Contemporary Mathematics

Rings of invariants and inseparable forms of algebras over the Steenrod algebra

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Dedicated to the memory of my mother, Dorthea Gray Wilkerson, 1925-1999.

Abstract. This paper establishes analogues of the the classical decompositions of a normal field extension in the context of graded integral domains with an unstable action of the mod p Steenrod algebra. These decompositions have in later work proved to have interesting topological consequences.

The concept of the maximal torus plays a key role in the classification of compact connected Lie groups. Likewise the role of the cohomology of the classifying space BT^n of the n-torus in the study of characteristic classes has been long understood via the splitting theorems. With the work of Adams-Wilkerson, [**1**], it was realized that the cohomology rings $H^*(BT^n, F_p)$ as algebras over the Steenrod algebra A_p were universal in a strong technical sense. This feature was presaged by (Serre, [**18**]), (Quillen, [**15, 16**]), and (Wilkerson, [**19**]), and is greatly extended by later work of Carlsson, Miller, Lannes, Zarati, Schwartz, and Dwyer-Wilkerson.

The Adams-Wilkerson work provides an algebraic analogue of the existence of a maximal torus, in a suitable category of algebras with an action of the Steenrod algebra. More precisely, it provides for each graded F_p- algebra R^* which is an integral domain of finite transcendence degree n over F_p and equipped with an "unstable" action of the mod p Steenrod algebra A_p, an embedding

$$t_{R^*} : R^* \longrightarrow H^*(BT^n, F_p)$$

which is an A_p-map of graded algebras and is an invariant of R^* and its A_p-action. Furthermore, $H^*(BT^n, F_p)$ is algebraic and normal over R^*, via the morphism t_{R^*}.

In this paper, we denote $H^*(BT^n, F_p)$ together with its A_p-action as $S[V^{\#}]$, the symmetric algebra on the dual of the $n-$dimensional F_p-vector space V, for V concentrated in degree 2 (or degree 1 for $p = 2$).

The author was supported in part by the National Science Foundation, the Wayne State Fund, The Institute for Advanced Studies of Hebrew University, and Johns Hopkins University.

The embedding t_{R^*} has additional properties, depending on the properties of R^* as an algebra over the Steenrod algebra:

PROPERTIES OF THE ADAMS-WILKERSON EMBEDDING:

a) $S[V^\#]$ is integral over R^* via t_R if and only R^* is a finitely generated F_p-algebra.

b) t_{R^*} is a separable extension on the fraction field level if and only if the Milnor primitives $\{P^{\Delta_i}\}$ in A_p give n linearly independent derivations on the graded fraction field FR^* of R^* (as they do on $S[V^\#]$). Equivalently, $(S[V^\#])^p \cap FR^* = (FR^*)^p$, if and only if the field extension is separable.

c) There is a subgroup $W(R^*)$ of $GL(V)$ such that $R^* = S[V^\#]^{W(R^*)}$, if and only if both conditions a) and b) hold, and, in addition, R^* is integrally closed in its field of fractions.

That is, under some general conditions, R^* is a ring of invariants. Conversely, any ring of invariants has the properties properties a)-c).[1]

The three conditions required to ensure that R^* is a ring of invariants may seem to lack topological motivation. However, the noetherian and integral closure requirements together have an intrinsic meaning in terms of the Steenrod algebra action:

THEOREM 1. *If R^* is a graded integral domain of finite transcendence degree with an unstable A_p-action, then R^* is integrally closed and finitely generated if and only if $R^* = Un(FR^*)$, the set of unstable elements in the graded field of fractions of R^*. If we consider R^* as a subring of $S[V^\#]$ via t_{R^*}, then $Un(FR^*) = S[V^\#] \cap FR^*$.*

Interpretation of the separability condition is harder, but algebraic examples at all primes and topological examples at the prime 2 show that it is a necessary complication. The inseparable forms portion of the title refers to a construction given here by which inseparable examples are made from separable examples. The goal is an algebraic reformulation of the separability condition so that it can be used more readily with topological techniques such as "mod p Hopf invariant one", for odd primes. Roughly stated, our aim in Theorem 2 below is to show that any R^* with $R^* = Un(FR^*)$ is a ring of invariants, not necessarily of $S[V^\#]$ itself, but of a certain "diagonal" subalgebra $D^*(R^*)$ of $S[V^\#]$.

The definition of this $D^*(R^*)$ is motivated by an analogy to elementary field theory. Recall that if $K \to L$ is a finite *normal* field extension, then there are two ways of decomposing this into sub-extensions. First, one could find the maximal inseparable extension J of K in L, so $K \to J \to L$. Then $J \to L$ is Galois, with Galois group $W = Aut_K(L) = Aut_J(L)$, so that $J = L^W$. This is roughly the procedure in [**1**], with allowances made there for ambient Steenrod actions. A second method is more important for this paper – form the maximal *separable* extension I of K in L , where $K \to I \to L$. Here $K \to I$ is separable, and $J \to L$

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[1] We ask the reader to note that in Adams-Wilkerson, [**1**], the statement of Theorem 1.2 should include the hypothesis that H^* is noetherian. This hypothesis was used but was inadvertently omitted. There is also an oversight in the statement of Theorem 1.9 – in order to prove that condition 1.2.2 is necessary, H^* should be assumed also to be integrally closed. An alternate formulation is to assume that condition 1.2.2 holds for all elements of the fraction field of H^*, as stated above in part b).

is purely inseparable. Notice that K is the intersection of L^W with I, or $K = I^W$, see for example, [Lang, Chapter VII, Section 7, Prop. 12, [**10**]].

Our Theorem 2 provides a ring level description of this process of taking a maximal separable extension. Somewhat surprisingly, the analogue of the maximal separable field extension I above is greatly restricted by the existence of the Steenrod algebra action. This restricted structure of $D^*(R^*)$ is the important for applications, [**2**], [**12**].

THEOREM 2. *Let* R^* *be a sub*$-A_p$*-algebra of* $S[V^\#]$ *such that* $FR^* \to FS[V^\#]$ *is algebraic. Define a filtration* $\{U_i(R^*)\}$ *of* $V^\#$ *by the rule that* y *is in* $U_i(R^*)$ *if and only if* y^{p^i} *is separable over* FR^*. *Let* $D^*(R^*)$ *be the "diagonal " subalgebra of* $S[V^\#]$ *generated by* $\{U_i(R^*)^{p^i}\}$. *Let* I^* *be the maximal separable graded field extension of* FR^* *in* $FS[V^\#]$. *Then*

a) $R^* \to D^*(R^*)$ *is separable and* $D^*(R^*) \to S[V^\#]$ *is purely inseparable.*
b) $D^*(R^*) = Un(I^*)$ *and* $U_i(R^*) = \{y | y^{p^i} \in I^*\}$.
c) $D^*(R^*) \approx \otimes_i (S[U_i(R^*)/U_{i-1}(R^*)])^{p^i}$ *as* A_p-*algebras.*
d) $Aut_{R^*}(S[V^\#]) = Aut_{R^*}(D^*(R^*)) = Aut_{FR^*}(FS[V^\#])$
e) For $W(R) = Aut_{R^*}(S[V^\#]),\ W(R)(U_i(R^*)) \subseteq U_i(R^*)$.
f) $D^*(R^*)^{W(R^*)} = Un(FR^*)$.

Note that although $V^\#$ has a filtration by the $W(R^*)-$vector spaces $\{U_i(R^*)\}$, the filtration need not split over $W(R^*)$, so that $V^\#$ is not necessarily isomorphic to $\oplus U_{i+1}/U_i$ as $W(R^*)$ vector spaces.

In summary, $Un(FR^*)$ is explicitly determined by two pieces of data:
1) The subgroup $W(R^*)$ of $GL(V^\#)$
2) The $W(R^*)$ filtration $\{U_i(R^*)\}$ of $V^\#$, together with the implicit exponents p^i.

This leads to a constructive classification of all noetherian integrally closed unstable domains of a given trancendence degree. In Section Five we give some examples of this classification.

These methods also answer a question of Mitchell and Stong:

THEOREM 3. *If the polynomial algebra* $R^* = F_p[y_1, \ldots, y_n]$ *has an unstable action of* A_p, *then the Adams-Wilkerson embedding* t_{R^*} *factors through* $D^*(R^*) \approx \mathbf{F}_p[z_1, \ldots, z_n]$ *where* $|z_i| = 2p^{N_i}$. *Since* $R^* \to D^*(R^*)$ *is separable, the Jacobian* $\mid \partial y_i/\partial z_j \mid$ *is nonzero for homogeneous algebra bases* $\{z_i\}$ *of* $D^*(R^*)$ *and* $\{y_j\}$ *of* R^*.

Theorem 2 does not directly answer the problem of realizing these rings as the cohomology rings of topological spaces, but it does express the maximum algebraic content obtainable from the A_p-action. Dwyer, Miller, and Wilkerson have used Theorem 2 together with applications of the Sullivan Conjecture technology of Miller and Lannes to prove very strong uniqueness and non-realizability results for classifying spaces. Theorem 4 below from [**2**] is a sampler that points out that separability is forced by topological considerations.

THEOREM 4. [**2**] *If* X *is a simply connected CW complex such that* $R^* = H^*(X, \mathbf{F}_p)$ *is a integrally closed finitely generated graded integral domain of trancendence degree* n, *then for* $p > 2$

$$H^*(X, \mathbf{F}_p) \approx S[V^\#]^{W(R^*)}.$$

Furthermore, the homotopy type of X *determines a lift of the inclusion of* $W(R^*) \to GL(V)$ *up to the general linear group of rank* n *over the* $p-$*adic integers.*

The proof of Theorem 4 is beyond the scope of this paper, but we give here just a brief hint as to the role that Theorem 2 plays in [**2**]. If $R^* = H^*(X)$, then by a fundamental result of Lannes,[**11**], the embedding $t_{R^*} : R^* \to S[V^\#] \to H^*(BV)$ can be topologically realized by a continuous function $f : BV \to X$. The component of the function space $Map(BV, X)_f$ has cohomology $T^V_{f^*}(R^*) = D^*(R^*)$, where $T^V_{f^*}(\_)$ is a summand of the $T-$functor of Lannes. Accepting this, "mod p Hopf invariant one" shows that the exponents in the definition of $D^*(R^*)$ are all zero, since it is the cohomology of a space. Thus $Map(BV, X)_f \approx BT^n$. That is, [**2**] carries out the Adams-Wilkerson program on the space level and obtains new information about separability and lifting of automorphism groups to characteristic zero.

There is another approach to the "inseparability" questions treated in this paper. In Quillen, [**15**], the concept of an $F-$isomorphism provided a convenient treatment of $p-$th powers. Quillen's main theorems have been treated more abstractly (for unstable algebras rather than equivariant cohomology) in work of Rector, [**17**] and Lam, [**9**]. We recall the principal technical result of Lam, and reinterpret it in terms of Theorem 2.

Theorem 5. *1) Let* R^* *be a sub-*A_p-*algebra of* $S[V^\#]$ *such that* $S[V^\#]$ *is algebraic over* R^*. *Suppose in addition* R^* *is p-closed, that is* $(S[V^\#])^p \cap R^* = (R^*)^p$. *Then the top Dickson invariant element*

$$c_0 = \prod_{y \in V^\# - \{0\}} y$$

lies in R^*.

2) If $R^* \to S[V^\#]$ *is algebraic but not necssarily p-closed, then there exists* $N \geq 0$ *such that* $(c_0)^{p^N} \in R^*$.

3) If $z = c_0^{p^N}$ *is inverted to form* $S_z^{-1}R^*$, *then* $Un(FR^*) = Un(S_z^{-1}R^*)$.

Part 3) of Theorem 5 combined with the "localization" invariance properties of the Lannes $T-$functor established in [**2**], gives the computation that each component of Lannes $T-$functor corresponding to a *monomorphism* $\psi : R^* \to H^*(BV)$ is $D^*(R^*)$, even in the absence of the noetherian or integrally closed requirements on R^*. Of course, if we add the topological input that $R^* = H^*(X)$ for some space X, then by D-M-W, [**2**], in fact $D^*(R^*) = H^*(BT^n)$ for some n.

The author would like to thank S. Mitchell and R. Stong for their interest and correspondence on this problem. The statement of Theorem 3 is due to them, and their questions led to the formulation of Theorem 2 from an earlier version of the results of Section Two. The author would also like to thank J.F. Adams for helpful comments on the organization of the proofs.

1. The Maximal Separable Extension

Our technique is to apply field level arguments, and then use the $Un-$functor of taking the subalgebra of unstable elements to recover ring level results. A form of $Un-$functor appeared in Wilkerson, [**20**]. It has also found use in [**1**], [**3**], and [**4**].

PROPOSITION 1.1. *Let M^* be an evenly graded module over A_p. Define the graded vector space $Un(M^*)$ as*

$$Un(M^*)_{2i} = \{m \in M_{2i} | P^j m = 0, \forall j > i\}$$

Then

a) $Un(M^*)$ is closed under the A_p- action on M^*, and is an unstable A_p-module.

b) If M^* is a localization of some unstable A_p-algebra R^*, then $Un(M^*)$ is an unstable A_p-algebra.

c) $Un(FS[V^{\#}]) = S[V^{\#}]$

d) If R^* is an unstable integral domain, then $u \in FR^*$ and u integral over R^* implies that $u \in Un(FR^*)$.

LEMMA 1.2. *If R^* is a graded unstable integral domain of finite transcendence degree with an unstable A_p-action, then R^* is integrally closed and finitely generated as an algebra if and only if $R^* = Un(FR^*)$, the set of unstable elements in the graded field of fractions of R^*.*

PROPOSITION 1.3. *Let R^* be an A_p-sub-algebra of $S[V^{\#}]$ such that $S[V^{\#}]$ is algebraic over R^*. Define S^*_R as the set of elements in $S[V^{\#}]$ which are separable over R^*. Then S^*_R is a A_p-sub-algebra of $S[V^{\#}]$ which contains R^*. In addition,*

a) The fraction field FS^*_R of S^*_R is the maximal subfield I^* of the fraction field of $S[V^{\#}]$ which is separable over R^*.

b) $R^* \to S^*_R$ is separable, and S^*_R is maximal for this property.

c) $S^*_R = Un(FS^*_R) = Un(I^*)$ and hence is integrally closed and noetherian.

d) $S^*_R \to S[V^{\#}]$ is purely inseparable (for each x in $S[V^{\#}]$ there exists an s such that $x^{p^s} \in S^*_R$).

PROOF. Proof of Prop. 1.3 □

Let I^* be the maximal separable extension of FR^* in $FS[V^{\#}]$. Then $S^*_R = Un(I^*)$, since $Un(FS[V^{\#}]) = S[V^{\#}]$. It follows that S^*_R is maximal among subalgebras of $S[V^{\#}]$ separable over R^*. By I.1, S^*_R is integrally closed and noetherian, since $S^*_R = Un(FS^*_R)$. Now if $u \in S[V^{\#}]$, there exists $N \geq 0$ so that $u^{p^N} \in I^*$. Since this power of u is unstable, it is actually in S^*_R.

It remains to show that $FS^*_R = I^*$. Let u in $FS[V^{\#}]$ be separable over R^*, with minimal separable equation $g(u) = 0 = a_0 u^N + \ldots + a_N$ with a_i in R^*. Multiply by a_0^{N-1} to obtain an integral equation for $a_0 u = x$ over R^*. Then x is in $S[V^{\#}]$, since it is integral over R^*. Also, x is separable over R^*. Hence $x \in Un(I^*) = S^*_R$. Thus putting $y = a_0$, we obtain $u = x/y$, with x and y in S^*_R. That is, $FS^*_R = I^*$.

PROOF. Proof of Lemma 1.2 □

The favor of these arguments is similar to those in Wilkerson, [**20**]. From [**1**], R^* is noetherian if and only if $S[V^{\#}]$ is integral over R^* via t_R. Also, since $Un(FS[V^{\#}]) = S[V^{\#}]$, for any subfield L^*, $Un(L^*) = L^* \cap S[V^{\#}]$.

Now assume $R^* = Un(FR^*)$. If $u \in FR^*$ is integral over R^*, then $u \in Un(FR^*) = R^*$. That is, R^* is integrally closed. Next if $y \in V^{\#}$ has $x = v^{p^N}$ separable over FR^*, x satisfies a minimal equation of the form

$$X^r + c_1 X^{r-1} \ldots + c_r = 0$$

with the coefficients $\{c_j\}$ in FR^*. But since the extension is normal, and has one root x in $S[V^{\#}]$, it factors completely, with all roots in $S[V^{\#}]$. Hence the coefficients $\{c_i\}$ are unstable, since they are polynomials in the roots. Thus x is integral over $Un(FR^*) = R^*$, $V^\#$ is integral over R^*, and so is $S[V^\#]$. From property a) of the introduction, R^* is finitely generated.

Conversely, assume that R^* is noetherian and integrally closed. We need to show that $R^* = Un(FR^*)$. Let $u \in Un(FR^*)$. Then $u \in S[V^\#]$, and hence is integral over R^* since R^* is noetherian. Hence u is in the integral closure of R^* in its field of fractions. But we assumed that R^* is integrally closed, so $u \in R^*$. That is, $Un(R^*) \subseteq R^*$.

Proof. Proof of Proposition 1.1 □

If the action of the Bockstein were non-zero, the appropriate definition of an "unstable" element would be more complicated. The concept of "unstable" involves recursion, since if u is unstable, one also wants θu to be unstable, for any $\theta \in A_p$. However, Lemma 2.6 from [**1**], shows that this recursion is automatic if only the reduced power Steenrod operations are considered. If M^* is an algebra over A_p, then $Un(M^*)$ is also an algebra over A_p, and of course, $Un(M^*)$ is also an unstable module. However, $Un(M^*)$ need not be an unstable algebra, since this requires also the condition $P^i m_{2i} = m^p$ for each m. Although this is true for case b) , see [**20**], it is not true in general.

Finally, let $u = x/y$ in FR^*. Let

$$u^N + c_1 u^{N-1} \dots c_0 = 0$$

be the monic equation of integrality, so that $c_i \in R^*$. Apply the total Steenrod operation P_T to the equation. Multiply through by $P_T(y)^N$, and compare coefficients of powers of T. One sees that the coefficients for $P_T(u)$ vanish above half the dimension of u. A form of this argument appears in Wilkerson, [**20**].

2. Jacobson Differential Correspondence

The proof of Theorem 2 requires a study of the subfields intermediate between the fraction field $FS[V^\#]$ of $S[V^\#]$ and some p^s-th power of this field. The setting of inseparable field extensions has has a rich algebraic structure, and there is a large established theory, see Winter, Chapters 5 and 6 for a survey. Let $K \to L$ be an inseparable extension – the theory revolves around the correspondence between the subfield K and the subring $End_K(L)$ of additive functions from L into itself which are K-linear. In the case of our interest $L = FS[V^\#]$, the $L-$span of the Steenrod operations form a large part of the endomorphism ring. One in principle could work through the general theory, and prove Theorem 2 by characterizing the endomorphisms of $S[V^\#]$ which are linear over R^* as those which are linear over the diagonal algebra $D^*(R^*)$. This would be in the spirit of section 5 of [**1**] and indeed was the original intent of this paper.

However, we can apply a simpler theory. The general approach to inseparable Galois theory sketched above was designed to generalize to larger exponents the exponent 1 correspondence of Jacobson, ([**7**], Volume III, Chapter 4, page 186):

Jacobson Differential Correspondence

Let L be a field of characteristic p. There is a 1-1 correspondence between
1) subfields K of L which contain L^p and which have finite codimension in L ($dim_K(L) < \infty$),

and
2) finite dimensional L-subspaces of Der(L) which are closed under commutators and p-th powers (L-restricted Lie subalgebras of Der(L)).

The correspondence is

$$K \rightsquigarrow Der_K(L)$$

and

$$\mathfrak{B} \rightsquigarrow L_{\mathfrak{B}},$$

the constant field of $\mathfrak{B}$*. Here* $dim_K(L) = p^{dim_L Der_K(L)}$

The present proof of Theorem 2 uses an induction step provided by the treatment of p-th powers by the Steenrod algebra, together with a version of the results of section 5 of [**1**]], interpreted by the Jacobson Differential Correspondence. Recall that the Milnor primitive $\{P^{\Delta(i)}\}$ from A_p acts as a derivation of degree $2(p^i - 1)$ on any algebra over A_p, [**1**].

PROPOSITION 2.1. *Suppose* $S[V^{\#}]^{p^N} \to R^* \to S[V^{\#}]$*, are monomorphisms of unstable* A_p-*algebras, for some* $N \geq 0$*, and that* $R^* = Un(FR^*)$*. Define* $M^* = R^* \cap (S[V^{\#}])^p$*, so* $(FR^*)^p \to FM^* \to FR^*$*. Then*
a) The Milnor primitives $\{P^{\Delta(i)}\}$ *span the graded derivations of* FR^* *into* FR^* *which vanish on* FM^**,* $Der_{FM^*}(FR^*)$*.*
b) $Rank_{FR^*} Span_{FR^*}\{P^{\Delta(i)}\} = rank_{F_p}(V \cap R^*) = n_R$*.*
c) $dim_{FM^*}(FR^*) = p^{n_R}$

The A_p–action is crucial at this point. For example, consider the elements $x = t_1^2, y = t_2^2, u = t_1t_2$ in $\mathbf{F}_2[t_1, t_2]$. Let R^* be the sub-algebra generated by these elements. Then R^* is not closed under the Steenrod operations, but it is finitely generated and integrally closed. However, $V \cap R^* = 0$ even though $rank_{FR^*} Der_{FM^*}(FR^*) = 1$. In general, there are many more intermediate rings between $S[V^{\#}]^p$ and $S[V^{\#}]$ than those predicted by Proposition 2.1. The Steenrod action forces a more "linear" structure than the general theory can see.

LEMMA 2.2. *If* $p = 2$ *or we restrict to evenly graded objects, then given an* $\mathbf{F}_p-$*restricted graded Lie algebra* $\mathfrak{L}$ *and an action of* $\mathfrak{L}$ *on a commutative graded field* K*, the tensor product* $K \otimes_{\mathbf{F}_p} \mathfrak{L}$ *is a graded* $K-$*Lie algebra closed under brackets and* $p-$*th powers, and it inherits an action on* K*.*

PROOF. Proof of Prop. 2.1.a By 2.2, the FR^*-span of $\{P^{\Delta(i)}\}$ in $Der(FR^*)$ is a FR^*-restricted Lie subalgebra. Since by [**1**] the Milnor primitives span the derivations of $FS[V^{\#}]$, an element of $FS[V^{\#}]$ is a $p-$th power in $FS[V^{\#}]$ if and only if it is annihilated by each $P^{\Delta(i)}$. That is, the constant field of the FR^*-span of $\{P^{\Delta(i)}\}$ acting on FR^* is FM^*, the intersection of FR^* with the $p-$th powers from $FS[V^{\#}]$. By the Jacobson Correspondence, the Milnor primitives span $Der_{FM^*}(FR^*)$. □

PROOF. Proof of Proprositions 2.1.b and 2.1.c

By 5.1 of [**1**], there exists for R^* an n_R such that any distinct n_R of the $\{P^{\Delta(i)}\}$ are linearly independent over FR^* in $Der(FR^*)$, and any $n_R + 1$ are linearly dependent. This holds even if the "grading" derivation $\tilde{P}^0$ is included. Here $\tilde{P}^0 x_{2n} = nx$. Hence

$$Rank_{FR^*} Span_{FR^*}\{P^{\Delta(i)}\} = n_R.$$

For any non-trivial equation of linear dependence with $n_R + 1$ terms

$$\delta = c_0 \tilde{P}^0 + \ldots + c_r P^{\Delta(n_R)}$$

all of the coefficients $\{c_i\}$ are non-zero, since otherwise the choice of n_R would be contradicted. If $v \in V_0 = V \cap R^*$, then $\delta v = 0 = c_0 v + \ldots + c_r v^{p^r}$ for $r = n_R$. Hence $dim_{\mathbf{F}_p} V_0 \leq n_R$. However, by 5.2.(i-ii) of [**1**], all solutions of $\{c_0 X + \ldots + c_r X^{p^r}\}$ are in $S[V^{\#}]$. Hence $dim_{\mathbf{F}_p} V_0 = n_R$, since the equation has no repeated roots. $\square$

PROOF. Proof of 2.2

One takes the natural bracket definition :

$$[aD, b\delta] = ab[D, \delta] + aD(b)\delta - b\delta(a)D$$

and extends by bilinearity, keeping the action of $\mathfrak{L}$ on K in mind. The $p-$th power operation requires more work. In fact there are two non-trivial formulas which are needed:

1) Given $a \in K$ and $D \in \mathfrak{L}$, there exist $b, c \in K$ so that $(aD)^p = bD + cD^p$.

This is attributed to Hochshild in exercise E.5.10, page 125 of Winter, [**21**]. This allows us to define the $p-$th power on monomials from $K \otimes \mathfrak{L}$.

2) If $a, b \in K$ and $D, \delta \in \mathfrak{L}$, we need a formula for $(aD + b\delta)^p$ in terms of brackets and $p-$th powers. Then induction on the number of summands in $\Sigma a_i D_i$ provides a general definition of the $p-$th power map on $K \otimes \mathfrak{L}$. The needed formula is provided in Jacobson, [**6**], page 187:

In the free associative algebra $\mathbf{F}_p < X, Y >$,

$$(X + Y)^p = X^p + Y^p + \sum_1^{p-1} S_i(X, Y)$$

where the S_i are (non-commutative) polynomials expressible in terms of iterated commutators in X and Y. $\square$

Notice that even if the original Lie algebra $\mathfrak{L}$ had zero $p-$th powers and zero Lie brackets (as in the case of application to the Lie algebra spanned by the $\{P^{\Delta(i)}\}$), the same is not necessarily true for the semi-tensor product Lie algebra constructed via these formulas.

REMARK 2.3. If the reader prefers an absolutely minimal path to the proof of Theorem B, the crucial fact established in this section is that

$$dim_{FM^*}(FR^*) = p^{dim_{\mathbf{F}_p}(V \cap R^*)}$$

This can be deduced in the case above from a result of Gerstenhaber and Zaromp quoted in exercise E.5.9 of [**21**], using the derivations $\{P^{\Delta(i)}\}$ for their $\{D_i\}$. The crucial property needed is closure under the $p-$th power map, and this is clear for the $\{P^{\Delta(i)}\}$

3. Proof of Theorem 2 in the purely inseparable case

By the results of Section One, we can replace the arbitrary R^* by its maximal separable closure S_{R^*}. This leaves some details about the automorphism groups to be sorted out in Section Four, but the harder work is here, to show that if the extension $R^* \to S[V^\#]$ is purely inseparable and $R^* = Un(FR^*)$, then $R^* = D^*(R^*)$. The method of proof is an induction using the results of Section Two.

Recall that the exponent of an inseparable extension is the smallest integer e such that for any x in $FS[V^\#]$, x^{p^e} is in FR^*.

The induction will be on the exponent $e(R)$ of the inseparable extension $FR^* \to FS[V^\#]$. The exponent 0 case is covered by the hypothesis $R^* = Un(FR^*)$, while the exponent 1 case is essentially Proposition 2.1.

Exponent 1 Case: $(FS[V^\#])^p \to FR^* \to FS[V^\#]$. The filtration is $U_0(R^*) = V \cap R^*$ and $U_1(R^*) = V$. We have

$$(S[V^\#])^p \to D^*(R^*) \to R^* \to S[V^\#]$$

But by Proposition 2.1, the dimension of F^*R^* and that of $FD^*(R^*)$ over $FS[V^\#]^p$ are each p^{n_R}. Hence the two fields coincide.

Special Case: If R^* is entirely contained in $S[V^\#]^p$, we can replace R^* by the algebra of its p-th roots in $S[V^\#]$, $\sqrt[p]{R^*}$. $e(\sqrt[p]{R^*} = e(R) - 1$ and $\sqrt[p]{R^*} = UnF\sqrt[p]{R^*}$ so the induction hypothesis gives that $\sqrt[p]{R^*} = D^*(\sqrt[p]{R^*})$. Since $(\sqrt[p]{R^*})^p = R^*$ as A_p−algebras, $R^* = (D^*(\sqrt[p]{R^*}))^p = D^*(R^*)$.

Induction Step: If $V_0 = V \cap R^* \neq 0$, then we use $R^{*''} = R^* \cap S[V^\#]^p$. We need to show first that $R^{*\prime\prime} \to D^*(R^*)$. Now $1 \leq e(R'') \leq max(e(R), 1)$. If it is strictly less than $e(R)$, we appeal to the induction hypothesis to conclude that $R^{*''} = D^*(R^{*''})$ If the exponent is still $e(R)$, we obtain the same isomorphism from the special case conclusion. Thus we obtain $R^{*''} \to D^*(R^*)$. Then we have

$$R^{*^p} \to R^{*''} \to D^*(R^*) \to R^*,$$

and we must show that $D^*(R^*) = R^*$.

We again appeal to Proposition 2.1. Since $U_0 = V \cap R^* = V \cap D^*(R^*)$ by definition of $D^*(R^*)$, we have

$$dim_{FR^{*''}}(FD^*(R^*)) = dim_{FR^{*''}}(FD^*(R^*))$$

Thus $FD^*(R^*) = FR^*$ and $R^* = D^*(R^*)$.

Remark 3.1. The use of the Jacobson Correspondence in Proposition 2.1 to compute the relative derivations $Der_{FR^{*''}}(FR^*)$ is very helpful. If the full Lie algebra of derivations $Der(FR^*)$ were known, one could seek to explicitly compute $Der_{FR^{*''}}(FR^*)$. However, while it is true that $Der(FR^*)$ can be spanned by linear combinations of Steenrod operations, more than just the Milnor primitives are required in general.

For example, if

$$R^* = F_2[t_1, t_2^2] = \mathbf{F}_2[x, y]$$

then the standard partial derivative basis for $Der(FR^*)$ is given by

$$\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} = \begin{bmatrix} x^{-2}Sq^1 \\ (y^4 + x^4y^2)^{-1}(Sq^{(0,2)} - x^4Sq^2) \end{bmatrix}$$

in terms of Steenrod operations. In this case, the constant field is $(FR^*)^p$, $R^{*''} = \mathbf{F}_2[x^2, y]$, and

$$Der_{FR^{*''}}(FR^*) = FR^*\{Sq^1\} = FR^*\{\partial/\partial x\},$$

by direct computation. However, by Proposition 2.1 one achieves the same result by observing that with $V \cap R^* = \{0, x\}$, any Milnor primitive, since it is non-zero on x will span the relative derivations. Such direct computation has its drawbacks for more complicated examples.

4. Proofs of Theorems 2, 3, and 5

We are left to provide proofs for those statements in Theorem 2 involving the automorphism groups. It is not obvious that the automorphisms computed in various possible ambient categories should agree. The essential point is that these are relative automorphisms that do preserve all structures on a large sub-object. Again these arguments are similar to those in Wilkerson, [**20**].

PROPOSITION 4.1. *Let* R^* *be a sub-*A_p*-algebra of* $S[V^\#]$ *such that* $S[V^\#]$ *is algebraic over* R^**. Let* L^* *be a graded field in* $FS[V^\#]$*. Then*

a) If $i : FR^* \to L$ *is a separable extension of graded fields, then* L *has a unique* A_p *action respecting the Cartan formula and such that* i *is a map of* A_p*-algebras. Thus any automorphism of* L^* *which is the identity on* R *respects this* A_p*-structure.*

b) If $j : FR^* \to FS[V^\#]$ *is purely inseparable, then any graded field automorphism of* $FS[V^\#]$ *which restricts to an* A_p*-automorphism of* FR^* *is itself already an* A_p*-automorphism of* $FS[V^\#]$*.*

Hence, for $S_{R^*} = D^*(R^*)$ *as in Section Three*

c) $Aut_{R^*}(D^*(R^*)) = Aut_{FR^*}(FD^*(R^*))$*.*

d) $Aut_{D^*(R^*)}(S[V^\#]) = Aut_{FD^*(R^*)}(FS[V^\#]) = Id$ *and*

e) $Aut_{R^*}(S[V^\#]) = Aut_{FR^*}(FS[V^\#]) = Aut_{R^*}(D^*(R^*))$ *and these preserve Steenrod operations.*

PROPOSITION 4.2. *Let* $U_0 \subseteq U_1 \ldots \subseteq V$ *be an increasing filtration of* $V^\#$*, and* $D*$ *the associated diagonal algebra. Then the following groups coincide:*

a) $\mathbb{G}_1$ *: the subgroup of* $GL(V^\#)$ *which respects the filtration.*

b) $\mathbb{G}_2$ *: the gradation respecting, Steenrod action preserving algebra automorphisms of* D^**.*

3) $\mathbb{G}_3$ *: the gradation respecting, Steenrod action preserving field automorphisms of* FD^**.*

We remark that for $S[V^\#]$ itself the automorphism groups in several different plausible categories coincide. However this property is not inherited for the "diagonal" subalgebras D^*:

EXAMPLE 4.3. A gradation preserving algebra automorphism of D^* need not respect the Steenrod algebra action.

Details of 4.3:
Let $D^* = \mathbf{F}_2[t_1, t_2, t_3^2]$, and define $\psi(t_1) = t_1, \psi(t_2) = t_2$, but $\psi(t_3^2) = t_3^2 + t_1 t_2$. Extend this to all of D^*. Then ψ does not commute with Sq^1 on t_3^2.

Thus Theorem 4.1.a has non-trivial content. If one observes that $D^*(R^*)$ is also a Hopf-algebra, then it is true that Hopf-algebra automorphisms of $D^*(R^*)$ respect the Stennrod algebra action.

Proof. Proof of Theorem 2

We first observe that if R^* is replaced by its maximal separable extension in $S[V^\#]$, $S^*_{R^*}$, then Section II proves Theorem 2 in this special case, with $W(S^*_{R^*}) = \{Id\}$. However, it is clear that the filtrations and diagonal subalgebras defined by R^* and $S^*_{R^*}$ agree. It remains to check that the action of $W(R^*)$ preserves the filtration on V. But if y^{p^i} is separable over R^*, then any Galois conjugate of it is also separable over R^*. The identification of the various possible definitions of the automorphism groups is done in 4.1. ☐

Proof. Proof of Theorem 3

Let $R^* = \mathbf{F}_p[y_1, \ldots, y_n]$ be the polynomial algebra, and $D^*(R^*)$ its maximal separable extension in $S[V^\#]$. Any derivation of R^* or its fraction field FR^* into $FD^*(R^*)$ extends uniquely to a derivation on $FD^*(R^*)$, since the extension is separable. More explicitly, if $\delta \in Der(FR^*)$ and $\alpha \in FD^*(R*)$ has minimal separable equation over FR^* of

$$f(\alpha) = \alpha^N + b_1\alpha^{N-1} + \ldots + b_0$$

then

$$\delta\alpha = -(\sum \alpha^{N-i}\delta b_i)/f'(\alpha)$$

Since f is separable, there are no repeated roots and $f(\alpha) \neq 0$. In the case at hand of polynomial algebras, the partial derivatives in $\{y_i\}$ and $\{z_j\}$ form bases respectively for $Der(FR^*)$ and $Der(FD^*(R^*))$. But from the above $\{\partial/\partial y_i\}$ is also a basis for $Der(FD^*(R^*))$ over $FD^*(R^*)$. The Jacobian $|\partial y_i/\partial z_j|$ is the determinant of the change of basis matrix for these two bases of $Der(FD^*(R^*))$ over $FD^*(R^*)$, and hence must be non-zero. ☐

Proof. Proof of Prop.4.2 Certainly, $G_1 \subseteq G_2 \subseteq G_3$ We know that $D^* = Un(FR^*)$. Since G_3 induces an automorphism of D^*, evidently $G_3 \approx G_2$. But the induced automorphism on D^* extends uniquely to $S[V^\#]$, by the rule that if $v^{p^N} \in D^*$, then $\psi(v) = (\psi v^{p^N})^{1/p^N}$. Since $p-$th roots are unique if they exist, this extension is well defined, since the question of whether an element is a $p-$th power is detectable by the Steenrod action. Hence $G_1 \approx G_2$. Since $Un(FD^*) = D^*$, one has $G_2 = G_3$ immediately. ☐

Proof. Proof of Theorem V

Statement 1) is directly from Lam, [**9**]

For 2) consider the $p-$th root closure of $(R^*)''$ in $S[V^\#]$. That is, $y \in (R^*)''$ if and only if there exists some $N \geq 0$ with $y^{p^N} \in R^*$. Then by definition, $(S[V^\#])^p \cap (R^*)'' \subseteq ((R^*)'')^p$ and 1) applies. That is, $c_0 \in (R^*)''$, and there exists $N \geq 0$ with $c_0^{p^N} \in R^*$. This argument holds for any non-zero ideal of R^* which is closed under the Steenrod algebra action also.

To prove 3), let $\overline{R^*} = Un(FR^*)$. If $y \in \overline{R^*}$ consider the conductor ideal $C(y, R^*) \subseteq R^*$ consisting of those elements $r \in R^*$ for which $ry \in R^*$. Let $S(y, R^*)$ be the radical of this ideal, that is, the set of all elements which have a power in $C(y, R^*)$. Finally, define

$$S(\overline{R^*}, R^*) = \bigcap\nolimits_{y \in \overline{R^*}} S(y, R^*)$$

Then we claim that $S(\overline{R^*}, R^*)$ is a non-zero ideal in R^* which is closed under the action of the Steenrod algebra. If so, then by part 2) of this theorem, some power of c_0 falls in $S(\overline{R^*}, R^*)$. That is,

$$S_z^{-1}\overline{R^*} = S_z^{-1}R^*$$

It remains to check that $S(\overline{R^*}, R^*)$ is closed under the Steenrod algebra action. It suffices to check this for each $S(y, R^*)$. (this need not be true for each $C(y, R^*)$). Let $|y| = 2M$, and $r \in C(y, R^*)$. Now use the Cartan formula to compute

$$P^{ip^M} r^{p^M} y = (P^{ip^M} r^{p^M})y + \sum_{j>0} P^{jp^M} r^{p^M} P^{(i-j)p^M} y = (P^i r)^{p^M} y$$

The sparseness of the first expansion is due to the treatment of $p-$th powers by the Steenrod algebra. But moreover, since y is unstable of dimension $2M$, each term on the right hand side except the first vanishes, since $(i-j)p^M > M$. This demonstrates that if $r \in C(y, R^*)$, then $P^i r \in S(y, R^*)$, and by replacing r by a suitable $p-$th power, that $P^i S(y, R^*) \subseteq S(y, R^*)$.

Finally, to show that $S(\overline{R^*}, R^*) \neq 0$, since $\overline{R^*}$ is notherian, we can choose a finite set $\{y_k\}$ of algebra generators for $\overline{R^*}$. Then for each k, there exist $w_k, x_k \in R^*$ such that $y_k = x_k/w_k$. Then the element

$$\Delta = \prod w_k$$

is in $S(\overline{R^*}, R^*)$. □

5. Discussion and Examples

In this section, we make some comments on the algebraic classification of unstable domains and give some illustrations of the filtration structure pointed out by Theorem 2.

The special case of polynomial algebras has great historical interest. The application of Theorem 2 to this case is a success, since Theorem IV from [D-M-W] gives very strong restrictions on such algebras to be topologically realizable. In particular, the filtration produced in Theorem 2 must degenerate to $U_0(R^*) = V$ for odd primes p.

However from a strictly algebraic viewpoint, there is still a nagging question as to whether the property of the W-action having a polynomial algebra as its ring of invariants is inherited by the action on the diagonal algebras $D^*(R^*)$, and vice versa. More precisely,

CONJECTURE 5.1. *Let W be a subgroup of $GL(V^\#)$ such that W restricts to an action on D^*, a diagonal subalgebra of $S[V^\#]$. Then $(D^*)^W$ is a polynomial algebra if and only if $S[V^\#]^W$ is a polynomial algebra.*

This conjecture is easily verified if the order of W is prime to p, since in that case the ring of invariants is a polynomial algebra if and only if W is generated by generalized reflections.

Without attempting to restrict to just polynomial algebras, Theorem 2 provides give a construction and classification of all A_p - integral domains which are noetherian and integrally closed. One can view this in two steps:

1) Choose a particular subgroup $W \subseteq GL(V)$. This generates the separable example as $S[V^\#]^W$

2) The inseparable forms of this algebra W are are then parameterized by W-invariant filtrations of V and choices of exponents, or equivalently, a “diagonal”

algebra D^* such that the action of W on $S[V^{\#}]$ restricts to D^*. So the inseparable form is $R^* = (D^*)^W$.

Notice that if the representation of W has no proper invariant subspaces, then there are always only the "standard" copies $(S[V^{\#}]^{p^N})^W$ associated with W. Therefore, non-trivial examples of Theorem 2 must use non-trivial $W-$ filtrations on V.

EXAMPLE 5.2. If one restricts the search to classical rationally irreducible Weyl groups, the examples are rather restricted. Various forms of G/C', where C' is a finite central subgroups, give non-trivial W filtrations. There are two interesting examples for which the filtrations do not involve a trivial submodule in the associated graded. These occur for $W(G_2)$ at the prime 3 and $W(F_4)$ at the prime 2. We treat the G_2 example in detail. For $p = 3$, this Weyl group provides an example of a proper invariant subspace which has a non-trivial W-action. The Weyl group of G_2 is the dihedral group D_{12} of order 12. The mod 3 cohomology of BG_2 is a ring of invariants which provides a good illustration of two possible viewpoints of the Steenrod algebra action. We can compare the internal view provided by the explicit Steenrod algebra action with the external view provided by the Adams-Wilkerson embedding and its associated Weyl group action. The filtration is easiest to see in terms of the W-action, but it is also visible indirectly in the formulas detailing the Steenrod algebra action.

We take the repesentation of D_{12} to be determined by the two reflections

$$\alpha = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

in $GL_2(\mathbb{Z})$.

So over F_3, there are generators for the invariants

$$x_4 = t_1^2,$$

and

$$< y_{12} = (t_2(t_1+t_2)(t_1-t_2))^2 > .$$

There is only one nontrivial stable subspace for D_{12}, $\{0, t_1, -t_1\}$, and it has no D_{12} complement. This filtration corresponds to the family of A_3-polynomial algebras parametrized by $N, M \geq 0$

$$(F_3[x_4, y_{12}^{3^N}])^{3^M}$$

in

$$F_3[x_4, y_{12}] = H^*(BG_2, F_3).$$

Here the A_3 -actions can be specified as

$$P^1 x = -x^2, P^{3^N} y^{3^N} = x^{3^N} y^{3^N}, P^{3^{N+1}} y^{3^N} = y^{3^N}(x^{3^{N+1}} - y^{3^N}).$$

and analogues for $M > 0$. On the other hand one sees from these formulas that one can not introduce a x^{3^M} instead of x without also substituting y^{3^M} for y. That is,

$$F_3[x^{3^M}, y^{3^N}]$$

is not closed under the A_3 -action if $N < M$. This is reflected on the vector space level by the fact that there is only one proper invariant subspace. In particular, there is no $W-$splitting of the filtration in this case.

For G_2 for $p > 3$, any proper invariant subspace would also be a direct summand, but in fact, there are no invariant 1 dimensional subspaces for $p > 3$. Hence for $p > 3$ the only inseparable forms of $F_p[x_4, y_{12}]$ have the form $F_p[x_4, y_{12}]^{p^N}$, for N non-negative.

The matrices of the $W(G_2)$ action at the prime 3 can be pictured as

$$\begin{bmatrix} GL_1(\mathbf{F}_3 & \mathbf{X} \\ \mathbf{0} & GL_1(\mathbf{F}_3) \end{bmatrix}$$

in which $\mathbf{0}$ and $\mathbf{X}$ are (1×1) $\mathbf{F}_3$-matrices.

The related $W(F_4)$ example at $p = 2$ has a similar description as

$$\begin{bmatrix} GL_2(\mathbf{F}_2) & \mathbf{X} \\ \mathbf{0} & GL_2(\mathbf{F}_2) \end{bmatrix}$$

in which $\mathbf{0}$ and $\mathbf{X}$ are (2×2) $\mathbf{F}_2$-matrices. The analysis is quite similar to the G_2 example. One could extend these examples to other primes p and n, but these two are natural because of the Lie connection.

Example 5.3. We now sketch the classification of noetherian integrally closed unstable domains of rank 2 in $\mathbf{F}_2[t_1, t_2]$. We define some useful classes in this ambient polynomial ring in order to ease the notation later.

$$w_1 = t_1 + t_2, w_2 = t_1 t_2$$

are the Steifel-Whitney classes, and the Dickson invariants are

$$c_0 = t_1 t_2 (t_1 + t_2), c_1 = t_1^2 + t_2^2 + t_1 t_2$$

Finally

$$u = t_1^3 + t_2^3 + t_1^2 t_2$$

.

We now provide the classification by listing the Weyl groups, filtrations, and exponents that are possible for a given group.

1) $W = (id)$. All subspaces are invariant. The algebras contructed by choices of filtrations and exponents are isomorphic to $\mathbf{F}_2[t_1^{2^i}, t_2^{2^j}]$.

2) $W = Z/2Z$. Up to conjugation, we can use the representation

$$\tau = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

There is one non-trivial subspace, spanned by w_1 . Hence the examples corresponding to $0 \subset V_0 \subset V$ and a choice of exponents N and M give rise to algebras isomorphic to

$$\mathbf{F}_2[w_1^{2^N}, w_2^{2^M}] \subseteq \mathbf{F}_2[w_1, w_2]$$

with $M \geq N$.

3) $W = Z/3Z$. Up to conjugation, we can use the representation

$$\rho = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

There are no non-trivial $W-$invariant subspaces. In this case the invariants are not polynomial. In fact,

$$R^* = S[V^{\#}]^W = \mathbf{F}_2[c_0, c_1, u]/(u^2 + c_0 u + c_0^2 + c_1^3).$$

Any inseparable examples have the form $(R^*)^{2^N}$.

4) $W = GL(V)$ and we can use as generators for the representation the matrices for ρ and τ in 2) and 3) above. Again, there are no proper invariant subspaces, so the only inseparable forms are $\mathbf{F}_2[c_0, c_1]^{2^N}$.

From the inseparability viewpoint, only 1) and 2) are interesting. The reader might wish to attempt this classification of rank 2 unstable polynomial algebras over $\mathbf{F}_2$ by direct computation instead of this contruction suggested by Theorem 2.

One special case of 1) and 2) that arises in the classification of the equivariant cohomology rings associated to involutions on cohomology projective planes is

Proposition 5.4. *If the polynomial algebra* $\mathbf{F}_2[x_1, y_m]$ *with* $|x_1| = 1, |y_m| = m$ *has an unstable* A_2-*action then* $m = 2^N$ *and either*

a)

$$\mathbf{F}_2[x_1, y_{2^N}] \approx \mathbf{F}_2[t_1, (t_2)^{2^N}], |t_j| = 1$$

or

b)

$$\mathbf{F}_2[x_1, y_{2^N}] \approx \mathbf{F}_2[w_1, (w_2)^{2^{N-1}}] \subseteq H^*(BO(2))$$

In case a) $Sq^k y = 0, 0 < k < 2^N$

In case b) $Sq^k y = 0, 0 < k < 2^{N-1}$ *and* $Sq^{2^{N-1}} y = x^{2^{N-1}} y$

Of course, the formulas for the Steenrod operations in case b) can be perturbed slightly if one replaces the generator y with $y + x^m$.

Historical Notes:

This paper was originally accepted by J. F. Adams for the Proceedings of the Cambridge Philosophical Society. In the aftermath of his untimely death, the paper slipped through the cracks at the author's end and the final copy was never delivered to the journal.

On the other hand, it has been available as a preprint since 1984 and in electronic form on the Hopf Archive since 1991. The paper has been referenced, e.g. (Mitchell-Stong, Aguade, Dwyer-Miller-Wilkerson), and extensively excerpted in the book of Kane, [**8**]. Related material in a more general context has been developed by M. Neusel [**13, 14**].

The localization approach is related to the Lannes T - functor, e.g. [**4**]. One might therefore ask if there are alternative T-theoretic proofs of these theorems. To the best of the author's knowledge, there are not easy T-replacements for the totality of these. However, certainly, there are nice restatements and interpretations of most of Theorem 2.

The author has resisted making substantive changes to the paper of record on Hopf. One concession was to change the original notation for $H^*(BT^n, \mathbf{F}_p)$ to match that of the recent paper of Dwyer and Wilkerson, [**5**], $S[V^\#]$. A second has been to supply more comments in this last section.

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