$$

where all the arrows are the obvious ones. The rows are surjective, while the right hand column is an isomorphism, by Proposition 9.5: hence it is enough to show that the rows are injective, after inverting $(p-1)!$.

The first step is to observe that the restriction homomorphism $\mathrm{A}^*_{\mathrm{PGL}_p}(\mathfrak{sl}_p \setminus \{0\}) \to \mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(\mathfrak{sl}_p \setminus \{0\})$ is injective, by Totaro's Theorem 9.3. Next, the restriction homomorphisms $\mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(\mathfrak{sl}_p \setminus \{0\}) \to \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(\mathfrak{sl}_p \setminus \{0\})$ and $\mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(D^*_p) \to \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(D^*_p)$ become injective after inverting $(p-1)!$. So it is enough to show that the restriction homomorphisms $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(\mathfrak{sl}_p \setminus \{0\}) \to \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(\mathfrak{sl}^*_p)$ and $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(D_p \setminus \{0\}) \to \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(D^*_p)$ are injective.

Lemma 10.2. *Suppose that W is a representation of $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}$, and U an open subset of $W \setminus \{0\}$. Assume that*

(a) *the restriction of W to $\mathrm{C}_p \times \boldsymbol{\mu}_p$ splits as a direct sum of 1-dimensional representations $W = L_1 \oplus \cdots \oplus L_r$, in such a way that the characters $\mathrm{C}_p \times \boldsymbol{\mu}_p \to \mathbb{G}_{\mathrm{m}}$ describing the action of $\mathrm{C}_p \times \boldsymbol{\mu}_p$ on the L_i are all distinct, and each $L_i \setminus \{0\}$ is contained in U, and*

(b) *U contains a point that is fixed under $\mathrm{T}_{\mathrm{PGL}_p}$.*

Then the restriction homomorphism $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(W \setminus \{0\}) \to \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(U)$ is an isomorphism.

Proof. First of all, let us show that $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(W \setminus \{0\}) \to \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(U)$ is an isomorphism. Denote by D the complement of U in $W \setminus \{0\}$, with its reduced scheme structure. Let P be the projectivization of W, and call $\overline{U}$ and $\overline{D}$ the (respectively open and closed) subschemes of P corresponding to U and D.

We have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(\overline{D}) & \longrightarrow & \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(P) & \longrightarrow & \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(\overline{U}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(D) & \longrightarrow & \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(W \setminus \{0\}) & \longrightarrow & \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(U) & \longrightarrow & 0
\end{array}$$

where the columns are surjective pullbacks, and the rows are exact. It follows that is enough to show that the composite

$$\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(\overline{D}) \longrightarrow \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(P) \longrightarrow \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(W \setminus \{0\})$$

is 0, or, equivalently, that any element of the kernel of $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(P) \to \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(\overline{U})$ maps to 0 in $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(W \setminus \{0\})$. Denote by $q_i \in P$ the rational point corresponding to L_i.

Denote by $\ell_i \in \mathrm{A}^1_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ the first Chern class of the character $\mathrm{C}_p \times \boldsymbol{\mu}_p \to \mathbb{G}_{\mathrm{m}}$ describing the action of $\mathrm{C}_p \times \boldsymbol{\mu}_p$ on L_i, and $h \in \mathrm{A}^1_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ the first Chern class of the sheaf $\mathcal{O}(1)$ on P. We have presentations

$$\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(P) = \mathbb{Z}[\xi, \eta, h]/\bigl(p\xi, p\eta, (h - \ell_1) \dots (h - \ell_r)\bigr)$$

and

$$\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(W \setminus \{0\}) = \mathbb{Z}[\xi, \eta]/(p\xi, p\eta, \ell_1 \dots \ell_r),$$

and a commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}[\xi, \eta, h]/\bigl(p\xi, p\eta, (h - \ell_1) \dots (h - \ell_r)\bigr) & \longrightarrow & \mathbb{Z}[\xi, \eta]/(p\xi, p\eta, \ell_1 \dots \ell_r) \\
\downarrow & & \downarrow \\
\mathbb{F}_p[\xi, \eta, h]/\bigl((h - \ell_1) \dots (h - \ell_r)\bigr) & \longrightarrow & \mathbb{F}_p[\xi, \eta]/(\ell_1 \dots \ell_r)
\end{array}$$

in which the first row is the map that sends h to 0, and corresponds to the pullback.

The restriction homomorphism $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(P) \to \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(q_i) = \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ sends h into ℓ_i. But $\overline{U}$ contains all the q_i, so the kernel K of the restriction $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(P) \to \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(\overline{U})$ is contained in the intersection of the ideals $(h - \ell_i)$. In the polynomial ring $\mathbb{F}_p[\xi, \eta, h]$, however, the intersection of the ideals $(h - \ell_i)$ is the ideal generated by the product of the $h - \ell_i$, because $\mathbb{F}_p[\xi, \eta, h]$ is a unique factorization domain, and the $h - \ell_i$ are pairwise non-associated primes. Hence the image of an element of K is 0 in $\mathbb{F}_p[\xi, \eta]/(\ell_1 \dots \ell_r)$; but the homomorphism

$$\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(W \setminus \{0\}) \longrightarrow \mathbb{F}_p[\xi, \eta]/(\ell_1 \dots \ell_r)$$

is an isomorphism in positive degree, and from this the statement follows.

Now consider the restriction homomorphism

$$\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(W \setminus \{0\}) \longrightarrow \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(U).$$

Denote by γ the top Chern class of W in $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$; the kernel of the surjective pullback $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \to \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(W \setminus \{0\})$ is the ideal generated by γ, and we need to show that the kernel of the pullback $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \to \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(U)$ is also the ideal generated by γ.

Denote by R the image of $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ in $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p} = \mathbb{Z}[\xi, \eta]/(p\xi, p\eta)$; this is the subring generated by ξ and the image of σ_p, that is $\eta^p - \xi^{p-1}\eta$, by Lemma 6.6.

Take some u in the kernel of $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \to \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}(U)$. Since $\mathrm{T}_{\mathrm{PGL}_p}$ has a fixed point in U, the pullback $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}} \to \mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}(U)$ is an isomorphism; hence u is contained in the kernel of the restriction $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \to \mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$. This kernel is the ideal $\xi\, \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$, which is a vector space over the field $\mathbb{F}_p$, with a basis consisting of the elements $\xi^i \sigma_p^j$, with $i > 0$ and $j \geq 0$. The homomorphism $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \to \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ sends $\xi^i \sigma_p^j$ into $\xi^i(\eta^p - \xi^{p-1}\eta)^j$. The two elements ξ and $\eta^p - \xi^{p-1}\eta$ are algebraically independent in $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$, so the ideal $\xi\, \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ map isomorphically onto the ideal ξR. Hence it is enough to show that u maps into the ideal γR. But u maps into the ideal $\gamma\, \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$, because by hypothesis maps into 0 in $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}(W \setminus \{0\})$, so we will be done once we have shown that $\gamma R = R \cap \gamma\, \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$.

For this purpose, consider the diagram

$$\begin{array}{ccc} R & \rightarrowtail & \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p} \\ \downarrow & & \downarrow \\ \mathbb{F}_p[\xi, \eta^p - \xi^{p-1}\eta] & \rightarrowtail & \mathbb{F}_p[\xi, \eta] \end{array}$$

where the horizontal arrows are inclusions and the vertical arrows are isomorphisms in positive degree. It suffices to prove that

$$\gamma \mathbb{F}_p[\xi, \eta^p - \xi^{p-1}\eta] = \mathbb{F}_p[\xi, \eta^p - \xi^{p-1}\eta] \cap \gamma \mathbb{F}_p[\xi, \eta];$$

but this follows from the fact that the extension $\mathbb{F}_p[\xi, \eta^p - \xi^{p-1}\eta] \subseteq \mathbb{F}_p[\xi, \eta]$ is faithfully flat, since it is a finite extension of regular rings.

This concludes the proof of the Lemma. ♠

The Lemma applies to the case $W = D_p$ and $W = \mathfrak{sl}_p$. In the first case this is straightforward; in the second one it follows from Lemma 5.1. ♠

11. The classes ρ and β

In this section we construct the classes $\rho \in \mathrm{A}^{p+1}_{\mathrm{PGL}_p}$ and $\beta \in \mathrm{H}^3_{\mathrm{PGL}_p}$.

Proposition 11.1. *There exists a unique torsion class $\rho \in \mathrm{A}^{p+1}_{\mathrm{PGL}_p}$, whose image in $\mathrm{A}^{p+1}_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ equals $r = \xi\eta(\xi^{p-1} - \eta^{p-1})$.*

Furthermore we have $\rho^{p-1} = \mathrm{c}_{p^2-1}(\mathfrak{sl}_p) \in \mathrm{A}^*_{\mathrm{PGL}_p}$.

Proof. Uniqueness is obvious from Proposition 9.4.

Let us construct a p-torsion element $\overline{\rho} \in \mathrm{A}^{p+1}_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ that maps to r in $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$.

Consider the element $-\xi\phi(\sigma_p) = \xi\, \mathrm{c}_p(V) \in \mathrm{A}^{p+1}_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$; by Lemma 6.6, its restriction to $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ is r. It is p-torsion, because ξ is p-torsion; hence it maps to 0 in $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$. Since the torsion part of $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ injects into $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$, and the image of $\xi\, \mathrm{c}_p(V)$ in $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ is invariant under $\mathbb{F}_p^* \ltimes \mathbb{F}_p$, it follows that $\xi\, \mathrm{c}_p(V)$ is also invariant under $\mathbb{F}_p^* \ltimes \mathbb{F}_p$.

By Proposition 8.1, there exists a p-torsion class $\overline{\rho} \in \mathrm{A}^{p+1}_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ whose image in $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ is r. By Proposition 10.1, there exists a p-torsion element $\rho \in \mathrm{A}^{p+1}_{\mathrm{PGL}_p}$ whose image in $\mathrm{A}^{p+1}_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ has the form $\overline{\rho} + \mathrm{c}_{p-1}(D_p)\sigma$ for a certain class $\sigma \in \mathrm{A}^2_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$.

The image of ρ in $\left(\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}\right)^{\mathrm{SL}_2(\mathbb{F}_p)} = \mathbb{Z}[q, r]$ must be an integer multiple ar of r, for reasons of degree. The image of $\mathrm{c}_{p-1}(D_p)$ is $-\xi^{p-1}$; hence by mapping into $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ we get an equality

$$ar = r - \xi^{p-1} h \in \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p},$$

where $h \in \mathrm{A}^2_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ is the image of σ. From this equality it follows easily that a is 1 and h is 0, and therefore ρ maps to r.

To check that $\rho^{p-1} = \mathrm{c}_{p^2-1}(\mathfrak{sl}_p)$, observe that both members of the equality are 0 when restricted to $\mathrm{T}_{\mathrm{PGL}_p}$; hence, by Proposition 9.4, it is enough to show that the restriction of $\mathrm{c}_{p^2-1}(\mathfrak{sl}_p) = \mathrm{c}_{p^2-1}(\mathfrak{gl}_p)$ to $\mathrm{A}^{p^2-1}_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ equals r^{p-1}; and this follows from Lemma 5.6. ♠

Corollary 11.2. *The restriction homomorphism* $\mathrm{A}^*_{\mathrm{PGL}_p} \to \left(\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}\right)^{\mathrm{SL}_2(\mathbb{F}_p)}$ *is surjective.*

Proof. The ring $\left(\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}\right)^{\mathrm{SL}_2(\mathbb{F}_p)}$ is generated by q and r. The class $-\mathrm{c}_{p^2-p}(\mathfrak{sl}_p)$ restricts to q, by Lemma 5.6, while ρ restricts to r. ♠

Remark 11.3. The class ρ gives a new invariant for sheaves of Azumaya algebras of prime degree. Let X be a scheme of finite type over k, and let $\mathcal{A}$ be a sheaf of Azumaya algebras of degree p. This corresponds to a PGL_p-torsor $E \to X$; and according to a result of Totaro (see [13] and [1]), we can associate to the class $\rho \in \mathrm{A}^{p+1}_{\mathrm{PGL}_p}$ and the PGL_p-torsor E a class $\phi(\mathcal{A}) \in \mathrm{A}^{p+1}(X)$ (where by $\mathrm{A}^*(X)$ we mean the bivariant ring of X, see [3]). Since by definition $\mathcal{A}$ is the vector bundle associated with E and the representation $\mathfrak{gl}_p$ of PGL_p, we have the relation

$$\rho(\mathcal{A})^{p-1} = \mathrm{c}_{p^2-1}(\mathcal{A}) \in \mathrm{A}^{p^2-1}(X).$$

Remark 11.4. The class ρ depends on the choice of the primitive p^{th} root of 1 that we have denoted by ω. If we substitute ω^i for ω, then the new class ρ is $i\rho$.

For the class β, one possibility it to obtain it as the Brauer class of the canonical PGL_p-principal bundle, as explained in the Introduction. Another possibility is to define it via a transgression homomorphism, as follows. There is a well known Hochshild–Serre spectral sequence

$$E_2^{ij} = \mathrm{H}^i_{\mathrm{PGL}_p} \otimes \mathrm{H}^j_{\mathbb{G}_\mathrm{m}} \Longrightarrow \mathrm{H}^{i+j}_{\mathrm{GL}_p}$$

from which we get an exact sequence

$$\mathrm{H}^2_{\mathrm{GL}_p} \longrightarrow \mathrm{H}^2_{\mathbb{G}_\mathrm{m}} \longrightarrow \mathrm{H}^3_{\mathrm{PGL}_p} \longrightarrow \mathrm{H}^3_{\mathrm{T}_{\mathrm{PGL}_p}} = 0;$$

and $\mathrm{H}^2_{\mathbb{G}_\mathrm{m}}$ is the infinite cyclic group generated by the first Chern class t of the identity character $\mathbb{G}_\mathrm{m} = \mathbb{G}_\mathrm{m}$, while $\mathrm{H}^2_{\mathrm{GL}_p}$ is the cyclic group generated by the first Chern class of the determinant $\mathrm{GL}_p = \mathbb{G}_\mathrm{m}$, whose image in $\mathrm{H}^2_{\mathbb{G}_\mathrm{m}}$ is pt. Hence $\mathrm{H}^3_{\mathrm{PGL}_p}$ is the cyclic group of order p generated by the image of t. We define $\beta \in \mathrm{H}^3_{\mathrm{PGL}_p}$ to be this image.

The odd dimensional cohomology $\mathrm{H}^{\mathrm{odd}}_{\mathrm{PGL}_p}$ maps to 0 in $\mathrm{H}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$; hence, according to Proposition 9.4, maps injectively into $\mathrm{H}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$. By the results of Section 5, we have that $\mathrm{H}^3_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, hence the restriction homomorphism $\mathrm{H}^3_{\mathrm{PGL}_p} \to \mathrm{H}^3_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$ is an isomorphism; and the image of β generated $\mathrm{H}^3_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$. From Proposition 5.9 we obtain the following.

Corollary 11.5. *The restriction homomorphism* $\mathrm{H}^*_{\mathrm{PGL}_p} \to \left(\mathrm{H}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\right)^{\mathrm{SL}_2(\mathbb{F}_p)}$ *is surjective.*

12. The splitting

In this section we prove Theorem 3.2.

Consider the embeddings

$$\begin{array}{ccc} \boldsymbol{\mu}_p & \hookrightarrow & \mathrm{T}_{\mathrm{PGL}_p} \\ \downarrow & & \downarrow \\ \mathrm{C}_p \times \boldsymbol{\mu}_p & \hookrightarrow & \mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p} \end{array}$$

which induce a diagram of restriction homomorphisms

$$\begin{array}{ccc} \mathrm{A}^*_{\mathrm{S}_p\ltimes\mathrm{T}_{\mathrm{PGL}_p}} & \longrightarrow & \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p} \\ \downarrow & & \downarrow \\ \mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p} & \longrightarrow & \mathrm{A}^*_{\boldsymbol{\mu}_p}\,. \end{array}$$

Lemma 12.1. *The induced homomorphism*

$$\ker\bigl(\mathrm{A}^*_{\mathrm{S}_p\ltimes \mathrm{T}_{\mathrm{PGL}_p}} \to \mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\bigr) \longrightarrow \ker\bigl(\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p} \to \mathrm{A}^*_{\boldsymbol{\mu}_p}\bigr)$$

is surjective.

Proof. We will prove surjectivity in two steps; first we will show that the map is surjective when tensored with $\mathbb{Z}[1/p]$, then that is surjective when tensored with $\mathbb{Z}[1/(p-1)!]$.

For the first case, notice that $\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p} \otimes \mathbb{Z}[1/p]$ is 0 in positive degree, while in degree 0 there is nothing to prove; so what we are really trying to show is that $\mathrm{A}^*_{\mathrm{S}_p\ltimes \mathrm{T}_{\mathrm{PGL}_p}} \otimes \mathbb{Z}[1/p] \to \bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p} \otimes \mathbb{Z}[1/p]$ is surjective.

Consider the subgroup $\mathrm{S}_{p-1} \subseteq \mathrm{S}_p$ of the Weyl group of PGL_p, consisting of permutations of $\{1,\dots,p\}$ leaving p fixed.

Lemma 12.2. *The restriction homomorphism* $\mathrm{A}^*_{\mathrm{S}_{p-1}\ltimes \mathrm{T}_{\mathrm{PGL}_p}} \to (\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}})^{\mathrm{S}_{p-1}}$ *is surjective.*

Proof. There is an isomorphism $\mathrm{T}_{\mathrm{GL}_{p-1}} \simeq \mathrm{T}_{\mathrm{PGL}_p}$, defined by

$$(t_1,\dots,t_{p-1}) \mapsto (t_1,\dots,t_{p-1},1)$$

that is S_{p-1}-equivariant, and therefore induces an isomorphism of the semidirect product $\mathrm{S}_{p-1}\ltimes \mathrm{T}_{\mathrm{PGL}_p}$ with the normalizer $\mathrm{S}_{p-1}\ltimes \mathrm{T}_{\mathrm{GL}_{p-1}}$ of the maximal torus in GL_{p-1}. Hence it is enough to show that $\mathrm{A}^*_{\mathrm{S}_{p-1}\ltimes \mathrm{T}_{\mathrm{GL}_{p-1}}} \to (\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_{p-1}}})^{\mathrm{S}_{p-1}}$ is surjective; but the composite

$$\mathrm{A}^*_{\mathrm{GL}_{p-1}} \longrightarrow \mathrm{A}^*_{\mathrm{S}_{p-1}\ltimes \mathrm{T}_{\mathrm{GL}_{p-1}}} \longrightarrow (\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_{p-1}}})^{\mathrm{S}_{p-1}}$$

is an isomorphism, and this proves what we want. ♠

Take an element $u \in \bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p}$; according to the Lemma above, there is some $v \in \mathrm{A}^*_{\mathrm{S}_{p-1}\ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ such that $\mathrm{res}^{\mathrm{S}_{p-1}\ltimes \mathrm{T}_{\mathrm{PGL}_p}}_{\mathrm{T}_{\mathrm{PGL}_p}} v = u$. Consider the element

$$w \overset{\text{def}}{=} \mathrm{tsf}^{\mathrm{S}_{p-1}\ltimes \mathrm{T}_{\mathrm{PGL}_p}}_{\mathrm{S}_p\ltimes \mathrm{T}_{\mathrm{PGL}_p}} v \in \mathrm{A}^*_{\mathrm{S}_p\ltimes \mathrm{T}_{\mathrm{PGL}_p}};$$

to compute its restriction to $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$ we use Mackey's formula (Proposition 4.4). The double quotient $\mathrm{T}_{\mathrm{PGL}_p}\backslash \mathrm{S}_p\ltimes \mathrm{T}_{\mathrm{PGL}_p}/\mathrm{S}_{p-1}\ltimes \mathrm{T}_{\mathrm{PGL}_p}$ has p element, and we may take C_p as a set of representatives. Then the formula gives us that the restriction of w to $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$ is

$$\sum_{s\in \mathrm{C}_p} s\,\mathrm{res}^{\mathrm{S}_{p-1}\ltimes \mathrm{T}_{\mathrm{PGL}_p}}_{\mathrm{T}_{\mathrm{PGL}_p}} v = pu.$$

If we invert p, this shows that u is in the image of $\mathrm{A}^*_{\mathrm{S}_p\ltimes \mathrm{T}_{\mathrm{PGL}_p}}$, and completes the proof of the first step.

For the second step, take some $u \in L$. According to Proposition 7.1 (e) there exists v in the kernel of the restriction homomorphism $\mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \to \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ whose restriction to $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$ is u. Consider the element

$$w \overset{\mathrm{def}}{=} \operatorname{tsf}^{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} v.$$

I claim that w is in K. In fact the restriction of w to $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$ is $(p-1)!v = -v$, and thefore further restricting it to $\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}$ sends it to 0.

The double quotient $\mathrm{T}_{\mathrm{PGL}_p} \backslash \mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p} / \mathrm{S}_{p-1} \ltimes \mathrm{T}_{\mathrm{PGL}_p}$ has $(p-1)!$ elements, and a set of representatives is given by S_{p-1}. Hence according to Mackey's formula we have that the restriction of w to $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$ is

$$\sum_{s \in \mathrm{S}_{p-1}} s \operatorname{res}^{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}_{\mathrm{T}_{\mathrm{PGL}_p}} v = (p-1)!u$$

and this completes the second step in the proof of Lemma 12.1. ♠

Similarly, there is a diagram of restriction homomorphisms

$$\begin{array}{ccc} \mathrm{A}^*_{\mathrm{PGL}_p} & \longrightarrow & \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p} \\ \big\downarrow & & \big\downarrow \\ \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p} & \longrightarrow & \mathrm{A}^*_{\boldsymbol{\mu}_p}\,. \end{array}$$

Lemma 12.3. *The homomorphism*

$$\ker\left(\mathrm{A}^*_{\mathrm{PGL}_p} \to \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}\right) \longrightarrow \ker\left(\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p} \to \mathrm{A}^*_{\boldsymbol{\mu}_p}\right)$$

induced by restriction is an isomorphism.

Proof. Injectivity follows from Proposition 9.4.

As in the previous case, we show surjectivity first after inverting p, and then after inverting $(p-1)!$.

As before, we have $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p} \otimes \mathbb{Z}[1/p] = \mathbb{Z}[1/p]$, so we only need to check that $\mathrm{A}^*_{\mathrm{PGL}_p} \otimes \mathbb{Z}[1/p] \to \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p} \otimes \mathbb{Z}[1/p]$ is surjective. This follows from Lemma 12.1 and from Corollary 9.6.

Now we invert $(p-1)!$. Choose an element

$$u \in \ker\Big(\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p} \to \mathrm{A}^*_{\boldsymbol{\mu}_p}\Big) \otimes \mathbb{Z}[1/(p-1)!];$$

by Lemma 12.1, we can choose

$$u' \in \ker\left(\mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \to \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}\right) \otimes \mathbb{Z}[1/(p-1)!]$$

mapping to u in $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$. By Proposition 10.1, we can write

$$u' = v + \mathrm{c}_{p-1}(D_p)w,$$

where v is in $\mathrm{A}^*_{\mathrm{PGL}_p} \otimes \mathbb{Z}[1/(p-1)!]$ and w is in $\mathrm{A}^*_{\mathrm{S}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}} \otimes \mathbb{Z}[1/(p-1)!]$. The image of $\mathrm{c}_p(D_p)$ in $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$ is 0, because $\mathrm{T}_{\mathrm{PGL}_p}$ acts trivially on D_p; so the image of v in $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$ equals u. But there is no reason why v should map to 0 in $\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$.

Let us denote by $\overline{v}$ and $\overline{w}$ the images of v and w respectively in

$$\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\otimes\mathbb{Z}[1/(p-1)!] = \mathbb{Z}[1/(p-1)!][\xi,\eta]/(p\xi,p\eta);$$

the restriction of $\mathrm{c}_{p-1}(D_p)$ equals $-\xi^{p-1}$, so we have $\overline{v}-\xi^{p-1}\overline{w}=0$. On the other hand $\overline{v}$ is contained in

$$\big(\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\big)^{\mathrm{SL}_2(\mathbb{F}_p)}\otimes\mathbb{Z}[1/(p-1)!] = \mathbb{Z}[1/(p-1)!][q,r]/(pq,pr);$$

since $\overline{v}$ is contained in the ideal of $\mathbb{Z}[1/(p-1)!][\xi,\eta]/(p\xi,p\eta)$ generated by ξ, and the images of q and r in

$$\mathbb{Z}[1/(p-1)!][\xi,\eta]/(\xi,p\eta) = \mathbb{Z}[1/(p-1)!][\eta]/(p\eta)$$

are η^{p^2-p} and 0, we see that $\overline{v}$ is a multiple of r; hence we can write $\overline{v} = r\phi(q,r)$, where ϕ is a polynomial with coefficients in $\mathbb{Z}[1/(p-1)!]$. Set

$$v' = v-\rho\phi(-\mathrm{c}_{p^2-p},\rho);$$

then v' restricts to 0 in $\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$, and its image in $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$ equals the image of v, which is u, because ρ maps to 0.

This concludes the proof of Lemma 12.3. ♠

Set $K=\ker\big(\mathrm{A}^*_{\mathrm{PGL}_p}\to\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\big)$ and $L=\ker\big(\big(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\big)^{\mathrm{S}_p}\to\mathrm{A}^*_{\boldsymbol{\mu}_p}\big)$. The induced homomorphism $K\to L$ is an isomorphism, according to Lemma 12.3.

Consider the subring $\mathbb{Z}\oplus L\subseteq\big(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\big)^{\mathrm{S}_p}$; Proposition 7.1 (e) gives us a copy $\mathbb{Z}\oplus K$ of it inside $\mathrm{A}^*_{\mathrm{PGL}_p}$. To finish the proof of Theorem 3.2 we need to extend this splitting to all of $\big(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\big)^{\mathrm{S}_p}$. According to Proposition 3.1, we have that $\big(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\big)^{\mathrm{S}_p}$ is generated as an algebra over $\mathbb{Z}\oplus L$ by the single element δ. We need to find a lifting for δ; this is provided by the following lemma.

Lemma 12.4. *The restriction of* $\mathrm{c}_{p^2-p}(\mathfrak{sl}_p)\in\mathrm{A}^*_{\mathrm{PGL}_p}$ *to* $\mathrm{A}^{p^2-p}_{\mathrm{T}_{\mathrm{PGL}_p}}$ *equals* δ.

Proof. We use the notation in the beginning of Section 3. The representations $\mathfrak{sl}_p$ and $\mathfrak{gl}_p=\mathfrak{sl}_p\oplus k$ of PGL_p have the same Chern classes. If $V=k^n$ is the standard representation of GL_p, then $\mathfrak{gl}_p = V\otimes V^\vee$ has total Chern class

$$\mathrm{c}_t(\mathfrak{gl}_p)=\prod_{i,j}\big(1+t(x_i-x_j)\big)=\prod_{i\neq j}\big(1+t(x_i-x_j)\big)$$

in $\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}$; but $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\subseteq\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}$, so the thesis follows. ♠

Set $\delta_1=\mathrm{c}_{p^2-p}(\mathfrak{sl}_p)\in\mathrm{A}^*_{\mathrm{PGL}_p}$. We consider the subring $(\mathbb{Z}\oplus K)[\delta_1]$ of $\mathrm{A}^*_{\mathrm{PGL}_p}$; to finish the proof of the theorem we have left to show that it maps injectively into $(\mathbb{Z}\oplus L)[\delta]=\big(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\big)^{\mathrm{S}_p}$.

Let us take a homogeneous element $x \in (\mathbb{Z} \oplus K)[\delta_1]$ that maps to 0 in $\big(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\big)^{\mathrm{S}_p}$; according to Proposition 9.4, to check that it is 0 it is enough to prove that it restricts to 0 into $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$. Write

$$x = a_0 + a_1\delta_1 + a_2\delta_1^2 + a_3\delta_1^3 + \cdots .$$

The a_i of positive degree are in K, and therefore map to 0 in $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$ by definition; so there can be at most one term that does not map to zero, and that has to be of the form $h\delta_1^d$, where h is an integer. However, the restriction of x to $\mathrm{A}^*_{\boldsymbol{\mu}_p} = \mathbb{Z}[\eta]/(p\eta)$ is zero, and since δ_1 restricts to a nonzero multiple of η^{p^2-p} we see that h must be divisible by p. This proves that $h\delta_1^d$ also restricts to 0 in $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$, and completes the proof of the theorem.

Remark 12.5. The splitting $\big(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\big)^{\mathrm{S}_p} \to \mathrm{A}^*_{\mathrm{PGL}_p}$ that we have constructed is not compatible with the splitting $\big(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\big)^{\mathrm{C}_p} \to \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}}$ constructed in Section 7, in the sense that the diagram

$$\begin{array}{ccc}
\big(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\big)^{\mathrm{S}_p} & \longrightarrow & \mathrm{A}^*_{\mathrm{PGL}_p} \\
\big\downarrow & & \big\downarrow \\
\big(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\big)^{\mathrm{C}_p} & \longrightarrow & \mathrm{A}^*_{\mathrm{C}_p \ltimes \mathrm{T}_{\mathrm{PGL}_p}},
\end{array}$$

where the rows are the splittings and the columns are restrictions, does *not* commute.

13. The proofs of the main Theorems

Let us prove Theorem 3.3.

First of all, let us check that ρ generates $\mathrm{A}^*_{\mathrm{PGL}_p}$ as an algebra over $\big(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\big)^{\mathrm{S}_p}$. Take a homogeneous element $\alpha \in \mathrm{A}^*_{\mathrm{PGL}_p}$. The image of $\delta \in \big(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\big)^{\mathrm{S}_p}$ in $\mathrm{A}^*_{\mathrm{PGL}_p}$ is $\mathrm{c}_{p^2-p}(\mathfrak{sl}_p)$, by construction; and this maps to $-q$ in $\mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$, by Lemma 5.6. So there is a polynomial $\phi(x, y)$ with integer coefficients such that $\alpha - \phi(\delta, \rho)$ is in the kernel of the restriction homomorphism $\mathrm{A}^*_{\mathrm{PGL}_p} \to \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$; but this kernel is in the image of $\big(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\big)^{\mathrm{S}_p}$, again by construction; and this shows that $\mathrm{A}^*_{\mathrm{PGL}_p}$ is generated by ρ as an algebra over $\big(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\big)^{\mathrm{S}_p}$.

The relations given in the statement are satisfied. We have $p\rho = 0$ by construction. Furthermore, by construction the splitting $\big(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\big)^{\mathrm{S}_p} \to \mathrm{A}^*_{\mathrm{PGL}_p}$ sends the kernel of the homomorphism $\big(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\big)^{\mathrm{S}_p} \to \mathrm{A}^*_{\boldsymbol{\mu}_p}$ into the kernel of $\mathrm{A}^*_{\mathrm{PGL}_p} \to \mathrm{A}^*_{\mathrm{C}_p \times \boldsymbol{\mu}_p}$; hence, if $u \in \ker\big(\big(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\big)^{\mathrm{S}_p} \to \mathrm{A}^*_{\mathrm{PGL}_p}\big)$ we have that $u\rho \in \mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$ goes to 0 in $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$, because ρ is torsion, and to $\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$. Hence $u\rho = 0$, because of Proposition 9.4.

Let x be an indeterminate, I the ideal in the polynomial algebra

$$\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p}[x]$$

generated by px and by the polynomials ux, where u is in the kernel of the restriction homomorphism $\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p} \to \mathrm{A}^*_{\boldsymbol{\mu}_p}$; we need to show that the homomorphism $\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p}[x]/I \to \mathrm{A}^*_{\mathrm{PGL}_p}$ that sends x to ρ is an isomorphism. Pick a polynomial $\phi \in \bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p}[x]$ such that $\phi(\delta) = 0$ in $\mathrm{A}^*_{\mathrm{PGL}_p}$. After modifying it by an element of I, we may assume that is of the form $\alpha + \psi(\delta, \rho)$, where α is in the the kernel of $\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p} \to \mathrm{A}^*_{\boldsymbol{\mu}_p}$, while ψ is a polynomial in two variables with coefficients in $\mathbb{F}_p$. Since the images of δ and ρ in $\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$, that are q and r, are linearly independent in $\mathbb{F}_P[\xi, \eta]$, we see that ψ must be 0. Hence $\alpha = 0$ in $\mathrm{A}^*_{\mathrm{PGL}_p}$; but since $\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p}$ injects inside $\mathrm{A}^*_{\mathrm{PGL}_p}$, we have that $\phi(x) = 0$, as we want.

Next we prove Theorem 3.4. We start by proving Corollary 3.5, that says that the cycle homomorphism $\mathrm{A}^*_{\mathrm{PGL}_p} \to \mathrm{H}^{\mathrm{even}}_{\mathrm{PGL}_p}$ is an isomorphism.

Call K and L, respectively, the kernels of the restriction homomorphisms $\mathrm{A}^*_{\mathrm{PGL}_p} \to \bigl(\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\bigr)^{\mathrm{SL}_2(\mathbb{F}_p)}$ and $\mathrm{H}^{\mathrm{even}}_{\mathrm{PGL}_p} \to \bigl(\mathrm{H}^{\mathrm{even}}_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\bigr)^{\mathrm{SL}_2(\mathbb{F}_p)}$; we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & \mathrm{A}^*_{\mathrm{PGL}_p} & \longrightarrow & \bigl(\mathrm{A}^*_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\bigr)^{\mathrm{SL}_2(\mathbb{F}_p)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & \mathrm{H}^{\mathrm{even}}_{\mathrm{PGL}_p} & \longrightarrow & \bigl(\mathrm{H}^{\mathrm{even}}_{\mathrm{C}_p\times\boldsymbol{\mu}_p}\bigr)^{\mathrm{SL}_2(\mathbb{F}_p)} & \longrightarrow & 0
\end{array}$$

with exact rows. The right hand column is an isomorphism, because of Propositions 5.4 and 5.9. The group L injects into

$$\ker\Bigl(\bigl(\mathrm{H}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p} \to \mathrm{H}^*_{\boldsymbol{\mu}_p}\Bigr) = \ker\Bigl(\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p} \to \mathrm{A}^*_{\boldsymbol{\mu}_p}\Bigr),$$

because of Proposition 9.4; on the other hand the restriction homomorphism

$$K \longrightarrow \ker\bigl(\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p} \to \mathrm{A}^*_{\boldsymbol{\mu}_p}\bigr)$$

is an isomorphism, because of Lemma 12.3. This proves that $K \to L$ is an isomorphism, and this proves Corollary 3.5.

To show that ρ and β generate $\mathrm{H}^*_{\mathrm{PGL}_p}$ as an algebra over $\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p}$, take a homogeneous element $\alpha \in \mathrm{A}^*_{\mathrm{PGL}_p}$. The element ρ generates $\mathrm{H}^{\mathrm{even}}_{\mathrm{PGL}_p}$, because of Theorem 3.3 and the fact above.

If α is a homogeneous element of odd degree in $\mathrm{H}^*_{\mathrm{PGL}_p}$, its image in $\mathrm{H}^{\mathrm{odd}}_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$ can be written in the form $\phi(q, r)s$, where ϕ is an integral polynomial, by Proposition 5.9. Then $\alpha - \phi(-\delta, \rho)\beta$ maps to 0 in $\mathrm{H}^{\mathrm{odd}}_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$. On the other hand $\mathrm{H}^{\mathrm{odd}}_{\mathrm{PGL}_p}$ injects into $\mathrm{H}^{\mathrm{odd}}_{\mathrm{C}_p\times\boldsymbol{\mu}_p}$, by Proposition 9.4, and this completes the proof that ρ and β generated.

To prove that the given relations generated the ideal of relations is straightforward, and left to the reader.

Finally, let us prove Theorem 3.6.

Since the homomorphisms $\mathrm{A}^*_{\mathrm{PGL}_p}\otimes\mathbb{Q} \to \mathrm{A}^*_{\mathrm{SL}_p}\otimes\mathbb{Q}$ and $\mathrm{A}^*_{\mathrm{PGL}_p}\otimes\mathbb{Q} \to \mathrm{A}^*_{\mathrm{SL}_p}\otimes\mathbb{Q}$ are isomorphisms, the ranks of $\mathrm{A}^i_{\mathrm{PGL}_p}$ and $\mathrm{H}^i_{\mathrm{PGL}_p}$ equal the ranks of $\mathrm{A}^i_{\mathrm{SL}_p}$ and $\mathrm{H}^i_{\mathrm{SL}_p}$. The ranks of the $\mathrm{H}^i_{\mathrm{PGL}_p}$ are 0 when i is odd; while for any $m \geq 0$ the rank of $\mathrm{A}^m_{\mathrm{PGL}_p} \simeq \mathrm{H}^{2n}_{\mathrm{PGL}_p}$ equal the number of monomials of degree m in $\sigma_2, \ldots, \sigma_p$. Such a monomial $\sigma_2^{d_2}\ldots p^{d_p}$ can be identified with a partition $\langle 2^{d_2}\ldots p^{d_p}\rangle$ of m, so this rank is the numbero of partitions of m with numbers between 2 and p.

On the other hand it follows from Theorem 3.3 that the torsion part of $\mathrm{A}^*_{\mathrm{PGL}_p}$ is a vector space over the field $\mathbb{F}_p$, with a basis given by the elements $\delta^i\rho^j$, where $i \geq 0$ and $j > 0$. Similarly, from Corollary 3.5 we see that the same elements form a basis for $\mathrm{H}^{\mathrm{even}}_{\mathrm{PGL}_p}$, while $\mathrm{H}^{\mathrm{odd}}_{\mathrm{PGL}_p}$ is an $\mathbb{F}_p$-vector space with a basis formed by the elements $\delta^i\rho^j\beta$, where $i \geq 0$ and $j \geq 0$.

The theorem follows easily from these facts.

14. On the ring $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$

If T is a torus, we denote by $\widehat{T}$ the group of characters $T \to \mathbb{G}_{\mathrm{m}}$. We have a homomorphism of $\widehat{T}$ into the additive group A^*_T that sends each character into its first Chern class: and this induced an isomorphism of the symmetric algebra $\mathrm{Sym}_{\mathbb{Z}}\,\widehat{T}$ with A^*_T.

In this section we study the ring of invariants $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$. It is convenient to view $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$ as a subring of $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\right)^{\mathrm{S}_p}$; this last ring is generated by the symmetric functions $\sigma_1, \ldots, \sigma_p$ of the first Chern characters $x_1, \ldots, x_p$ of the projections $\mathrm{T}_{\mathrm{GL}_p} \to \mathbb{G}_{\mathrm{m}}$.

If we tensor $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$ with $\mathbb{Z}[1/p]$, then we get a polynomial ring; and it is easy to exhibit generators. The homomorphism of groups of characters

$$\widehat{\mathrm{T}}_{\mathrm{PGL}_p} \longrightarrow \widehat{\mathrm{T}}_{\mathrm{SL}_p}$$

induced by the projection $\mathrm{T}_{\mathrm{SL}_p} \to \mathrm{T}_{\mathrm{PGL}_p}$ is injective, with cokernel $\mathbb{Z}/p\mathbb{Z}$; hence it becomes an isomorphism when tensored with $\mathbb{Z}[1/p]$. Hence

$$\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}\otimes\mathbb{Z}[1/p] \longrightarrow \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}}\right)^{\mathrm{S}_p}\otimes\mathbb{Z}[1/p]$$

is an isomorphism.

According to Lemma 7.2, the ring $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}}\right)^{\mathrm{S}_p}$ is a quotient

$$\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}}\right)^{\mathrm{S}_p}/(\sigma_1) = \mathbb{Z}[\sigma_1, \ldots, \sigma_p]/(\sigma_1) = \mathbb{Z}[\tau_2, \ldots, \tau_p],$$

where we have denoted by τ_i the image of σ_i in $\mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}}$. One way to produce elements of $\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p}$ is to write down explicitly the elements corresponding to the σ_i in the isomorphism

$$\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p} \otimes \mathbb{Z}[1/p] \simeq \mathbb{Z}[1/p][\sigma_2, \dots, \sigma_p]$$

and then clear the denominators.

The composite

$$\begin{array}{ccccc} \mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}} \otimes \mathbb{Z}[1/p] & \hookrightarrow & \mathrm{A}^*_{\mathrm{T}_{\mathrm{GL}_p}} \otimes \mathbb{Z}[1/p] & \longrightarrow & \mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}} \otimes \mathbb{Z}[1/p] \\ & & \| & & \| \\ & & \mathbb{Z}[1/p][x_1, \dots, x_p] & \longrightarrow & \mathbb{Z}[1/p][x_1, \dots, x_p]/(\sigma_1) \end{array}$$

is an isomorphism, and the inverse $\mathbb{Z}[1/p][x_1, \dots, x_p]/(\sigma_1) \to \mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}} \otimes \mathbb{Z}[1/p]$ is obtained by sending x_i to $x_i - \frac{1}{p}\sigma_1$. We need to compute the image of the σ_k in $\mathrm{A}^*_{\mathrm{PGL}_p} \otimes \mathbb{Z}[1/p] \subseteq \mathbb{Z}[1/p][\sigma_1, \dots, \sigma_p]$, and this is given by the following formula (the one giving the Chern classes of the tensor product of a vector bundle and a line bundle).

Lemma 14.1. *If t is an indeterminate, we have*

$$\begin{aligned} \sigma_k(x_1 + t, \dots, x_p + t) &= \sum_{i=0}^{k} \binom{p-k+i}{i} t^i \sigma_{k-i} \\ &= \sigma_k + (p-k+1)t\sigma_{k-1} + \binom{p-k+2}{2} t^2 \sigma_{k-2} \\ &\qquad + \dots + \binom{p-1}{k-1} t^{k-1}\sigma_1 + \binom{p}{k} t^k. \end{aligned}$$

in $\mathbb{Z}[x_1, \dots, x_p, t]$, for $k = 0, \dots, p$.

Proof. This follows by comparing terms of degree k in the equality

$$\begin{aligned} \sum_{i=0}^{p} (1+t)^i \sigma_{p-i} &= \prod_{i=1}^{p} (1 + t + x_i) \\ &= \sum_{i=0}^{p} \sigma_i(x_1 + t, \dots, x_p + t). \end{aligned}$$ ♠

If we subtitute $-\frac{1}{p}\sigma_1$ for t we obtain the images of the τ_k in $\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p} \otimes \mathbb{Z}[1/p]$; we denote them by γ'_k. In order to get elements of $\bigl(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\bigr)^{\mathrm{S}_p}$, we can clear the denominators in the γ'_i; by a straightforward calculation we can check that

$$\begin{aligned} \gamma_k &= p^{k-1}\gamma'_k \\ &= \sum_{i=0}^{k-2} (-1)^i p^{k-i-1} \binom{p-k+i}{i} \sigma_{k-i}\sigma_1^i + (-1)^{k-1}\frac{k-1}{k}\binom{p-1}{k-1}\sigma_1^k \end{aligned}$$

for $k = 2, \ldots, p-1$, while

$$\begin{aligned}\gamma_p &= p^p \gamma'_p \\ &= \sum_{i=1}^{p-2} (-1)^i p^{p-i} \sigma_{p-i} \sigma_1^i + (p-1)\sigma_1^p.\end{aligned}$$

From the discussion above we get that $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p} \otimes \mathbb{Z}[1/p]$ is a polynomial ring over $\mathbb{Z}[1/p]$ over $\gamma_2, \ldots, \gamma_p$. However, the γ_i cannot generate $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$ integrally, because all of them are in the kernel of the homomorphism $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p} \to \mathrm{A}^*_{\boldsymbol{\mu}_p}$, while $\delta \in \left(\mathrm{A}^{p^2-p}_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$ is not.

When $p = 3$ the situation is simple. The following result was proved by Vezzosi.

Theorem 14.2 ([15, Lemma 3.2])**.**

$$\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_3}}\right)^{\mathrm{S}_3} = \mathbb{Z}[\gamma_2, \gamma_3, \delta]/\left(27\delta - 4\gamma_2^3 - \gamma_3^2\right).$$

Proof. What follows is essentially the argument given in the proof of Lemma 3.2 in [15]. We have

$$\gamma_2 = 3\sigma_2 - \sigma_1^2$$

and

$$\gamma_3 = 27\sigma_3 - 9\sigma_1\sigma_2 + 2\sigma_1^3.$$

Let us express δ as a rational polynomial in γ_2 and γ_3. This is most easily done after projecting into

$$\mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}} = \mathbb{Z}[x_1, x_2, x_3]/(\sigma_1)$$

since we know that $\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}$ injects inside $\mathrm{A}^*_{\mathrm{T}_{\mathrm{SL}_p}}$. Since δ is the opposite of the classical discriminant $\prod_{1 \le i < j \le 3}^3 (x_i - x_j)^2$, the image of δ in $\mathbb{Z}[x_1, x_2, x_3]/(\sigma_1)$ equals $4\sigma_2^3 + 27\sigma_3^2$. The images of γ_2 and γ_3 in $\mathbb{Z}[x_1, x_2, x_3]/(\sigma_1)$ are $3\sigma_2$ and $27\sigma_3$; hence we get the formula

$$27\delta = 4\gamma_2^3 + \gamma_3^2$$

showing that the relation in the statement of the theorem holds.

We will show that if $\phi \in \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_3}}\right)^{\mathrm{S}_3} \subseteq \mathbb{Z}[\sigma_1, \sigma_2, \sigma_3]$ is such that 3ϕ is in $\mathbb{Z}[\gamma_2, \gamma_3, \delta]$, then ϕ is also in $\mathbb{Z}[\gamma_2, \gamma_3, \delta]$. This implies that $\mathbb{Z}[\gamma_2, \gamma_3, \delta] = \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_3}}\right)^{\mathrm{S}_3}$, because we know that γ_2 and γ_3 generate $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_3}}\right)^{\mathrm{S}_3} \otimes \mathbb{Z}[1/3]$, hence if $\phi \in \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_3}}\right)^{\mathrm{S}_3}$ then $3^n\phi \in \mathbb{Z}[\gamma_2, \gamma_3, \delta]$ for sufficiently large n, and we can proceed by descending induction on n.

Write

$$3\phi = p(\gamma_2, \gamma_3, \delta) \in \mathbb{Z}[\sigma_1, \sigma_2, \delta] \subseteq \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_3}}\right)^{\mathrm{S}_3};$$

for an integral polynomial p. The image of $p(\gamma_2, \gamma_3, \delta)$ in the polynomial ring $\mathbb{F}_3[\sigma_1, \sigma_2, \sigma_3]$ is 0; the images of γ_2, γ_3 and δ in $\mathbb{F}_3[\sigma_1, \sigma_2, \sigma_3]$ are $-\sigma_1^2$, $-\sigma_1^3$

and σ_2^3 respectively; and the ideal of relations between these three polynomials is generated by $(-\sigma_2)^3 + (-\sigma^3)^2$, which is the image in $\mathbb{F}_3[\sigma_1, \sigma_2, \sigma_3]$ of

$$4\gamma_2^3 + \gamma_3^2 = 27\delta.$$

Hence there are two integral polynomials q and r such that we can write

$$\begin{aligned} 3\phi &= p(\sigma_2, \sigma_3, \delta) \\ &= 3q(\sigma_2, \sigma_3, \delta) + 27\delta \cdot r(\sigma_2, \sigma_3, \delta). \end{aligned}$$

Dividing by 3 we see that ϕ is in $\mathbb{Z}[\sigma_2, \sigma_3, \delta] \subseteq \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_3}}\right)^{\mathrm{S}_3}$; and this concludes the proof that γ_2, γ_3 and δ generate.

Consider the surjective ring homomorphism

$$\mathbb{Z}[x_2, x_3, y] \longrightarrow \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_3}}\right)^{\mathrm{S}_3}$$

that sends x_i to γ_i and y to δ; call I its kernel. We know that

$$(27y - 4x_2^3 - x_3^2) \subseteq I.$$

After tensoring with $\mathbb{Z}[1/3]$, both rings

$$\mathbb{Z}[x_2, x_3, y]/(27y - 4x_2^3 - x_3^2) \quad \text{and} \quad \left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_3}}\right)^{\mathrm{S}_3}$$

become polynomial rings $\mathbb{Z}[1/3][x_2, x_3]$; hence, if $f \in I$, some multiple $3^n f$ is in $(27y - 4x_2^3 - x_3^2)$. But this implies that f is in $(27y - 4x_2^3 - x_3^2)$ because 3 is a prime in the unique factorization domain $\mathbb{Z}[x_2, x_3, y]$ and does not divide $27y - 4x_2^3 - x_3^2$, so

$$(27y - 4x_2^3 - x_3^2) = I,$$

as claimed. ♠

From this, Theorem 3.7 follows easily.

As p grows, the calculations become very complicated very quickly. The obvious generalization of the result above, that $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_p}}\right)^{\mathrm{S}_p}$ is generated by the γ_i and δ, fails badly. When p is larger than 3, it is not hard to see that they fail to generate already in degree 4. When $p = 5$ the ring $\left(\mathrm{A}^*_{\mathrm{T}_{\mathrm{PGL}_5}}\right)^{\mathrm{S}_5}$ has 9 generators, in degrees 2, 3, 4, 5, 6, 7, 9, 12 and 20; with some pain, it is possible to write them down explicitly. The generators in degree 2 and 3 are γ_2 and γ_3. With more work it shoud also be possible to find the relations among them.

There are other approaches to calculations other than the one given here for $p = 3$; but none of them seem to give a lot of information in the general case.

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