

EXACTNESS OF HOMOTOPY FUNCTORS OF SPACES

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ABSTRACT. We will provide an analysis of the generalized Atiyah–Hirzebruch spectral sequence (GAHSS), which was introduced by Hakim-Hashemi and Kahn. To do so, we introduce a new class of functors, called n -exact functors, which are analogous to Goodwillie’s n -excisive functors. In the study of these functors, we introduce a new spectral sequence, the homological Barratt–Goerss spectral sequence (HBGSS), which has properties similar to those of the classical Barratt–Goerss Spectral Sequence on homotopy. We close by giving an identification of the E^2 term of the GAHSS in the case of 2-exact functors on Moore spaces.

1. INTRODUCTION

Given a functor F from the category of pointed finite CW complexes to itself and a finite CW complex X , there is a generalized Atiyah–Hirzebruch spectral sequence (GAHSS) converging to $H_*(FX)$, where FX is filtered by applying F to the CW structure of X . In [13], Hakim-Hashemi and Kahn study the GAHSS, and identify the E^2 term in the event that F is *homotopy exact*, which will be defined below. They also determine a criterion on F for determining when the spectral sequence collapses. The motivation for their work can be traced back to the famous theorem of Dold’s which states that the homology of the symmetric product $SP^k X$ depends only on the homology of the space X [6]. The GAHSS can be thought of as a rough comparison of the homology of X with the homology of FX .

This article continues the study of the GAHSS a step further. We will first classify homotopy exact functors, and then generalize the concept of exactness to define n -exact functors. This definition is similar to Goodwillie’s definition of n -excisiveness [10]. If X is a Moore space and $n = 2$, identification of the E^2 term is possible by using Baues’ quadratic tensor product [2, 3].

Unfortunately, the difficulties seen in extending the work of [13] lead to the conclusion that the filtration of the spectral sequence is too coarse to be useful, and another path must be taken to obtain further progress. The author has derived a more tractable spectral sequence containing similar information which will be studied in a future article.

Most of the results in this work originally appeared in the author’s Ph.D. thesis, conducted under the supervision of Donald W. Kahn at the University of Minnesota. The author would also like to thank Paul Goerss and Stewart Priddy of Northwestern University for helpful suggestions (mathematical and otherwise) during the writing of this article.

We begin by setting some basic notation and terminology. We will work in the category CW_* of finite pointed CW complexes and basepoint preserving maps.

2000 *Mathematics Subject Classification.* Primary 55P65, Secondary 55T25.

Let $*$ denote the zero object in \mathcal{CW}_* . We denote the space of all maps in \mathcal{CW}_* from X to Y by $\text{map}_*(X, Y)$. The space $\text{map}_*(X, Y)$ is topologized by giving it the compact–open topology. We will use $X \simeq Y$ to indicate that X and Y are homotopy equivalent.

Unless otherwise stated, we will use the word "functor" to refer to an endofunctor of \mathcal{CW}_* . We assume that all functors are continuous, meaning that the natural function

$$\text{map}_*(X, Y) \rightarrow \text{map}_*(FX, FY)$$

is continuous when the mapping spaces are given the compact–open topology. Furthermore, functors will be assumed to be reduced, that is $F(*) \simeq *$. As we are working with reduced functors in \mathcal{CW}_* , we assume that all homology theories are reduced.

We now present the GAHSS. If X is a CW complex, let X^p denote the p –skeleton of X . We may filter FX as follows:

$$* = F(X^{-1}) \subset F(X^0) \subset F(X^1) \subset \cdots \subset F(X).$$

This filtration gives rise to a filtered chain complex, which in turn yields a spectral sequence converging to $H_*(FX)$ (actually, we can get one converging to $E_*(FX)$ where E is an arbitrary generalized homology theory). If F is the identity functor, it is easy to see that we obtain the classical Atiyah–Hirzebruch spectral sequence. In fact, the filtration gives rise to the cellular chain complex for X . There are other cases when $H_* \circ F$ is itself a generalized homology theory. In these cases, we again obtain the classical AHSS. This case was studied by Hakim-Hashemi and Kahn, and requires the concept of a homotopy exact functor. We review their work now.

Let $C(f, g)$ denote the double mapping cylinder of f and g . We recall that given a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array},$$

we have a natural map $\Phi: \text{hocolim}(Y \xleftarrow{f} X \xrightarrow{g} Z) \simeq C(f, g) \rightarrow W$. If Φ is a homotopy equivalence, the square is said to be *homotopy co-Cartesian* (or simply *co-Cartesian*).

Definition 1.1. A functor F is said to be *homotopy exact* if whenever \mathfrak{X} is a co-Cartesian square, the diagram $F\mathfrak{X}$ is co-Cartesian as well.

Example 1.2. The functor $F(X) = X \wedge K$ for a fixed space K is homotopy exact. In fact, as we will see, up to natural equivalence, this is the only possibility.

Remark 1.3. We note that the concept of homotopy exactness doesn't require the functor F to be reduced. In this event, we get a number of additional examples. In particular, if we fix a space K , then $F(X) = X \vee K$ and $F(X) = X \times K$ are both homotopy exact, as is the Borel construction $F(X) = X \times_G EG$ for any group G . Details of these examples are presented in [12]. Although these examples are interesting in their own right, we will not consider them in this example.

The major result of [13] involving homotopy exact functors was

Theorem 1.4. *If F is a homotopy exact functor, then given any generalized homology theory E_* , $E_* \circ F$ is itself a generalized homology theory. Furthermore, in this case, the GAHSS for X coincides with the classical AHSS for $E_* \circ F$.*

This theorem will guide the present work in two ways. In the next section, we will identify homotopy exact functors. In subsequent sections, we will generalize the concept of homotopy exactness and analyze the GAHSS in this situation.

2. HOMOTOPY EXACT FUNCTORS

The major result of this section is the classification of homotopy exact functors. In particular, if F is homotopy exact, then F is naturally equivalent to the functor $X \wedge F(S^0)$.

We recall that we require all functors to be continuous. With this assumption, we may define a map $\eta_F: X \wedge F(Y) \rightarrow F(X \wedge Y)$ for any spaces X and Y . When it is clear from the context, we will use η to denote this map. Note that we are not yet assuming that F is homotopy exact. The map is given by $\eta(x \wedge y) = F(i_x)(y)$ where $i_x: Y \rightarrow X \wedge Y$ is the map sending y to $x \wedge y$. We first note that this map is in fact continuous.

Lemma 2.1. *η is continuous for any choice of F , X or Y*

Proof. Continuity of the functor F means that the map

$$\text{map}_*(Y, X \wedge Y) \rightarrow \text{map}_*(FY, F(X \wedge Y))$$

is continuous, and hence, so is $X \rightarrow \text{map}_*(Y, X \wedge Y) \rightarrow \text{map}_*(FY, F(X \wedge Y))$, where $X \rightarrow \text{map}_*(Y, X \wedge Y)$ is the adjoint of the identity map $X \wedge Y \rightarrow X \wedge Y$. This map obviously sends x to i_x . We notice that the composite then sends x to $F(i_x)$, and its adjoint is η . Since η is in the image of the adjunction, it must be continuous. \square

The virtue of η is that it provides a natural transformation. In particular, we see

Lemma 2.2. *The map η is natural in X , Y and F .*

Proof. This can be proved by a straight-forward diagram chase. Details are left to the reader. \square

This brings us to the main result of this section.

Theorem 2.3. *Let F be a reduced homotopy exact functor. Then the natural transformation $\eta: X \wedge FY \rightarrow F(X \wedge Y)$ is a homotopy equivalence.*

Proof. Suppose that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

is co-Cartesian. If $\tau: F \rightarrow G$ is a natural transformation of homotopy exact functors such that τ is a homotopy equivalence on X , Y and Z , then τ must be an equivalence on W as well, as we have equivalences

$$F(W) \xleftarrow{\sim} C(F(f), F(g)) \xrightarrow{\sim} C(G(f), G(g)) \xrightarrow{\sim} G(W)$$

where $\tau: F(W) \rightarrow G(W)$ is the composite.

We know that any space X in \mathcal{CW}_* can be built from $*$ and S^0 by finitely many iterations of mapping cones and wedges, so by the preceding argument, it suffices to prove the theorem for the cases $X = *$ and $X = S^0$. However, both of these cases are obvious from the definition of η . \square

By setting $Y = S^0$, we then immediately get the desired corollary.

Corollary 2.4. *A functor F is homotopy exact if and only if F is naturally equivalent to the functor $X \wedge F(S^0)$.*

Remark 2.5. We recall that a commutative square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

is *Cartesian* if the natural map $X \rightarrow \text{holim}(B \rightarrow D \leftarrow C)$ is a homotopy equivalence.

Goodwillie defines a functor L to be *linear* if whenever \mathfrak{X} is a co-Cartesian square, the square $L(\mathfrak{X})$ is Cartesian [9]. (Recall that we are assuming L reduced). We note as well that if F is homotopy exact, the composite functor $L \circ F$ is obviously linear. In particular, this holds for $Q \circ F$, where $Q = \Omega^\infty \Sigma^\infty$. Any linear functor L is known to be naturally equivalent to the functor $\Omega^\infty(X \wedge L(S))$, where S is the sphere spectrum. Additionally, $\pi_* \circ L$ is a homology theory. Thus, the preceding theorem may be regarded as an unstable version of the result on linear functors.

The practical consequence of Corollary 2.4 is that for homotopy exact functors F , the identification given in Theorem 1.4 is less useful than was initially anticipated. In fact, the GAHSS reduces to the Künneth Theorem in this case.

3. n -EXACT FUNCTORS

To make further progress in our analysis of the GAHSS, we will develop the connection between homotopy exact and linear functors seen at the end of the preceding section. In order to do so, we must initially recall some notions and results from [10].

Let S be a set. We may consider its power set $\mathcal{P}(S)$ as a poset, and hence a small category, in the usual way. A functor $\mathfrak{X}: \mathcal{P}(S) \rightarrow \mathcal{CW}_*$ is said to be an S -cube. If $S = \{1, 2, 3, \dots, n\}$, we refer to \mathfrak{X} simply as an n -cube. A 0-cube is simply a space, i.e., the functor corresponding to the set $S = \emptyset$.

Given an S -cube \mathfrak{X} , we are interested in two related functors. Let $\mathcal{P}_0(S)$ be the full subcategory of $\mathcal{P}(S)$ with objects all non-empty subsets of S , and let $\mathcal{P}_1(S)$ the full subcategory consisting of all proper subsets. Let \mathfrak{X}_i denote the composite functor $\mathcal{P}_i(S) \rightarrow \mathcal{P}(S) \rightarrow \mathcal{CW}_*$. We denote the homotopy limit of \mathfrak{X}_0 by $h_0(\mathfrak{X})$ and the homotopy colimit of \mathfrak{X}_1 by $h_1(\mathfrak{X})$. For any S -cube \mathfrak{X} , there are natural maps $a: \mathfrak{X}(\emptyset) \rightarrow h_0(\mathfrak{X})$ and $b: h_1(\mathfrak{X}) \rightarrow \mathfrak{X}(S)$. We say that \mathfrak{X} is (*homotopy*) *Cartesian* (resp. (*homotopy*) *co-Cartesian*) if the map a is a homotopy equivalence (resp. b is a homotopy equivalence). For later use, we also recall that \mathfrak{X} is k -Cartesian (resp. k -co-Cartesian) if the map a is k -connected (resp. b is k -connected).

Finally, an S -cube \mathfrak{X} is *strongly co-Cartesian* if every 2-face of \mathfrak{X} is co-Cartesian. It is easy to see that an n -cube \mathfrak{X} is strongly co-Cartesian if and only if every m -face of \mathfrak{X} with $2 \leq m \leq n$ is co-Cartesian. In particular, \mathfrak{X} itself is co-Cartesian.

We recall that Goodwillie defines a functor F to be *n -excisive* if given any strongly co-Cartesian $(n+1)$ -cube \mathfrak{X} , the diagram $F(\mathfrak{X})$ is Cartesian. It is immediate from the definition that 1-excisiveness is the same as linearity. Following this lead, we define a functor F to be *n -exact* if for any strongly co-Cartesian $(n+1)$ -cube \mathfrak{X} , the diagram $F(\mathfrak{X})$ is co-Cartesian. Homotopy exactness then corresponds to 1-exactness. We will occasionally have reason to refer to functors which aren't reduced. The definition can be used without change in that case, although some properties won't apply.

We note the following results, which may be proved in a similar manner to the analogous results in [10].

Theorem 3.1. *Let F be $(n-1)$ -exact. Then F is n -exact.*

Theorem 3.2. *Let F be a functor from $CW_*^m \rightarrow CW_*$ such that F is n_i -exact when considered as a functor in the i -th variable. If $\Delta: CW_* \rightarrow CW_*^m$ is the diagonal functor, then the composite $F \circ \Delta$ is n -exact, where $n = n_1 + \dots + n_m$.*

We note that this gives many examples of n -exact functors, as seen in this example.

Example 3.3. The following are all examples of n -exact functors.

- (1) The n -fold smash product $X^{\wedge n} = X \wedge \dots \wedge X$.
- (2) The n -fold Cartesian product X^n .
- (3) If G is a subgroup of the symmetric group Σ_n , then the functor X^n/G is n -exact. In particular, the symmetric product functor SP^n is n -exact.

Some aspects of the Goodwillie calculus are also found in n -exact functors, although no corresponding tower has been found.

Given a functor F , we will define cross-effect functors $\hat{c}_i F$ for $i \geq 1$. These functors differ from the usual cross-effect, such as the one used by Johnson [15], but they have similar properties. We will occasionally assume in the following that a functor F takes values in simply-connected spaces. If this fails to be the case, we will replace F by either $\Sigma \circ F$ or $\Sigma^2 \circ F$, whichever is required to meet the hypothesis. As our ultimate goal is the calculation of $H_*(F(X))$, this presents no problem.

Recall from Goodwillie [10] that given a sequence of spaces $\mathcal{S} = X, X_1, \dots, X_n$ and cofibrations $X \rightarrow X_i$ for $i \geq 1$, we can construct a strongly co-Cartesian n -cube $\mathfrak{X}_{\mathcal{S}}$, called a *pushout cube*. If $T \subset \{1, \dots, n\}$, we can define $\mathfrak{X}_{\mathcal{S}}(T)$ to be the union of the X_i along X for each $i \in T$. We agree to set $\mathfrak{X}_{\mathcal{S}}(\emptyset) = X$. Every strongly co-Cartesian cube admits an equivalence from a pushout cube.

To define the cross-effect, we let $n \geq 2$. Given a sequence of n spaces $\mathcal{S} = X_1, \dots, X_n$, we define a strongly co-Cartesian n -cube $\hat{\mathfrak{X}}_{\mathcal{S}}$ by setting it to be the pushout cube $\mathfrak{X}_{\mathcal{T}}$ for the sequence $\mathcal{T} = *, X_1, \dots, X_n$.

Remark 3.4. If $n = 1$, we obtain the unique map $* \rightarrow X_1$. If $n = 2$, this is simply the co-Cartesian square

$$\begin{array}{ccc}
* & \longrightarrow & X_1 \\
\downarrow & & \downarrow \\
X_2 & \longrightarrow & X_1 \vee X_2
\end{array}$$

If $\mathfrak{X} \rightarrow \mathfrak{Y}$ is a map of $(n-1)$ -cubes (i.e., a natural transformation of functors), it may be considered as an n -cube. Thus, if \mathfrak{Z} is an n -cube, we can write $\mathfrak{Z} = \mathfrak{X} \rightarrow \mathfrak{Y}$ (actually, we can do so in n different ways).

If $n \geq 2$, let \mathcal{T} be the collection of $n-1$ spaces X_1, \dots, X_{n-1} . We can then easily see that $\hat{\mathfrak{X}}_{\mathcal{S}} = \hat{\mathfrak{X}}_{\mathcal{T}} \rightarrow (\hat{\mathfrak{X}}_{\mathcal{T}} \vee X_n)$.

We may now define the cross-effect.

Definition 3.5. If F is a functor, we define the n th cross effect of F as the functor

$$\hat{c}r_n F(X_1, X_2, \dots, X_n) = \text{hocofib}(h_1(F(\hat{\mathfrak{X}}_{\mathcal{S}})) \rightarrow F(\text{hocolim } \hat{\mathfrak{X}}_{\mathcal{S}})),$$

where the given map is the natural one, and \mathcal{S} is the sequence X_1, X_2, \dots, X_n . Note that we have $\hat{c}r_1 F(X) \simeq F(X)$.

We note the following immediate facts about $\hat{c}r_n F$, which are analogous to properties of the usual cross-effect.

Recall that the *total cofiber* $\tilde{c}\mathfrak{X}$ of an n -cube \mathfrak{X} is defined iteratively. For a 0-cube $\mathfrak{X} = X$, we define $\tilde{c}\mathfrak{X}$ to be the space X . If \mathfrak{X} is an n -cube with $n \geq 1$, we may view \mathfrak{X} as a map of $(n-1)$ -cubes $\mathfrak{X}' \rightarrow \mathfrak{X}''$. The total cofiber $\tilde{c}\mathfrak{X}$ is then merely the homotopy cofiber of the induced map $\tilde{c}\mathfrak{X}' \rightarrow \tilde{c}\mathfrak{X}''$. It is known that the homotopy type of the total cofiber doesn't depend on the directions chosen to decompose \mathfrak{X} . See, for example, Goodwillie [10].

Proposition 3.6. *The cross-effect $\hat{c}r_n F$ satisfies all of the following properties.*

- (1) *Let $\mathcal{S} = X_1, \dots, X_n$. Then $\hat{c}r_n F(X_1, \dots, X_n) \simeq \tilde{c}(F(\hat{\mathfrak{X}}_{\mathcal{S}}))$.*
- (2) *The cross-effect $\hat{c}r_n F(X_1, \dots, X_n)$ is homotopy equivalent to the homotopy cofiber of the map*

$$\hat{c}r_{n-1} F(X_1, \dots, \hat{X}_i, \dots, X_n) \vee \hat{c}r_{n-1} F(X_1, \dots, X_i \hat{+} 1, \dots, X_n) \rightarrow \hat{c}r_{n-1} F(X_1, \dots, X_i \vee X_{i+1}, \dots, X_n)$$

for any $1 \leq i < n$.

- (3) *$\hat{c}r_n F$ is symmetric. That is, for any permutation σ in the symmetric group Σ_n , we then have a homotopy equivalence*

$$\hat{c}r_n F(X_1, \dots, X_n) \rightarrow \hat{c}r_n F(X_{\sigma(1)}, \dots, X_{\sigma(n)}).$$

- (4) *$\hat{c}r_n F$ is reduced in every variable.*
- (5) *If F is n -exact then $\hat{c}r_{n+1} F$ vanishes.*
- (6) *There is a natural isomorphism*

$$H_*(F(X \vee Y)) \cong H_*(FX) \oplus H_*(FY) \oplus H_*(\hat{c}r_2 F(X, Y)).$$

- (7) *Let F be n -exact with $n > 1$. Then $\hat{c}r_2 F$ is $(n-1)$ -exact in each variable. More generally, $\hat{c}r_i F$ is $(n-i+1)$ -exact in each variable. We then see that we must have $\hat{c}r_n F(X_1, \dots, X_n) \simeq X_1 \wedge \dots \wedge X_n \wedge \hat{c}r_n F(S^0, \dots, S^0)$.*

Proof. (1) and (2) are immediate from the description of the total cofiber given by Goodwillie.

(3) is obvious from the fact that the total cofiber is independent of the directions chosen to take the homotopy cofibers.

We prove (4) by induction on n . If $n = 0$, this is by assumption. If $n > 0$, we may assume by (3) that $X_1 = *$. By the description of the cross effect in (2), $\hat{c}r_n F(*, X_2, \dots, X_n)$ is given by the homotopy cofiber of the map

$$\hat{c}r_{n-1} F(*, X_3, \dots, X_n) \vee \hat{c}r_{n-1} F(X_2, \dots, X_n) \rightarrow \hat{c}r_{n-1} F(X_2, \dots, X_n).$$

By the inductive hypothesis, the summand $\hat{c}r_{n-1} F(*, X_3, \dots, X_n)$ is contractible, hence the map is a homotopy equivalence, and the homotopy cofiber is contractible, as desired.

To prove (5), take any sequence of spaces $\mathcal{S} = X_1, \dots, X_{n+1}$. If F is n -exact, then the $(n+1)$ -cube $F(\hat{\mathcal{X}}_{\mathcal{S}})$ is co-Cartesian. Thus, it has trivial total cofiber.

To show (6), we consider the following sequence of spaces, which is a homotopy cofibration sequence by the definition of cross-effect.

$$FX \vee FY \rightarrow F(X \vee Y) \rightarrow \hat{c}r_2 F(X, Y)$$

The long exact homology sequence of this cofibration sequence can be seen to split. Note that, as FX and FY are both retracts of $F(X \vee Y)$, we have a natural map $H_*(F(X \vee Y)) \rightarrow H_*(FX) \oplus H_*(FY)$ which gives the desired split exact sequence.

The proof of (7) can be found later in this section. \square

Toward the end of proving the last part of the preceding Proposition, we give a series of technical lemmas.

Lemma 3.7. *Let \mathfrak{Z} be an n -cube, with $n \geq 2$. Choose a decomposition of \mathfrak{Z} as above, and write $\mathfrak{Z} = \mathfrak{X} \rightarrow \mathfrak{Y}$, where \mathfrak{X} and \mathfrak{Y} are $(n-1)$ -cubes. \mathfrak{Z} is co-Cartesian if and only if the diagram*

$$\begin{array}{ccc} h_1(\mathfrak{X}) & \longrightarrow & h_1(\mathfrak{Y}) \\ \downarrow & & \downarrow \\ \text{colim } \mathfrak{X} & \longrightarrow & \text{colim } \mathfrak{Y} \end{array}$$

is co-Cartesian.

Proof. Assume that \mathfrak{Z} is co-Cartesian. Let $S = \{1, 2, 3, \dots, n\}$, and let \mathfrak{U} represent the restriction of \mathfrak{Z} to the full subcategory of $\mathcal{P}_1(S)$ which omits the set $\{1, 2, 3, \dots, n-1\}$. We can then refer to results presented by Goodwillie [10] to obtain a pushout square.

$$\begin{array}{ccc} h_1(\mathfrak{X}) & \longrightarrow & \text{hocolim}(\mathfrak{U}) \\ \downarrow & & \downarrow \\ \text{hocolim}(\mathfrak{X}) & \longrightarrow & h_1(\mathfrak{Z}) \end{array}$$

By naturality, we may extend this diagram to

$$\begin{array}{ccc} h_1(\mathfrak{X}) & \longrightarrow & \text{hocolim}(\mathfrak{U}) \\ \downarrow & & \downarrow \\ \text{hocolim}(\mathfrak{X}) & \longrightarrow & h_1(\mathfrak{Z}) \\ \downarrow \wr & & \downarrow \wr \\ \text{colim}(\mathfrak{X}) & \longrightarrow & \text{colim}(\mathfrak{Z}) \end{array}$$

As the lower vertical maps are equivalences, the square made up of the outer vertices is co-Cartesian. We then obtain a commutative diagram

$$\begin{array}{ccc}
 h_1(\mathfrak{X}) & \xrightarrow{\quad} & \text{hocolim}(\mathfrak{U}) \\
 \downarrow & \searrow & \nearrow \sim \\
 & h_1(\mathfrak{Y}) & \\
 \text{colim}(\mathfrak{X}) & \xrightarrow{\quad} & \text{colim}(\mathfrak{Z}) \\
 & \searrow & \nearrow \sim \\
 & \text{colim}(\mathfrak{Y}) &
 \end{array}$$

The fact that the two arrows pointing upward are equivalences then tells us that the desired diagram is co-Cartesian.

Conversely, we can reverse the process to find that the square

$$\begin{array}{ccc}
 \text{hocolim}(\mathfrak{X}) & \longrightarrow & h_1(\mathfrak{Z}) \\
 \downarrow \wr & & \downarrow \\
 \text{colim}(\mathfrak{X}) & \longrightarrow & \text{colim}(\mathfrak{Z})
 \end{array}$$

is co-Cartesian. As the left vertical map is an equivalence, we can conclude that so is the right vertical map. As $\text{colim}(\mathfrak{Z}) = \mathfrak{Z}(S)$, this gives the desired result. \square

Let $\mathfrak{Z} = \mathfrak{X} \rightarrow \mathfrak{Y}$ be a map of n -cubes. We may define an auxiliary n -cube $\text{hocofib}(\mathfrak{Z})$ by setting its value on a set S to be the homotopy cofiber of the map $\mathfrak{X}(S) \rightarrow \mathfrak{Y}(S)$. The cube $\text{hocofib}(\mathfrak{Z})$ obviously depends on the choice of decomposition of \mathfrak{Z} as a map $\mathfrak{X} \rightarrow \mathfrak{Y}$, but this will always be clear from the context. We note then the following lemma

Lemma 3.8. *Let $\mathfrak{Z} = \mathfrak{X} \rightarrow \mathfrak{Y}$ be a map of n -cubes. If \mathfrak{Z} is co-Cartesian, then so is $\text{hocofib}(\mathfrak{Z})$. Conversely, if $\text{hocofib}(\mathfrak{Z})$ is co-Cartesian and every space in \mathfrak{Z} is simply connected, then \mathfrak{Z} is co-Cartesian.*

Proof. Assume that \mathfrak{Z} is co-Cartesian. Then the square

$$\begin{array}{ccc}
 \text{hocolim}(\mathfrak{X}_1) & \longrightarrow & \text{hocolim}(\mathfrak{Y}_1) \\
 \downarrow & & \downarrow \\
 \text{colim } \mathfrak{X} & \longrightarrow & \text{colim } \mathfrak{Y}
 \end{array}$$

is co-Cartesian by the above lemma. Take the homotopy cofibers of the rows. These are homotopy equivalent by the fact that the square is co-Cartesian. The homotopy cofiber of the top row is $\text{cofib}(h_1(\mathfrak{X}) \rightarrow h_1(\mathfrak{Y}))$. As homotopy colimits commute, this is precisely the same as $h_1(\text{hocofib}(\mathfrak{Z}))$. The homotopy cofiber of the bottom row is $\text{colim}(\text{hocofib}(\mathfrak{Z}))$, by definition. We thus have an equivalence

$$h_1(\text{hocofib}(\mathfrak{Z})) \xrightarrow{\sim} \text{colim}(\text{hocofib}(\mathfrak{Z})),$$

hence $\text{hocofib}(\mathfrak{Z})$ must be co-Cartesian.

Conversely, if $\text{hocofib}(\mathfrak{Z})$ is co-Cartesian, the homotopy cofibers of the rows in the above diagram are homotopy equivalent. As each of the spaces involved is simply connected, this implies that the diagram is co-Cartesian. By the previous lemma, we have that \mathfrak{Z} is co-Cartesian. \square

Given a functor F , define a functor F_Y by setting

$$F_Y(X) = \text{hocofib}(F(X) \rightarrow F(X \vee Y)).$$

We note that we obviously have $\hat{c}r_2F(X, Y) \simeq \text{hocofib}(F(Y) \rightarrow F_Y(X))$.

Proof of Proposition 3.6 (7). By symmetry, we only need prove that the functor $\hat{c}r_2F(X, Y)$ is $(n-1)$ -exact as a functor of the first variable. The other statements then follow immediately. Let \mathfrak{X} be a strongly co-Cartesian n -cube. Then the n -cube $\mathfrak{X} \vee Y$ obtained by wedging Y to each space in \mathfrak{X} is also strongly co-Cartesian. Thus, we also have that the $(n+1)$ -cube $\mathfrak{Y} = \mathfrak{X} \rightarrow \mathfrak{X} \vee Y$ is strongly co-Cartesian. Since F is n -exact, $F(\mathfrak{Y})$ is co-Cartesian as well. We can also view this cube as $F(\mathfrak{X}) \rightarrow F(\mathfrak{X} \vee Y)$. Since this is co-Cartesian, the n -cube $\text{hocofib}(F(\mathfrak{Y}))$ given along the above map is also co-Cartesian. It is obvious from the definitions that this n -cube is identical to $F_Y(\mathfrak{X})$. Hence, F_Y is an $(n-1)$ -exact functor. Now, let $*$ represent the n -cube composed entirely of one-point spaces. This is obviously strongly co-Cartesian, so we have a map $F_Y(*) \rightarrow F_Y(\mathfrak{X})$ of co-Cartesian n -cubes. By the above comment, we see that $\text{hocofib}(F_Y(*) \rightarrow F_Y(\mathfrak{X}))$ is actually $\hat{c}r_2F(\mathfrak{X}, Y)$. By the previous lemma, this is co-Cartesian, so we are done. \square

Remark 3.9. Baues includes a similar proof for the cross-effect of a 2-exciseive functor in [3].

We now construct a spectral sequence which is similar to the one Barratt introduced in [1]. The best reference for this can be found in Goerss [8].

Let \mathcal{S}_k be the sequence X, X_1, \dots, X_k . We assume that we have a collection of cofibrations $f_i: X \rightarrow X_i$. Let \mathcal{T}_k be the sequence $CX, C_{f_1}, C_{f_2}, \dots, C_{f_k}$, where CX is the cone on X and C_{f_i} represents the homotopy cofiber of f_i . Notice that if we set $\mathcal{S}_{k+1} = X, X_1, \dots, X_k, CX$, then the pushout cube $\mathfrak{X}_{\mathcal{S}_{k+1}}$ is the same as the map of pushout cubes $\mathfrak{X}_{\mathcal{S}_k} \rightarrow \mathfrak{X}_{\mathcal{T}_k}$. Moreover, the cube $\mathfrak{X}_{\mathcal{T}_k}$ is equivalent to the cube $\hat{\mathfrak{X}}_{\mathcal{T}'_k}$ for the sequence $\mathcal{T}'_k = C_{f_1}, C_{f_2}, \dots, C_{f_k}$.

Fix a sequence of spaces and cofibrations $\mathcal{S}_k = X, X_1, \dots, X_k$ as above. Define $\mathcal{S}_i = X, X_1, \dots, X_k, CX, \dots, CX$ for any $i > k$, where there are $i-k$ copies of CX . Let \mathcal{T}_i be defined similarly to above for $i \geq k$. As above, we then have for any $i \geq k$ a co-Cartesian cube

$$\mathfrak{X}_{\mathcal{S}_{i+1}} = \mathfrak{X}_{\mathcal{S}_i} \rightarrow \mathfrak{X}_{\mathcal{T}_i}.$$

Given a functor F , and $i \geq k$, we define $\tilde{F}_i = \tilde{c}(F(\mathfrak{X}_{\mathcal{S}_i}))$ and $F_i = \tilde{c}(F(\mathfrak{X}_{\mathcal{T}_i}))$. By definition of the total cofiber, this means that for every $i \geq k$, we have a homotopy cofibration sequence

$$\tilde{F}_i \rightarrow F_i \rightarrow \tilde{F}_{i+1}.$$

We can use these cofibration sequences to define an exact couple and spectral sequence.

Set $D_{p,q} = H_q \tilde{F}_{-p}$ for $p+k \leq 0$ and $D_{p,q} = H_{p+q-k} \tilde{F}_{-k}$ for $p+k \geq 0$. Set $E_{p,q} = H_q F_{-p}$ for $p+k \leq 0$ and $E_{p,q} = 0$ otherwise. These definitions give an exact couple. We see that the E^1 term of the corresponding spectral sequence will be made up of cross-effects.

Notice that for any q , we have a sequence of maps in homology

$$\cdots \rightarrow H_{q+i+1} \tilde{F}_{k+i+1} \rightarrow H_{q+i} \tilde{F}_{k+i} \rightarrow \cdots \rightarrow H_q \tilde{F}_k$$

This implies that we can filter $H_q \tilde{F}_k$ by the subgroups $\text{im}(H_{q+i} \tilde{F}_{k+i} \rightarrow H_q \tilde{F}_k)$. In the event that F is n -exact, all but finitely many of these subgroups are trivial,

and the spectral sequence will converge to this filtration. In particular, we have the following.

Theorem 3.10 (Homological Barratt–Goerss spectral sequence). *If F is an n -exact functor and \mathcal{S}_k is a sequence as given above, then there is a spectral sequence with the following properties:*

- (1) *The spectral sequence is a second quadrant sequence of homological type*
- (2) *The E^1 term is completely contained in the band $-n \leq p \leq -k$. Specifically, we have*

$$E_{p,q}^1 = H_q(\hat{c}r_{-p}F(C_{f_1}, \dots, C_{f_k}, \Sigma X, \dots, \Sigma X))$$

for $-n \leq p \leq -k$.

- (3) *The spectral sequence converges to the above filtration of $H_*\tilde{F}_k$. Here the diagonal $p + q = a$ actually provides the filtration for $H_{a+k}\tilde{F}_k$.*

We will concentrate on the spectral sequence following from the sequence $\mathcal{S} = X, Y$, where $f: X \rightarrow Y$ is a cofibration. In the event that F is n -exact, we notice that we have convergence to a filtration of the homology of the homotopy cofiber of the map $Ff: FX \rightarrow FY$. In particular, if we take $Y = CX$, we obtain a spectral sequence converging to the homology of ΣFX . In this case, we also have the identification of the E^1 term as

$$E_{p,q}^1 \cong H_q(\hat{c}r_{-p}F(\Sigma X, \dots, \Sigma X)).$$

The spectral sequence we obtain is then an obvious analogue of the Barratt–Goerss desuspension spectral sequence.

4. CONSEQUENCES OF THE HBGSS

The spectral sequence of Theorem 3.10 has strong consequences for the behavior of an n -exact functor F . The first of these applications looks at constraints that the HBGSS puts on the homology of FX .

We will say that a space X is m -acyclic if $H_i(X) = 0$ for all $i \leq m$. Every space is considered to be (-1) -acyclic. Given an n -exact functor F , we will let $c(F)$ be the largest k such that $\hat{c}r_j F(S^0, \dots, S^0)$ is k -acyclic for all j .

Lemma 4.1. *Suppose that F is an n -exact functor. Let $m > 0$, and let $f: X \rightarrow Y$ be a cofibration. We suppose that ΣX and C_f are both m -acyclic. Additionally, suppose that FX is $(m + c(F))$ -acyclic and C_{Ff} is $(m + c(F) + 1)$ -acyclic. Then $F(C_f)$ is itself $(m + c(F) + 1)$ -acyclic.*

Proof. We use induction on n . The result is obvious for all m if $n = 1$, as then $F(C_f) \simeq C_{Ff}$. We now assume the result is true for $(n - 1)$ -exact functors. For $2 \leq r \leq n$, we examine the column $E_{-r,*}^1$ of the HBGSS corresponding to the sequence $\mathcal{S} = X, Y$. This spectral sequence converges to $H_*(C_{Ff})$. We know that $E_{-r,q}^1 = H_q(\hat{c}r_r F(C_f, \Sigma X, \dots, \Sigma X))$ and that $\hat{c}r_r F$ is $(n - (r - 1))$ -exact, so we can assume the result for $\hat{c}r_r F$. Since $\hat{c}r_r F$ is a functor in r variables, we can apply the inductive hypothesis on each variable successively, and we find that $\hat{c}r_r F(C_f, \Sigma X, \dots, \Sigma X)$ is at least $(rm + r + c(F))$ -acyclic. In particular, the E^1 term vanishes at and below the line $q = -(m + 1)p + c(F)$, except for possibly in the column $p = -1$. If we do have a $q \leq m + c(F) + 1$ with $E_{-1,q}^1 \neq 0$, this term will survive to E^∞ and we find that $H_q(C_{Ff}) \neq 0$, contradicting the hypothesis.

Thus, the smallest possible q with $E_{-1,q}^1 \neq 0$ is $q = m + c(F) + 2$, which completes the proof, as $E_{-1,q}^1 = H_q(F(C_f))$. \square

Lemma 4.2. *Let F be n -exact. Then $F(S^m)$ is $(m + c(F))$ -acyclic.*

Proof. The proof is by induction on m . We begin with the case $m = 0$, which is obvious from the definitions. Assume the result for S^m . Then, as we have a cofibration sequence $S^m \rightarrow CS^m \rightarrow S^{m+1}$, we satisfy the hypotheses of Lemma 4.1. Hence, we can conclude that FS^{m+1} is $(m + c(F) + 1)$ -acyclic. \square

Lemma 4.3. *If F is n -exact then $F(\bigvee_{\alpha \in A} S^m)$ is $(m + c(F))$ -acyclic for $m \geq 0$ and for any finite A .*

Proof. The homology of $F(\bigvee_{\alpha \in A} S^m)$ splits as a direct sum of the form

$$\bigoplus H_q(\hat{c}r_i F(S^m, \dots, S^m)).$$

By the preceding lemma, we can assume that these are all trivial for $q \leq m + c(F)$, hence the result follows. \square

These lemmas put us in a position to give the first of our results on the homology of FX . We take here the usual convention that any space is (-1) -connected.

Theorem 4.4. *If F is n -exact and X is m -connected, then FX is $(m + 1 + c(F))$ -acyclic for any $m \geq -1$. In particular, FX is always m -acyclic.*

Proof. The proof is by induction on the skeleton of X . We may assume that X^i is contractible for $i \leq m$. We then let $i = m + 1$. Since X^{m+1} is homotopy equivalent to a wedge of copies of S^{m+1} , we can apply Lemma 4.3 to see that FX^{m+1} must be $(m + 1 + c(F))$ -acyclic. If $i > m + 1$, we may assume that FX^{i-1} is $(m + 1 + c(F))$ -acyclic. Since X^i is equivalent to the homotopy cofiber of a map $\bigvee S^{i-1} \rightarrow X^{i-1}$, we can apply Lemma 4.1 to show that FX^i is itself $(m + 1 + c(F))$ -acyclic. \square

Obviously, if FX happens to be simply connected, we can make the stronger assertion that FX is m -connected. This will be the case whenever F takes simply connected spaces to simply connected spaces. It is immediate from the classification that any 1-exact functor satisfies this condition, and a simple van Kampen argument suffices to prove it for 2-exact functors. At this point, it is unknown whether this holds for n -exact functors with $n \geq 2$.

As the above proof indicates, we can certainly draw stronger conclusions about $H_*(FX)$. We observe that the vanishing line in Lemma 4.1 leads immediately to the following Freudenthal-type theorem.

Theorem 4.5. *Let F be n -exact, and X an m -connected space. Then there is an isomorphism*

$$H_i(FX) \xrightarrow{\sim} H_{i+1}(F(\Sigma X))$$

valid for all $i \leq 2m + c(F)$. Moreover, the map

$$H_{2m+1+c(F)}(FX) \rightarrow H_{2m+2+c(F)}(F(\Sigma X))$$

is onto.

These results mimic the classical results on homotopy groups. In fact, the similarity runs even deeper, as we can extend our results into the “metastable” range to obtain an EHP-type sequence. The fact that this is true shouldn’t be a surprise,

as our spectral sequence is merely a homological version of the classical homotopy spectral sequence.

Theorem 4.6. *Let F be n -exact, and X an m -acyclic space. Then there is a long exact sequence*

$$(1) \quad H_{3m+c(F)+2}(F(\Sigma X)) \xrightarrow{H} H_{3m+c(F)+2}(\hat{c}r_2F(\Sigma X, \Sigma X)) \rightarrow \cdots \rightarrow H_i(FX) \\ \xrightarrow{E} H_{i+1}(F(\Sigma X)) \xrightarrow{H} H_{i+1}(\hat{c}r_2F(\Sigma X, \Sigma X)) \xrightarrow{P'} H_{i-1}(FX) \rightarrow \cdots$$

Proof. In the region $q \leq 3m + c(F)$, the HBGSS is concentrated in the columns $p = -2$ and $p = -1$. Standard spectral sequence arguments then suffice to derive the long exact sequence. \square

In addition, we note that, by applying Theorem 4.5 to $\hat{c}r_2F(\Sigma X, \Sigma X)$ in each variable, we obtain an isomorphism

$$\sigma: H_{i-1}(\hat{c}r_2F(X, X)) \xrightarrow{\sim} H_{i+1}(\hat{c}r_2F(\Sigma X, \Sigma X))$$

on this range. Thus, the long exact sequence of Theorem 4.6 can be rewritten as

$$(2) \quad H_{3m+c(F)+2}(F(\Sigma X)) \xrightarrow{H'} H_{3m+c(F)}(\hat{c}r_2F(X, X)) \rightarrow \cdots \rightarrow H_i(FX) \\ \xrightarrow{E} H_{i+1}(F(\Sigma X)) \xrightarrow{H'} H_{i-1}(\hat{c}r_2F(X, X)) \xrightarrow{P} H_{i-1}(FX) \rightarrow \cdots$$

Corollary 4.7. *If F is 2-exact, the above EHP-type sequence applies in all degrees.*

Proof. The HBGSS is entirely concentrated in the columns $p = -1$ and $p = -2$ in this case. \square

We now identify the maps appearing in this sequence. It's obvious from the construction that the map $E: H_{i-1}(FX) \rightarrow H_i(F(\Sigma X))$ is induced by the natural map $\Sigma FX \rightarrow F(\Sigma X)$. The maps H and P are less obvious, but are still exactly what they should be in the cases we're interested in.

Theorem 4.8. *Let F be a 2-exact functor which takes values in simply-connected spaces. Let $\mu: \Sigma X \rightarrow \Sigma X \vee \Sigma X$ denote the usual comultiplication map on ΣX . The maps H , H' , P and P' found in the long exact sequences (1) and (2) are given as follows.*

- (1) $H: H_n(F(\Sigma X)) \rightarrow H_n(\hat{c}r_2F(\Sigma X, \Sigma X))$ is the composite

$$H_n(F(\Sigma X)) \xrightarrow{H_n(F(\mu))} H_n(F(\Sigma X \vee \Sigma X)) \twoheadrightarrow H_n(\hat{c}r_2F(\Sigma X, \Sigma X)),$$

where the surjection is the projection induced by the natural splitting.

- (2) $H' = \sigma \circ H$.
 (3) $P: H_n(\hat{c}r_2F(X, X)) \rightarrow H_n(FX)$ is the composite

$$H_n(\hat{c}r_2F(X, X)) \hookrightarrow H_n(F(X \vee X)) \xrightarrow{H_n(F\nabla)} H_n(FX),$$

where the inclusion is given by the natural splitting and $\nabla: X \vee X \rightarrow X$ is the folding map.

- (4) $P' = P \circ \sigma^{-1}$.

Proof. This is essentially Proposition 3.5 in Baues [3]. \square

The bounds on the above results are sharp. As an example, we take $FX = X \wedge X$ and $X = S^m$. Since $c(F) = -1$, Theorem 4.5 gives the trivial result

$$H_i(S^{2m}) \xrightarrow{\sim} H_{i+1}(S^{2m+2})$$

for $i < 2m$. Also, we obtain the trivial surjection

$$H_{2m}(S^{2m}) \twoheadrightarrow H_{2m+1}(S^{2m}).$$

In this case, we have $\hat{c}r_2 F(X, Y) = (X \wedge Y) \vee (Y \wedge X)$, and Theorem 4.6 reduces to the short exact sequence

$$0 \rightarrow H_{2n+2}(S^{2n}) \rightarrow H_{2n+2}(S^{2n+2} \vee S^{2n+2}) \rightarrow H_{2n}(S^{2n}) \rightarrow 0.$$

By the identifications given above, we see that the first map must be the diagonal, and the second map subtraction.

A more interesting example comes from taking $F = SP^2$, the symmetric square. We recall (see Hatcher, Example 4K.5, [14]) that the homology of the symmetric square of S^n is the following. For $n \geq 2$, we get the formula

$$H_*(SP^2(S^n)) \cong H_*(S^n) \oplus H_*(\Sigma^{n+1} \mathbb{R}P^{n-1}).$$

As in the case of $X \wedge X$, we see that $c(F) = -1$, so Theorem 4.5 gives an isomorphism

$$H_i(SP^2(S^n)) \xrightarrow{\sim} H_{i+1}(SP^2(S^{n+1}))$$

valid for $i < 2n$, as can be seen from the formula. We also have the trivial surjection

$$0 = H_{2n}(SP^2(S^n)) \rightarrow H_{2n+1}(SP^2(S^{n+1})) = 0$$

if n is odd, and a surjection

$$\mathbb{Z} \cong H_{2n}(SP^2(S^n)) \rightarrow H_{2n+1}(SP^2(S^{n+1})) \cong \mathbb{Z}/2$$

if n is even.

The potentially non-trivial portion of the EHP sequence is

$$\begin{aligned} 0 \rightarrow H_{2n+2}(SP^2(S^{n+1})) &\xrightarrow{H'} H_{2n}(\hat{c}r_2 SP^2(S^n, S^n)) \\ &\xrightarrow{P} H_{2n}(SP^2(S^n)) \rightarrow H_{2n+1}(SP^2(S^{n+1})) \rightarrow 0. \end{aligned}$$

As we saw above, this sequence depends on whether n is odd or even. We note that $\hat{c}r_2 SP^2(X, Y) \simeq X \wedge Y$. If n is odd, we obtain the sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{H'} \mathbb{Z} \rightarrow 0 \rightarrow 0 \rightarrow 0$$

If n is even, we obtain

$$0 \rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{P} \mathbb{Z} \xrightarrow{E} \mathbb{Z}/2 \rightarrow 0.$$

Hence, we can identify P as multiplication by 2. This information, together with the identification of the GAHSS for 2-exact functors on Moore spaces in Section 5 allows for the determination of $SP^2(M)$ for any Moore space M (cf. [17, 16]).

In addition to the above theorems, the HBGSS also gives a Whitehead-type uniqueness theorem. In particular, for n -exact functors F and G , the naturality of the spectral sequence together with induction on n obviously implies the following theorem.

Theorem 4.9. *Let $\tau: F \rightarrow G$ be a natural transformation of n -exact functors taking values in simply-connected spaces. If for any $i \geq 1$, the induced map*

$$\tau: \hat{c}r_i F(S^0, \dots, S^0) \rightarrow \hat{c}r_i G(S^0, \dots, S^0)$$

is a homotopy equivalence, then τ is a natural equivalence of functors.

Proof. Our proof is by induction on n . For $n = 1$, the theorem is obvious. Assume the conclusion for $(n - 1)$ -exact functors. By applying the inductive hypothesis to each cross-effect functor, we may assume that we have a natural equivalence

$$\tau: \hat{c}r_i F \rightarrow \hat{c}r_i G$$

for each $i \geq 2$.

It obviously suffices to show that whenever τ is an equivalence for X and Y , then it is for $X \vee Y$ and C_f for any map $f: X \rightarrow Y$.

By the natural splitting given in Part 6 of Proposition 3.6, the conclusion holds for $X \vee Y$. It only remains to show it for C_f .

Let \mathcal{E} denote the exact couple of the spectral sequence converging to C_{Ff} , and let \mathcal{E}' denote the exact couple giving the spectral sequence converging to C_{Gf} . We will denote the individual groups of the latter exact couple by $E'_{*,*}$ and $D'_{*,*}$. The transformation τ induces a map of exact couples

$$\mathcal{E} \rightarrow \mathcal{E}'$$

which, by the inductive hypothesis, gives an isomorphism

$$E_{*,*} \rightarrow E'_{*,*}$$

except possibly for $p = -1$. These isomorphisms, together with the fact that $D_{p,*} = D'_{p,*} = 0$ for $p < -n$ imply that $D_{n-1,*} \cong E_{n,*}$ and $D'_{n-1,*} \cong E'_{n,*}$. Thus, τ induces an isomorphism

$$D_{n-1,*} \cong D'_{n-1,*}$$

A five lemma argument then implies that we have isomorphisms

$$D_{p,*} \cong D'_{p,*}$$

for $p \leq -2$, and hence, for all p , since we know that we have a homotopy equivalence $C_{Ff} \rightarrow C_{Gf}$ by assumption. Another five lemma argument then implies that τ induces an isomorphism

$$H_*(F(C_f)) = E_{-1,*} \xrightarrow{\sim} E'_{-1,*} = H_*(G(C_f)),$$

which finished the proof. \square

Remark 4.10. Analogously to the classical Whitehead theorem, it's not enough to show F and G with homotopy equivalent cross-effects on S^0 . The homotopy equivalence must be given by a natural transformation. We will now construct a pair of 2-exact functors with homotopy equivalent cross-effects on S^0 which are not naturally equivalent.

Our construction of the functors begins with the following observation.

Lemma 4.11. *Let F and G be reduced n -exact functors taking values in simply connected spaces, and let $\tau: F \rightarrow G$ be a natural transformation. Then $H = \text{hocofib}(\tau)$ is again an n -exact functor, and the cross-effect $\hat{c}r_i H(X_1, X_2, \dots, X_i)$ is the homotopy cofiber of the induced map*

$$\hat{c}r_i F(X_1, X_2, \dots, X_i) \rightarrow \hat{c}r_i G(X_1, X_2, \dots, X_i).$$

Proof. We first show that H is n -exact. Let \mathfrak{X} be a strongly co-Cartesian $(n+1)$ -cube. Then the cubes $F(\mathfrak{X})$ and $G(\mathfrak{X})$ are co-Cartesian, as F and G are n -exact. Then, by Lemma 3.8, we see that $H(\mathfrak{X}) = \text{hocofib}(F(\mathfrak{X}) \rightarrow G(\mathfrak{X}))$ is co-Cartesian as well.

It remains to show that the cross-effects are as indicated. By iteration, it suffices to prove this for the first cross-effect. To that order, we examine the following diagram, which is commutative by the naturality of τ . The non-trivial maps are the obvious ones.

$$\begin{array}{ccccc} * & \leftarrow & G(X) \vee G(Y) & \rightarrow & G(X \vee Y) \\ \uparrow & & \uparrow & & \uparrow \\ * & \leftarrow & F(X) \vee F(Y) & \rightarrow & F(X \vee Y) \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & * & \longrightarrow & * \end{array}$$

If we take the homotopy colimits of the columns of this diagram, we obtain

$$* \leftarrow H(X) \vee H(Y) \rightarrow H(X \vee Y)$$

which has homotopy colimit $\hat{c}r_2 H(X, Y)$. Similarly, if we take the homotopy colimits of the rows, we obtain

$$\hat{c}r_2 G(X, Y) \leftarrow \hat{c}r_2 F(X, Y) \rightarrow *$$

which has homotopy colimit $\text{hocofib}(\hat{c}r_2 F(X, Y) \rightarrow \hat{c}r_2 G(X, Y))$. As homotopy colimits commute, we have the natural equivalence

$$\hat{c}r_2 H(X, Y) \simeq \text{hocofib}(\hat{c}r_2 F(X, Y) \rightarrow \hat{c}r_2 G(X, Y)),$$

which is what we claimed. \square

We recall the map $\eta_F: X \wedge F(S^0) \rightarrow F(X)$ defined at the beginning of Section 2. Note that if F is an n -exact functor, then, by the preceding lemma, the functor $\text{hocofib}(\eta)$ is n -exact as well. Let F' denote $\text{hocofib}(\eta_F)$ for the functor $F(X) = \Sigma^2(X \wedge X)$ and let G' denote $\text{hocofib}(\eta_G)$ for the functor $G(X) = \Sigma^2(SP^2(X) \vee SP^2(X))$. Since the original functors are obviously 2-exact, so are F' and G' . Additionally, since the source of η is 1-exact, its cross-effects vanish, and the cross-effects of F' and G' are merely $\hat{c}r_2 F'(S^0, S^0) \simeq \hat{c}r_2 F(S^0, S^0)$ and $\hat{c}r_2 G'(S^0, S^0) \simeq \hat{c}r_2 G(S^0, S^0)$.

We now show that F' and G' satisfy have isomorphic homology on 0-spheres, but not in a natural manner. We initially note that $F'(S^0)$ and $G'(S^0)$ are contractible, since they are the homotopy cofibers of homotopy equivalences. We now must only show that $\hat{c}r_2 F(S^0, S^0) \simeq \hat{c}r_2 G(S^0, S^0)$, since all higher cross-effects are trivial by 2-exactness. We have $\hat{c}r_2 F(X, Y) \simeq \Sigma^2((X \wedge Y) \vee (Y \wedge X))$. Similarly, the cross-effect of $\Sigma^2 \circ SP^2$ can be seen to be $\Sigma^2(X \wedge Y)$, so the wedge of two copies will have cross-effect $\Sigma^2(X \wedge Y) \vee \Sigma^2(X \wedge Y)$.

Finally, we note that F' and G' are not equivalent functors. Take $X = S^{n-1}$ with $n > 2$. Note that, by the definition of F' , the sequence

$$0 \rightarrow H_{2n}(S^{2n}) \rightarrow H_{2n}(F'(S^{n-1})) \rightarrow 0$$

is exact. Similarly, the sequence

$$0 \rightarrow H_{2n-2}(SP^2(S^{n-1})) \oplus H_{2n-2}(SP^2(S^{n-1})) \rightarrow H_{2n}(G'(S^{n-1})) \rightarrow 0$$

is exact. Thus, for $n \geq 2$, we see that $F'(S^n)$ and $G'(S^n)$ differ, hence F' and G' are not naturally equivalent.

We now turn our attention to the analyticity of n -exact functors. For this, we recall more terminology and notation from [10]. We let κ be an integer with $\kappa \geq -1$, and use \mathcal{C} to denote the class of all strongly co-Cartesian $(n+1)$ -cubes \mathfrak{X} such that for any $s \in \{1, \dots, n+1\}$, we have that $\mathfrak{X}(\emptyset) \rightarrow \mathfrak{X}(\{s\})$ is k_s -connected with $k_s \geq \kappa$. A functor F is then said to be $E_n(c, \kappa)$ if $F(\mathfrak{X})$ is $(-c + \sum k_s)$ -Cartesian for all $\mathfrak{X} \in \mathcal{C}$.

Definition 4.12. A functor F is *stably n -excisive* if there are numbers c and κ such that F is $E_n(c, \kappa)$. F is *ρ -analytic* if there is some number q such that F is $E_n(n\rho - q, \rho + 1)$ for all $n \geq 1$.

The condition that F be ρ -analytic is known to be sufficient for the Goodwillie tower of F to converge to F for ρ -connected spaces [11]. Using analogous methods to the proof of Theorem 4.4, we can use the HBGSS to prove

Theorem 4.13. *Any n -exact functor which takes values in simply-connected spaces is 1-analytic.*

Let \mathfrak{X} be an S -cube with $|S| = m \geq 1$. Given a subset $T \subset S$, we may define the T -face of \mathfrak{X} , denoted $\partial^T \mathfrak{X}$, to be the T -cube defined by

$$\partial^T \mathfrak{X}(U) = \mathfrak{X}(U)$$

for $U \subset T$. Suppose that we have a function $k: \mathcal{P}(S) \rightarrow \mathbb{Z}$ having the properties that $\partial^T \mathfrak{X}$ is $k(T)$ -co-Cartesian and $k(U) \leq k(T)$ whenever $U \subset T$. We allow the possibility that $k(T) = \infty$.

Let \mathcal{T} be the set of all partitions of S by non-empty sets. Given a partition $T \in \mathcal{T}$, define

$$l(T) = 1 - m + \sum_{T_\alpha \in T} k(T_\alpha).$$

Lemma 4.14. *Under the above hypotheses, \mathfrak{X} is l -Cartesian, where*

$$l = \min_{T \in \mathcal{T}} l(T).$$

Proof. This is Theorem 2.5 in [10]. □

Proof of Theorem 4.13. (Cf. Theorem 2.3 in [10]). Let \mathfrak{X} be a strongly co-Cartesian m -cube with $m \geq 1$, and set $S = \{1, 2, \dots, m\}$. We will use $\mathfrak{X}(i)$ to refer to the space $\mathfrak{X}(\{i\})$ for readability. Let F be an n -exact functor taking values in simply-connected spaces. Suppose that the map

$$\mathfrak{X}(\emptyset) \rightarrow \mathfrak{X}(s)$$

is s -connected for all $s \in S$. If $m = 1$, we can apply Theorem 4.4 and the reasoning in the proof of Lemma 4.1 to see that the map

$$F(\mathfrak{X}(\emptyset)) \rightarrow F(\mathfrak{X}(1))$$

is k_1 -connected, hence we may assume $m > 1$.

If $m > n+1$, all T -faces with $n+1 \leq |T| < m$ are co-Cartesian, as F is n -exact, hence $k(T) = \infty$. We may then restrict ourselves to looking at partitions of S into sets with no more than n elements. We then look at the connectivity of $\tilde{c}(F(\partial^T \mathfrak{X}))$ for all $T \subset S$ with $j = |T| \leq n$.

Since all strongly co-Cartesian cubes admit equivalences from pushout cubes we may assume that the cube $\partial^T \mathfrak{X}$ is a pushout cube for the sequence $S =$

$\mathfrak{X}(\emptyset), \mathfrak{X}(s_1), \dots, \mathfrak{X}(s_j)$, where $T = \{s_1, \dots, s_j\}$. Define F_i as we did in constructing the HBGSS, and let $\text{conn}(F_i)$ denote the connectivity of F_i . Set

$$k' = \min\{\text{conn}(F_{j+i}) \mid i \geq 1\},$$

and set $t = \text{conn}(\mathfrak{X}(\emptyset))$. By applying the reasoning in Lemma 4.1, we can see that

$$\text{conn}(F_{j+i}) \geq j + i - 1 + \sum_{s \in T} k_s + it.$$

Thus, we know that

$$k' \geq j + \sum_{s \in T} k_s$$

and the map $\tilde{c}(F(\partial^T \mathfrak{X})) \rightarrow F_j$ must be at least k' -connected.

As $\text{conn}(F_j) \geq (j-1) + \sum_{s \in T} k_s$, we can say the same for

$$k(T) = \text{conn}(\tilde{c}(F(\partial^T \mathfrak{X}))).$$

Let $T = \{T_\alpha\}$ be a partition of S with $1 \leq |T_\alpha| \leq n$ for every T_α . Thus we have

$$\sum_{\alpha} k(T_\alpha) \geq \left(\sum_{\alpha} |T_\alpha| \right) - |T| + \sum_{s \in S} k_s,$$

which tells us that

$$\min \left\{ \sum_{\alpha} k(T_\alpha) \right\} \geq m - m + \sum_{s \in S} k_s = \sum_{s \in S} k_s.$$

Hence by Lemma 4.14, $F(\mathfrak{X})$ is k -Cartesian, where

$$k = (1 - m) + \sum_{s \in S} k_s,$$

and F is $E_m(m-1, 1)$ for all $m \geq 1$, hence 1-analytic, as claimed. \square

5. THE GAHSS FOR 2-EXACT FUNCTORS

In this section, we review Baues' work with quadratic \mathbb{Z} -modules and the quadratic tensor product, and use this to examine the E^2 term of the GAHSS in the case where F is 2-exact and X is a Moore space $M(A, n)$ with $n \geq 2$. References for these topics are [2, 3]. We assume that all Moore spaces are simply-connected.

Let T be a functor from a category \mathcal{C} with a zero object and coproducts into an Abelian category \mathcal{A} . Suppose that $T(0) = 0$. We can then define the cross-effect of T by

$$T(X|Y) = \ker\{(Tr_1, Tr_2): T(X \vee Y) \rightarrow T(X) \times T(Y)\}$$

where r_1 and r_2 are the obvious retractions. We obtain the usual splitting property, so

$$T(X \vee Y) \cong T(X) \oplus T(Y) \oplus T(X|Y).$$

Such a functor T is *linear* if the cross-effect $T(X|Y) = 0$ for all X and Y . T is said to be *quadratic* if $T(X|Y)$ is linear in each variable. Baues' study of quadratic functors led to his introduction of quadratic modules and tensor products.

Definition 5.1. A *quadratic \mathbb{Z} -module* (henceforth simply *quadratic module*), written

$$M = M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e,$$

consists of a pair of \mathbb{Z} -modules M_e and M_{ee} , together with a pair of homomorphisms $H: M_e \rightarrow M_{ee}$ (the *Hopf map*) and $P: M_{ee} \rightarrow M_e$ (the *Whitehead product map*), such that $HPH = 2H$ and $PHP = 2P$.

Remark 5.2. It is possible to (and Baues does) define a quadratic A -module for an arbitrary Abelian group A . We are not interested in that development here.

Definition 5.3. If M and N are quadratic modules, a *morphism* $f: M \rightarrow N$ is a pair of homomorphisms $f_e: M_e \rightarrow N_e$ and $f_{ee}: M_{ee} \rightarrow N_{ee}$ making all the relevant diagrams commute.

We may define direct sums, direct products, kernels and cokernels “coordinate-wise” in the obvious manner. It is then straight-forward to verify that the category \mathcal{QM} of quadratic modules is an Abelian category.

The reason for the names of the maps in a quadratic module M becomes a little more apparent when examining situations where quadratic modules arise. Baues has shown how to derive quadratic modules in a natural way from quadratic functors $\mathcal{C} \rightarrow \mathcal{Ab}$, where \mathcal{C} is an additive category. In particular, if we take \mathcal{C} to be the category of simply-connected homotopy co-commutative cogroups, the construction applies. If X is a two-fold suspension, then X is obviously such a space, so all of the spaces S^n are in \mathcal{C} for $n \geq 2$. Given any element X of \mathcal{C} , we have maps

$$X \xrightarrow{\mu} X \vee X \xrightarrow{\nabla} X$$

where μ is the comultiplication map and ∇ is the folding map. Applying any functor F then gives the obvious sequence

$$FX \xrightarrow{F(\mu)} F(X \vee X) \xrightarrow{F(\nabla)} FX.$$

On applying homology, we obtain

$$H_*(F(X \vee X)) \cong H_*(FX) \oplus H_*(FX) \oplus H_*(\hat{c}r_2F(X, X)),$$

by Part 6 of Proposition 3.6. Let $r: H_*(F(X \vee X)) \rightarrow H_*(\hat{c}r_2F(X, X))$ be the restriction map, and $i: H_*(\hat{c}r_2F(X, X)) \rightarrow H_*(F(X \vee X))$ the injection. We may then define maps $H = r \circ H_*(F(\mu))$ and $P = H_*(F(\nabla)) \circ i$. Baues proves

Theorem 5.4. *The data*

$$H_*(FX) \xrightarrow{H} H_*(\hat{c}r_2F(X, X)) \xrightarrow{P} H_*(FX)$$

*defines a quadratic module, which we denote $H_*F\{X\}$.*

Remark 5.5. The maps H and P in Theorem 5.4 are the same as the maps H and P in the EHP-type sequences of Theorem 4.6.

Given an Abelian group A and a quadratic module M , we may define the *quadratic tensor product* $A \otimes M$ as follows. Let G be the free Abelian group generated by all the symbols $a \otimes m$ and $[a, b] \otimes n$, where $a, b \in A$, $m \in M_e$ and $n \in M_{ee}$. $A \otimes M$ is the quotient of G given by imposing the following set of relations, where $a, a', b, b' \in A$, $m, m' \in M_e$ and $n, n' \in M_{ee}$.

$$(1) \ a \otimes (m + m') = a \otimes m + a \otimes m'$$

- (2) $[a + a', b] \otimes n = [a, b] \otimes n + [a', b] \otimes n$
- (3) $[a, b + b'] \otimes n = [a, b] \otimes n + [a, b'] \otimes n$
- (4) $[a, b] \otimes (n + n') = [a, b] \otimes n + [a, b] \otimes n'$
- (5) $(a + a') \otimes m = a \otimes m + a' \otimes m + [a, a'] \otimes H(m)$
- (6) $[a, a] \otimes n = a \otimes P(n)$

Remark 5.6. The following facts are obvious from the above definition.

- (1) The quadratic tensor product is quadratic in the first variable and linear in the second. The cross-effect $(A|B) \otimes M$ is simply the usual tensor product $A \otimes B \otimes M_{ee}$.
- (2) Any Abelian group G can be treated as a quadratic module

$$M = G \rightarrow 0 \rightarrow G.$$

Then for any Abelian group A , the quadratic tensor product $A \otimes M$ is the same as the usual tensor product $A \otimes G$.

For any quadratic module M , we can use the above description to see that we have a natural isomorphism $\mathbb{Z} \otimes M \cong M_e$. Since the quadratic tensor product is quadratic in the first variable, we can use the above identification of the cross-effect to show that we have a natural isomorphism

$$\mathbb{Z}^n \otimes M \cong (M_e)^n \oplus (M_{ee})^{n(n-1)/2}.$$

Given an Abelian group A , let Δ denote the diagonal map $A \rightarrow A \oplus A$, and let $\nabla: A \oplus A \rightarrow A$ denote the addition map. For any quadratic functor $T: \mathcal{A}b \rightarrow \mathcal{A}b$, we obtain a quadratic module

$$T\{\mathbb{Z}\} = T(\mathbb{Z}) \xrightarrow{H} T(\mathbb{Z}|\mathbb{Z}) \xrightarrow{P} T(\mathbb{Z})$$

where H is the composite

$$T(\mathbb{Z}) \xrightarrow{T(\Delta)} T(\mathbb{Z} \oplus \mathbb{Z}) \rightarrow T(\mathbb{Z}|\mathbb{Z})$$

and P is the composite

$$T(\mathbb{Z}|\mathbb{Z}) \rightarrow T(\mathbb{Z} \oplus \mathbb{Z}) \xrightarrow{T(\nabla)} T(\mathbb{Z}).$$

Here, the unlabeled maps denote the obvious projection and inclusion respectively.

For any such functor $T: \mathcal{A}b \rightarrow \mathcal{A}b$, there is a map $\lambda: A \otimes T\{\mathbb{Z}\} \rightarrow T(A)$ called the *tensor approximation* of T . The definition of this map can be found in [2]. If A is finitely generated and free, then λ is an isomorphism. It is not an isomorphism in general, however.

The following results on quadratic tensor products appear in the literature.

Theorem 5.7. *The quadratic tensor product is right exact in the first variable.*

Remark 5.8. We take here a non-additive definition of right exactness, given by Baues in [2]. An equivalent definition is given by Bouc [5]. Suppose that

$$A_1 \rightarrow A_0 \rightarrow A \rightarrow 0$$

is an exact sequence of Abelian groups, and $T: \mathcal{A}b \rightarrow \mathcal{A}b$ an arbitrary functor with $T(0) = 0$. We say that T is *right exact* if the sequence

$$F(A_1|A_0) \oplus F(A_1) \rightarrow F(A_0) \rightarrow F(A) \rightarrow 0$$

is exact. This is obviously a generalization of the notion of right exactness of a linear functor.

Given an Abelian group A and a quadratic module M , we may take a free resolution of A

$$A_1 \rightarrow A_0 \rightarrow A \rightarrow 0.$$

Using the simplicial methods of Dold and Puppe for derived functors of a non-linear functor [7], we obtain a chain complex C_* given by

$$(A_1|A_1) \otimes M \rightarrow ((A_1|A_0) \otimes M) \oplus (A_1 \otimes M) \rightarrow A_0 \otimes M.$$

By right-exactness, the 0th homology group of C_* is simply $A \otimes M$. The 1st and 2nd homology groups of C_* are the *quadratic torsion products*. We define $A *' M = H_1(C_*)$ and $A *'' M = H_2(C_*)$. In the event that $M = G \rightarrow 0 \rightarrow G$ is simply an Abelian group, $A *'' M$ is trivial and $A *' M = \text{Tor}(A, G)$, the usual torsion product of Abelian groups. As with the quadratic tensor product, both quadratic torsion products are quadratic in the first variable and linear in the second. The identification of $A \otimes M$ for any finitely generated free Abelian A given above makes calculation of the quadratic tensor and torsion products fairly easy.

At this point, for a 2-exact functor F , we will use Q_i to denote the composite $H_i \circ F$, so $Q_i(X) = H_i(FX)$ and $Q_i(X, Y) = H_i(FX, FY)$. We will write $Q_i\{X\}$ for the quadratic module denoted $H_i F\{X\}$. We will use $Q_i(X|Y)$ to denote $H_i(\hat{c}r_2 F(X, Y))$. This is easily shown to be equivalent to the definition of cross-effect for the functor Q_i given above. Analogously to the tensor approximation given above, we have a natural homomorphism $\lambda: A \otimes Q_{p+q}\{S^p\} \rightarrow Q_{p+q}(M(A, p))$ for any finitely generated Abelian group A with $p \geq 2$. This map is defined as follows. Let $a, b \in A$, $m \in Q_{p+q}(S^p)$ and $n \in Q_{p+q}(S^p|S^p)$. We may view a and b as elements of $\pi_p(M(A, p)) = A$, and so we can set

$$\begin{aligned} \lambda(a \otimes m) &= Q_{p+q}(a)(m) \\ \lambda([a, b] \otimes n) &= PQ_{p+q}(a|b)(n) \end{aligned}$$

If A is free Abelian, then λ is an isomorphism. We denote the cokernel of λ by ${}_\lambda Q_{p+q}(M(A, p))$. Proposition 3.12 of [3] shows that for any finitely generated Abelian group A , we have a natural exact sequence

$$\begin{aligned} 0 \rightarrow A *' Q_{p+q}\{S^p\} \rightarrow {}_\lambda Q_{p+q+1}(M(A, p)) \rightarrow A *'' Q_{p+q-1}\{S^p\} \\ \rightarrow A \otimes Q_{p+q}\{S^p\} \xrightarrow{\lambda} Q_{p+q}(M(A, p)) \rightarrow {}_\lambda Q_{p+q}(M(A, p)) \rightarrow 0 \end{aligned}$$

We are now in position to compute the E^2 term of the GAHSS in the event that F is 2-exact and X is a Moore space.

Let M denote the Moore space $M(A, p)$. If $A_1 \rightarrow A_0 \rightarrow A$ is a free resolution of A , we set $Y = M(A_0, p)$ and $X = M(A_1, p)$. We may then take Y to be the p -skeleton of M , and the Moore space is the mapping cone of $X \rightarrow Y$. We note that a consequence is that the E^1 term of the GAHSS is restricted to two columns. In particular, we have that the only non-trivial differential

$$E_{p+1, q}^1 = Q_{p+q+1}(M, Y) \xrightarrow{\partial} Q_{p+q}(Y) = E_{p, q}^1.$$

Examine the following diagram, which appears in a somewhat different form in [3].

$$\begin{array}{ccc}
A_0 \otimes Q_{p+q+1}\{S^p\} & \xlongequal{\quad} & Q_{p+q+1}(Y) \\
\downarrow & & \downarrow i \\
A \otimes Q_{p+q+1} & \xrightarrow{\lambda} & Q_{p+q+1}(M) \\
& & \downarrow j \\
& & Q_{p+q+1}(M, Y) \\
& & \downarrow \partial \\
A_0 \otimes Q_{p+q}\{S^p\} & \xlongequal{\quad} & Q_{p+q}(Y) \\
\downarrow & & \downarrow i \\
A \otimes Q_{p+q}\{S^p\} & \xrightarrow{\lambda} & Q_{p+q}(M) \\
& & \downarrow j \\
& & Q_{p+q}(M, Y)
\end{array}$$

The column is exact, as it is part of the obvious long exact sequence in homology. From this diagram, we can see that

$$\operatorname{im} \lambda = \operatorname{im} i = \ker j \cong \operatorname{coker} \partial = E_{p,q}^2.$$

Similarly, we obtain

$$\lambda Q_{p+q+1}(M(A, p)) = \operatorname{coker} i \cong \operatorname{im} j = \ker \partial = E_{p+1,q}^2.$$

Hence, we obtain the following identification of the the E^2 term of the GAHSS.

Theorem 5.9. *Using the above notation, the E^2 term of the GAHSS for a 2-exact functor on a Moore space is given by the exact sequence.*

$$\begin{aligned}
0 \rightarrow A *' Q_{p+q}\{S^p\} \rightarrow E_{p+1,q}^2 \rightarrow A *'' Q_{p+q-1}\{S^p\} \\
\rightarrow A \otimes Q_{p+q}\{S^p\} \xrightarrow{\lambda} E_{p,q}^2 \rightarrow 0
\end{aligned}$$

In the event that $A *'' Q_{p+q-1}\{S^p\} = 0$ (which happens on a regular basis in practice), we get the obvious identifications

$$\begin{aligned}
E_{p,q}^2 &\cong A \otimes Q_{p+q}\{S^p\} \\
E_{p+1,q}^2 &\cong A *' Q_{p+q}\{S^p\}
\end{aligned}$$

Remark 5.10. Further progress on the GAHSS along the lines of this article seems unlikely. The differential in the E^1 term of the GAHSS contains quite a bit of information, and appears to be virtually in any case more difficult than the present. There is hope for a similar program, however. We note that we may define an ad hoc spectral sequence by setting

$$\begin{aligned}
E_{p,q}^2 &= A \otimes Q_{p+q}\{S^p\} \\
E_{p+1,q}^2 &= A *' Q_{p+q}\{S^p\} \\
E_{p+2,q}^2 &= A *'' Q_{p+q}\{S^p\}
\end{aligned}$$

and taking $d : E_{p+2,q-1}^2 = A *'' Q_{p+q-1}\{S^p\} \xrightarrow{d} A \otimes Q_{p+q}\{S^p\} = E_{p,q}^2$ to be the only non-trivial differential, where d is the map in the statement of Theorem 5.9. It is obvious that the E^3 term of this spectral sequence is the same as the E^2 term of the GAHSS, and convergence is the same. This suggests looking for a natural construction of a spectral sequence whose E^2 term agrees with the E^2 term of the ad hoc spectral sequence. From that point, the generalization to n -exact functors is clear by using the extension of quadratic modules found in [4]. Such a spectral sequence does exist, and will be the subject of a future article.

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