

ON CO- H -MAPS TO THE SUSPENSION OF THE PROJECTIVE PLANE

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ABSTRACT. We study co- H -maps from a suspension to the suspension of the projective plane and provide examples of non-suspension 3-cell co- H -spaces.

1. INTRODUCTION

A co- H -space is a pointed space X which admits a comultiplication $\mu': X \rightarrow X \vee X$. A suspension is a co- H -space, but conversely it may not be true in general. When p is an odd prime, it was known [3, p.444] that the two-cell complex $S^3 \cup_\alpha e^{2p+1}$ is a non-suspension co- H -space, where α is a non-trivial element in $\pi_{2p}(S^3)$ of order p . For the case where $p = 2$, there are no non-suspension two-cell co- H -spaces by considering the EHP -sequences. John Harper asked whether there are non-suspension co- H -spaces which has the cell-structure $\Sigma\mathbb{R}P^2 \cup_f e^n$ for some attaching map $f \in \pi_{n-1}(\Sigma\mathbb{R}P^2)$. In this article, we will show that, for any element $\alpha \in \pi_n(S^3)$ of order 2, there exists a correspondent element $f \in \pi_{n+1}(\Sigma\mathbb{R}P^2)$ such that the three-cell complex $\Sigma\mathbb{R}P^2 \cup_f e^{n+2}$ is a non-suspension co- H -space. This provides (infinitely many) examples of non-suspension 2-local three-cell co- H -spaces. One of such examples is $\Sigma\mathbb{R}P^2 \cup_f e^6$ for some $f \in \pi_5(\Sigma\mathbb{R}P^2)$ and this is the only non-suspension co- H -space among the complexes $X = \Sigma\mathbb{R}P^2 \cup e^n$ with $n \leq 6$. We should point out that very few examples of non-suspension 2-local co- H -spaces were known. John Harper pointed out to the author that he was able to construct one such an example by attaching a cell to $\Sigma\mathbb{C}P^2$. On the other hand, we do not see any such examples in published references. Below we describe the results in more detail.

Lemma 2.6 shows that, for $n \geq 4$, $\Sigma\mathbb{R}P^2 \cup_f e^{n+1}$ is a non-suspension co- H -space if and only if $f: S^n \rightarrow \Sigma\mathbb{R}P^2$ is a co- H -map. Thus the problem on the co- H -spaces is reduced to whether a map $f: S^n \rightarrow \Sigma\mathbb{R}P^2$ is a co- H -map. Co- H -maps have been much studied (see for instance [1, 2, 3, 4, 10, 11, 12, 16, 17]). The Hopf invariant is a basic tool. The basic ideas are as follows.

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Let Z be a co- H -space and let $f: \Sigma Y \rightarrow Z$ be a map. Let $f': Y \rightarrow \Omega Z$ be the adjoint of f . Then f is a co- H -map if and only if the composite

$$Y \xrightarrow{f'} \Omega Z \xrightarrow{\Omega \mu'} \Omega(Z \vee Z) \xrightarrow{H} \Omega \Sigma(\Omega Z \wedge \Omega Z)$$

is null homotopic, where H is the Hopf map. (We will go over this in detail in Section 2.) Let $F_H(Z)$ be the homotopy fibre of the composite

$$\Omega Z \xrightarrow{\Omega \mu'} \Omega(Z \vee Z) \xrightarrow{H} \Omega \Sigma(\Omega Z \wedge \Omega Z).$$

An equivalent statement is that $f: \Sigma Y \rightarrow Z$ is a co- H -map if and only if the map $f': Y \rightarrow \Omega Z$ lifts to $F_H(Z)$. Thus the study of co- H -maps is essentially equivalent to study the homotopy theory of $F_H(Z)$.

Now we consider our case where $Z = \Sigma \mathbb{R}P^2$. We write $P^n(2)$ for $\Sigma^{n-2} \mathbb{R}P^2$. In our notation, $P^3(2) = \Sigma \mathbb{R}P^2$. Let $\mathbb{R}P_a^b = \mathbb{R}P^b / \mathbb{R}P^{a-1}$ and let $X\langle n \rangle$ be the n -connected cover of a space X . According to [20], the homotopy fibre of the inclusion $P^3(2) \hookrightarrow BSO(3)$ is $\Sigma \mathbb{R}P_2^4 \vee P^6(2)$ and so there is a fibre sequence

$$SO(3) \longrightarrow \Sigma \mathbb{R}P_2^4 \vee P^6(2) \longrightarrow P^3(2).$$

It follows that there is a fibre sequence

$$\Omega(P^3(2)\langle 2 \rangle) \xrightarrow{\partial} S^3 \longrightarrow \Sigma \mathbb{R}P_2^4 \vee P^6(2) \longrightarrow P^3(2)\langle 2 \rangle,$$

where the map $S^3 \rightarrow \Sigma \mathbb{R}P_2^4 \vee P^6(2)$ is of degree 4 into the bottom cell of target space. This fibre sequence induces a splitting of $\Omega^3 P^3(2)$ (see [20]). Let $S^3\{2\}$ be the homotopy fibre of degree 2 map from S^3 to S^3 . Our main result is as follows.

Theorem 1.1. *Let $\partial: \Omega(P^3(2)\langle 2 \rangle) \rightarrow S^3$ be defined above.*

1) *The composite*

$$F_H(P^3(2))\langle 1 \rangle \longrightarrow \Omega(P^3(2)\langle 2 \rangle) \xrightarrow{\partial} S^3$$

lifts to $S^3\{2\}$ and

2) *Let $\theta: F_H(P^3(2))\langle 1 \rangle \rightarrow S^3\{2\}$ be a resulting lifting. Then θ has a cross-section. Thus $S^3\{2\}$ is a retract of the universal cover of $F_H(P^3(2))$.*

Since the space $S^3\{2\}$ is indecomposable, this theorem determines the “smallest retract” of the universal cover of $F_H(P^3(2))$ which contains the bottom cell.

Corollary 1.2. *Let Y be a simply connected space and let $g: Y \rightarrow S^3$ be a map. Then there exists a co- H -map $f: \Sigma Y \rightarrow P^3(2)$ such that the composite*

$$Y \xrightarrow{f'} (\Omega P^3(2))\langle 1 \rangle \simeq \Omega(P^3(2)\langle 2 \rangle) \xrightarrow{\partial} S^3$$

is homotopic to g if and only if the homotopy class $[g]$ is of order 2 in the group $[Y, S^3]$.

In particular, we have

Corollary 1.3. *Let $\alpha \in \pi_n(S^3)$ be an element of order 2. Then there is a map $f: S^{n+1} \rightarrow P^3(2)$ such that*

1) *the composite*

$$S^n \xrightarrow{f'} \Omega(P^3(2)\langle 2 \rangle) \xrightarrow{\partial} S^3$$

is a representative for the element $\alpha \in \pi_n(S^3)$ and

2) *the three-cell complex $P^3(2) \cup_f e^{n+2}$ is a non-suspension co- H -space.*

Conversely, suppose that $P^3(2) \cup_f e^{n+2}$ is a co- H -space. Then the composite

$$S^n \xrightarrow{f'} \Omega(P^3(2)\langle 2 \rangle) \xrightarrow{\partial} S^3$$

is of order 2 in $\pi_n(S^3)$.

Our answer to Harper's question is as follows.

Corollary 1.4. *There are infinitely many non-suspension co- H -spaces which admits a cell structure $(\Sigma\mathbb{R}P^2) \cup e^n$.*

The ideas for proving Theorem 1.1 are as follows. Assertion (1) follows from some standard arguments in homotopy theory. We introduce “*combinatorial calculations*” for the Hopf map to prove assertion (2). These combinatorial methods were introduced by Fred Cohen in [6] to attack the Barratt conjecture and have been applied to the James-Hopf maps [21].

One may push Harper's question further to ask how to classifying all of co- H -spaces which admits a cell structure $P^3(2) \cup e^n$. By assuming $\pi_*(S^3)$, this general question is reduced to how to determine the kernel of the composite

$$\pi_*(\Omega(\Sigma\mathbb{R}P_2^4 \vee P^6(2))) \longrightarrow \pi_*(\Omega P^3(2)) \xrightarrow{H_*} \pi_*(\Omega\Sigma(\Omega P^3(2) \wedge \Omega P^3(2))).$$

So far it is unknown whether there are non-trivial elements in this kernel except for lower homotopy groups. On the other hand, a “big part” of $\pi_*(\Sigma\mathbb{R}P_2^4 \vee P^6(2))$ does not belong to this kernel (see Section 4). However the stable homotopy type of co- H -spaces $P^3(2) \cup e^n$ can be classified by assuming the homotopy groups.

Let $k \geq 1$. Recall that $\pi_{k+4}(P^{k+3}(2)) = \mathbb{Z}/4$ and its generator is represented by the map $\bar{\eta}: S^3 \rightarrow \Omega^{k+1}P^{k+3}(2)$ such that the composite

$$S^3 \xrightarrow{\bar{\eta}} \Omega^{k+1}P^{k+3}(2) \xrightarrow{\text{pinch}} \Omega^{k+1}S^{k+3}$$

represents the element $\eta \in \pi_{k+4}(S^{k+3})$. Let V_n^k be the image of the homomorphism

$$\{\alpha \in \pi_n(S^3) \mid 2\alpha = 0\} \subseteq \pi_n(S^3) \xrightarrow{\bar{\eta}_*} \pi_n(\Omega^{k+1}P^{k+3}(2)).$$

Let \mathcal{C}_n^k be the set of the homotopy type of the spaces $\Sigma^k X$, where X runs over all co- H -space which admits a cell structure $P^3(2) \cup e^n$.

Theorem 1.5. *Let $1 \leq k \leq \infty$ and let $n \geq 4$. Then \mathcal{C}_{n+2}^k is isomorphic to V_n^k .*

This shows that certain stable complex of the form $P^3(2) \cup e^n$ can be desuspensionable to an unstable non-suspension co- H -space $P^3(2) \cup e^n$. For low dimensional cases, we are able to determine the group V_n^k by computing the homotopy groups. However the determination of V_n^k for general n is out of reach under current technology.

Let $\Omega_0 X$ be the path-connected component of ΩX which contains the base-point.

Theorem 1.6. *Let $j: P^3(2) \rightarrow \Sigma \mathbb{R}P^4$ be the inclusion. Then the composite*

$$\Omega_0 F_H(P^3(2)) \longrightarrow \Omega_0^2 P^3(2) \xrightarrow{\Omega^2 j} \Omega_0^2 \Sigma \mathbb{R}P^4$$

is null homotopic.

This gives a relation between co- H -spaces $P^3(2) \cup e^n$ and $\Sigma \mathbb{R}P^4$.

Corollary 1.7. *Let $X = P^3(2) \cup e^n$ be a co- H -space with $n \geq 4$. Then the inclusion $P^3(2) \rightarrow \Sigma \mathbb{R}P^4$ factors through X .*

The map $F_H(P^3(2)) \rightarrow \Omega P^3(2)$ admits an exponent of 2 after looping.

Theorem 1.8. *The map*

$$\Omega F_H(P^3(2)) \longrightarrow \Omega^2 P^3(2)$$

is of order 2 in the group $[\Omega F_H(P^3(2)), \Omega^2 P^3(2)]$.

Corollary 1.9. *Let $f: \Sigma^2 Y \rightarrow P^3(2)$ be a co- H -map. Then f is of order at most 2 in the group $[\Sigma^2 Y, P^3(2)]$.*

Corollary 1.10. *If $X = P^3(2) \cup_f e^{n+1}$, then the attaching map $f: S^n \rightarrow P^3(2)$ extends to a map $\bar{f}: P^{n+1}(2) \rightarrow P^3(2)$ and X is the $(n+1)$ -skeleton of the homotopy cofibre of \bar{f} .*

Standard notations of homotopy theory will be directly used: $X^{(n)}$ for n -fold self smash product of X , F_f for the homotopy fibre of a map f , C_f for the homotopy cofibre of a map f , $J(X)$ for the James construction and $\{J_n(X)\}$ for the James filtration. In addition, Toda's notations [18] for elements in the homotopy groups of spheres will be used without explanation. Every space is localized at 2 in this article. The mod 2 homology of X is denoted by $H_*(X)$.

The article is organized as follows. In Section 2, we give some preliminary lemmas. We introduce combinatorial calculations for the Hopf map in Section 3. The proof of Theorem 1.1 is given in this section. In Section 4, we give further properties of the space $F_H(P^3(2))$. The proofs of Theorems 1.5, 1.6 and 1.8 are given in Section 5. In Section 6, we discuss some examples.

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2. PRELIMINARY LEMMAS

2.1. Desuspensions. A map $f: \Sigma X \rightarrow \Sigma Y$ is called *desuspensionable* if there is a map $g: X \rightarrow Y$ such that $f \simeq \Sigma g$.

Lemma 2.1. *Let Y be a path-connected finite dimensional CW-complex and let $f: S^n \rightarrow \Sigma Y$ be a map with $n \geq \dim Y + 1$. Then the homotopy cofibre C_f is homotopy equivalent to a suspension if and only if f is desuspensionable.*

Proof. Clearly $C_f \simeq \Sigma C_g$ if $f \simeq \Sigma g$ for some $g: S^{n-1} \rightarrow Y$. Conversely, suppose that $C_f \simeq \Sigma Z$ for some space Z . Since $n \geq \dim Y + 1$, Y is the $\dim Y$ -skeleton of Z and $Z \simeq Y \cup_g e^n$ for certain attaching map $g: S^{n-1} \rightarrow Y$. Let $j: \Sigma Y \rightarrow C_f$ be the inclusion. Then there is a homotopy commutative diagram

$$\begin{array}{ccccc} F_j & \xrightarrow{i} & \Sigma Y & \xrightarrow{j} & C_f \\ \uparrow h & & \parallel & & \\ S^n & \xrightarrow{f} & \Sigma Y & & \end{array}$$

where h is of degree ± 1 into the bottom cell of F_j . Let $\theta: \Sigma Z \rightarrow C_f$ be a homotopy equivalence. We may assume that $\theta|_{\Sigma Y}: \Sigma Y \rightarrow \Sigma Y$ is homotopic to the identity map. Let $\theta': Z \rightarrow \Omega C_f$ be the adjoint map of θ . Then there is a homotopy commutative diagram

$$\begin{array}{ccccc} \Omega F_j & \xrightarrow{\Omega i} & \Omega \Sigma Y & \xrightarrow{\Omega j} & \Omega C_f \\ \uparrow \tilde{h} & & \uparrow & & \uparrow \theta' \\ S^{n-1} & \xrightarrow{g} & Y & \longrightarrow & Z, \end{array}$$

where \tilde{h} is of degree ± 1 into the bottom cell of ΩF_j , and hence the result. \square

Note. In general, it is possible that the homotopy cofibre of a non-desuspensionable map is still a suspension. An example is the cofibre sequence $P^4(2) \xrightarrow{f} P^3(2) \longrightarrow \Sigma \mathbb{R}P^4$, where the attaching map f is *not* desuspensionable.

Lemma 2.2. *Let $E: \mathbb{R}P^2 \rightarrow \Omega P^3(2) = \Omega \Sigma \mathbb{R}P^2$ be the canonical inclusion. Then*

$$\Omega^2 E: (\Omega^2 \mathbb{R}P^2)\langle 1 \rangle \longrightarrow (\Omega^3 P^3(2))\langle 1 \rangle$$

is null homotopic. In particular, $E_: \pi_n(\mathbb{R}P^2) \rightarrow \pi_{n+1}(P^3(2))$ is the trivial homomorphism for $n \geq 4$.*

Proof. According to [19], $\pi_3(P^3(2)) \cong \mathbb{Z}/4$. A generator α for $\pi_3(P^3(2))$ is represented by the composite

$$S^3 \xrightarrow{\eta} S^2 \hookrightarrow P^3(2)$$

and 2α is represented by the adjoint of the composite

$$S^2 \xrightarrow{q} \mathbb{R}P^2 \xrightarrow{E} \Omega P^3(2),$$

where $q: S^2 \rightarrow \mathbb{R}P^2$ is the canonical quotient map. Thus there is a homotopy commutative diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{[2]} & S^2 \\ \downarrow E \circ q & & \downarrow \alpha' \\ \Omega P^3(2) & \xlongequal{\quad} & \Omega P^3(2). \end{array}$$

The assertion follows from the fact that

- 1) $\Omega q: \Omega S^2 \rightarrow \Omega_0 \mathbb{R}P^2$ is a homotopy equivalence;
- 2) $\Omega[2]: (\Omega S^2)\langle 1 \rangle \simeq \Omega S^3 \rightarrow (\Omega S^2)\langle 1 \rangle \simeq \Omega S^3$ is homotopic to the power map 4.
- 3) the power map 4: $(\Omega^2 S^2)\langle 1 \rangle \simeq \Omega^2(S^3\langle 3 \rangle) \rightarrow (\Omega^2 S^2)\langle 1 \rangle \simeq \Omega^2(S^3\langle 3 \rangle)$ is null homotopic [5].

□

Note. The map E_* in low dimensional cases are given by

- 1) $E_*: \pi_1(\mathbb{R}P^2) = \mathbb{Z}/2 \rightarrow \pi_2(P^3(2)) = \mathbb{Z}/2$ is an isomorphism;
- 2) the image of $E_*: \pi_2(\mathbb{R}P^2) = \mathbb{Z} \rightarrow \pi_3(P^3(2)) = \mathbb{Z}/4$ is $\mathbb{Z}/2$;
- 3) $E_*: \pi_3(\mathbb{R}P^2) \rightarrow \pi_4(P^3(2))$ is the trivial map.

Corollary 2.3. *Let $X = P^3(2) \cup_f e^n$ with $n > 4$. If f is essential, then X is not a suspension.*

Corollary 2.4. *Let Y be a 2-connected space and let $f: \Sigma^2 Y \rightarrow P^3(2)$. If f is essential, then f is not desuspensionable.*

2.2. Co- H -spaces and Co- H -maps.

Lemma 2.5. *Let Y be a simply connected finite dimensional CW-complex and let $f: S^n \rightarrow Y$ be a map. Suppose that $n > \dim Y \geq 2$. Then the homotopy cofibre C_f of the map f is a co- H -space if and only if there is a comultiplication of Y such that f is a co- H -map.*

Proof. Clearly if f is a co- H -map with respect to a comultiplication of Y , then C_f is a co- H -space (see, for instance, [1]). Conversely suppose that C_f is a co- H -space. Let $\mu': C_f \rightarrow C_f \vee C_f$ be a comultiplication. Since $Y \vee Y$ is the n -skeleton of $C_f \vee C_f$ and $\dim Y < n$, the map $\mu': C_f \rightarrow C_f \vee C_f$ induces a (unique up to homotopy) map $\mu': Y \rightarrow Y \vee Y$. Let F be the homotopy fibre of the map $Y \vee Y \rightarrow C_f \vee C_f$. By computing the first homology group of F , $S^n \vee S^n$ is the n -skeleton of F and the map $F \rightarrow Y \vee Y$ restricted to $S^n \vee S^n$ is given by $f \vee f$ up to homotopy. Thus f is a co- H -map and hence the result. \square

We regard $P^3(2) = \Sigma\mathbb{R}P^2$ as a co- H -space under the canonical comultiplication.

Lemma 2.6. *Let $n \geq 4$ and let $f: S^n \rightarrow P^3(2)$ be an essential map. Then $X = P^3(2) \cup_f e^{n+1}$ is a nonsuspension co- H -space if and only if f is a co- H -map.*

Proof. By Corollary 2.3, X is not a suspension. Consider the fibre sequence

$$\Sigma\Omega P^3(2) \wedge \Omega P^3(2) \longrightarrow P^3(2) \vee P^3(2) \longrightarrow P^3(2) \times P^3(2).$$

There are two comultiplications on $P^3(2)$ which are given by the canonical comultiplication $\mu': P^3(2) \rightarrow P^3(2) \vee P^3(2)$ and the composite

$$\tilde{\mu}: P^3(2) \xrightarrow{\mu'} P^3(2) \vee P^3(2) \xrightarrow{T} P^3(2) \vee P^3(2),$$

where $T(x, y) = (y, x)$. It follows that f is a co- H -map with respect to μ' if and only if f is a co- H -map with respect to $\tilde{\mu}$. The assertion follows from Lemma 2.5 now. \square

2.3. The Hopf Invariant. Let X and Y be path-connected spaces. Recall that [9, 14] there is a fibre sequence

$$\Sigma\Omega X \wedge \Omega Y \xrightarrow{\phi} X \vee Y \xrightarrow{q} X \times Y$$

and the adjoint $\phi': \Omega X \wedge \Omega Y \rightarrow \Omega(X \vee Y)$ is the Samelson product $[i_1, i_2]$, where $i_1: \Omega X \rightarrow \Omega(X \vee Y)$ and $i_2: \Omega Y \rightarrow \Omega(X \vee Y)$ are the canonical inclusions. Let θ_X and $\theta_Y: X \vee Y \rightarrow X \vee Y$ be the maps defined by the composites

$$X \vee Y \xrightarrow{\text{pinch}} X \hookrightarrow X \vee Y \quad \text{and} \quad X \vee Y \xrightarrow{\text{pinch}} Y \hookrightarrow X \vee Y,$$

respectively. Let $\tilde{H}: \Omega(X \vee Y) \rightarrow \Omega(X \vee Y)$ be a map such that the homotopy class

$$[\tilde{H}] = [\text{id}][\theta_Y]^{-1}[\theta_X]^{-1}$$

in the group $[\Omega(X \vee Y), \Omega(X \vee Y)]$. Clearly the following statements holds:

- 1) The composite $q \circ \tilde{H}: \Omega(X \vee Y) \rightarrow \Omega X \times \Omega Y$ is null homotopic and so the map \tilde{H} lifts to the fibre $\Omega\Sigma(\Omega X \wedge \Omega Y)$ up to homotopy.

- 2) Let $H: \Omega(X \vee Y) \rightarrow \Omega\Sigma(\Omega X \wedge \Omega Y)$ be a (unique up to homotopy) homotopy lifting of \tilde{H} . Then the composite

$$\Omega\Sigma(\Omega X \wedge \Omega Y) \xrightarrow{\Omega\phi} \Omega(X \vee Y) \xrightarrow{H} \Omega\Sigma(\Omega X \wedge \Omega Y)$$

is homotopic to the identity map.

- 3) There is a fibre sequence

$$\Omega X \times \Omega Y \xrightarrow{i_1 \cdot i_2} \Omega(X \vee Y) \xrightarrow{H} \Omega\Sigma(\Omega X \wedge \Omega Y).$$

The map $H: \Omega(X \vee Y) \rightarrow \Omega\Sigma(\Omega X \wedge \Omega Y)$ is called a *Hopf map*. The following proposition is well-known (see, for instance, [1, 4, 10]).

Proposition 2.7. *Let Z be a path connected co- H -space and let $f: \Sigma Y \rightarrow Z$ be any map. Then f is a co- H -map if and only if the composite*

$$Y \xrightarrow{f'} \Omega Z \xrightarrow{\Omega\mu'} \Omega(Z \vee Z) \xrightarrow{H} \Omega\Sigma(\Omega Z \wedge \Omega Z)$$

is null homotopic, where f' is the adjoint map of f .

Let Z be a co- H -space. The composite

$$\Omega Z \xrightarrow{\Omega\mu'} \Omega(Z \vee Z) \xrightarrow{H} \Omega\Sigma(\Omega Z \wedge \Omega Z)$$

is called a *Hopf map* for the co- H -space Z and we abbreviate H for this map. Note that the Hopf map $H: \Omega Z \rightarrow \Omega\Sigma(\Omega Z \wedge \Omega Z)$ depends on the choice of comultiplications on Z . Let $F_H(Z)$ be the homotopy fibre of the Hopf map $H: \Omega Z \rightarrow \Omega\Sigma(\Omega Z \wedge \Omega Z)$ with induced map $\lambda = \lambda_Z: F_H(Z) \rightarrow \Omega Z$. This gives a homotopy functor F_H from co- H -spaces to spaces. By the definition, there is a homotopy pull-back diagram

$$\begin{array}{ccc} F_H(Z) & \xrightarrow{(\lambda, \lambda)} & \Omega Z \times \Omega Z \\ \downarrow \lambda & \text{pull} & \downarrow i_1 \cdot i_2 \\ \Omega Z & \xrightarrow{\Omega\mu'} & \Omega(Z \vee Z). \end{array}$$

For a co- H -space Z , the map $[k]: Z \rightarrow Z$ of degree k is defined to be the composite

$$Z \xrightarrow{\mu'_k} \bigvee_{j=1}^k Z \xrightarrow{\text{fold}} Z,$$

where μ'_k is a k -fold comultiplication. For an H -space X , the power map $k: X \rightarrow X$ of degree k is the composite

$$X \xrightarrow{\Delta} \prod_{j=1}^k X \xrightarrow{\mu_k} X,$$

where μ_k is a k -fold multiplication. (**Note.** The maps $[k]$ and k depend on the choices of k -fold comultiplication of Z and k -fold multiplication of X , respectively.) The following lemma will be useful.

Lemma 2.8. *Let $[k]: Z \rightarrow Z$ be any map of degree k . Then*

$$\Omega([k]) \circ \lambda \simeq k \circ \lambda: F_H(Z) \longrightarrow \Omega Z.$$

The proof is immediate.

3. PROOF OF THEOREM 1.1

3.1. Combinatorial Calculations for the Hopf Invariant. In this subsection, we give some methods how to construct new co- H -maps by given some co- H -maps under certain conditions.

Let Z be a path-connected co- H -space and let X and Y be path connected spaces. Let $f: \Sigma X \rightarrow Z$ and $g: Y \rightarrow Z$ be co- H -maps and let $f': X \rightarrow \Omega Z$ and $g': Y \rightarrow \Omega Z$ be the adjoint map of f and g , respectively. Let n and m be any positive integers. The group $K_{n,m}(X, Y, f, g, Z)$ is defined to be the subgroup

$$[X^n \times Y^m, \Omega Z]$$

generated by the elements x_i and y_j for $1 \leq i \leq n$ and $1 \leq j \leq m$, where x_i and y_j are represented by the composite

$$X^m \times Y^n \xrightarrow{\pi_i} X \xrightarrow{f'} \Omega Z \quad \text{and} \quad X^m \times Y^n \xrightarrow{\pi_{n+j}} Y \xrightarrow{g'} \Omega Z,$$

where π_k is the k -th coordinate projection. Observe that the element $x_1 \dots x_n y_1 \dots y_m$ is represented by the composite

$$X^n \times Y^m \xrightarrow{\text{pinch}} J_n(X) \times J_m(Y) \hookrightarrow J(X) \times J(Y) \simeq \Omega \Sigma X \times \Omega \Sigma Y \xrightarrow{\Omega f \times \Omega g} \Omega Z \times \Omega Z \xrightarrow{\mu} \Omega Z.$$

Let s_1 and $s_2: Z \longrightarrow Z \vee Z$ be the first and the second coordinate inclusions, respectively. Let $K_{n,m}(X, Y, f, g, Z \vee Z)$ be the subgroup of

$$[X^n \times Y^m, \Omega(Z \vee Z)]$$

generated by the elements $x_{\epsilon,i}$ and $y_{\epsilon,j}$ for $\epsilon = 1, 2$, $1 \leq i \leq n$ and $1 \leq j \leq m$, where $x_{\epsilon,i}$ and $y_{\epsilon,j}$ are represented by the composites

$$X^n \times Y^m \xrightarrow{\pi_i} X \xrightarrow{f'} \Omega Z \xrightarrow{\Omega i_\epsilon} \Omega(Z \vee Z) \quad \text{and} \quad X^n \times Y^m \xrightarrow{\pi_{n+j}} Y \xrightarrow{g'} \Omega Z \xrightarrow{s_\epsilon} \Omega(Z \vee Z),$$

respectively.

Lemma 3.1. *Suppose that $f: \Sigma X \rightarrow Z$ and $g: \Sigma Y \rightarrow Z$ are co- H -maps. Then the map $\Omega\mu': \Omega Z \rightarrow \Omega(Z \vee Z)$ induces a groups homomorphism*

$$d^1: K_{n,m}(X, Y, f, g, Z) \longrightarrow K_{n,m}(X, Y, f, g, Z \vee Z)$$

for any n and m with the following formula

$$d^1(x_i) = x_{1,i}x_{2,i} \quad \text{and} \quad \Omega\mu'_*(y_j) = y_{1,j}y_{2,j}.$$

for any i and j .

The proof follows from the definition.

Note. The group $K_{n,m}(X, Y, f, g, Z)$ is a modification of the Cohen group K_n introduced in [6]. This group is free in general, for instance, $X = Y = \mathbb{C}P^\infty$, $Z = \Sigma\mathbb{C}P^\infty$ and $f = g = \text{id}$. We are particularly interested in the case where $X = \mathbb{R}P^2$, $Y = S^2$, $Z = P^3(2)$, $f = \text{id}$ and g is the suspension of the canonical quotient $S^2 \rightarrow \mathbb{R}P^2$. In this case, we will obtain some special properties.

Observe that there is a short exact sequence of groups

$$[W, \Omega\Sigma(\Omega Z \wedge \Omega Z)] \hookrightarrow [W, \Omega(Z \vee Z)] \twoheadrightarrow [W, \Omega Z \times \Omega Z]$$

for any space W . Let $\tilde{H}: \Omega(Z \vee Z) \rightarrow \Omega(Z \vee Z)$ be the map defined in Subsection 2.3. Then the image of the function

$$\tilde{H}_*: [W, \Omega(Z \vee Z)] \longrightarrow [W, \Omega(Z \vee Z)]$$

lies in the subgroup $[W, \Omega\Sigma(\Omega Z \wedge \Omega Z)]$. Thus the image of the function

$$[W, \Omega Z] \xrightarrow{\Omega\mu'_*} [W, \Omega(Z \vee Z)] \xrightarrow{\tilde{H}_*} [W, \Omega(Z \vee Z)]$$

lies in the subgroup $[W, \Omega\Sigma(\Omega Z \wedge \Omega Z)]$ and so it induces a function

$$[W, \Omega Z] \longrightarrow [W, \Omega\Sigma(\Omega Z \wedge \Omega Z)]$$

which is the same as the function induced by the Hopf map

$$H: \Omega Z \rightarrow \Omega\Sigma(\Omega Z \wedge \Omega Z)$$

by the definition. The map Ωi_1 and $\Omega i_2: \Omega Z \rightarrow \Omega(Z \vee Z)$ induce group homomorphisms d^0 and $d^2: K_{n,m}(X, Y, f, g, Z) \rightarrow K_{n,m}(X, Y, f, g, Z \vee Z)$, respectively, with $d^0(x_i) = x_{1,i}$, $d^0(y_j) = y_{1,j}$, $d^2(x_i) = x_{2,i}$ and $d^2(y_j) = y_{2,j}$. By the definition of the map \tilde{H} , we have

Lemma 3.2. *The Hopf map H induces a function*

$$\delta: K_{n,m}(X, Y, f, g, Z) \rightarrow K_{n,m}(X, Y, f, g, Z \vee Z)$$

such that

$$\delta(w) = d^1(w)d^2(w)^{-1}d^0(w)^{-1}$$

for any word $w \in K_{n,m}(X, Y, f, g, Z)$.

Example. Let $Y = *$ and let $Z = \Sigma X$ with $f = \text{id}$. Suppose that X is conilpotent, that is, the reduced diagonal $\bar{\Delta}: X \rightarrow X \wedge X$ is null homotopic. Let $w = x_1x_2$. Then

$$\begin{aligned} \delta(w) &= x_{1,1}x_{2,1}x_{1,2}x_{2,2}(x_{2,1}x_{2,2})^{-1}(x_{1,1}x_{1,2})^{-1} \\ &= x_{1,1}x_{2,1}x_{1,2}x_{2,1}^{-1}x_{1,2}^{-1}x_{1,1}^{-1} = [x_{1,1}, [x_{2,1}, x_{1,2}]] [x_{2,1}, x_{1,2}], \end{aligned}$$

where $[a, b] = aba^{-1}b^{-1}$. The element $[x_{1,1}, [x_{2,1}, x_{1,2}]] = 1$ because it is represented by the composite

$$X^2 \xrightarrow{\text{pinch}} X^{(2)} \xrightarrow{\bar{\Delta} \wedge \text{id}} X^{(3)} \xrightarrow{\beta} J(X \vee X)$$

for certain 3-fold Samelson product β . Recall that $\Sigma q_n: \Sigma X^n \rightarrow \Sigma J_n(X)$ has a cross-section. It follows that

$$q_n^*: [J_n(X), \Omega W] \longrightarrow [X^n, \Omega W]$$

is a monomorphism for any W and so the above formula shows that the composite

$$J_2(X) \longrightarrow J(X) \xrightarrow{H} \Omega \Sigma(J(X) \wedge J(X))$$

is homotopic to the composite

$$J_2(X) \xrightarrow{\text{pinch}} X^{(2)} \xrightarrow{\tau} X^{(2)} \hookrightarrow \Omega \Sigma(J(X) \wedge J(X))$$

when X is conilpotent, where $\tau(a \wedge b) = b \wedge a$.

Now we consider the special case where $X = \mathbb{R}P^2$, $Y = S^2$, $Z = P^3(2)$, $f = \text{id}$ and $g = \Sigma q$, where $q: S^2 \rightarrow \mathbb{R}P^2$ is the canonical quotient map. We abbreviate $K_{n,m}(\mathbb{R}P^2, S^2, \text{id}, \Sigma q, P^3(2))$ and $K_{n,m}(\mathbb{R}P^2, S^2, \text{id}, \Sigma q, P^3(2) \vee P^3(2))$ to $K_{n,m}(S^2, q, P^3(2))$ and $K_{n,m}(S^2, q, P^3(2) \vee P^3(2))$, respectively, when there are no confusions.

Lemma 3.3. *In the group $K_{n,m}(S^2, q, P^3(2) \vee P^3(2))$, the following relations hold*

$$[x_{\epsilon,i}, y_{\epsilon',j}] = [y_{\epsilon,i}, y_{\epsilon',j}] = 1$$

for $\epsilon, \epsilon' = 1, 2$, $1 \leq i \leq n$ and $1 \leq j \leq m$.

Proof. It suffices to show that $[x_{\epsilon,i}, y_{\epsilon',j}] = 1$. We may assume that $n = m = 1$. We only prove that $[x_{1,1}, y_{2,1}] = 1$. The other cases follow from the same lines. Observe that the element $[x_{1,1}, y_{2,1}]$ is represented by the composite

$$\mathbb{R}P^2 \times S^2 \xrightarrow{\text{pinch}} \mathbb{R}P^2 \wedge S^2 \xrightarrow{\text{id} \wedge q} \mathbb{R}P^2 \wedge \mathbb{R}P^2 \xrightarrow{[i_1, i_2]} J(\mathbb{R}P^2 \vee \mathbb{R}P^2).$$

The adjoint map of this composite is given by the composite

$$\Sigma(\mathbb{R}P^2 \times S^2) \xrightarrow{\text{pinch}} \Sigma\mathbb{R}P^2 \wedge S^2 \xrightarrow{\Sigma \text{id} \wedge q} \Sigma\mathbb{R}P^2 \wedge \mathbb{R}P^2 \xrightarrow{[i_1, i_2]^*} \Sigma(\mathbb{R}P^2 \vee \mathbb{R}P^2),$$

where $[i_1, i_2]^*$ is the adjoint map of $[i_1, i_2]$. Recall that $\Sigma q: S^3 \rightarrow P^3(2)$ is homotopic to the composite $S^3 \xrightarrow{2} S^3 \xrightarrow{\eta} S^2 \hookrightarrow P^3(2)$. Thus there is a homotopy commutative diagram

$$\begin{array}{ccccc} S^5 & \xleftarrow{\text{pinch}} & \Sigma\mathbb{R}P^2 \wedge S^2 & \xrightarrow{\Sigma \text{id} \wedge q} & \Sigma\mathbb{R}P^2 \wedge \mathbb{R}P^2 \\ \downarrow \eta & & \downarrow [2] & & \parallel \\ S^4 \hookrightarrow & \mathbb{R}P^2 \wedge S^3 & & \Sigma\mathbb{R}P^2 \wedge \mathbb{R}P^2 \\ \downarrow \eta & \downarrow \mathbb{R}P^2 \wedge \eta & & \parallel \\ S^3 \hookrightarrow & \mathbb{R}P^2 \wedge S^2 \hookrightarrow & \Sigma\mathbb{R}P^2 \wedge \mathbb{R}P^2. \end{array}$$

The assertion follows from the following lemma. □

Lemma 3.4. *The composite*

$$P^5(2) \xrightarrow{\text{pinch}} S^5 \xrightarrow{\eta^2} S^3 \hookrightarrow P^4(2)$$

is null homotopic.

Proof. By direct calculation, $\pi_5(P^4(2)) = \pi_5^s(P^4(2)) = \mathbb{Z}/4$. It follows that the composite $S^5 \xrightarrow{\eta^2} S^3 \hookrightarrow P^4(2)$ is divisible by 2 and hence the result. □

Let μ_q be the composite

$$J(\mathbb{R}P^2) \times J(S^2) \xrightarrow{\text{id} \times \Omega \Sigma q} J(\mathbb{R}P^2) \times J(\mathbb{R}P^2) \xrightarrow{\mu} J(\mathbb{R}P^2).$$

Lemma 3.5. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} J(\mathbb{R}P^2) \times J(S^2) & \xrightarrow{\mu_q} & J(\mathbb{R}P^2) \\ \downarrow \pi_1 & & \downarrow H \\ J(\mathbb{R}P^2) & \xrightarrow{H} & J((J(\mathbb{R}P^2))^{(2)}). \end{array}$$

Proof. Consider the function

$$\delta: K_{n,m}(S^2, q, P^3(2)) \longrightarrow K_{n,m}(S^2, q, P^3(2) \wedge P^3(2)).$$

By Lemma 3.3, we have

$$\begin{aligned} \delta(x_1 x_2 \dots x_n y_1 y_2 \dots y_m) &= x_{1,1} x_{2,1} \dots x_{1,n} x_{2,n} y_{1,1} y_{2,1} \dots y_{1,m} y_{2,m} \\ &\cdot (y_{2,1} y_{2,2} \dots y_{2,m})^{-1} (x_{2,1} x_{2,2} \dots x_{2,n})^{-1} (y_{1,1} y_{1,2} \dots y_{1,n})^{-1} (x_{1,1} x_{1,2} \dots x_{1,n})^{-1} \\ &= \delta(x_1 x_2 \dots x_n) \delta(y_1 y_2 \dots y_m) = \delta(x_1 x_2 \dots x_n). \end{aligned}$$

Thus there is a homotopy commutative diagram

$$\begin{array}{ccc} J_n(\mathbb{R}P^2) \times J_m(S^2) & \xrightarrow{\mu_q} & J(\mathbb{R}P^2) \\ \downarrow \pi_1 & & \downarrow H \\ J_n(\mathbb{R}P^2) & \xrightarrow{H} & J((J(\mathbb{R}P^2))^{(2)}) \end{array}$$

for any n and m . The assertion follows from the fact that

$$\lim^1_{n,m} [J_n(X) \times J_m(Y), \Omega Z] = 0$$

for any spaces X , Y and Z by the suspension splitting theorem for $J(X) \times J(Y)$. \square

Corollary 3.6. *Let $f: \Sigma Y \rightarrow P^3(2)$ be any co- H -map. Then the composite*

$$Y \times J(S^2) \xrightarrow{f' \times \Omega \Sigma q} \Omega P^3(2) \times \Omega P^3(2) \xrightarrow{\mu} \Omega P^3(2) \xrightarrow{H} \Omega \Sigma((\Omega P^3(2))^{(2)})$$

is null homotopic.

3.2. Proof of Theorem 1.1. Let $j: P^3(2) \rightarrow BSO(3)$ be the inclusion.

Lemma 3.7. *The composite*

$$F_H(P^3(2)) \xrightarrow{\lambda} \Omega P^3(2) \xrightarrow{\Omega j} SO(3) \xrightarrow{2} SO(3)$$

is null homotopic.

Proof. By Lemma 2.8, $\Omega[2] \circ \lambda = 2 \circ \lambda$. Recall that the degree 2 map $[2]: P^3(2) \rightarrow P^3(2)$ is homotopic to the composite

$$P^3(2) \xrightarrow{\text{pinch}} S^3 \xrightarrow{\eta} S^2 \hookrightarrow P^3(2).$$

Thus the composite $j \circ [2]: P^3(2) \rightarrow BSO(3)$ is null homotopic because $\pi_3(BSO(3)) = \pi_2(SO(3)) = 0$. It follows that

$$2 \circ \Omega j \circ f = \Omega j \circ 2 \circ f = \Omega j \circ \Omega[2] \circ f = \Omega(j \circ [2]) \circ f \simeq *$$

and hence the result. \square

Let $\phi: P^4(2) \rightarrow P^3(2)$ be the map in the cofibre sequence

$$\mathbb{R}P^2 \hookrightarrow \mathbb{R}P^4 \longrightarrow P^4(2) \xrightarrow{\phi} P^3(2).$$

Lemma 3.8. *Let $\phi: P^4(2) \rightarrow P^3(2)$ be the map defined above. Then*

- 1) ϕ restricted to S^3 is homotopic to $\Sigma q: S^3 \rightarrow P^3(2)$ and
- 2) ϕ is a co- H -map

Proof. Assertion (1) is obvious. (2). Let $\phi': P^3(2) \rightarrow \Omega P^3(2)$. Then

$$\phi'_*: \bar{H}_*(P^3(2)) \rightarrow \bar{H}_*(\Omega P^3(2))$$

is zero because $H_*(\Omega P^3(2)) \rightarrow H_*(\Omega \Sigma \mathbb{R}P^4)$ is a monomorphism. By Assertion (1), the map ϕ restricted to S^3 is a co- H -map and so there is a homotopy commutative diagram

$$\begin{array}{ccc} \Omega P^3(2) & \xrightarrow{H} & \Omega \Sigma((\Omega P^3(2))^{(2)}) \\ \uparrow \phi' & & \uparrow \bar{\phi} \\ P^3(2) & \xrightarrow{\text{pinch}} & S^3. \end{array}$$

Since $\phi'_*: H_3(P^3(2)) \rightarrow H_3(\Omega P^3(2))$ is zero, the homomorphism

$$\bar{\phi}_*: H_3(S^3) \rightarrow H_3(\Omega \Sigma(\Omega P^3(2))^{(2)})$$

is zero or the homotopy class $[\bar{\phi}] \in \pi_3(\Omega \Sigma(\Omega P^3(2))^{(2)})$ lies in the Hurewicz kernel. Now we compute the homotopy group

$$\pi_3(\Omega \Sigma(\Omega P^3(2))^{(2)}) = \pi_4(\Sigma(\mathbb{R}P^2)^{(2)}) \oplus \pi_4(\Sigma(\mathbb{R}P^2)^{(3)})^{\oplus 2} = \pi_4(\Sigma(\mathbb{R}P^2)^{(2)}) \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

The cofibre sequence

$$\Sigma \mathbb{R}P^4 \longrightarrow \Sigma \mathbb{R}P^2 \wedge \mathbb{R}P^2 \longrightarrow S^4$$

induces an exact sequence on low homotopy groups

$$\pi_5(S^4) = \mathbb{Z}/2 \xrightarrow{0} \pi_4(\Sigma \mathbb{R}P^4) \rightarrow \pi_4(\Sigma(\mathbb{R}P^2)^{(2)}) \xrightarrow{0} \pi_4(S^4)$$

and so

$$\pi_4(\Sigma(\mathbb{R}P^2)^{(2)}) \cong \pi_4(\Sigma \mathbb{R}P^4) \cong \pi_4(P^3(2)) = \mathbb{Z}/4.$$

The generator α_2 for $\pi_3(\Omega \Sigma(\mathbb{R}P^2)^{(2)}) = \mathbb{Z}/4$ has the nontrivial Hurewicz image in $H_3(\Omega \Sigma(\mathbb{R}P^2)^{(2)})$. It follows that the kernel of the Hurewicz map

$$\pi_3(\Omega \Sigma(\Omega P^3(2))^{(2)}) \longrightarrow H_3(\Omega \Sigma(\Omega P^3(2))^{(2)})$$

is $\mathbb{Z}/2$ generated by $2\alpha_2$. Thus the map

$$\bar{\phi}: S^3 \longrightarrow \Omega\Sigma(\Omega P^3(2))^{(2)}$$

is divisible by 2 and hence the result. \square

Proof of Theorem 1.1. (1). By Lemma 3.7, there is homotopy commutative diagram

$$\begin{array}{ccc} F_H(P^3(2)) & \xrightarrow{\lambda} & \Omega P^3(2) \\ \downarrow & & \downarrow \\ SO(3)\{2\} & \longrightarrow & SO(3). \end{array}$$

Assertion (1) follows by taking the universal covers.

(2). Let $\phi: P^4(2) \rightarrow P^3(2)$ be the map in Lemma 3.8. Consider the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccccc} \Omega S^3 & \longrightarrow & S^3\{2\} & \longrightarrow & S^3 & \xrightarrow{[2]} & S^3 \\ \downarrow \Omega g & & \downarrow \bar{g} & & \downarrow & & \downarrow g \\ \Omega P^4(2) & \xlongequal{\quad} & \Omega P^4(2) & \longrightarrow & * & \longrightarrow & P^4(2). \end{array}$$

Since the fibre sequence

$$\Omega S^3 \longrightarrow S^3\{2\} \longrightarrow S^3$$

is principal, there is a right $J(S^2)$ -action

$$\mu: S^3\{2\} \times J(S^2) \longrightarrow S^3\{2\}$$

with a homotopy commutative diagram

$$\begin{array}{ccc} S^3\{2\} \times J(S^2) & \xrightarrow{\mu} & S^3\{2\} \\ \downarrow \bar{g} \times \Omega g & & \downarrow \bar{g} \\ J(P^3(2)) \times J(P^3(2)) & \xrightarrow{\mu} & J(P^3(2)). \end{array}$$

Let $\tilde{s}: S^3\{2\} \rightarrow J(\mathbb{R}P^2)$ be the composite

$$S^3\{2\} \xrightarrow{\bar{g}} J(P^3(2)) \xrightarrow{\Omega\phi} J(\mathbb{R}P^2).$$

It follows that there is a homotopy commutative diagram

$$\begin{array}{ccc} S^3\{2\} \times J(S^2) & \xrightarrow{\mu} & S^3\{2\} \\ s \times J(q) \downarrow & & \downarrow \tilde{s} \\ J(\mathbb{RP}^2) \times J(\mathbb{RP}^2) & \xrightarrow{\mu} & J(P^3(2)). \end{array}$$

By Corollary 3.6, the composite

$$P^3(2) \times J(S^2) \xrightarrow{\mu} S^3\{2\} \xrightarrow{\tilde{s}} J(\mathbb{RP}^2) \xrightarrow{H} \Omega\Sigma(J(\mathbb{RP}^2))^{(2)}$$

is null homotopic. By the suspension splitting of $S^3\{2\}$, the map

$$\mu^*: [S^3\{2\}, \Omega W] \longrightarrow [P^3(2) \times J(S^2), \Omega W]$$

is a monomorphism for any W . Thus the composite

$$S^3\{2\} \xrightarrow{\tilde{s}} J(\mathbb{RP}^2) \xrightarrow{H} \Omega\Sigma(J(\mathbb{RP}^2))^{(2)}$$

is null homotopic and so the map \tilde{s} lifts to $F_H(P^3(2))$. Let $\bar{s}: S^3\{2\} \rightarrow F_H(P^3(2))$ be a lifting of \tilde{s} . Since $S^3\{2\}$ is simply connected, the map \bar{s} lifts to the universal cover $F_H(P^3(2))\langle 1 \rangle$ and let $s: S^3\{2\} \rightarrow F_H(P^3(2))\langle 1 \rangle$ be a lifting of \bar{s} . Then composite

$$S^3\{2\} \xrightarrow{s} F_H(P^3(2))\langle 1 \rangle \xrightarrow{\theta} S^3\{2\}$$

is a homotopy equivalence because it induces an isomorphism on H_3 of the atomic space $S^3\{2\}$. The assertion follows. \square

4. FURTHER PROPERTIES OF THE SPACE $F_H(P^3(2))$

Let $\bar{\mu}': P^3(2) \rightarrow BSO(3) \vee BSO(3)$ be the composite

$$P^3(2) \xrightarrow{\mu'} P^3(2) \vee P^3(2) \hookrightarrow BSO(3) \vee BSO(3).$$

Then there is a homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc} \Sigma\mathbb{RP}_2^4 \vee P^6(2) & \longrightarrow & P^3(2) & \hookrightarrow & BSO(3) \\ \downarrow \bar{\phi} & & \downarrow \bar{\mu}' & & \downarrow \Delta \\ \Sigma SO(3) \wedge SO(3) & \longrightarrow & BSO(3) \vee BSO(3) & \hookrightarrow & BSO(3) \times BSO(3). \end{array}$$

Lemma 4.1. *The induced map in mod 2 homology*

$$\bar{\phi}_*: H_*(\Sigma\mathbb{RP}_2^4 \vee P^6(2)) \rightarrow H_*(\Sigma SO(3) \wedge SO(3))$$

is a monomorphism.

Proof. The assertion follows by comparing the Serre cohomology spectral sequences for the fibre sequences in the following homotopy commutative diagram

$$\begin{array}{ccccc} SO(3) & \longrightarrow & \Sigma\mathbb{RP}_2^4 \vee P^6(2) & \longrightarrow & P^3(2) \\ \downarrow \Delta & & \downarrow \bar{\phi} & & \downarrow \bar{\mu}' \\ SO(3) \times SO(3) & \longrightarrow & \Sigma SO(3) \wedge SO(3) & \longrightarrow & BSO(3) \vee BSO(3). \end{array}$$

□

Lemma 4.2. *There is a homotopy decomposition*

$$\Sigma SO(3) \wedge SO(3) \simeq X^7 \vee P^6(2) \vee P^6(2),$$

where X^7 is the homotopy cofibre of the composite

$$S^6 \xrightarrow{\nu'} S^3 \hookrightarrow \Sigma\mathbb{RP}^2 \wedge \mathbb{RP}^2.$$

Proof. Consider the cofibre sequence

$$\Sigma SO(3) \wedge S^2 \xrightarrow{\Sigma \text{id} \wedge q} \Sigma SO(3) \wedge \mathbb{RP}^2 \hookrightarrow \Sigma SO(3) \wedge SO(3),$$

where $q: S^2 \rightarrow \mathbb{RP}^2$ is the quotient map. By the proof of Lemma 3.3, the map $\Sigma \text{id} \wedge q: \Sigma\mathbb{RP}^2 \wedge S^2 \rightarrow \Sigma\mathbb{RP}^2 \wedge \mathbb{RP}^2$ is null homotopic. It follows that the 6-skeleton of $\Sigma SO(3) \wedge SO(3)$ is homotopic to $\Sigma\mathbb{RP}^2 \wedge \mathbb{RP}^2 \vee P^6(2) \vee P^6(2)$. Recall that $\Sigma^2 SO(3) \simeq P^4(2) \vee S^5$. Let $r: \Sigma^2 SO(3) \rightarrow P^4(2)$ be a retraction and let $s: S^5 \rightarrow \Sigma^2 SO(3)$ be a cross-section of the pinch map. It suffices to determine the composite

$$f: S^6 \xrightarrow{\Sigma s} \Sigma SO(3) \wedge S^2 \xrightarrow{\Sigma \text{id} \wedge q} \Sigma SO(3) \wedge \mathbb{RP}^2.$$

Consider the homotopy commutative diagram

$$\begin{array}{ccccc}
S^6 & \xrightarrow{\Sigma s} & \Sigma SO(3) \wedge S^2 & \xrightarrow{\Sigma \text{id} \wedge q} & \Sigma SO(3) \wedge \mathbb{R}P^2 \\
\downarrow [2] & & \downarrow [2] & & \parallel \\
S^6 & \xrightarrow{\Sigma s} & SO(3) \wedge S^3 & & \Sigma SO(3) \wedge \mathbb{R}P^2 \\
\downarrow g & & \downarrow \text{id} \wedge \eta & & \parallel \\
P^4(2) \vee S^5 \simeq & & SO(3) \wedge S^2 \hookrightarrow & \longrightarrow & \Sigma SO(3) \wedge \mathbb{R}P^2.
\end{array}$$

Observe that the adjoint map $(\text{id} \wedge \eta)'$ is homotopic to the composite

$$SO(3) \wedge S^2 \xrightarrow{\text{id} \wedge \eta} SO(3) \wedge J(S^1) \xrightarrow{\theta} J(SO(3) \wedge S^1) \simeq \Omega(SO(3) \wedge S^2),$$

where

$$\theta(y \wedge (x_1 x_2 \dots x_n)) = (y \wedge x_1)(y \wedge x_2) \dots (y \wedge x_n).$$

By computing homology for the commutative diagram

$$\begin{array}{ccc}
SO(3) \times S^1 \times S^1 & \longrightarrow & (SO(3) \times S^1) \times (SO(3) \times S^1) \\
\downarrow & & \downarrow \\
SO(3) \wedge J_2(S^1) & \xrightarrow{\theta} & J_2(SO(3) \wedge S^1),
\end{array}$$

the map

$$(\text{id} \wedge \eta)'_*: H_5(\Sigma^2 SO(3)) = \mathbb{Z}/2 \rightarrow H_5(\Omega \Sigma^2 SO(3))$$

is a monomorphism and so the homotopy class

$$[g'] \in \pi_5(\Omega(P^4(2) \vee S^5)) = \pi_5(\Omega P^4(2)) \oplus \pi_5(\Omega S^5) = \pi_5(\Omega P^4(2)) \oplus \mathbb{Z}/2$$

has a nontrivial image in mod 2 homology. Let $[g'] = \alpha_1 + \alpha_2$ with $\alpha_1 \in \pi_5(\Omega P^4(2))$ and $\alpha_2 \in \pi_5(\Omega S^5)$. According to [8], the element α_1 generates a $\mathbb{Z}/4$ summand in $\pi_5(\Omega P^4(2))$. Let Z be the homotopy fibre of the composite

$$P^4(2) \xrightarrow{\text{pinch}} S^4 \hookrightarrow BS^3.$$

Then Z is the homotopy cofibre of the map $S^6 \xrightarrow{(2, \nu')} S^6 \vee S^3$, see [19]. It follows that $\pi_6(Z) = \mathbb{Z}/8$ and $\pi_6(P^4(2)) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$. Thus there is a homotopy commutative

diagram

$$\begin{array}{ccc} S^6 & \xrightarrow{g \circ [2]} & P^4(2) \vee S^5 \\ \downarrow \nu' & & \parallel \\ S^3 & \hookrightarrow & P^4(2) \vee S^5 \end{array}$$

and hence the result. \square

Note. This lemma was also known by Mukai [15]. Recall that the 7-skeleton of $Sp(2)$ is $S^3 \cup_{\nu'} e^7$. This lemma shows that there is a map from the 7-skeleton of $Sp(2)$ to $\Sigma SO(3) \wedge SO(3)$ which induces a monomorphism in mod 2 homology. According to [19],

$$\pi_6(\Sigma \mathbb{R}P^2 \wedge \mathbb{R}P^2) = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

and the composite $S^6 \xrightarrow{\nu'} S^3 \hookrightarrow \Sigma \mathbb{R}P^2 \wedge \mathbb{R}P^2$ is essential. Thus X^7 is indecomposable.

Let $\bar{\phi}: \Sigma \mathbb{R}P^4_2 \vee P^6(2) \rightarrow \Sigma SO(3) \wedge SO(3)$ be the map defined in Lemma 4.1.

Lemma 4.3. *The map*

$$\bar{\phi} \wedge \text{id}: (\Sigma \mathbb{R}P^4_2 \vee P^6(2)) \wedge P^3(2) \longrightarrow \Sigma SO(3) \wedge SO(3) \wedge P^3(2)$$

has a retraction. In particular, $\Sigma \mathbb{R}P^4_2 \wedge P^3(2)$ is a retract of $X^7 \wedge P^3(2)$.

Proof. Recall that $\Sigma^2 SO(3) \simeq S^5 \vee P^4(2)$. The assertion follows from the homotopy decomposition [8]

$$\Sigma(\mathbb{R}P^2)^{(3)} \simeq \Sigma \mathbb{C}P^2 \wedge \mathbb{R}P^2 \vee P^6(2) \vee P^6(2) \simeq \Sigma \mathbb{R}P^4_2 \wedge \mathbb{R}P^2 \vee P^6(2).$$

\square

Let \tilde{F}_H be the space in the homotopy pull-back diagram

$$\begin{array}{ccc} \tilde{F}_H & \xrightarrow{\psi} & \Omega(\Sigma \mathbb{R}P^4_2 \vee P^6(2)) \\ \downarrow & \text{pull} & \downarrow \\ F_H(P^3(2)) & \longrightarrow & \Omega P^3(2) \end{array}$$

and let T^5 be the homotopy fibre of the pinch map $\Sigma \mathbb{R}P^4_2 \rightarrow P^5(2)$.

Proposition 4.4. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} \tilde{F}_H & \xrightarrow{\psi} & \Omega(\Sigma\mathbb{R}P_2^4 \vee P^6(2)) \\ \parallel & & \uparrow \\ \tilde{F}_H & \longrightarrow & \Omega T^5. \end{array}$$

Proof. Consider the homotopy commutative diagram

$$\begin{array}{ccccc} \Omega(\Sigma\mathbb{R}P_2^4 \vee P^6(2)) & \longrightarrow & \Omega P^3(2) & \xrightarrow{H} & \Omega\Sigma(\Omega P^3(2) \wedge \Omega P^3(2)) \\ \downarrow \Omega\bar{\phi} & & \downarrow \Omega\bar{\mu}' & & \downarrow \\ \Omega\Sigma(SO(3) \wedge SO(3)) & \longrightarrow & \Omega(BSO(3) \vee BSO(3)) & \xrightarrow{\tilde{H}} & \Omega\Sigma(SO(3) \wedge SO(3)), \end{array}$$

where the composite of the maps in the bottom row is the identity map. Let $F_{\bar{f}}$ be the homotopy fibre of the composite

$$\bar{f}: \Sigma\mathbb{R}P_2^4 \vee P^6(2) \xrightarrow{\bar{\phi}} \Sigma SO(3) \wedge SO(3) \xrightarrow{r} X^7 \vee P^6(2),$$

where r is a choice of the retractions such that

$$\bar{f} \simeq f_1 \vee \text{id}: \Sigma\mathbb{R}P_2^4 \vee P^6(2) \longrightarrow X^7 \vee P^6(2)$$

and f_1 induces a monomorphism in mod 2 homology. Then there is a homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc} \tilde{F}_H & \longrightarrow & \Omega(\Sigma\mathbb{R}P_2^4 \vee P^6(2)) & \longrightarrow & \Omega\Sigma(\Omega P^3(2) \wedge \Omega P^3(2)) \\ \downarrow & & \parallel & & \downarrow \\ \Omega F_{\bar{f}} & \longrightarrow & \Omega(\Sigma\mathbb{R}P_2^4 \vee P^6(2)) & \xrightarrow{\Omega\bar{f}} & \Omega(X^7 \vee P^6(2)). \end{array}$$

Consider the commutative diagram of fibre sequences

$$\begin{array}{ccccc} \Sigma(\Omega\Sigma\mathbb{R}P_2^4) \wedge \Omega P^6(2) & \longrightarrow & \Sigma\mathbb{R}P_2^4 \vee P^6(2) & \longrightarrow & \Sigma\mathbb{R}P_2^4 \times P^6(2) \\ \downarrow \Sigma\Omega f_1 \wedge \text{id} & & \downarrow \bar{f} & & \downarrow f_1 \times \text{id} \\ \Sigma(\Omega X^7) \wedge \Omega P^6(2) & \longrightarrow & X^7 \vee P^6(2) & \longrightarrow & X^7 \times P^6(2). \end{array}$$

By Lemma 4.3, the composite

$$\begin{aligned} \Sigma(\Omega\Sigma\mathbb{R}P_2^4) \wedge \Omega P^6(2) &\simeq \Sigma \bigvee_{i,j=1}^{\infty} (\mathbb{R}P_2^4)^{(i)} \wedge (P^5(2))^{(j)} \xrightarrow{\Sigma\Omega f_1 \wedge \text{id}} \Sigma(\Omega X^7) \wedge \Omega P^6(2) \\ &\hookrightarrow \Sigma(\Omega\Sigma(SO(3) \wedge SO(3))) \wedge \Omega P^6(2) \simeq \Sigma \bigvee_{i,j=1}^{\infty} (SO(3))^{(2i)} \wedge (P^5(2))^{(j)} \end{aligned}$$

has a retraction. It follows that the composite

$$\Omega F_{\bar{f}} \longrightarrow \Omega(\Sigma\mathbb{R}P_2^4 \vee P^6(2)) \xrightarrow{\tilde{H}} \Omega\Sigma((\Omega\Sigma\mathbb{R}P_2^4) \wedge \Omega P^6(2)) \times \Omega P^6(2)$$

is null homotopic and so there is a homotopy commutative diagram

$$\begin{array}{ccc} \Omega F_{\bar{f}} & \longrightarrow & \Omega(\Sigma\mathbb{R}P_2^4 \vee P^6(2)) \\ \downarrow & & \uparrow \cup \\ \Omega F_{f_1} & \longrightarrow & \Omega\Sigma\mathbb{R}P_2^4, \end{array}$$

where F_{f_1} is the homotopy fibre of $f_1: \Sigma\mathbb{R}P_2^4 \rightarrow X^7$. By Lemma 4.2, the pinch map $\Sigma\mathbb{R}P_2^4 \rightarrow P^5(2)$ factors through X^7 . Thus there is a homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc} F_{f_1} & \xlongequal{\quad} & F_{f_1} & & \\ \downarrow & & \downarrow & & \\ T^5 & \longrightarrow & \Sigma\mathbb{R}P_2^4 & \longrightarrow & P^5(2) \\ \downarrow & & \downarrow f_1 & & \parallel \\ F_q & \longrightarrow & X^7 & \xrightarrow{q} & P^5(2) \end{array}$$

and hence the result. \square

Corollary 4.5. *The kernel of the composite*

$$\pi_*(\Omega(\Sigma\mathbb{R}P_2^4 \vee P^6(2))) \longrightarrow \pi_*(\Omega P^3(2)) \xrightarrow{H_*} \pi_*(\Omega\Sigma(\Omega P^3(2) \wedge \Omega P^3(2)))$$

lies in the image of the homomorphism $\pi_(\Omega T^5) \longrightarrow \pi_*(\Omega(\Sigma\mathbb{R}P_2^4 \vee P^6(2)))$.*

5. PROOFS OF THEOREMS 1.5, 1.6 AND 1.8

Proof of Theorem 1.6. Let $\theta: F_H(P^3(2))\langle 1 \rangle \rightarrow S^3\{2\}$ be the map in Theorem 1.1 and let $j: F_\theta \rightarrow F_H(P^3(2))\langle 1 \rangle$ be the map in the fibre sequence

$$F_\theta \xrightarrow{j} F_H(P^3(2))\langle 1 \rangle \longrightarrow S^3\{2\}.$$

Then the map j lifts to $F_{\tilde{H}}$, where $F_{\tilde{H}}$ is defined in Proposition 4.4. Consider the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc} \Sigma\mathbb{R}P_2^4 \vee P^6(2) & \xrightarrow{\bar{\phi}} & \Sigma SO(3) \wedge SO(3) & \equiv & \Sigma SO(3) \wedge SO(3) \\ \downarrow & & \downarrow & & \downarrow \\ P^3(2) & \hookrightarrow & \Sigma SO(3) & \xrightarrow{\bar{\mu}'} & BSO(3) \vee BSO(3) \\ \downarrow & & \downarrow & & \downarrow \\ BSO(3) & \equiv & BSO(3) & \xrightarrow{\Delta} & BSO(3) \times BSO(3). \end{array}$$

It follows that there is a homotopy commutative diagram

$$\begin{array}{ccccc} F_{\tilde{H}} & \longrightarrow & \Omega(\Sigma\mathbb{R}P_2^4 \vee P^6(2)) & \longrightarrow & \Omega\Sigma(\Omega P^3(2) \wedge \Omega P^3(2)) \\ \downarrow & & \parallel & & \downarrow \\ \Omega F_{\bar{\phi}} & \longrightarrow & \Omega(\Sigma\mathbb{R}P_2^4 \vee P^6(2)) & \longrightarrow & \Omega\Sigma(SO(3) \wedge SO(3)) \end{array}$$

and so the composite

$$F_{\tilde{H}} \longrightarrow \Omega(\Sigma\mathbb{R}P_2^4 \vee P^6(2)) \longrightarrow \Omega P^3(2) \longrightarrow \Omega\Sigma SO(3) \longrightarrow \Omega\Sigma\mathbb{R}P^4$$

is null homotopic. In particular, the composite

$$F_\theta \xrightarrow{j} F_H(P^3(2)) \longrightarrow \Omega P^3(2) \longrightarrow \Omega\Sigma\mathbb{R}P^4$$

is null homotopic. Let $\phi: P^4(2) \rightarrow P^3(2)$ be the map in Lemma 3.8. By the proof of Theorem 1.1, there is homotopy commutative diagram

$$\begin{array}{ccc} S^3\{2\} & \xrightarrow{s} & F_H(P^3(2)) \\ \downarrow & & \downarrow \\ \Omega P^4(2) & \xrightarrow{\Omega\phi} & \Omega P^3(2) \longrightarrow \Omega\Sigma\mathbb{R}P^4, \end{array}$$

where the bottom sequence of the looping of the cofibre sequence and so the composite

$$S^3\{2\} \xrightarrow{s} F_H(P^3(2)) \longrightarrow \Omega P^3(2) \longrightarrow \Omega\Sigma\mathbb{R}P^4$$

is null homotopic. The assertion follows from the fact that the map

$$\Omega S^3\{2\} \times \Omega F_\theta \xrightarrow{\Omega s \cdot \Omega j} \Omega_0 F_H(P^3(2))$$

is a homotopy equivalence. □

Proof of Theorem 1.5. It suffices to show that the assertion holds for $k = 1$. We use the notations in the proof of Theorem 1.6. By the proof of Theorem 1.6, the composite $F_{\tilde{H}} \rightarrow \Omega P^3(2) \rightarrow \Omega\Sigma\mathbb{R}P^3$ is null homotopic. It follows that the adjoint $\Sigma F_{\tilde{H}} \rightarrow P^3(2)$ is null homotopic after suspension or the composite

$$F_{\tilde{H}} \longrightarrow \Omega P^3(2) \longrightarrow \Omega^2 P^4(2)$$

is null homotopic. Thus the image of $\pi_*(F_H(P^3(2)))$ in $\pi_*(\Omega^2 P^4(2))$ is the same as that of $\pi_*(S^3\{2\})$ in $\pi_*(\Omega^2 P^4(2))$.

Since $\Sigma^2\mathbb{R}P^3 \simeq S^4 \vee P^5(2)$, there is a homotopy commutative diagram of cofibre sequences

$$\begin{array}{ccccc} S^4 & \longrightarrow & * & \longrightarrow & S^5 \\ \downarrow & & \downarrow & & \downarrow \\ P^5(2) & \xrightarrow{\Sigma\phi} & P^4(2) & \longrightarrow & \Sigma^2\mathbb{R}P^4 \\ \downarrow & & \parallel & & \downarrow \\ S^5 & \xrightarrow{f} & P^4(2) & \longrightarrow & X. \end{array}$$

Since $Sq_*^2: H_6(X) \rightarrow H_4(X)$ is an isomorphism, $f = \bar{\eta}$ represents a generator for $\pi_5(P^4(2)) = \mathbb{Z}/4$ and so there is a homotopy commutative diagram

$$\begin{array}{ccc} S^3\{2\} & \longrightarrow & \Omega P^3(2) \\ \downarrow & & \downarrow \\ S^3 & \xrightarrow{\bar{\eta}} & \Omega P^4(2). \end{array}$$

Now let $X = P^3 \cup_f e^{n+2}$ be a co- H -space such that $\Sigma X \not\cong P^4(2) \vee S^{n+3}$. Then there is map $g: S^n \rightarrow S^3\{2\}$ such that f is homotopic to the composite

$$S^n \xrightarrow{g} S^3\{2\} \longrightarrow \Omega P^3(2).$$

By the homotopy commutative diagram

$$\begin{array}{ccc} S^3\{2\} & \longrightarrow & \Omega(P^3(2)\langle 2 \rangle) \\ \downarrow & & \downarrow \partial \\ S^3 & \xlongequal{\quad} & S^3, \end{array}$$

the composite $\bar{g}: S^n \xrightarrow{g} S^3\{2\} \longrightarrow S^3$ is uniquely determined by f up to homotopy. Observe that the map \bar{g} is of order 2. This sets up a one-to-one correspondence between \mathcal{C}_{n+2}^1 and V_n^1 and hence the result. \square

Proof of Theorem 1.8. We use notations in the proof of Theorem 1.6. By Lemma 2.8, it suffices to show that the composite

$$\Omega F_H(P^3(2)) \longrightarrow \Omega^2 P^3(2) \xrightarrow{\Omega^2[2]} \Omega^2 P^3(2)$$

is null homotopic. By Theorem 1.1 and Proposition 4.4, it suffices to show that

- 1) the composite $S^3\{2\} \longrightarrow \Omega P^3(2) \xrightarrow{\Omega[2]} \Omega P^3(2)$ is null homotopic and
- 2) the composite $\Omega T^5 \longrightarrow \Omega P^3(2) \xrightarrow{\Omega[2]} \Omega P^3(2)$ is null homotopic.

By using the fact that $[2]: P^3(2) \rightarrow P^3(2)$ is homotopic to the composite

$$P^3(2) \xrightarrow{\text{pinch}} S^3 \xrightarrow{\eta} S^2 \hookrightarrow P^3(2),$$

the second statement above follows from the homotopy commutative diagram

$$\begin{array}{ccc} P^3(2) & \xrightarrow{\text{pinch}} & S^3 \\ \uparrow & & \uparrow \tilde{\eta} \\ \Sigma\mathbb{R}P^4_2 & \xrightarrow{\text{pinch}} & P^5(2), \end{array}$$

where $\tilde{\eta}$ is the extension of $\eta: S^4 \rightarrow S^3$. Consider the homotopy commutative diagram

$$\begin{array}{ccccccc} \Omega P^3(2) & \xrightarrow{\text{pinch}} & \Omega S^3 & \xrightarrow{\Omega\eta} & \Omega S^2 & \hookrightarrow & \Omega P^3(2) \\ \uparrow & & \uparrow \eta & & & & \\ S^3\{2\} & \longrightarrow & S^3 & & & & \end{array}$$

The first statement above follows from that the map $S^4 \xrightarrow{\eta^2} S^2 \hookrightarrow P^3(2)$ is divisible by 2 in the group $\pi_4(P^3(2)) = \mathbb{Z}/4$ and hence the result. \square

6. EXAMPLES

In this section, we discuss complexes $P^3(2) \cup e^n$ for small n until we get the first example of non-suspension co- H -spaces. Let $\{u, v\}$ be a basis for $\bar{H}_*(\mathbb{R}P^2)$ with $Sq_*^1 v = u$. Note that $H_*(\Omega P^3(2)) \cong T(u, v)$. Let ι_n be a generator for $H_n(S^n)$.

Recall that $\pi_3(P^3(2)) = \mathbb{Z}/4$. Thus there are only two complexes $P^3(2) \cup e^4$ which are given by $\Sigma\mathbb{R}P^3$ and $A^4 = \mathbb{C}P^2 \cup_{[2]} e^3$. Clearly A^4 is not a co- H -space because it has a nontrivial cup product.

Consider the complexes $P^3(2) \cup e^5$. Recall that $\pi_5(P^3(2)) = \mathbb{Z}/4$. A generator is represented by the map $\delta: S^4 \rightarrow P^3(2)$ such that adjoint map $\delta': S^3 \rightarrow \Omega P^3(2)$ has the property that $\delta'_*(\iota_3) = [u, v]$. Let $A^5 = P^3(2) \cup_\delta e^5$ and let $B^5 = P^3(2) \cup_{2\delta} e^5$. Since $2\delta: S^4 \rightarrow P^3(2)$ is homotopic to the composition

$$S^4 \xrightarrow{\eta} S^3 \xrightarrow{\eta} S^2 \hookrightarrow P^3(2),$$

the complex $B^5 \simeq P^3(2) \cup_{\eta^2} e^5$. Clearly both A^5 and B^5 are not co- H -spaces by checking the Hopf invariants. The complex A^5 has the following special property.

Theorem 6.1. *Let A^3 be defined as above. Then*

- 1) *The mod 2 cohomology algebra of A^5 is isomorphic to the exterior algebra $E(x, y)$ with $|x| = 2$ and $Sq^1 x = y$;*

- 2) In $H^*(A^5)$, $Sq^2y = xy$;
 3) There is a 2-local fibre sequence

$$SU(3) \longrightarrow A^5 \longrightarrow BSO(3).$$

Proof. (1). Observe that the map $H_*(\Omega P^3(2)) \rightarrow H_*(\Omega A^3)$ sends $[u, v]$ to zero. Assertion (1) follows by considering the Serre spectral sequence for the fibre sequence $\Omega A^5 \longrightarrow * \longrightarrow A^5$. Assertion (2) follows from the fact that the composite

$$S^4 \xrightarrow{\delta} P^3(2) \xrightarrow{\text{pinch}} S^3$$

is η .

(3). Consider the homotopy commutative diagram

$$\begin{array}{ccc} P^3(2) & \hookrightarrow & A^5 \\ \downarrow & & \downarrow \\ BSO(3) & \equiv & BSO(3). \end{array}$$

Let F be the homotopy fibre of the map $A^5 \longrightarrow BSO(3)$. By assertion (1), $H_*(\Omega A^5)$ is the polynomial algebra generated by u and v . Since $\Omega F \longrightarrow \Omega A^5 \longrightarrow SO(3)$ is a multiplicative fibre sequence and $H_*(\Omega A^5) \rightarrow H_*(SO(3))$ is onto, $H_*(\Omega F)$ is the polynomial algebra generated by u^2 and v^2 . It follows that $H^*(F)$ is the exterior algebra generated by x' and y' with $|x'| = 3$ and $y' = Sq^2x'$. This shows that the 5-skeleton of F is $\Sigma\mathbb{C}P^2$ and so $F = \Sigma\mathbb{C}P^2 \cup_f e^8$ for some map $f: S^7 \rightarrow \Sigma\mathbb{C}P^2$. Let $\{a, b\}$ be a basis for $\bar{H}_*(\mathbb{C}P^2)$ with $Sq_*^2b = a$. Since $H_*(\Omega F)$ is the polynomial algebra generated by a and b , $f'_*(\iota_6) = [a, b]$ in $H_*(\Omega\Sigma\mathbb{C}P^2) = T(a, b)$, where $f': S^6 \rightarrow \Omega\Sigma\mathbb{C}P^2$ is the adjoint map of f . Let $g: S^7 \rightarrow \Sigma\mathbb{C}P^2$ be the attaching map for $SU(3)$. Then

$$g_*: \pi_7(S^7) \longrightarrow \pi_7(\Sigma\mathbb{C}P^2)$$

is an epimorphism because $\pi_7(SU(3)) = 0$ (see [13, pp. 970]). Observe that $[f']$ has non-trivial Hurewicz image in $H_*(\Omega\Sigma\mathbb{C}P^2)$. The homotopy class $[f] = k[g]$ for some $k \not\equiv 0 \pmod{2}$ in $\pi_7(\Sigma\mathbb{C}P^2)$ and hence the result. \square

By using the fact that the map

$$\Omega_0^3 P^3(2) \rightarrow \Omega^2(SO(3)\langle 3 \rangle) \simeq \Omega^2(S^3\langle 3 \rangle)$$

has a cross-section, we have

Corollary 6.2. *There is a homotopy decomposition localized at 2*

$$\Omega_0^3 A^3 \simeq \Omega_0^3 SU(3) \times \Omega^2(S^3\langle 3 \rangle).$$

In particular, the torsion of $\pi_*(A^5)$ has a bounded exponent and so the Moore conjecture holds for the 3-cell complex A^5 .

Now consider the complexes $P^3(2) \cup e^6$. By using the fibre sequence,

$$\Sigma\mathbb{R}P_2^4 \vee P^6(2) \longrightarrow P^3(2) \longrightarrow BSO(3),$$

we obtain $\pi_5(P^3(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Thus, up to homotopy, there are eight complexes $P^3(2) \cup e^6$, where one of them is $P^3(2) \vee S^6$. It is a routine exercise to check that the kernel of

$$H_*: \pi_4(\Omega P^3(2)) \rightarrow \pi_4(\Omega\Sigma(\Omega P^3(2) \wedge \Omega P^3(2)))$$

is $\mathbb{Z}/2$ and so there is a unique (up to homotopy) non-suspension co- H -space among the complexes $P^3(2) \cup_f e^6$, where the attaching map f is given as follows. Let ϕ be the map in Lemma 3.8 and let $\bar{\eta}: S^5 \rightarrow P^4(2)$ be the generator for $\pi_5(P^4(2)) = \mathbb{Z}/4$. Then $f = \phi \circ \bar{\eta}$.

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