ALGEBRAIC BP-THEORY AND NORM VARIETIES

NOBUAKI YAGITA

ABSTRACT. Let X be a smooth variety over a field k of $ch(k) = 0$. For a fixed prime p, the algebraic BP-theory $ABP^{*,*'}(X)$ is the algebraic version of the topological BP-theory. Given a nonzero symbol $a \in K^M_{n+1}(k)/p$, the norm variety V_a is a variety such that $a = 0 \in K^M_{n+1}(k(V_a))/p$ and $V_a(\mathbb{C}) = v_n$. In this paper, we mainly study $ABP^{*,*'}(V_a)$ for p odd prime case.

1. INTRODUCTION

A.Suslin and V.Voevodsky constructed and developed the motivic cohomology theory $H^{*,*'}(X;\mathbb{Z}/p)$ for algebraic sets over the base field k. This theory is the counter part in algebraic geometry of the usual mod p singular cohomology in algebraic topology. Let $ch(k) = 0$ and fix an embedding $k \subset \mathbb{C}$. As the counter part of the complex cobordism theory $MU^*(X)$, Voevodsky [Vo1,2] defined the algebraic cobordism theory $MGL^{*,*'}(X)$ and used it in the first proof of the Milnor conjecture.

Given a nonzero symbol $a \in K^M_{n+1}(k)/p$, the norm variety V_a is a variety such that $a = 0 \in K^M_{n+1}(k(V_a))/M$ and $V_a(\mathbb{C}) = v_n$. Here v_n is the $2(p^n - 1)$ complex manifold generating

$$\mathbb{Z}_{(p)}[v_1, v_2, ...] \cong BP^*(pt.) \subset MU^*(pt.)_{(p)} \cong MGL^{2*,*}(Spec(k))_{(p)}$$

the coefficient ring of the BP-theory in algebraic topology.

For $p = 2$, we can take the norm variety by the smallest neighbor Q_a of the Pfister quadric defined by a. Voevodsky proved [Vo1] the Milnor conjecture by studying cohomology operations on $H^{*,*'}(Q_a;\mathbb{Z}/2)$. Moreover $MGL^{2*,*}(Q_a)$ is studied by Vishik and Yagita [Vi-Ya].

Recently Rost ([Ro],[Su-Jo]) announced the constructions of the norm variety V_a also for p odd, and Voevodsky ([Vo4]) gives the proof of the Bloch-Kato conjecture (which is the odd prime version of the Milnor conjecture) by studying $H^{*,*'}(V_a;\mathbb{Z}/p)$.

2000 *Mathematics Subject Classification.* Primary 14C15, 57T25; Secondary 55R35, 57T05.

Key words and phrases. motivic cobordism, Chow ring, motivic cohomology, BP-theory, Atiyah-Hirzebruch spectral sequence.

In this paper we write down the properties of $ABP^{*,*'}(X)$ which is the algebraic counter part of BP-theory. For example we give a construction of the Atiyah-Hirzebruch spectral sequence for $ABP^{*,*}(X;\mathbb{Z}/p)$; its existence (of MGL-version) was announced by Hopkins and Morel more that several years before, however any proof (or even statement) does not appear yet. We study the cohomology operations, products and Gysin maps explicitly in $ABP^{*,*'}$-theory. Using these results, we compute $ABP^{*,*'}(V_a)$ which extends some parts of the results by Vishik and Yagita to odd p cases.

2. COHOMOLOGY OPERATIONS

Let p be a fixed prime number. Let k be a field with $ch(k) = 0$, which contains a primitive p-th root of unity. In this paper, the $mod(p)$ motivic cohomology $H^m_{Zar}(X;\mathbb{Z}/p(n))$ is written by $H^{m,n}(X;\mathbb{Z}/p)$. We fix an emmbedding $k \subset \mathbb{C}$ and denote by $t_{\mathbb{C}}$ the realization map

$$t_{\mathbb{C}} : H^{*,*'}(X;\mathbb{Z}) \to H^*(X(\mathbb{C});\mathbb{Z})$$

where the right hand side is the usual (singular) cohomology of the $\mathbb{C}$-rational points of X.

In motivic $mod(p)$ cohomology, we have the Bockstein and the reduced powers operations

$$(2.1) \qquad \beta : H^{*,*'}(X;\mathbb{Z}/p) \to H^{*+1,*'}(X;\mathbb{Z}/p),$$

$$(2.2) \qquad P^i : H^{*,*'}(X;\mathbb{Z}/p) \to H^{*+2(p-1)i,*'+(p-1)i}(X;\mathbb{Z}/p)$$

which are compatible with the usual Bockstein and the reduced powers operations via the realization map $t_{\mathbb{C}}$.

Let $\tau \in H^{0,1}(pt.;\mathbb{Z}/p) \cong \mathbb{Z}/p$ and $\rho \in H^{1,1}(pt.;\mathbb{Z}/p) \cong k^*/(k^*)^p$ be elements corresponding to the primitive root ζ of unity. Then $\beta\tau = \rho$. Reduced powers operations have the following properties (Lemma 9.7,Lemma 9.8 in [Vo4]),

$$(2.3) \quad P^0 = Identity, \quad P^n(x) = x^p \quad if \quad x \in H^{2n,n}(X;\mathbb{Z}/p),$$

$$(2.4) \quad P^i(x) = 0 \quad if \ x \in H^{m,n}(X;\mathbb{Z}/p), \quad i > m-n \ and \ i \geq n.$$

When $p > 2$, the Cartan formula

$$P^i(xy) = \sum_{0\leq j\leq i} P^j(x)P^{i-j}(x)$$

and the Adem relations are also satisfied as the topological cases. However when $p = 2$ we need some modification for τ and ρ ($P^i = Sq^{2i}$

and $\beta = Sq^1$). For example

$$(2.5)\quad Sq^{2i}(uv) = \sum_{0\le i\le i} Sq^{2j}(u)Sq^{2i-2j}(v) + \tau \sum_{0\le j\le i-1} Sq^{2j+1}(u)Sq^{2i-2j-1}(v).$$

Moreover we have the Milnor operation

$$(2.6)\quad Q_i : H^{*,*'}(X;\mathbb{Z}/p) \to H^{*+2p^i-1,*+p^i-1}(X;\mathbb{Z}/p).$$

When $p \ge 3$, we have $Q_0 = \beta$ and $Q_{i+1} = [Q_i, P^{p^i}]$. But for $p = 2$ the above property holds only with $mod(\rho)$ (see [Vo3] for details). We note $Q_i^2 = 0$ and $Q_iQ_j = -Q_jQ_i$. But Q_i is not a derivation when $\rho \ne 0$ and $p = 2$ (while it is a derivation whenever $p \ge 3$). In fact from Proposition 13.4 in [Vo3],

$$(2.7)\quad Q_i(xy) = Q_i(x)y + xQ_i(y) + \rho Q_{i-1}(x)Q_{i-1}(y)$$

$$+\rho \sum a_{JJ'}Q_J(x)Q_{J'}(y)$$

where $a_{IJ} \in H^{*,*'}(pt.;\mathbb{Z}/2)$, $|a_{JJ'}| > 0$ and $Q_J = Q_0^{\epsilon_0}...$ for $J = (\epsilon_0,...,\epsilon_{i-1})$, $\epsilon_j = 0$ or 1.

For a non zero element x in $H^{m,n}(X;\mathbb{Z}/p)$ or each cohomology operation (or differential in the spectral sequence), we define the weight and the difference by

$$w(x) = 2s.deg(x) - f.deg(x) = 2n - m$$

$$d(x) = f.deg(x) - s.deg(x) = m - n$$

(here $f.deg(x)$ (resp. $s.deg(x)$) is the first degree (resp. second degree) of x) so that if X is smooth, then

$$w(x) \ge 0, \quad d(x) \le dim(X).$$

We also note $w(\beta) = -1$, $w(P^i) = 0$, $w(Q_i) = -1$.

The solution of the Bloch-Kato conjecture by Voevodsky implies

$$H^{*,*'}(X;\mathbb{Z}/p) \cong H^*_{et}(X;\mathbb{Z}/p) \quad for\ * \le *',$$

$$H^{*,*}(pt.;\mathbb{Z}/p) \cong K^M_*(k)/p \cong H^*_{et}(pt.;\mathbb{Z}/p).$$

Since $d(x) \le 0$ for non zero $x \in H^{*,*'}(pt.;\mathbb{Z}/p)$, we have

Lemma 2.1. $H^{*,*'}(pt.;\mathbb{Z}/p) \cong \mathbb{Z}/p[\tau] \otimes K^M_*(k)/p$.

Corollary 2.2. *Let $p \ge 3$. In $H^{*,*'}(pt.;\mathbb{Z}/p)$, we see $Q_i(x) = 0$ and $P^j(x) = 0$ for all $i, j \ge 1$.*

Proof. By dimensional reason, $P^n(x) = 0$ for $x \in H^{n,n}(pt.;\mathbb{Z}/p) \cong K_n^M(k)$ or $x = \tau$. When $p > 2$, the Cartan formula holds, hence $P^n(x) = 0$ for all $x \in H^{*,*'}(pt;\mathbb{Z}/p)$ and $n > 0$. Thus we see also $Q_n(x) = 0$ for $n > 0$. □

Remark. However when $p = 2$, in general, $P^n(x) \neq 0$ and $Q_n(x) \neq 0$ for $x \in H^{*,*'}(pt,;\mathbb{Z}/p)$ which, for example, see §14 bellow.

V.Voevodsky ([Vo3,4] in particular Lemma 2.2 in [Vo4]) and G.Powell [Po] showed that the mod p motivic Steenrod algebra $A_p^{*,*'}$ is generated as an $H^{*,*'}(pt,\mathbb{Z}/p)$-module by products of P^i and β. Moreover they also prove

$$(2.8) \quad A_p^{*,*'} \cong H^{*,*'}(pt;\mathbb{Z}/p) \otimes RP \otimes \Lambda(Q_0, Q_1, ...)$$

where RP is the $\mathbb{Z}/p$-module generated by products of reduced powers $P^{i_1}....P^{i_n}$ (without the Bockstein). Hence each element a in $A_p^{*,*'}$ is represented by

$$a = \sum a_{IJ} P^I Q_J = \sum a_{IJ} P^{i_1}...P^{i_n} Q_0^{\epsilon_0}...Q_m^{\epsilon_m}$$

with $I = (i_1, ..., i_n)$, $i_k > 0$, and $J = (\epsilon_0, ...\epsilon_m)$, $\epsilon_i = 0$ *or* 1, and $a_{IJ} \in H^{*,*'}(pt.;\mathbb{Z}/p)$.

In this paper, we assume and consider (generalized) cohomology theories in stable homotopy categories which hold the following Whitehead type theorem. Let X and Y be connected. If $f : X \to Y$ is a map such that for each extension F of k, there is the isomorphism

$$f^*|_F : H^{*,*'}(Y|_F;\mathbb{Z}_{(p)}) \cong H^{*,*'}(X|_F;\mathbb{Z}_{(p)}) \qquad X|_F = X \otimes F,$$

then f induces the equivalence in this (satble homotopy) category over the field k.

3. ABP THEORIES

Recall that $MU^*(-)$ is the complex cobordism theory defined in the usual (topological) spaces and

$$MU^* = MU^*(pt.) = \mathbb{Z}[x_1, x_2, ...] \quad |x_i| = -2i.$$

Here each x_i is represented by sum of hypersurfaces of $dim(x_i) = 2i$ defined by polynomials with the coeffients in $\mathbb{Z}$, in some product of complex projective spaces.

Let $MGL^{*,*'}(-)$ be the motivic cobordism theory defined by Voevodsky. By the Thom isomorphism, it is easily proved that ([Hu-Kr], [Ve]) that there is the $H^{*,*'}(pt)$-module isomorphism

$$H^{*,*'}(MGL) \cong H^{*,*'}(BGL) \cong H^{*,*'}(pt)[c_1, c_2, ...] \quad with\ |c_i| = 2i.$$

This isomorphism induces the $A_p^{*,*'}$-module isomorphism

$$H^{*,*'}(MGL;\mathbb{Z}/p) \cong H^{*,*'} \otimes RP \otimes \mathbb{Z}/p[m_i|i \neq p^j - 1]$$

with $H^{*,*} = H^{*,*'}(pt.;\mathbb{Z}/p)$ and $|m_i| = 2i$.

Let us write by AMU the spectrum $MGL_{(p)}$ representing the motivic cobordism theory, i.e., $MGL^{*,*'}(-)_{(p)} = AMU^{*,*'}(-)$. Since $MGL^{*,*'}(X)$ is a multiplicative cohomology theory, we know it is an $MGL^{*,*'}(pt.)$-algebra. Moreover we can embeds MU^* into $MGL^{2*,*}(pt.)$ ([Vo1]). Hence $MGL^{*,*}(X)$ is also an MU^*-algebra.

Given a regular sequence $S_n = (s_1, ..., s_n)$ with $s_i \in MU^*_{(p)}$, we can inductively construct the AMU-module spectrum by the cofibering of spectra

$$(3.1) \quad \mathbb{T}^{-1/2|s_i|} \wedge AMU(S_{i-1}) \xrightarrow{\times s_i} AMU(S_{i-1}) \to AMU(S_i)$$

where $\mathbb{T} = \mathbb{A}/(\mathbb{A} - \{0\} \cong S^1_t \wedge S^1_s$ is the Tate object. It is also immediate that $t_{\mathbb{C}}(AMU(S_n)) \cong MU(S_n)$ with

$$MU(S_n)^* = MU^*/(Ideal(S_n)).$$

Recall $BP^* \cong \mathbb{Z}_{(p)}[v_1, ...]$ with identifying $v_i = x_{p^i-1}$. We can construct spectra

$$ABP = AMU(x_i|i \neq p^j - 1), \;\; AP(n), \;\; Ak(n), \;\; AH\mathbb{Z}, \;\; AH\mathbb{Z}/p$$

so that $t_{\mathbb{C}}(Ah) \cong h$ for $h = BP, P(n), ...$ Here $P(n)^* = BP^*/(p, ..., v_{n-1})$ and $k(n)^* = \mathbb{Z}/p[v_n] \cong BP^*/(p, ..\hat{v}_n, ...)$.

For $S = (v_{i_1}, ..., v_{i_n})$, let us write

$$ABP(S) = AMU(S \cup \{x_i|i \neq p^j\})$$

so that $t_{\mathbb{C}}(ABP(S)) = BP(S)$ with $BP(S)^* = BP^*/(S)$. By using the long exact sequence induced from (3.1)

Lemma 3.1. *([Ya4]) Let $S = (v_{i_1}, ..., v_{i_n})$. Then*

$$H^{*,*'}(ABP(S);\mathbb{Z}/p) \cong H^{*,*'}(pt.;\mathbb{Z}/p) \otimes H^*(BP(S);\mathbb{Z}/p)$$

$$\cong H^{*,*'}(pt.;\mathbb{Z}/p) \otimes RP \otimes \Lambda(Q_{i_1}, ..., Q_{i_n}).$$

By the above lemma for $S = (p, v_1, ...)$ and the Whitehead type theorem, we see that in the $\mathbb{A}^1$-stable homotopy category,

$$(3.2) \quad H_{\mathbb{Z}/p} \cong AH\mathbb{Z}/p = AMU(p, x_1, x_2, ...),$$

e.g., $AH\mathbb{Z}/p^{*,*'}(X) \cong H^{*,*'}(X;\mathbb{Z}/p)$. More strongly, Hopkins-Morel showed that $AH\mathbb{Z} \cong H_{\mathbb{Z}}$, namely, $AH\mathbb{Z}^{*,*'}(X) \cong H^{*,*'}(X,\mathbb{Z})$; the motivic cohomology.

Theorem 3.2. *([Ya4],[Ho-Mo]) Let $Ah = ABP(S)$ for $S = (v_{i_1}, v_{i_2}, ...)$. Then there is the Atiyah-Hirzebruch spectral sequence*

$$E(Ah)_2^{(m,n,2n')} = H^{m,n}(X; h^{2n'}) \Longrightarrow Ah^{m+2n',n+n'}(X)$$

with the differential $\quad d_{2r+1} : E_{2r+1}^{(m,n,2n')} \to E_{2r+1}^{(m+2r+1,n+r,2n'-2r)}$.

Remarks.

1) The cohomology $H^{m,n}(X, h^{2n'})$ here is the usual motivic cohomology with coefficients in the abelian group $h^{2n'}$, e.g., if $h^{2n'}$ is $\mathbb{Z}/p$-module, then $H^{m,n}(X; h^{2n'}) \cong H^{m,n}(X; \mathbb{Z}/p) \otimes h^{2n'}$. In particular, if X is smooth, then $E_r^{m,n,2n'} \cong 0 \quad for\ m > 2n$.
2) The convergence in AHss means that there is the filtration
$$Ah^{*,*'}(X) = F_0^{*,*'} \supset F_1^{*,*'} \supset F_2^{*,*'} \supset ...$$
such that $F_i^{*,*'}/F_{i+1}^{*,*'} \cong E_\infty^{*+2i,*'+i,-2i}$.
3) Let $S \subset R = (v_{j_1}, ...)$. Then the induced map $ABP(S) \to ABP(R)$ of spectra induces the natural $BP^*_{(p)}$- module map of AHss
$$E(ABP(S))_r^{*,*,*} \to E(ABP(R))_r^{*,*,*}.$$

From the above theorem and dimensional reason, we see

$$(3.3) \quad ABP(S)^{2*,*}(pt) \cong BP(S)^* = BP^*/(S).$$

From (1), we also have for smooth X,

$$(3.4) \quad ABP(S)^{2*,*}(X) \otimes_{BP^*} \mathbb{Z}_{(p)} \cong H^{2*,*}(X) \cong CH^*(X)_{(p)}.$$

Let $v_j \notin S$. Consider the long exact sequence

$$\to ABP(S)^{2*,*}(X) \stackrel{v_j}{\to} ABP(S)^{2*,*}(X)$$
$$\to ABP(S, v_j)^{2*,*}(X) \to ABP(S)^{2*+1,*}(X) \to .$$

Here the last term is zero when X is smooth. Hence this case

$$ABP(S, v_j)^{2*,*}(X) \cong ABP(S)^{2*,*}(X)/(v_j).$$

More generally

Lemma 3.3. *Let X be smooth and $S \subset R \subset (p, v_1, ...)$. Then*

$$ABP(R)^{2*,*}(X) \cong ABP(S)^{2*,*}(X)/(R).$$

Of course, the case $R = (p, v_1, ...)$ of the above lemma is the isomorphism (3.4).

Here we give a proof of Theorem 2.1 for $ABP(S)^{*,*'}(X)$ with $p \in S$.

Let $I = (i_0, i_1, i_2, ...)$ with $i_0 = 0 < i_1 < i_2 < ...$ and $S_I = (p, v_{i_1}, v_{i_2}, ...)$. We consider the AHss for the theory $ABP(S_I)^{*,*'}(X)$.

We first study the construction for the following motivic Adams spectral sequence. We consider an (injective) resolution of $ABP(S_I)$ by the (motivic) Eilenberg-MacLane spectrum $H\mathbb{Z}/p$, namely, the sequence

$$(1) \quad ABP(S_I) \xrightarrow{\delta_0} H_1 \xrightarrow{d_1} H_2 \xrightarrow{d_2} H_3 \xrightarrow{d_3} H_4 ...$$

where H_i is the product of $\mathbb{T}^* H\mathbb{Z}/p$ and induced cohomology sequence is the resolution of $H^{*,*'}(ABP(S_I);\mathbb{Z}/p)$ over $A_p^{*,*'}$.

Consider sequences $R = (r_1, r_2, ...)$ with $r_i = 0$ for $i \in I$. Let $l(R) = \sum r_i$ and $|R| = \sum 2r_i(p^i - 1)$ (be finite). Let V_n be a $\mathbb{Z}/p$-vector space generated by sequences with $l(R) = n-1$. We can construct an resolution by

$$H_n = H\mathbb{Z}/p \otimes V_n \cong \prod_{l(R)=n-1,\ and\ r_i=0\ for\ i\in I} H\mathbb{Z}/p\{v_R\}$$

where $\{v_R\}$ is just the base with $|v_R| = |R|$. Hence

$$H^{*,*'}(H_n;\mathbb{Z}/p) \cong A_p^{*,*'} \otimes V_n \cong \otimes_{l(R)=n-1} A_p^{*,*'} \otimes \{v_R\}.$$

(Note that for fixed degree $(*,*')$, each $H^{*,*'}(H_n;\mathbb{Z}/p)$ is finite, while V_n itself is infinite dimensional vector space.) The differential map $d_n^* : H^{*,*'}(H_{n+1};\mathbb{Z}/p) \to H^{*+1,*'}(H_n;\mathbb{Z}/p)$ is give by

$$(2) \quad d_n^*(v_R) = \sum Q_i \otimes v_{R-\Delta_i} \quad \Delta_i = (0, ..., 0, \overset{i}{1}, 0, ...).$$

If there are maps d_n with (2), then by standard arguments, (2) induces the projective resolution of

$$H^{*,*'}(ABP(S_I);\mathbb{Z}/p) \cong H^{*,*'}(pt.;\mathbb{Z}/p) \otimes RP \otimes \Lambda(Q_i | i \in I)$$

for $A_p^{*,*'}$ which is isomorphic to

$$H^{*,*'}(H\mathbb{Z}/p;\mathbb{Z}/p) \cong H^{*,*'}(pt.;\mathbb{Z}/p) \otimes RP \otimes \Lambda(Q_0, ...).$$

(For topological case, this is well known by Milnor and Novikov. See page 513 in [Mi].)

Note that the homotopy group

$$Hom(H\mathbb{Z}/p, H\mathbb{Z}/p) \cong H^{*,*'}(H\mathbb{Z}/p;\mathbb{Z}/p) \cong A_p^{*,*'}.$$

Hence we have $Hom(H_n, H_{n+1}) \cong A_p^{*,*'} \otimes Hom(V_{n+1}, V_n)$. (Note also that for fixed (i,j) the $\mathbb{A}^1$-homotopy $[S^{i,j}H_n, H_{n+1}]$ is finite.) Hence there exists unique map $d_n : H_n \to H_{n+1}$ satisfying (2). Thus we have the resolution (1).

Let B_n be the cofiber of d_{n-1}, namely, $H_{n-1} \stackrel{d_{n-1}}{\to} H_n \stackrel{p_n}{\to} B_n$ is the cofiber sequence. Since $d_n d_{n-1} = 0$, there is a map $\delta_n : B_n \to H_{n+1}$ such that $\delta_n p_n = d_{n+1}$

$$\begin{array}{ccccccc} H_{n-1} & \xrightarrow{d_{n-1}} & H_n & \xrightarrow{d_n} & H_{n+1} & \xrightarrow{d_{n+1}} & H_{n+2} \\ & & p_n \downarrow & \nearrow \delta_n & & & \\ & & B_n & & & & \end{array}$$

Here $H^{*,*'}(B_n;\mathbb{Z}/p) \cong Kerd_{n-1}^* \cong Cokerd_{n+1}^*$. Hence the sequence $B_n \stackrel{\delta_n}{\to} H_{n+1} \stackrel{d_{n+1}}{\to} H_{n+2}$ is the cofiber sequence, by the Whitehead theorem for motivic theories, in fact, the induced cohomologies make exact sequence. Similarly, the exact sequence

$$0 \leftarrow Kerd_{n-1}^* \leftarrow H^{*,*'}(H_{n+1};\mathbb{Z}/p) \leftarrow Kerd_n^* \leftarrow 0$$

induces the cofiber sequence $S^{-1}B_n \stackrel{\delta_n}{\to} H_n \stackrel{p_{n+1}}{\to} B_{n+1} \stackrel{i_{n+1}}{\to} B_n$.

Thus we get the following diagram such that the triangles $\delta \searrow \stackrel{\stackrel{i}{\leftarrow}}{} \uparrow p$ is cofiber sequence and $\stackrel{p\uparrow\searrow\delta}{\rightarrow_d}$ is commutative

$$\begin{array}{ccccccc} ABP(S_I) = B_0 & \xleftarrow{i_1} & B_1 & \xleftarrow{i_2} & B_2 & \xleftarrow{i_3} & B_3 \\ & \delta_0 \searrow & p_1 \uparrow & \delta_1 \searrow & p_2 \uparrow & \delta_2 \searrow & p_3 \uparrow \\ & & H_1 & \xrightarrow{d_1} & H_2 & \xrightarrow{d_2} & H_3 \end{array}$$

Taking $Hom(X,-)$, we have the diagram of cohomology theories such that the triangles $\delta_* \searrow \stackrel{\stackrel{i_*}{\leftarrow}}{} \uparrow p_*$ is exact and $\stackrel{p_*\uparrow\searrow\delta_*}{\rightarrow_{d_*}}$ is commutative

$$\begin{array}{ccccccc} ABP(S_I)^{*,*'}(X) & \xleftarrow{i_{1*}} & B_1^{*,*'}(X) & \xleftarrow{i_{2*}} & B_2^{*,*'}(X) & \xleftarrow{i_{3*}} & B_3^{*,*'}(X) \\ & \delta_{0*} \searrow & p_{1*} \uparrow & \delta_{1*} \searrow & p_{2*} \uparrow & \delta_{2*} \searrow & p_{3*} \uparrow \\ & & H_1^{*,*'}(X) & \xrightarrow{d_{1*}} & H_2^{*,*'}(X) & \xrightarrow{d_{2*}} & H_3^{*,*'}(X) \end{array}$$

where $H_n^{*,*'}(X) \cong \oplus_{l(R)=n-1} H^{*,*'}(X;\mathbb{Z}/p)\{v^R\}$ with degree $|v^R| = -|v_R| = |v_1^{r_1} v_2^{r_2} ...|$.

For an element $x \in H_n^{*,*'}(X)$ with $p_{n*}(x) \in Image(i_{n+1*}...i_{n+r*})$, we can define the differential

$$d(A)_r(x) = \delta_{n+r*}(i_{n+1*}...i_{n+r*})^{-1} p_{n*}(x).$$

If we give th filtration by

$$Filt_s^{Adams}(\oplus_n H_n^{*,*'}(X)) = \oplus_{s \le n} H_n^{*,*'}(X;\mathbb{Z}/p).$$

Then we have the Adams spectral sequence converging $ABP(S_I)^{*,*'}(X)$, namely

$$Ext_{\Lambda(Q_j|j\notin I)}(\mathbb{Z}/p, H^{*,*'}(X;\mathbb{Z}/p)) \Longrightarrow ABP^{*,*'}(S_I)(X).$$

Here we give the another filtration

$$Filt_s^{AHss}(\oplus_n H_n^{*,*'}(X)) = \oplus_{s\geq |v^R|} H^{*,*'}(X;\mathbb{Z}/p)\{v^R\}.$$

Writing $v^R = v_1^{r_1}v_2^{r_2}...$, we have the isomorphism

$$Fil_s/Fil_{s-1} \cong H^{*,*'}(X;\mathbb{Z}/p)\otimes (BP/S_I)^s.$$

Thus we get the desired AHss converging $ABP(S_I)^{*,*'}(X)$.

Remark. V.Voevodsky define a slice filtration $f_q(E)$ for a spectrum E in the stable homotopy theory $SH(k)$ (for details, see Theorem 2.2 [Vo5])

$$E = f_0(E) \leftarrow f_1(E) \leftarrow f_2(E) \leftarrow ....$$

The slice $s_q(E)$ is defined by the cofibering

$$f_{q+1}(E) \to f_q(E) \to s_q(E)$$

and $s_q(E)$ belong to $\Sigma_{\mathbb{T}}^q SH(k)^{eff}$. Here $SH(k)^{eff}$ is the smallest triangulated subcategory in $SH(k)$ which is closed under direct sums and contains suspension spectra of spaces but not their $\mathbb{T}$-desuspensions.

The spectrum E is called slice wise cellular, if any $q\in\mathbb{Z}$ the slice $s_q(E)$ belongs to the smallest triangulated category of $SH(k)$ closed under direct sums which contain the spectrum $\Sigma_T^q H_{\mathbb{Z}}$. Note that when $ch(k) = 0$, the category of slice wise cellular spectra contains the sphere spectrum and therefore $\mathbb{T}$-cellular spectra (Corollary 4.5 in [Vo5]). Indeed $s_q(H\mathbb{Z}) = H\mathbb{Z}$ for $q = 0$ and $s_q(H\mathbb{Z}) = 0$ otherwise (Conjecture 1 in [Vo5]), and it is also proved $s_0(1) = H\mathbb{Z}$ where 1 is the sphere spectrum

If E is slice wise cellular, then we can construct the motivic Adams spectral sequence

$$Ext_{A_p^{*,*}}(H^{*,*}(E;\mathbb{Z}/p), H^{*,*'}(X;\mathbb{Z}/p)) \Longrightarrow [X,E]_{(p)}$$

similar to the case above ; exchanging $B_i \mapsto f_i(E)$, $H_i \mapsto s_{i-1}(E)$. We note that we can take $s_q(E) = \Sigma^{0,q}H\mathbb{Z}_{\Pi_q(E)}$ (see §5 in [Vo5]) where $H\mathbb{Z}_{\Pi_q(E)} = H\mathbb{Z}\otimes \Pi_q(E)$ in the Voevodsky's notation.

The above arguments for $ABP(S_I)$ showed that $ABP(S_I)$ is mod p slice wise cellular, namely,

$$s_q(ABP(S_I)) = \Sigma_{\mathbb{T}}^q H\mathbb{Z}/p_{BP/(S_I)^{-2q}} = \Sigma_{\mathbb{T}}^q H\mathbb{Z}/p\otimes BP/(S_I)^{-2q}$$

(Conjecture 5 in [Vo5]).

As a most simple example, we consider the case $S_I = (p, v_1, ..., \hat{v}_n, ...)$, namely, $BP(S_I) = k(n)$ the connected Morava K-theory

$$k(n)^* \cong BP^*/(p, v_1, ..., \hat{v}_n, ...) \cong \mathbb{Z}/p[v_n].$$

This case $H_i = H\mathbb{Z}/p\{v_{i\Delta_n}\}$. The map $f : X \to H\mathbb{Z}/p\{v_{i\Delta_n}\}$ represents the element $x \otimes v_n^i \in H_i^{*,*'}(X)$ if $f^*(v_{i\Delta_n}) = x \in H^{*,*'}(X;\mathbb{Z}/p)$. The differential $d_{2p^n-1}(x \otimes v_n^i)$ is represented by the composition map

$$X \xrightarrow{f} H\mathbb{Z}/p\{v_{i\Delta_n}\} \xrightarrow{d_i} H\mathbb{Z}/p\{v_{(i+1)\Delta_n}\}.$$

Here we have

$$f^* d_i^* v_{i+1\Delta_n} = f^*(Q_n \otimes v_{i\Delta_n}) = Q_n(x).$$

This means that $d_{2p^n-1}(x \otimes v_n^i) = Q_n(x) \otimes v_n^{i+1}$ in AHss.

Lemma 3.4. *The first nonzero differential in AHss for $Ak(n)^{*,*'}(X)$ is given by*

$$d_{2p^n+1}(x) = v_n \otimes Q_n(x).$$

From the above lemma and the exact sequence

$$\to Ak(n)^{*,*'}(X) \xrightarrow{v_n} Ak(n)^{*,*'}(X) \to H^{*,*'}(X;\mathbb{Z}/p) \to .$$

Lemma 3.5. *If $v_n y = 0 \in Ak(n)^{*,*'}(X)$, then there is $x \in H^{*,*'}(X;\mathbb{Z}/p)$ such that $Q_n(x) = \rho(y_n)$ where $\rho : ABP \to AH\mathbb{Z}/p$ is the natural (Thom) map.*

We also note the following lemma ([Ya2]).

Lemma 3.6. *If $\sum v_n y_n = 0 \in ABP^{*,*'}(X)$, then there is $x \in H^{*,*'}(X;\mathbb{Z}/p)$ such that $Q_n(x) = \rho(y_n)$ where $\rho : ABP \to AH\mathbb{Z}/p$ is the natural (Thom) map.*

Proof. (It is just the motivic version of the arguments of Tamanoi [Ta].) Let AL be the spectrum defined by the cofiber sequence

$$S_s^{-1}AL \xrightarrow{\Pi q_i} \Pi \mathbb{T}^{p^i-1}ABP \xrightarrow{\kappa} ABP \xrightarrow{\theta} AL$$

where the map κ is defined by

$$\Pi \mathbb{T}^{p^i-1}ABP \xrightarrow{\vee v_i} \bigvee ABP \xrightarrow{folding} ABP$$

so that $\kappa_*(b_0, b_1, ...) = \sum v_i b_i$ for $b_i \in ABP^{*,*'}(X)$.
Since $\kappa^* = 0$ on $H^{*,*'}(-;\mathbb{Z}/p)$, we have

$$0 \to H^{*-1,*}(\Pi \mathbb{T}^{p^i-1}ABP;\mathbb{Z}/p) \xrightarrow{\Pi q_i^*} H^{*,*'}(AL;\mathbb{Z}/p) \to H^{*,*'}(ABP;\mathbb{Z}/p) \to 0.$$

Recall $H^{*,*'}(ABP;\mathbb{Z}/p) \cong H^{*,*'}(pt.;\mathbb{Z}/p) \otimes RP$ (see Lemma 3.1). Hence the mod p cohomology is easily seen

$$H^{*,*'}(AL;\mathbb{Z}/p) \cong H^{*,*'}(pt.;\mathbb{Z}/p) \otimes RP \otimes \{1, q_0^*(1_0), q_1^*(1_1), ...\}$$

where $1_i \in H^{2p^i-1,p^i-1}(\mathbb{T}^{p^i-1}ABP;\mathbb{Z}/p)$. Here we can prove that $q_i^*(1_i) = Q_i(1)$ for $1 \in H^{0,0}(AL;\mathbb{Z}/p)$. Because this holds for topological case (see [Ta] for details), and the $w(x) = -1$ parts in $H^{*,*'}(AL;\mathbb{Z}/p))$ are written as $RP \otimes \{1, q_0^*(1_0), q_1^*(1_1), ...\}$ which maps injectivity to the topological case by the realization map $t_{\mathbb{C}}$.

Let $\eta : AL \to AH\mathbb{Z}/p$ be the map of spectra representing $1 \in H^{0,0}(AL;\mathbb{Z}/p)$. The above equality means

$$\rho q_i = Q_i\eta : AL \to S^{2p^i-1,p^i-1}AH\mathbb{Z}/p$$

as homotopy maps.

Suppose $\sum v_i y_i = 0 \in ABP^{*,*'}(X)$. Then $\kappa(\Pi(y_i)) = 0$. So there is $z \in AL^{*-1,*}(X)$ with $\Pi(q_i(z)) = \Pi(y_i)$. Take $x = \eta(z)$ and we get

$$\rho(y_i) = \rho q_i(z) = Q_i\eta(z) = Q_i(x).$$

□

Corollary 3.7. *Let $z \in E_\infty^{*,*',0} \subset H^{*,*'}(X;\mathbb{Z}/p)$ in AHss converging to $ABP^{*,*'}(X)$ such that $v_i z = 0 \in E_\infty^{*,*',*''}$. Then there is $x \in H^{*,*'}(X;\mathbb{Z}/p)$ such that $\sum v_n y_n = 0$ in $ABP^{*,*}(X)$ with $\rho(y_n) = Q_n(x)$ for all n and $z = \rho(y_i) = Q_i(x)$.*

4. COHOMOLOGY OPERATIONS IN $ABP^{*,*'}(-)$-THEORY

Recall that

$$(4.1) \quad H^{*,*'}(MGL) \cong H^{*,*'}(pt.;\mathbb{Z}) \otimes H^*(MU)$$

where additively $H^*(MU) \cong H^*(BU) \cong \mathbb{Z}[c_1, ...]$ and where c_i is the i-th Chern class with $deg(c_i) = (2i, i)$ in $H^{*,*}(MGL;\mathbb{Z})$. Since $w(c_i) = 0$, the elements c_i are infinite cycles in AHss for $X = MGL$

$$E(X)_2^{*,*',*''} = H^{*,*'}(X,\mathbb{Z}) \otimes MU^{*''} \Longrightarrow MGL^{*,*'}(X).$$

Hence we have the isomorphism of spectral sequences $E(MGL)_r^{*,*,*} \cong E(pt)_r^{*,*,*} \otimes H^*(MU)$. This means

$$MGL^{*,*'}(MGL) \cong MGL^{*.*'}(pt) \otimes H^*(MU).$$

Hence the Steenrod algebra of MGL-theory is generated as an $MGL^{*,*'}(pt)$-module by the Landweber-Novikov operation s_α corresponding $c^\alpha = c_1^{\alpha_1}c_2^{\alpha_2}...$ for $\alpha = (\alpha_1, \alpha_2, ...)$.

Lemma 4.1. *(Theorem 7.2 in [Ya4]) The theory $ABP^{*,*'}(-)$ is a multiplicative theory and there exists the map $ABP \to AMGL_{(p)}$ such that*

$$ABP^{*,*'}(X) \cong MGL^{*,*'}(X)_{(p)} \otimes_{MU^*_{(p)}} BP^*,$$

$$MGL^{*,*'}(X)_{(p)} \cong ABP^{*,*'}(X) \otimes_{BP^*} MU^*_{(p)}.$$

The Steenrod algebra of ABP-theory is generated as an $ABP^{*,*'}(pt)$-module by the Quillen operation r_α for $\alpha = (\alpha_1, ...)$.

Proposition 4.2. *Let us write* $\tilde{RP} \cong \mathbb{Z}_{(p)}\{r_\alpha | \alpha = (\alpha_1, \alpha_2, ...), \alpha_i \geq 0, \}$. *Then there is the isomorphism*

$$ABP^{*,*'}(ABP) \cong ABP^{*,*'} \otimes H^*(BP) \cong ABP^{*,*'}(pt) \otimes \tilde{RP}.$$

Remark. The Landweber-Novikov operation s_α is also defined as the cohomology operations in $ABP^{*.*'}(-)$ theory by

$$ABP \to MGL_{(p)} \stackrel{s_\alpha}{\to} MGL_{(p)} \to ABP.$$

We also use the same letter s_α for the operation in $ABP^{*,*'}(-)$. Then for each sequence $\alpha = (\alpha_1, \alpha_2, ...)$ such that $\alpha_i = 0$ if $i \neq p^k - 1$ for some k, the Landweber-Novikov operation generates $ABP^{*,*'}(ABP)$ as an $ABP^{*.*}(pt)$-module.

Each multiplicative operation $o(-)$ in $BP^*(X)$ theory is determined by an element

$$o(y) \in (BP^*[[y]])^2 \cong BP^2(\mathbb{C}P^\infty) \quad |y| = 2.$$

The total Quillen operation r_t (resp. s_t) in $BP^*(-)[t_1, t_2, ...]$ ($|t_i| = 2(p^i - 1)$) is defined by

$$r_t^{-1}(y) = \sum^{F_{BP}} t_n y^{p^n} \quad (resp.\ s_t^{-1}(y) = \sum t_n y^{p^n})$$

where $\sum^{F_{BP}}$ means sum of the formal group law of $BP^*(-)$ theory. Then r_α is defined

$$r_t(x) = \sum r_\alpha(x) t^\alpha \quad with\ t^\alpha = t_1^{\alpha_1} ...$$

and s_α is defined similarly.

We note that the Quillen operation r_α (and the Landweber-Novikov operation s_α) satisfies the Cartan formula

Lemma 4.3.

$$r_\alpha(x) = \sum_{\alpha = \alpha' + \alpha''} r_{\alpha'}(x) r_{\alpha''}(y).$$

Proof. The Cartan formula holds if

$$(*) \quad \mu^*(r_\alpha) = \sum_{\alpha = \alpha' + \alpha''} r_{\alpha'} \otimes r_{\alpha''}$$

for the coproduct map $\mu^* : ABP^{*,*'}(ABP) \to ABP^{*,*'}(ABP \wedge ABP)$. Here note for $X = ABP, ABP \wedge ABP$, we have $ABP^{*,*'}(X) \cong ABP^{*,*'}(pt) \otimes H^*(X(\mathbb{C}))_{(p)}$. In particular

$$ABP^{2*,*}(X) \cong BP^*(X(\mathbb{C})).$$

The Cartan formular holds in $BP^*(-)$ theory and the formula $(*)$ holds in BP^*-theory and so does in $ABP^{2*,*}(ABP \wedge ABP)$, indeed $r_\alpha \in ABP^{2*,*}(ABP)$. □

By the similar arguments, we have

Lemma 4.4. *$ABP^{*,*'}(ABP)$ is a $BP^*(BP)$-module.*

Recall that $H_*(BP) \cong \mathbb{Z}_{(p)}[m_1, m_2, ...]$ where $m_i = 1/(p^i)\mathbb{C}P^{p^i-1}$. The Quillen operation r_α on m_n is explicitly written.

Lemma 4.5. *(Quillen [Ha],[Ra])*

$$r_\alpha(m_n) = \begin{cases} m_i & if\ \alpha = p^i\Delta_{n-i} \quad for\ \Delta_{n-i} = (0, .., 0, \overset{n-i}{1}, 0, ..) \\ 0 & otherwise. \end{cases}$$

Hazewinkel showed the expression of v_n by m_i

$$v_n = pm_n - \sum_{1\le i\le n-1} m_i v_{n-i}^{p^i}$$

identifying $\pi_*(BP) = \mathbb{Z}_{(p)}[v_1, ...] \subset H_*(BP) = \mathbb{Z}_{(p)}[m_1, ...]$.

Let us write by I_n the ideal in BP^* generated by $(v_0, ..., v_{n-1})$. (Let $v_0 = p$.) One of the important properties of r_α is ;

Lemma 4.6. *([Ha],[Ra])*

$$r_\alpha(v_n) = \begin{cases} v_i \ \ mod(I_i) & if\ \alpha = p^i\Delta_{n-i} \\ 0 \ \ mod(I_\infty^2) & otherwise. \end{cases}$$

An Ideal J in BP^* is called invariant if it is so under the Quillen (or Landweber-Novikov) operations, i.e., $r_\alpha(J) \subset J$ for all α.

Lemma 4.7. *(prime invariant theorem [La]) If for $a \in BP^*$, the ideal $J = (I_n, a)$ is invariant, then $a = \lambda v_n^s\ mod(I_n)$ for $\lambda \in \mathbb{Z}/p$ and $s \ge 1$. In particular, prime invariant ideals are written as I_m for $m \ge 1$ or I_∞.*

One of examples of invariant ideals is following. For AHss converging $ABP^{*,*'}(X)$, define a filtration of the infinite term by

$$F_s(X) = \oplus_{*\ge s} E_\infty^{*,*',*''}.$$

Corollary 4.8. *If $x \in E_\infty^{2*,*',0}$ and $BP^*/J\{x\} \subset E_\infty^{*,*',*''}$ for some ideal J, then this ideal J is invariant.*

Proof. Let us write $x' \in ABP^{*,*'}(X)$ a corresponding element to $x \in E_\infty^{*,*',0}$. Let $a \in J$ so that $ax' = 0\ mod(F_{n+1})$. Then

$$0 \equiv r_\alpha(ax') = \sum_{\alpha=\alpha'+\alpha''} r_{\alpha'}(a) r_{\alpha''}(x') \equiv r_\alpha(a)x' \quad mod(F_{n+1})$$

since $r_{\alpha''}(x') \in F_{n+1}$ for $\alpha'' \neq 0$ by dimensional reason. Hence $r_\alpha(a)$ is also in J. □

5. $AP(n) = ABP(I_n)$ THEORIES

As the topological case, let us write

$$ABP(p, v_1, ..., v_{n-1})^{*,*'}(X) = AP(n)^{*,*'}(X)$$

e.g., $AP(0)^{*,*'}(X) = ABP^{*,*'}(X)$, $AP(1)^{*,*'}(X) = ABP^{*,*'}(X; \mathbb{Z}/p)$ and $AP(\infty)^{*,*'}(X) = H^{*,*'}(X; \mathbb{Z}/p)$.

We want show the Conner-Floyd type theorem for $AP(n)^{*,*}(-)$ and $AK(n)^{*,*}(-)$. For ease of notations, let us write $Q(n) = \Lambda(Q_0, ..., Q_n)$.

Lemma 5.1. *There is an $AP(n)^{*,*'}$-module isomorphism*

$$AP(n)^{*,*}(AP(n)) \cong AP(n)^{*,*} \otimes RP \otimes Q(n-1).$$

Proof. (Compare [Wu],[Ya1]) For the cofiber sequence

$$(1) \quad \mathbb{T}^{p^n-1} \wedge AP(n) \xrightarrow{v_n} AP(n) \to AP(n+1),$$

we get the long exact sequence

$$\xleftarrow{\delta_k} AP(n)^{*-2p^n+2,*-p^n+1}(AP(n)) \xleftarrow{v_n *} AP(n)^{*,*}(AP(n))$$

$$\longleftarrow AP(n)^{*,*}(AP(n+1)) \leftarrow .$$

By induction, we assume the isomorphism in the lemma for n. Let $\iota \in AP(n)^{0,0}(AP(n))$ represents the identity map of $AP(n)$. Then $v_n^* \iota = v_n \iota$. Let $x = \sum a r^I Q_J \iota \in AP(n)^{*,*}(AP(n))$ for $a \in AP(n)^{*,*'}$. Then we see

$$v_n^* r^I Q_J(\iota) = r^I Q_J(v_n^* \iota) = r^I Q_J(v_n \iota) = v_n(r^I Q_J \iota) \quad mod(I_n).$$

The last equation is shown by Lemma 5.2 bellow. Therefore we get $v_n^* x = v_n x$. Hence we have $AP(n)^{*,*}(AP(n+1)) \cong AP(n)^{*,*}(AP(n))/(v_n)$.

Next we consider (the Sullivan) exact sequence induced from (1)

$$AP(n)^{*+2p^n-1,*+2p^n-1}(AP(n+1)) \xrightarrow{v_n} AP(n)^{*,*}(AP(n+1))$$

$$\xrightarrow{\rho_n} AP(n+1)^{*,*}(AP(n+1)) \xrightarrow{\delta_n} .$$

Since $v_n = 0$ on $AP(n)^{*,*'}(AP(n+1)$, we get the isomorphism

$$AP(n+1)^{*,*}(AP(n+1)) \cong AP(n)^{*,*}(AP(n))/(v_n) \otimes \Lambda(Q_n)$$

identifying $Q_n = \delta_n \rho_n \ mod(Q(n-1))$. By induction on n, we get the lemma. □

Similarly, we get

$$AP^{*,*}(AP(n)^{\wedge s}) \cong AP(n)^{*,*'} \otimes (R \otimes Q(n-1))^{\otimes s}.$$

Lemma 5.2. *We can take r_I, Q_J which commute with v_n, as $AP(n)^{*,*'}$-module generators of $AP(n)^{*,*}(AP(n))$.*

Proof. Let us write by $\mu : ABP \wedge AP(n) \to AP(n)$ the product map. From Lemma 4.3 and the preceding lemma for n, we can write

$$\mu^*(r_I) = \sum_{I=I'+I''} r_{I'} \otimes r_{I''} + \sum a_{KLJ} r_K \otimes r_L Q_J$$

with $|K|,\ |J| > 0$. From Lemma 4.6, we still know that

$$r_I(v_n) = 0 \quad mod(I_n) \quad |I| \neq 0.$$

Recall $v_n x$ is defined by

$$\mathbb{T}^{p^n-1} \wedge AP(n) \stackrel{v_n \wedge x}{\to} ABP \wedge AP(n) \stackrel{\mu}{\to} AP(n).$$

Hence we have

$$r_I(v_n x) = \sum_{I=I'+I''} r_{I'}(v_n) r_{I''}(x) + \sum a_{KLJ} r_K(v_n) r_L Q_J(x)$$

which is equal to $v_n r_I(x)\ mod(I_n)$.

Define Q_i by the map

$$\rho_i \delta_i : AP(n)^{*,*'}(X) \stackrel{\delta_i}{\to} ABP(p, ..., \hat{v}_i, ... v_{n-1})^{*,*'}(X) \stackrel{\rho_i}{\to} AP(n)^{*,*'}(X).$$

Here the maps δ_i and ρ_i are $ABP^{*,*'}$-module maps. In particular $Q_i(v_n x) = v_n Q_i(x)$. □

We recall some arguments of Würgler([Wu]). Let us say that $x \in AP(n)^{*,*}(AP(n))$ is ABP-primitive if

$$\mu^*(x) = 1 \otimes x \quad in\ AP(n)^{*,*'}(ABP \wedge AP(n)).$$

We say that $x \in AP(n)^{*,*}(AP(n) \wedge AP(n))$ is $ABP \wedge ABP$-primitive if $\tilde{\mu}^*(x) = (1 \otimes 1) \otimes (x)$ for

$$\tilde{\mu} : ABP \wedge ABP \wedge AP(n) \wedge AP(n) \stackrel{1 \wedge T \wedge 1}{\to} ABP \wedge AP(n) \wedge ABP \wedge AP(n)$$

$$\stackrel{\mu \wedge \mu}{\to} AP(n) \wedge AP(n)$$

where T is the switch map. Similarly we can define $ABP^{\wedge s}$-primitivity for elements in $ABP(n)^{*,*'}(AP(n)^{\wedge s})$.

Of course $\iota \in AP(n)^{0,0}(AP(n))$ is ABP-primitive, and $\iota \otimes \iota \in AP(n)^{0,0}(AP(n)^2)$ is $ABP \wedge ABP$-primitive. Hereafter we simply write ι by 1.

Lemma 5.3. *If $p \geq 3$, then ABP-primitive elements in $AP(n)^{0,0}(AP(n))$ is additively generated by 1. Similarly if $p \geq 5$ and $i \leq 3$, then $ABP^{\wedge i}$-primitive elements in $AP(n)^{0,0}(AP(n)^{\wedge i})$ is additively generated by $1^{\otimes i}$.*

Proof. Each element $x \in AP(n)^{*,*'}(AP(n))$ is represented as a sum of $a_{IJ}r_IQ_J$. Then

$$\mu^*(x) = r_I \otimes a_{IJ}Q_J + ...$$

Hence if x is primitive, then $|I|$=0. Suppose that $|J| > 0$ (so $|a_J| < 0$) and

$$x = a_JQ_J \in AP(n)^{0,0}(AP(n)).$$

First note that

$$w(Q_J) \geq w(Q_0...Q_{n-1}) = -n, \quad \textit{and so} \quad w(a_J) \leq n.$$

In the E_∞-term of AHss for $AP(n)^{*,*}(pt.)$, let us write

$$[a_J] = \sum v_K \otimes a'_K \quad v_K \in P(n)^*,\ a'_K \in H^{*,*}(pt.;\mathbb{Z}/p).$$

Since $|a_J| < 0$, we see $v_K \in P(n)^-$, so $|v_K| \leq |v_n| = -2(p^n - 1)$. Moreover $w(a'_K) \leq n$ and so $|a'_K| \leq 2n$. Thus if $p \geq 3$, then

$$|x| = |v_Ka'_KQ_J|| \leq |v_n| + 2n + |Q_0...Q_{n-1}|$$

$$= 2(-p^n + 1 + ... + p^{n-1}) + 2 < 0.$$

This is a contradiction. Hence primitive elements are case $|I| = |J| = 0$, and the result follows from $AP(n)^{0,0}(pt.) \cong \mathbb{Z}/p$.

Each element x in $AP(n)^{*,*'}(AP(n) \wedge AP(n))$ is represented by a sum of

$$a \cdot r_IQ_J \otimes r_{I'}Q_{J'}.$$

The result for $ABP \wedge ABP$-primitive elements follows from that if $p \geq 5$, then

$$|x| \leq |v_n| + 4n + 2|Q_0...Q_{n-1}| = -2p^n + 4(1 + ... + p^{n-1}) < 0.$$

The $ABP^{\wedge 3}$-primitivity follows from the inequality

$$|v_n| + 6n + 3|Q_0...Q_{n-1}| = -2p^n + 6(1 + ... + p^{n-1}) < 0.$$

□

Theorem 5.4. *If $p \geq 3$, then the theory $AP(n)^{*,*'}(-)$ is independent of the choice of generator v_i. Moreover if $p \geq 5$, then there is the unique associative, commutative product compatible with the natural map $AP(n-1) \to AP(n)$.*

Proof. Let $I'_n = \{p', v'_1, ..., v'_n\}$ and $Ideal(I_n) = Ideal(I'_n)$. Let us write $AP'(n) = ABP(I'_n)$. Then we can see

$$AP'(n)^{*,*'}(AP(n)) \cong AP'(n)^{*,*'} \otimes R \otimes Q(n-1)$$

as the proof of Lemma 4.9. The element $1 \in AP'(n)^{0,0}(AP(n))$ represents the map $i : AP(n) \to AP'(n)$. Similarly we get the map $i' : AP(n) \to AP'(n)$. The fact that the composition $i'i$ is identity for $p \geq 3$ follows from the preceding lemma, namely, $i^*i'^*(1)$ is primitive which represents 1 on $AP'(n)^{0,0}(ABP)$. Such element must equal to 1.

Now we consider the product structure. The element $1 \otimes 1 \in AP(n)^{0,0}(AP(n) \wedge AP(n))$ represents the map

$$\mu : AP(n) \wedge AP(n) \to AP(n).$$

To see the commutativity, we consider the map

$$\mu' : AP(n) \wedge AP(n) \xrightarrow{T} AP(n) \wedge AP(n) \xrightarrow{\mu} AP(n).$$

We see both $\mu^*(1 \otimes 1)$ and $\mu'^*(1 \otimes 1)$ are $ABP \wedge ABP$-primitive, since so is $1 \otimes 1$. Hence from the preceding lemma, the both elements must be equal if $p \geq 5$. The associativity follows from the $ABP^{\wedge 3}$-primitivity, and the compatibility of the product for the natural map $AP(m) \to AP(n)$, $m \leq n$ follows from the ABP-primitivity. □

Corollary 5.5. *For $p \geq 5$, the AHss is a multipliticative spectral sequence. In particular there is the $P(n)^*$-algebra isomorphism*

$$grAP(n)^{*,*'}(pt.) \cong P(n)^{*,*'} \otimes H^{*,*'}(pt;\mathbb{Z}/p).$$

Proof. Consider AHss

$$E_2^{*,*'*''} = H^{*,*'}(pt.;\mathbb{Z}/p) \otimes P(n)^{*''} \Longrightarrow AP(n)^{*,*'}(pt.).$$

Recall that $H^{*,*'}(pt;\mathbb{Z}/p) \cong K_*^M(k)/p[\tau]$. By dimensional reason,

$$d_r(\tau) = 0, \quad and \quad d_r(x) = 0 \ \ for\ x \in K_*^M(k)/p.$$

Since the differential is a derivation, the AHss collapses. □

Corollary 5.6. *Let $p \geq 5$. Then we can take the cohomology operation Q_i and r_I on $AP(n)^{*,*}(-)$ such that*

(1) $Q_i^2 = 0,\ Q_iQ_j = -Q_jQ_i,\ \mu^*(Q_i) = Q_i \otimes 1 + 1 \otimes Q_i.$

(2) $$\mu^*(r_I) = \sum_{I=I'+I''} r_{I'} \otimes r_{I''} + \sum ar_KQ_L \otimes r_{K'}Q_{L'}$$

with $a \in I_\infty AP(n)^{*,*}$, $|K + L| > 0$ and $|K' + L'| > 0$.

Proof. Define $Q_i = \rho_i\delta_i$ as in the proof of Lemma 4.10. Then $Q_i^2 = 0$ and $Q_iQ_j = -Q_jQ_i$ are immediately seen. We can prove the primitivity from the fact that the element Q_i is generates the ABP-primitive elements in $AP(n)^{2p^i-1,p^i-1}(AP(n))$. This fact is proved by the arguments similar to the proof of Lemma 5.2. □

Suppose $p \geq 5$. Consider the filtration

$$0 = F_{-1} \subset F_0 \subset ... \subset F_\infty = AP(n)^{*,*'}(X)$$

with $F_i = \{\sum a_I x_I | a_I \in AP(n)^{*,*'},\ w(x_I) \leq i\}$. Then operations r_I act on F_i/F_{i-1} satisfying the Cartan formula from (2) in the preceding corollary. Therefore using the arguments in the proof of Corollary 4.8, we get the filtration

$$F_{i0} \subset F_{i1} \subset .... \subset F_{in} = F_i/F_{i-1}$$

such that $F_{is}/F_{i,s-1}$ is the $P(s)^*$-free module. Using this fact (for details, see [La], [Ya1]), we can prove ;

Lemma 5.7. *(Exact functor theorem for $AP(n)^{*,*'}$-theory) Let $p \geq 5$. Let G be a $P(n)^*$-module such that the map $v_m : G/I_m \to G/I_m$ are monic for all $m \geq n$. Then the functor*

$$AP(n)^{*,*'}(X) \mapsto AP(n)^{*,*'}(X) \otimes_{P(n)^*} G$$

is an exact functor (i.e., $AP(n)^{*,*'}(-) \otimes_{P(n)^*} G$ is the cohomology theory).

Corollary 5.8. *Let $p \geq 5$. Let $AK(n)^{*,*}(-) = [v_n^{-1}]Ak(n)^{*,*}(-)$ be the motivic Morava K-theory. Then we get the isomorphism*

$$AK(n)^{*,*}(X) \cong K(n)^* \otimes_{BP^*} AP(n)^{*,*}(X).$$

Proof. The operation r_I also acts on the AHss $E_r^{*,*,*}$ converging to $AP(n)^{*,*}(X)$. From the exact functor theorem, we know

$$E_r^{*,*,*} \mapsto E_r^{*,*,*} \otimes_{P(n)^*} K(n)^*$$

is the exact functor. Hence

$$H(E_r^{*,*,*}, d_r) \otimes_{P(n)^*} K(n)^* \cong H(E_r^{*,*,*} \otimes_{P(n)^*} K(n)^*, d_r \otimes 1).$$

Of course the left hand side is $E_{r+1}^{*,*,*} \otimes_{P(n)^*} K(n)^*$. By induction, we can show that the righthand side is isomorphic to the $r+1$-th term of AHss converging to $AK(n)^*(X)$ since

$$E_2^{*,*,*} \otimes_{P(n)^*} K(n)^* \cong H^{*,*}(X; \mathbb{Z}/p) \otimes K(n)^*$$

which is the E_2-term of AHss converging to $AK(n)^{*,*}(X)$. □

6. Homology theories and $ABP_{*,*'}(ABP)$

Since eighties, most Adams Novikov spectral sequences are studied for homology theories, but not cohomology theories. In this section, we remark a bit about the homology theories and the motivic version of the Adams-Novikov spectral sequence which are studied by Miller, Ravenel and Wilson [Mi-Ra-Wi].

First we recall the motivic homology for a smooth projective X. It is known by Suslin and Voevodsky that the motivic (co)homology theory holds the Poincare duality

$$-\cap u_X; H^{*,*'}(X;\mathbb{Z}/p) \cong H_{2d-*,d-*'}(X;\mathbb{Z}/p)$$

where $d = dim(X)$ and $u_X \in H_{2d,d}(X;\mathbb{Z}/p)$ is the fundamental class of X. Hence if $0 \neq x \in H_{*,*'}(X;\mathbb{Z}/p)$, then

$$f.deg(x) \leq 2d, \quad w(x) \leq 0, \quad and \quad d(x) \leq 0.$$

Let us write

$$H_{*,*'} = H_{*,*'}(pt.;\mathbb{Z}/p) \cong H^{-*,-*'}(pt.;\mathbb{Z}/p).$$

It is known (Conjecture [Vo5]) that the homology

$$H_{*,*'}(H\mathbb{Z}/p;\mathbb{Z}/p) \cong H_{*,*'} \otimes \bar{RP} \otimes \bar{\Lambda}$$

$$where \quad \bar{RP} = \mathbb{Z}/p[\xi_1, \xi_2, ...] \quad deg(\xi_i) = (2p^i - 2, p^i - 1)$$

$$\bar{\Lambda} = \Lambda(\tau_0, \tau_1, ...) \quad deg(\tau_i) = (2p^i - 1, p^i - 1).$$

As the cohomology cases ([Hu-Kr],[Ve]), we have

$$H_{*,*'}(MGL;\mathbb{Z}/p) \cong H_{*,*'} \otimes \bar{RP} \otimes \mathbb{Z}/p[m_i | i \neq p^i - 1].$$

Arguments similar to §3, we get

Lemma 6.1. *Let* $S = (v_{i_1}, ..., v_{i_s})$. *Then*

$$H^{*,*'}(ABP(S);\mathbb{Z}/p) \cong H_{*,*'} \otimes \bar{RP} \otimes \Lambda(\tau_{i_1}, ..., \tau_{i_s})$$

$$ABP(S)_{*,*'}(ABP(S)) \cong ABP_{*,*'} \otimes \bar{RP} \otimes \Lambda(\tau_{i_1}, ..., \tau_{i_s}).$$

Moreover we have the AHss for homology theory

Theorem 6.2. *Let* $Ah = ABP(S)$ *for* $S = (v_{i_1}, v_{i_2}, ...)$. *Then there is the Atiyah-Hirzebruch spectral sequence*

$$E(Ah)^2_{(m,n,2n')} = H_{m,n}(X; h_{2n'}) \Longrightarrow Ah_{m+2n',n+n'}(X)$$

with the differential $\quad d^{2r+1} : E^{2r+1}_{(m,n,2n')} \to E^{2r+1}_{(m-2r-1,n-r,2n'+2r)}$.

For ease of arguments, let B be ABP or $AP(n)$ for $p \geq 5$ so that they have the good product. Hence we also have the Kunneth map

$$B_{*,*'}(X) \otimes_{B_{*,*'}} B_{*,*'}(Y) \to BP_{*,*'}(X \times Y).$$

Since $|x| \leq 2d$ for nonzero $x \in H_{*,*'}(X;\mathbb{Z}/p)$, we see each element in $H_{2d,d}(X;\mathbb{Z}/p)$ is permanent in the above AHss. In particular we can take the fundamental class u_X also in $B_{2d,d}(X)$. Hence we can define the Poincare dual map $-\cap u_X : B^{*,*'}(X)\to B_{2d-*,d-*'}(X)$ by the map of spectra

$$x\cap u_X:\mathbb{T}^*\stackrel{u_X}{\to} X\wedge B\stackrel{\Delta\wedge 1}{\to} X\wedge X\wedge B\stackrel{1\wedge x\wedge 1}{\to} X\wedge B\wedge B\stackrel{1\wedge\mu_B}{\to} X\wedge B.$$

Theorem 6.3. *For smooth X, there holds the Poincare duality*

$$-\cap u_X : B^{*,*'}(X)\cong B_{2d-*,d-*'}(X).$$

Proof. Consider spectral sequences $E^r_{*,*,*}$ and $E_r^{*,*,*}$ which converge to $B_{*,*}(X)$ and $B^{*,*,*}(X)$ respectively. Since we can define the Poincare map in $B^{*,*}(-)$ theories, we can define the Poincare map of AHss's.

$$-\cap u_X : E_r^{*,*',*'}\to E^r_{2d-*,d-*',-*''}$$

The isomorphic of these maps follow from the isomorphic of E_2-terms, which follows from the Poincare duality of the motivic (co)homology theory. □

Lemma 6.4. $B_{*,*'}(B\times X)\cong B_{*,*'}(B)\otimes_{B_{*,*'}} B_{*,*'}(X)$.

Proof. First note that it is well known ([Vo3]) that the Kunneth formular

$$H_{*,*'}(Y\times X;\mathbb{Z}/p)\cong H_{*,*'}(Y;\mathbb{Z}/p)\otimes_{H_{*,*'}} H_{*,*'}(X;\mathbb{Z}/p)$$

satisfies for all X, when $Y=\mathbb{P}^n$. This induces that the Kunneth formular holds, when $Y=\mathbb{P}^\infty$, BGL, MGL. By induction on n we can prove the Kunneth formula for $AP(n)$,

$$H_{*,*'}(AP(n)\times X;\mathbb{Z}/p)\cong H_{*,*'}(AP(n);\mathbb{Z}/p)\otimes_{H_{*,*'}} H_{*,*'}(X;\mathbb{Z}/p).$$

The isomorphism in the lemma follows from AHss. □

Then we can define B_*-Adams Novikov resolution for X (see Definition 2.1.1 in [Ra]), that is the injective resolution

$$0\to B_{*,*'}(X)\to B_{*,*'}(B\wedge X)\to B_{*,*'}(B\wedge B\wedge X)\to ...$$

Thus we can construct the $B_{*,*'}$ Adams-Novikov spectral sequence ;

Theorem 6.5. *There is the spectral sequence*

$$E(B)^2_{s,*,*'}=Ext^s_{B_{*,*'}(B)}(B_{*,*'},B_{*,*'}(X))\Longrightarrow \pi_{*,*'}(X)_{(p)}$$

with the differential $d^r: E^r_{s,*,*'}\to E^r_{s+r,*,*'}$ and

$$gr(\pi_{*,*'}(X)_{(p)})\cong\oplus_s E_{s,*,*'}/E_{s+1,*,*'}.$$

Now we restrict the case that $B = ABP$ and X is celluler. Then for $K_s = ABP^{\wedge s} \wedge X$, we have isomorphisms

$$ABP_{*,*'}(K_s) \cong ABP_{*,*'} \otimes_{BP_*} ABP_{2*,*}(K_s)$$

$$\text{with} \quad ABP_{2*,*}(K_s) \cong BP_*(K_s(\mathbb{C})) \cong BP_{2*} \otimes (\bar{RP})^{\otimes s} \otimes H_{2*}(X(\mathbb{C}))_{(p)}.$$

The differentials in the $B_{*,*'}$-Adams-Novikov spectral sequence are defined by the alternated sum of the natural (diagonal) inclusions $i^j : K_s \to K_{s+1}$. The map $i^j_{*,*'}$ on $ABP_{2*,*}(K_s)$ is the just the map on $BP_{2*,*}(K_s(\mathbb{C}))$. Hence if $ABP_{*,*'}$ satisfies the condition of the exact functor theorem (Lemma 5.7), then

$$E(ABP)^2_{s,*,*'} \cong ABP_{*,*'} \otimes_{BP^*} Ext^s_{BP_*(BP)}(BP_*, BP_*(X(\mathbb{C})).$$

However the condition does not satisfied most cases. As for the case $B = AP(n)$, $n \geq 1$, we see $ABP_{2*,*}(K_s) \not\cong BP_{2*}(K(\mathbb{C}))$. We only know for $k = \mathbb{C}$.

Corollary 6.6. *Let X be celluler. Let $k = \mathbb{C}$, $p \geq 5$ and $n \geq 1$. Then*

$$E(AP(n))^2_{s,*,*'} \cong \mathbb{Z}/p[\tau] \otimes Ext^s_{P(n)*(P(n))}(P(n)_*, P(n)_*(X(\mathbb{C})).$$

7. Gysin maps

First we recall the Thom isomorphism. Let V be an m-dimensional vector bundle over X and $Th_X(V)$ be the induced Thom space. Then it is well known that there is the Thom isomorphism

$$Th : H^{*,*'}(X;\mathbb{Z}) \cong \tilde{H}^{*+2m,*'+m}(Th_X(V);\mathbb{Z}).$$

The element $Th(1) \in H^{2m,m}(Th_X(V))$ is called its Thom class and the above isomorphism is that of $H^{*,*'}(X;\mathbb{Z})$-modules ; the right hand module is a free $H^{*,*'}(X;\mathbb{Z})$-module generated by the Thom class $Th(1)$.

Lemma 7.1. *The Thom isomorphism also holds in $ABP^{*,*'}(X)$ for smooth X*

$$Th : ABP^{*,*'}(X) \cong ABP^{*+2m,*'+m}(Th_X(V)).$$

Proof. Consider $E(Th_X(V))_r$ (resp. $E(X)_r$) the AHss converging to $ABP^{*,*'}(Th_X(V))$ (resp. $ABP^{*,*'}(X)$). Since $w(Th(1)) = 0$, we see the Thom class $Th(1)$ is a permanent cycle in $E(Th_X(V))_r$. Then we see inductively that $E(TH_X(V))_r$ is the free $E(X)_r$-module generated by $Th(1)$. Hence we get the lemma. □

For a projective map $f : Y \to X$ of smooth projective varieties such that $c = codim_X(Y)$ is constant, we will define the Gysin map

$$f_* : ABP^{*,*'}(Y) \to ABP^{*+2c,*'+c}(X).$$

By the definition, the projective map is factored as

$$f : Y \xrightarrow{i} \mathbb{P}^m \times X \xrightarrow{p} X$$

where i is a closed embedding to the product $\mathbb{P}^m \times X$ and p is the projection.

For a close embedding $i : Y \to Z$ of $codim_Z(Y) = c$, we define the Gysin map i_* by

$$i_* : ABP^{*,*'}(Y) \cong ABP^{*+2c,*'+c}(Th_Y(N_{Z/Y})) \xrightarrow{q^*} ABP^{*+2c,*'+c}(Z)$$

where $N_{Z/Y}$ is the normal bundle of Y in Z and $q : Z \to Th_Y(N_{Z/Y})$ is the quotient map.

For $p : Z \times X \to X$, the Gysin map p_* is defined as follows. There is an m dimensional vector bundle V on Z with $dim(Z) = d$ (Theorem 2.11 [Vo3]) such that there is a map $i : \mathbb{T}^{m+d} \to Th_Z(V)$ having the property that the composition of maps

$$H^{2d,d}(Z) \cong H^{2(m+d),m+d}(Th_Z(V)) \xrightarrow{i^*} H^{2(m+d),m+d}(\mathbb{T}^{m+d}) \cong H^{0,0}(pt.) = \mathbb{Z}$$

coincides the degree map. Then we can define the Gysin map

$$p_* : ABP^{*,*'}(Z \times X) \cong ABP^{*+2m,*'+m}(Th_Z(V) \times X)$$

$$\xrightarrow{i^*} ABP^{*+2m,*'+m}(\mathbb{T}^{m+d} \times X) \cong ABP^{*-2d,*'-d}(X).$$

Of course for a projective map f, we define the Gysin map by $f_* = p_* i_*$.

Recall that we still defined the homological map $f_* : ABP_{*,*'}(Y) \to ABP_{*,*'}(X)$ in the preceding section. We can easily show that

$$f_*(y) \cap u_X = f_*(y \cap u_Y) \quad for\ y \in ABP^{*,*'}(Y)$$

by using the fact that $i_* u_Y = q^* Th_Y(1) \cap u_Z$ for an embedding and $p_*(Th_Z(1) \cap u_{X\times Z}) = u_X$ for a projection $p : X \times Z \to X$.

From Lemma 3.3, we know that

$$ABP^{2*,*}(X) \otimes_{BP^*} \mathbb{Z}/p \cong H^{2*,*}(X; \mathbb{Z}/p) \cong CH^*(X)/p.$$

By using the resolution of singularities, we can show that each element $x \in ABP^{2*,*}(X)$ is represented by $f_*(1_Y) = [f : Y \to X]$ such that $codim_X(Y) = c$ is constant and f is projective. (This fact is also showen by using the algebraic cobordism $\Omega^*(X)$ defined by Levine and Morel).

Recall that s_t (resp. c_t) is the total Landweber-Novikov operation (resp. total Chern class). Let us write

$$\nu_f = -f^*(T_X) + T_Y \in K(Y)$$

for the tangent bundles T_X and T_Y. Then on $ABP^{2*,*}(X)$, the Landweber-Novikov operations are written ([Q]) by

$$s_t(f_*(1_Y)) = f_*(c_t(\nu_f)).$$

Example. Consider the inclusion $i : \mathbb{P}^d \to \mathbb{P}^{d+1}$. Then the total Chern class of the normal bundle ν_i is

$$c_t^{-1}(\nu_i) = (\sum t_n y^{p^n-1}) \quad with\ e(\nu_i) = y$$

in fact $c_{\Delta_i}(L) = e(L)^{p^i-1}$ for line bundles L. On the other hand, the total Landweber-Novikov operation is

$$s_t^{-1}(i_*1) = s_t^{-1}(y) = \sum t_n y^{p^n}$$

from the definition of s_t (see the explanation before Lemma 4.3). Indeed, we show

$$i_*(c_t^{-1}(\nu_i)) = i_*(\sum t_n y^{p^n-1}) = \sum t_n y^{p^n}$$

since $i^*i_*(1) = e(\nu_i) = y$.

Lemma 7.2. *(Quillen [Q],[Me]) Let $x \in ABP^{2*,*}(Y)$ and $f : Y \to X$ be projective. Then $s_t(f_*(x)) = f_*(c_t(\nu_f)s_t(x))$.*

Proof. Let $x = [g : Z \to Y]$. By the definition

$$\nu_{fg} = -g^*f^*T_X + T_Z = g^*(-f^*T_X + T_Y) - g^*T_Y + T_Z = g^*\nu_f + \nu_g.$$

This implies $c_t(\nu_{fg}) = g^*(c_t(\nu_f))c_t(\nu_g)$. Hence we have

$$s_t(f_*x) = s_t(f_*g_*(1)) = f_*g_*(c_t(\nu_{fg}))$$

$$= f_*g_*(g^*(c_t(\nu_f)c_t(\nu_g)) = f_*(c_t(\nu_f)g_*(c_t(\nu_g)) = f_*(c_t(\nu_f)s_t(x)).$$

□

Let us write

$$I(X) = \pi_*ABP^{2*,*}(X) \subset ABP^{2*,*}(pt.) = BP^*.$$

From the Quillen's lemma, it is immediate

Lemma 7.3. *The ideal $I(X)$ is generated by elements x with $-dim(X) \leq |x| \leq 0$ as a BP^*-module. Moreover $I(X)$ is an invariant ideal of BP^*.*

Proof. Since $ABP^{2*,*}(X)$ is generated as a BP^* module by elements y with $0 \le |y| \le dim(X)$, we have the first statement. If $a \in I(X)$, then $a = \pi_*(x)$ for some $x = f_*(1_Y) = [f : Y \to X] \in ABP^{2*,*}(X)$. Then $s_t(a) = \pi_*(c_t(\nu_\pi)s_t(x)) \in I(X)[t]$. □

Let $\bar{k}$ be an algebraic closure of k. Let $i_{\bar{k}} : ABP^{*,*'}(X) \to ABP^{*,*'}(X|_{\bar{k}})$ be the induced map from the base change. Let $\bar{pt.} = Spec(\bar{k})$ and $j : \bar{pt.} \to X|_{\bar{k}}$ be an inclusion. Recall the definition ([Mo-Le]) of $deg(f) \in BP^*$;

$$deg(f)\{1_{\bar{pt.}}\} = j^* i_{\bar{k}} f_*(1_Y) \in ABP^{2d,d}(\bar{pt.}) \cong BP^{2d}\{1_{\bar{pt.}}\}.$$

(When $dim(X) = dim(Y)$, $deg(f) \in ABP^{0,0}(\bar{pt.}) \cong \mathbb{Z}_{(p)}$ is the usual degree of the map f. When $X = pt.$, we see $deg(f) = [Y]$.)

Let $\pi : X \to pt.$ is the projection. Let us write by $I^+(X)$ the sub BP^*-module of $I(X)$ generated by π_*-images of elements in $ABP^{2*,*}(X)$ of positive degree. The degree formula for the cobordism by Levine and Morel is following ;

Theorem 7.4. *(Levin-Morel [Le-Mo])*

$$s_\alpha[Y] - deg(f)s_\alpha[X] \in I^+(X).$$

Proof. Consider the element

$$z = f_*(1_Y) - deg(f)(1_X) \quad in\ ABP^{2d,d}(X).$$

Then by the definition, we see $j^*_{\bar{k}} i_{\bar{k}}(z) = 0$ since $j^*_{\bar{k}} i_{\bar{k}}(1_X) = 1_{\bar{pt.}}$. On the other hand, the kernel of the map $j^*_{\bar{k}} i_{\bar{k}}$ is the sub BP^*-module of $ABP^{2*,*}(X)$ generated by positive degree elements. Hence $\pi_*(z) \in I^+(X)$. Of course $\pi_*(z) = [Y] - deg(f)[X]$. From the proceeding lemma, we have the desired result. □

Let V be (a stable normal) m-dimensional bundle of X used to define the Gysin map ; there is the map $i : \mathbb{T}^{m'} \to Th_X(V)$, $m' = d+m$ which induces the degree on the motivic cohomology. Consider the cofibering

$$\to \mathbb{T}^{m'} \xrightarrow{i} Th_X(V) \xrightarrow{q} Th_X(V)/\mathbb{T}^{m'} \xrightarrow{\partial} S^{1,0}\mathbb{T}^{m'} \to,$$

and the induced long exact sequence on the mod p motivic cohomology.

The Thom class $Th(1) \in H^{2m,m}(Th_X(V); \mathbb{Z}/p)$ restricts to zero in $H^{*,*'}(\mathbb{T}^{m'}; \mathbb{Z}/p)$, and there exists $t \in H^{*,*'}(Th_X(V)/\mathbb{T}^{m'}; \mathbb{Z}/p)$ with $q^*(t) = Th(1)$. Let $\bar{\sigma} \in H^{2m'+1,m'}(Th_X(V); \mathbb{Z}/p)$ be the image of ∂^* of a generator σ of $H^{2m'+1,m}(S^{1,0}\mathbb{T}^{m'}; \mathbb{Z}/p)$.

Proposition 7.5. *(Lemma 4.1 in [Vo3])*

$$Q_n(t) = \lambda(deg(r_{\Delta_n}(X))/p)\bar{\sigma} \quad where\ \lambda \ne 0 \in \mathbb{Z}/p.$$

Proof. We consider the exact sequence for $Ak(n)^*(-)$ theory

$$\stackrel{\partial^*}{\leftarrow} Ak(n)^{*,*'}(\mathbb{T}^{m'}) \stackrel{i^*}{\leftarrow} Ak(n)^{*,*'}(Th_X(V)) \stackrel{q^*}{\leftarrow} Ak(n)^{*,*'}(Th_X(V)/\mathbb{T}^{m'}) \stackrel{\partial^*}{\leftarrow} .$$

Recall that the Gysin map is defined $\pi_* = i^*Th$ where $Th : Ak(n)^{*,*'}(X) \cong Ak(n)^{*+2m,*+m}(Th_X(V))$ is the Thom isomorphism. Hence we have the equivalents conditions

$$v_n \in I(X) \Longleftrightarrow v_n \in \pi_*(Ak(n)^{*,*'}(X))$$

$$\Longleftrightarrow v_n\sigma \in Im(i^*) \Longleftrightarrow v_n\bar{\sigma} = 0 \ \ in\ Ak(n)^*(Th_X(V))$$

$$\Longleftrightarrow Q_n z = \bar{\sigma} \quad for\ some\ z \ \ in\ H^{*,*'}(Th_X(V);\mathbb{Z}/p).$$

The last equivalence follows from Lemma 3.5. By dimensional reason, we can take $z = \lambda t,\ \lambda \neq 0 \in \mathbb{Z}/p$.

On the other hand

$$\pi_*(1_X) = [X] = v_n \Longleftrightarrow r_{\Delta_n}([X])/p \neq 0\ mod(p).$$

Since $k(n)^*$ is generated non-positive elements, $Ak(n)^{2*,*}(X)$ is generated by positive degree elements and 1_X. Hence we see

$$\pi_*(Ak(n)^{0,0}(X)) \ \ is\ generated\ by\ \ \pi_*(1_X) \quad mod(I_\infty^2).$$

Therefore $v_n \in I(X)$ if and only if $v_n = \lambda\pi_*(1_X)\ mod(I_\infty^2)$ for $\lambda \neq 0 \in \mathbb{Z}/p$. □

8. I_{n+1}-TORSION SPACES

Recall that $I_{n+1} = (p, v_1, ..., v_n)$. In this section, we consider I_{n+1}-torsion spaces and their applications according to V.Voevodsky. Recall that $BP\langle n\rangle^*(X)$ be the cohomology theory with the coefficient $BP\langle n\rangle^* = \mathbb{Z}_{(p)}[v_1, ..., v_n]$ so that $BP\langle -1\rangle^*(X) = H^*(X;\mathbb{Z}/p)$ and $BP\langle\infty\rangle^*(X) = BP^*(X)$.

Lemma 8.1. *([Ya4]) Let $E_r^{*,*',*''}$ be the AHss for $ABP\langle n\rangle^*(X)$. If $x = Q_n...Q_1Q_0x'$ in $H^{*,*'}(X;\mathbb{Z}/p)$, then $x \in E_\infty^{*,*',0}$ and x is I_{n+1}-torsion in $E_\infty^{*,*',*''}$.*

Proof. There is the cofiber map of spectra

$$\mathbb{T}^{p^k-1} \wedge ABP\langle k\rangle \xrightarrow{v_k} ABP\langle k\rangle \xrightarrow{\rho_k} ABP\langle k-1\rangle \xrightarrow{\delta_k}$$

Consider the Baas-Sullivan exact sequence, namely, the long exact sequence induced from the above cofiber map

$$\to ABP\langle k\rangle^{*+2p^k-2,*'+p^k-1}(X) \xrightarrow{v_k} ABP\langle k\rangle^{*,*'}(X) \xrightarrow{\rho_k}$$

$$ABP\langle k-1\rangle^{*,*'}(X) \xrightarrow{\delta_k} ABP\langle k\rangle^{*+2p^k-1,*'+p^k-1}(X) \to .$$

The induced map

$$Im(ABP\langle n-1\rangle^{*,*'}(X) \to H^{*,*'}(X;\mathbb{Z}/p)) \to H^{*,*'}(X:\mathbb{Z}/p)$$

defined by $\rho_0...\rho_{n-1}(x) \mapsto \rho_0...\rho_n\delta_n(x)$ represents the operation $Q_n + \sum a_{IJ}P^IQ_J$ with $a_{IJ} \in H^{plus,*}(X;\mathbb{Z}/p)$ and $J \geq 2$ from the topological case [Ya1] and (2.8).

By the Baas-Sullivan exact sequence, we can see that $x'' = \delta_n...\delta_0(x') \in ABP\langle n\rangle^{*,*'}(X)$ is I_{n+1}-torsion since the map δ_i is a map of ABP-module spectra. In particular, $x = Q_n...Q_0(x') = \rho_0...\rho_n(x'')$ is a permanent cycle in the spectral sequence

$$E(ABP\langle n\rangle)_2 = H^{*,*'}(X;BP\langle n\rangle^*) \Longrightarrow ABP\langle n\rangle^{*,*'}(X),$$

and $d_{2p^n-1}(y) = v_n \otimes x$ for $y = Q_{n-1}...Q_0(x')$. □

Compare with the spectral sequence

$$E(ABP)_2^{*,*,*} \cong H^{*,*'}(X;BP^*) \Longrightarrow ABP^{*,*'}(X).$$

Since $BP^* \cong BP\langle n\rangle^*$ for $* > -2p^{n+1}+2$, we can see that x is I_{n+1}-torsion also in $E(ABP)_{2p^n}^{*,*',*''}$.

Recall that $Q(n) = \Lambda(Q_0,...,Q_n)$. If $H^{*,*'}(X;\mathbb{Z}/p)$ is $Q(n)$-free and $H^{*,*'}(X)$ is just p-torsion, we have more strong results by using the Baas-Sullivan exact sequence in the proof of the preceding lemma.

Lemma 8.2. *([Ya4]) If $ABP\langle k\rangle^{*,*'}(X)$ is I_{k+1}-torsion for all $k \leq n$, then $H^{*,*'}(X;\mathbb{Z}/p)$ is a free $Q(n)$-module. If $H^{*,*'}(X;\mathbb{Z})$ has no (infinite) p-divisible elements, then the converse is also holds.*

Let the *Čech* complex $\check{C}(X)$ be the simplicial scheme such that $\check{C}(X)_n = X^{n+1}$ and the faces and degeneracy maps are given by partial projections and diagonals respectively ([Vo1,2]). One of the important properties of $\check{C}(X)$ is the following.

Lemma 8.3. *([Vo1,2]) Let X, Y be smooth schemes such that $Hom(Y,X) \neq \emptyset$. Then the projection $\check{C}(X)\times Y \to Y$ is a equivalence in $\mathbb{A}^1$-homotopy category.*

In the stable $\mathbb{A}^1$ homotopy category, define $\tilde{C}(X)$ by the following cofiber sequence

$$(5.1) \quad \tilde{C}(X) \to \check{C}(X) \to Spec(k).$$

Lemma 8.4. *(([Vo1]) Let $\pi : Y \to pt.$ be the projection and $\pi_*([Y]) = y$ in BP^*. Let $Ah = ABP(S_n)$ for some regular sequence S_n in BP^*. If $Hom(Y,X) \neq \emptyset$, then $Ah^{*,*'}(\tilde{C}(X))$ is y-torsion.*

Proof. Let $p : X \times Y \to X$ be the projection, and consider the composition map

$$p_*p^* : Ah^{*,*'}(X) \to Ah^{*,*'}(X \times Y) \to Ah^{*+|y|,*+1/2|y|}(X).$$

Here $p_*p^*(x) = yx$, indeed,

$$p_*p^*(x) = p_*(1_{X\times Y}p^*(x)) = p_*(1_{X\times Y})x = (y1_X)\cdot x.$$

But $Ah^*(\tilde{C}(X) \times Y) \cong 0$ since $Ah^{*,*'}(\check{C}(X) \times Y) \cong Ah^{*,*'}(Y)$. □

Recall that $I(X) = \pi_*(ABP^{2*,*}(X))$ for $\pi : X \to pt$.

Corollary 8.5. *If $v_n \in I(X)$, then $H^{*,*'}(\tilde{C}(X); \mathbb{Z}/p)$ is a free $Q(n)$-module.*

Proof. If there are maps $V_0 \to V_1 \to ... \to V_n \to X$ such that $t_{\mathbb{C}}(p_*[V_i]) = v_i$ for all $i \le n$, then we have the result. From Lemma 3.5, we know $r_{p^i\Delta_{n-i}}(v_n) = v_i$. . Since $I(X)$ is invariant ideal, we see that $v_i \in I(X)$ for all $i \le n$. This means the existence of V_i and above maps. □

9. Chow motive

For smooth X_1 and X_2, an element $\theta \in CH^{dim(X_2)}(X_1 \times X_2)$ can be viewed as a correspondence from X_1 to X_2. For more generally element $\theta \in CH^*(X_1 \times X_2)$ gives a homomorphism

$$f_\theta : H^{*,*'}(X_1; \mathbb{Z}/p) \to H^{*,*'}(X_2; \mathbb{Z}/p) \quad by \quad f_\theta(x) = pr_{2*}(pr_1^*(x) \cup \theta)$$

where pr_i are projections of $X_1 \times X_2$ onto X_i.

For $\theta \in CH^{dimX}(X \times X)$, the morphism $p_\theta = f_\theta$ is called a projector if $p_\theta \circ p_\theta = p_\theta$. The object of the Chow motive ($Chow^{eff}(k)$) are pairs (X, p) of smooth X and a projector $p = p_\theta$, and the morphisms are defined by morphisms f_θ (namely, the Chow motive is the pseudo abelian envelop of the category of correspondences). Objects (X, p) are simply called motives M which are direct summand of $M(X) = (X, id_X)$, and $H^{*,*'}(M; \mathbb{Z}/p)$ are defined as $Im(p)$.

Lemma 9.1. *Let M be a direct summand of $M(X)$ and p_θ be its projector, i.e., $p_\theta H^{*,*'}(X; \mathbb{Z}/p) = H^{*,*'}(M; \mathbb{Z}/p)$. Then p_θ commutes with Q_i. Hence $H^{*,*'}(M; \mathbb{Z}/p)$ has the natural $Q(\infty)$-module structure.*

Proof. Let $\theta \in CH^d(X \times X)$ with $dimX = d$. Then

$$p_\theta(Q_i(x)) = pr_{2*}(pr_1^*(Q_i(x)) \cdot \theta) = pr_{2*}(Q_i(pr_1^*(x) \cdot \theta)).$$

The last equation follows from $Q_i(\theta) = 0$ since $w(\theta) = 0$. Hence we have the desired result if $pr_{2*}Q_i = Q_ipr_{2*}$.

By definition of the Gysin map (recall §7), we know

$$pr_{2*}(x) = i^*(Th_X(1) \cdot x)$$

where $Th_X(1) \in H^{2m,m}(Th_X(V);\mathbb{Z}/p)$ is the Thom class for some bundle V over X and $i : \mathbb{T}^m \times X \subset Th_X(V) \times X$. Since $w(Th_X(1)) = 0$, we see $Q_i(Th_X(1) \cdot x) = Th_X(1) \cdot Q_i(x)$. Therefore we see that pr_{2*} commutes with Q_i. (Indeed, Q_i commutes with the Gysin maps.) □

Remark. The reduced powers P^i do not act naturally on $H^{*,*'}(M;\mathbb{Z}/p)$, see Lemma 9.2 bellow.

Let $A^*(X)$ be an oriented generalized cohomology theory on the category of smooth varieties X over k, in the sense of Panin [P]. The theories $CH^*(X)$ and $ABP^{2*,*}(X)$ are oriented generalized cohomology theories.

We can define the category of A-motive $M_A(k)$ as a pseudo abelian envelop of the category of A-correspondences Cor_A (of degree 0). Here objects in Cor_A are classes $[X]$ of smooth varieties and its morphisms are given by

$$Mor_{Cor_A}([X],[Y]) = A^{dim(X)}(X \times Y).$$

Theorem 9.2. *([Vi-Ya]) Let $\rho^A : A^*(X) \to CH^*(X)$ be a map of oriented cohomology theories such that ρ^A are epic and $Ker(\rho^A)$ are nilpotent for all X. Then ρ^A induces the natural 1 to 1 correspondence between the set of isomorphism classes of objects in $M_A(k)$ and $M_{CH}(k)$.*

The theory $ABP^{2*,*}(X)$ satisfies the assumption of the above theorem (with localized at p) from (3.4) and the fact that BP^* is generated by nonpositive degree elements.

Lemma 9.3. *(Karpenko-Merkurjev) [Ka-Me] For $x \in ABP^{2*,*}(X)$ and $\theta \in ABP^{2d,d}(X \times X)$, $d = dim(X)$, we have*

$$s_t(f_\theta(x)) = f_{s_t(\theta)}(s_t(x)c_t(\nu_X)).$$

Proof. From Lemma 7.2, we have

$$s_t(f_\theta(x)) = s_t(pr_{2*}(pr_1^*(x) \cdot \theta)) = pr_{2*}(s_t(pr_1^*(x)\theta)c_t(\nu_{pr_2})).$$

Here $c_t(\nu_{pr_2}) = pr_1^*(c_t(\nu_X))$. Hence the above element is

$$pr_{2*}(s_t(pr_1^*(x))s_t(\theta)pr_1^*(c_t(\nu_X)) = pr_{2*}(pr_1^*(s_t(x)c_t(\nu_X))s_t(\theta))),$$

which is $f_{s_t(\theta)}(s_t(x)c_t(\nu_X))$. □

Now we consider a hypersurface V in $\mathbb{P}^{d+1}$ of *degee* $= p$. Recall that

$$H^{*,*'}(\mathbb{P}^{d+1};\mathbb{Z}/p) \cong H^{*,*'}(pt.;\mathbb{Z}/p)[h]/(h^{d+2}).$$

We use the same letter $h \in H^{2,1}(V;\mathbb{Z}/p)$ which is the image of h by the map $H^{*,*'}(\mathbb{P}^{d+1};\mathbb{Z}/p) \to H^{*,*'}(V;\mathbb{Z}/p)$.

It is well known that $T_{\mathbb{P}^m} \oplus \epsilon \cong (m+1)O(1)$ where ϵ is a trivial line bundle. Hence there is the exact sequence of bundles

$$0 \to T_V \to T_{\mathbb{P}^{d+1}} \to O(p) \to 0.$$

Thus the total Chern class is $c_t(T_V) = (\sum h^{p^i-1}t_i)^{d+2}$.

Let us write simply

$$c_{t_i}(T_V) = (1 + h^{p^i-1})^{d+2}$$

letting $t_i = 1$ and $t_j = 0$ for $i \neq j$. Similarly define the total Landweber-Novikov operation $s_t^{-1}(-)$ by

$$s_{t_i}(x) = s_t^{-1}(x)|_{\{t_j=\delta_{ij}|j\geq 1\}}$$

such that $s(h) = s_0(h) + s_{\Delta_i}(h) = h + h^{p^i}$. By Lemma 7.2, for a projective map $f : Y \to X$, we have

$$s_{t_i}(f_*(x)) = f_*(c_{t_i}(f^*(T_X) - T_Y))s_{t_i}(x)).$$

Recall that $\bar{k}$ is an algebraic closure of k, $X|_{\bar{k}} = X \otimes_k \bar{k}$ and $i_{\bar{k}} : X \to X|_{\bar{k}}$ the extension map.

Lemma 9.4. *Let V have no k-rational point. Then $i_{\bar{k}}(h^d) = pw$ for a generator $w \in CH^d(V|_{\bar{k}})_{(p)} \cong \mathbb{Z}_{(p)}$.*

Proof. Since V has no k-rational points, the degree map $deg : CH^d(V)_{(p)} \to CH^0(pt.)_{(p)}$ is not epic. Hence from the commutative diagram

$$\begin{array}{ccc} CH^d(V) & \xrightarrow{deg} & p\mathbb{Z}_{(p)} \subset CH^0(pt)_{(p)} \cong \mathbb{Z}_{(p)} \\ {\scriptstyle i_{\bar{k}}}\downarrow & & =\downarrow \\ CH^d(V|_{\bar{k}}) & \xrightarrow{deg} & CH^0(\bar{pt})_{(p)} \cong \mathbb{Z}_{(p)} \end{array}$$

we see

$$(*) \qquad i_{\bar{k}}CH^d(X)_{(p)} \subset pCH^d(X|_{\bar{k}})_{(p)}.$$

On the other hand, for the emmbedding $i : V \to \mathbb{P}^{d+1}$, the normal bundle is $\nu_i = i^*O(p)$, so we see $i_*(1_V) = c_1(O(p)) = ph$. Hence

$$i_*(h^d) = h^d i_*(1) = ph^{d+1} \quad in\ CH^{d+1}(\mathbb{P}^{d+1}).$$

Now consider the composition map

$$CH^d(V) \stackrel{i_*}{\to} CH^{d+1}(\mathbb{P}^{d+1}) \stackrel{deg}{\to} CH^0(pt.).$$

Since $deg = \pi_*$ for the projection $\pi : X \to pt.$, it is immediate $deg|CH^d(V) = deg \circ i_*|CH^d(V)$. So we get $deg(h^d) = p$.

From $(*)$, we see that $i_{\bar{k}}(h^d) = pw$ for a generator w in $CH^d(V|_{\bar{k}})$. □

Lemma 9.5. *Let p_θ be a projector for $\theta \in CH^d(V \times V)$ for $dim(V) = d$ such that $p_\theta(h^d) = h^d$. Then for each $0 < p^s - 1 < d$, we see*

$$P^{\Delta_i}(p_\theta(h^{d-p^s+1})) = -h^d \quad \textit{in particular,}\ \ p_\theta(h^{d-p^s+1}) \neq 0.$$

Proof. Compare the equation given by Karpenko and Merkurjev

$$(*) \quad s_{t_s}(p_\theta(h^{d-i})) = p_{s_{t_s}(\theta)}(s_{t_s}(h^{d-i})c_{t_s}(-T_V)).$$

First we consider its right hand side term ;

$$s_{t_s}(h^{d-i}) = (h + h^{p^s})^{d-i} = h^{d-i}(1 + h^{p^s-1})^{d-i},$$

$$c_{t_s}(-T_V) = (1 + h^{p^s-1})^{-d-2}.$$

Here consider the case $i = p^s - 1$;

$s_{t_s}(h^{d-p^s+1})c(-T_V) = h^{d-p^s+1}(1 + h^{p^s-1})^{-p^s-1} = h^{d-p^s+1} + (-p^s - 1)h^d.$

Hence $P^{\Delta_s}(p_\theta(h^{d-p^s+1})) = -p_\theta(h^d) = -h^d$. □

Corollary 9.6. *If $d + 1 = 0\ mod(p)$, then $p_\theta(h^{d-p^s+1}) \neq h^{d-p^s+1}$.*

Proof. Note that

$$P^{\Delta_s}(h^{d-p^s+1}) = (d - p^s + 1)h^d,$$

which is zero if $d + 1 = 0\ mod(p)$. □

Remark. When $p \geq 3$, the norm variety described in the next sections are not hypersurfaces of $\mathbb{P}^{d+1}$.

10. Norm Variety

Recently, Voevodsky announced the proof of the Bloch-Kato conjecture for all odd primes [Vo4]. For non zero $a = \{a_0, ..., a_n\} \in K^M_{n+1}(k)/p$, Rost ([Ro3]) constructed the norm variety V_a such that

$$(1) \quad \pi_*[1_{V_a}] = V_a(\mathbb{C}) = v_n, \quad a = 0 \in K^M_{n+1}(k(V_a))/p$$

(2) the following sequence is exact

$$H_{-1,-1}(V_a \times V_a, \mathbb{Z}) \stackrel{pr_1 - pr_2}{\to} H_{-1,-1}(V_a; \mathbb{Z}) \to k^*.$$

Let us write $\chi_a = \check{C}(V_a)$. By the solution of Bloch-Kato conjecture, we see the exact sequence

$$(10.1) \quad 0 \to H^{*+1,*}(\chi_a; \mathbb{Z}/p) \stackrel{\times\tau}{\to} K^M_{*+1}(k)/p \to K^M_{*+1}(k(V_a))/p$$

identifying $H^{*+1,*+1}(\chi_a; \mathbb{Z}/p) \cong K^M_{*+1}(k)$. Since $a = 0 \in K^M_{n+1}(k(V_a))/p$, there is unique element $a' \in H^{n+1,n}(\chi_a; \mathbb{Z}/p)$ such that $\tau a' = a$.

Let M_a be the object in DM_-^{eff} defined by the following distinguished triangle

$$(10.2)\quad M(\chi_a(b_n))[2b_n] \to M_a \to M(\chi_a) \stackrel{\delta_a=Q_0...Q_{n-1}(a')}{\to} M(\chi_a)(b_n)[2b_n+1]$$

where $b_n = (p^n-1)/(p-1) = p^{n-1}+...+p+1$ so that $deg(\delta_a) = (2b_n+1, b_n)$. For $i<p$, define the symmetric powers

$$M_a^i = S^i(M_a) = q_i(M_a^{\otimes i}) \subset M_a^{\otimes i}$$

where $q_i(a) = (1/i!)\sum_{\sigma\in S_i}\sigma(a)$ for $a \in M_a^{\otimes i}$, and S_i is the symmetric group of i letters. One of the important results in [Vo4] Voevodsky proved is that M_a^{p-1} is a direct summand of a motive of V_a (for details see [Vo4]). Hence there are distinguished triangles ((5.5),(5.6) in [Vo4])

$$(10.3)\quad M_a^{i-1}(b_n)[2b_n] \to M_a^i \to M(\chi_a) \stackrel{s_i}{\to} M_a^{i-1}(b_n)[2b_n+1]$$

$$(10.4)\quad M(\chi_a)(b_ni)[2b_ni] \to M_a^i \to M_a^{i-1} \stackrel{r_i}{\to} M(\chi_a)(b_ni)[2b_ni+1].$$

Then we have the diagram

$$\begin{array}{ccccc} & & H^{*,*'}(\chi_a;\mathbb{Z}/p) & & \\ & & \downarrow r_{p-1}^* & & \\ H^{\sharp,\sharp'}(\chi_a;\mathbb{Z}/p) & \stackrel{s_{p-1}^*}{\longleftarrow} & H^{\natural,\natural'}(M_a^{p-2};\mathbb{Z}/p) & \longleftarrow & H^{\sharp-1,\sharp'}(M_a^{p-1};\mathbb{Z}/p) \\ & & \downarrow & & \\ & & H^{\natural,\natural'}(M_a^{p-1};\mathbb{Z}/p) & & \end{array}$$

where

$$(\sharp,\sharp') = (*+2(p^n+b_n), *'+p^n+b_n-1) = (*+2b_{n+1}, *'+b_{n+1}-1),$$

$$(\natural,\natural') = (*+2p^n-1, *'+p^n-1).$$

and the vertical and horizontal arrows are exact. From the result of Voevodsky, we know (Appendix in [S])

Lemma 10.1. *([Vo4]) For $x \in H^{*,*'}(\chi_a;\mathbb{Z}/p)$, we have*

$$s_{p-1}^*r_{p-1}^*(x) = \lambda Q_0Q_1...Q_n(a') \cup x \quad \lambda \neq 0 \in \mathbb{Z}/p.$$

Corollary 10.2. *The following map*

$$Q_0...Q_n(a') \cup - : H^{*,*'}(\chi_a;\mathbb{Z}/p) \to H^{\sharp,\sharp'}(\chi_a;\mathbb{Z}/p)$$

is surjective (resp. isomorphic) if the difference $*-*' \geq 0$ i.e., $\sharp-\sharp' \geq b_{n+1}-1$ (resp. $*-*'>0$ i.e., $\sharp-\sharp' > b_{n+1}-1$).

Proof. Let the difference $*-*'\geq 0$. Since M_{p-1} is a direct summand of the motive of V_a, we see

$$H^{\sharp,\sharp'}(M_a^{p-1};\mathbb{Z}/p)=0,\quad H^{\natural,\natural'}(M_a^{p-1};\mathbb{Z}/p)=0$$

since their difference is larger than $p^n-1=dim(V_a)$. Hence we know the subjectivity of $s_{p-1}^*r_{p-1}^*$. When the difference $*-*'>0$, we get moreover

$$H^{\sharp-1,\sharp'}(M_a^{p-1};\mathbb{Z}/p)=0,\quad H^{\natural-1,\natural'}(M_a^{p-1};\mathbb{Z}/p)=0,$$

by the same reasons. Thus we see the injectivity. □

Denote by $k(Q_a)$ the function field of Q_a and by $(Q_a)_0$ the set of closed points of Q_a. The main theorem of the paper ([Or-Vi-Vo]) by Orlov,Vishik and Voevodsky is the following (for $p=2$).

Theorem 10.3. *([Or-Vi-Vo]) For any $a=\{a_0,...,a_n\}\in K_*^M(k)/p$, the following sequence is exact*

$$\amalg_{x\in(V_a)_0}K_*^M(k(x))/p\overset{Tr_{k(x)/k}}{\longrightarrow}K_*^M(k)/p\overset{a}{\to}K_{*+n+1}^M(k)/p\to K_{*+n+1}^M(k(V_a))/p.$$

Outline of proof. (See for the case $*=1$, A.1 in [Su-Jo]) This is just odd primes p version of the arguments of the proof by Orlov, Vishik and Voevodsky. From arguments by Voevodsky ([Vo4],the main theorem in Appendix in [Su-Jo]), we see the exact sequence

$$(10.5)\quad \amalg_{x\in(V_a)_0}K_*^M(k(x))/p\overset{Tr_{k(x)/k}}{\longrightarrow}K_*^M(k)/p\overset{\delta_a}{\to}H^{*+2b_n+1,*+b_n}(\chi;\mathbb{Z}/p).$$

The last map δ_a is epic by the following reason. Consider the composition

$$K_*^M(k)/p\overset{\delta_a}{\to}H^{*+2b_n+1,*+b_n}(\chi;\mathbb{Z}/p)\overset{Q_n}{\to}H^{2pb_n+2,pb_n}(\chi_a;\mathbb{Z}/p).$$

Since $Q_n\delta_a=Q_n...Q_0(a')$, we see that $Q_n\delta_a$ is epic from the above lemma. Since $H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/p)$ is $\Lambda(Q_n)$-free we see that Q_n above is injective. Thus we show that the map δ_a in the above sequence is epic.

We also know that the map

$$(10.6)\quad K_*^M(k)/p\overset{\delta_a}{\to}H^{*+2b_n+1,*+b_n}(\chi;\mathbb{Z}/p)$$

$$\overset{(Q_{n-1}...Q_0)^{-1}}{\longrightarrow}H^{*+n+1,*+n}(\chi_a;\mathbb{Z}/p)\overset{\times\tau}{\to}K_{*+n+1}^M(k)/p$$

is the multiplication with a because V_a is a splitting variety of a. Thus we get the exact sequence. □

Corollary 10.4. *(Theorem 2.10 in [Or-Vi-Vo]) For each $0\neq h\in K_n^M(k)/p$, there is a field E/k and a nonzero pure symbol $a\in K_n^M(k)/p$ such that $h|_E=a|_E$ in $K_n^M(E)$.*

Proof. Let $h = b_1 + ... + b_l$ and each b_i a pure symbol for $1 \le i \le l$. Let V_{b_i} be the norm varieties and $E_i = k(V_{b_1} \times ... \times V_{b_i})$. Then of course $h|_{E_l} = 0$. Take i such that $h|_{E_{i-1}} \neq 0$ but $h|_{E_i} = 0$. Then from the above theorem,

$$Ker(K_n^M(E_{i-1})/p \to K_n^M(E_i)/p)) = b_i K_0^M(E_{i-1})/p.$$

Hence $h|E_{i-1} = \lambda b_i | E_i$ for $\lambda \neq 0 \in \mathbb{Z}/p$. □

Theorem 10.5. *(For $p = 2$, [Ya4]) Let $0 \neq a = (a_0, ..., a_n) \in K_{n+1}^M(k)/p$. Then there is a $K_*^M(k) \otimes Q(n)$-modules isomorphism*

$$H^{*,*'}(\tilde{\chi}_a; \mathbb{Z}/p) \cong K_*^M(k)/(Ker(a)) \otimes Q(n) \otimes \mathbb{Z}/p[\xi_a]\{a'\}$$

where $\xi_a = Q_n Q_{n-1} .... Q_0(a')$ and $deg(a') = (n+1, n)$.

Proof. Recall the difference $d(x) = f.deg(x) - s.deg(x)$. Hence if $0 \neq x \in H^{*,*'}(\tilde{\chi}_a; \mathbb{Z}/2)$, then $d(x) > 0$. We prove the theorem by induction on $d(t)$ for a $Q(n)$-module generator t in $H^{*,*'}(\tilde{\chi}_a; \mathbb{Z}/p)$. From (10.6) we already know that

$$K_*^M(k)/p \xrightarrow{\delta_a} H^{*+2b_n+1,*+b_n+1}(\chi_a; \mathbb{Z}/p) \xrightarrow{(Q_{n-1}...Q_0)^{-1}} H^{*+n+1,*+n}(\chi_a; \mathbb{Z}/p)$$

is an epimorphism, indeed, the map δ_a is epic from Corollary 10.2 and the map $Q_{n-1}...Q_0$ is isomorphic since $H^{*,*'}(\tilde{\chi}_a; \mathbb{Z}/2)$ is $Q(n)$-free. The composition of the above map with $H^{*+n+1,*+n}(\tilde{\chi}_a; \mathbb{Z}/p) \to K_{*+n+1}^M(k)/2$ is multiplying a from (10.6). Since the last map is monic from (10.1), we see that

$$H^{*+n+1,*+n}(\tilde{\chi}_a; \mathbb{Z}/p) \cong K_*^M(k)/(Ker(a))\{a\} \subset K_{*+n+1}^M(k).$$

Hence we get the case $d(t) = 1$.

Suppose that the isomorphism in the theorem holds for degree $d(x) < d$. Let $t \in H^{*,*'}(\tilde{\chi}_a; \mathbb{Z}/p)$ be a smallest weight element such that it is a $Q(n)$-module generator with $d(t) = d$. Then we see

$$d(Q_0...Q_n t) = p^n + p^{n-1} + ... + 1 + d > b_{n+1} + 1.$$

From Corollary 10.2, there is an element $y \in H^{*,*'}(\tilde{\chi}_a; \mathbb{Z}/p)$ such that

$$Q_0...Q_n(t) = s_{p-1}^* r_{p-1}^*(y) = \xi_a \cup y.$$

Since Q_i is a derivation (with some modification for $p = 2$).

$$\xi_a \cup Q_i(y) = Q_0...Q_n(a') \cup Q_i(y)$$

$$= Q_i(Q_0...Q_n(a') \cup y) = Q_i(Q_0...Q_n(t)) = 0$$

for $i \le n$. Since the map multiplying ξ_a is injective (indeed, isomorphic) for $H^{*,*'}(\tilde{\chi}_a; \mathbb{Z}/p)$ from Corollary 10.2, we see

$$Q_i(y) = 0 \quad for\ all\ 0 \le i \le n.$$

The fact that $H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/p)$ is $Q(n)$-free implies that

$$y = Q_nQ_{n-1}...Q_0(y') \quad for\ some\ y' \in H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/p).$$

Since we have

$$Q_0...Q_n(t-\xi_a y') = Q_0...Q_n(t) - \xi_a Q_0...Q_n(y') = Q_0...Q_n(t) - \xi_a y = 0,$$

the element $t-\xi_a y'$ is not a $Q(n)$-module generator. Hence we can take $Q(n)$-module generator $\xi_a y'$ instead of t.

Of course $d(y') = d(t) - d(\xi_a) < d(t)$. By induction on d, we see that $H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/p)$ is generated as a $K_*^M(k)\otimes Q(n)$ module by $\mathbb{Z}/p[\xi_a]\{a'\}$. The theorem follows from also the fact that the multiplying ξ_a is isomorphic. □

Remark. For $n=1$ case of the above theorem is known by A.Suslin.

Lemma 10.6. *If* $* < 4b_n$, *then* $H^{*,*'}(M_a;\mathbb{Z}/p) \cong H^{*,*'}(M_a^{p-1};\mathbb{Z}/p)$. *Moreover if* $* < 2b_n$, *then* $H^{*,*'}(M_a;\mathbb{Z}/p) \cong H^{*,*'}(\chi_a;\mathbb{Z}/p)$.

Proof. Since $H^{*,*'}(\chi_a;\mathbb{Z}/p) \cong 0$ for $* < 0$, we get this lemma from (10.2),(10.4). For example, (10.4) induces the long exact sequence

$$\leftarrow H^{*-2b_ni,*'-b_n}(\chi_a;\mathbb{Z}/p) \leftarrow H^{*,*'}(M_a^i;\mathbb{Z}/p) \leftarrow H^{*,*'}(M_a^{i-1};\mathbb{Z}/p) \leftarrow,$$

which induces the isomorphism $H^{*,*'}(M_a^i;\mathbb{Z}/p) \cong H^{*,*'}(M_a^{i-1};\mathbb{Z}/p)$ for the first degree $* < 2b_ni$. □

Let us consider the following triangular domain generated by bidegree

$$D_i = \{deg(x)|w(x) \geq 0,\ f.deg(x) < 2b_ni,\ d(x) > b_n(i-1)\}$$

and $D = \cup_{j=1}^{p-1} D_j$.

Lemma 10.7. *Let us write* $K = K_*^M(k)/(Ker(a))$. *For bidegree* $(*,*') \in D$ *defined above, we have the* K*-module (but not ring) isomorphism,*

$$H^{*,*'}(M_a^{p-1};\mathbb{Z}/p) \cong K[t]/(t^{p-1})\otimes Q(n-1)\{a'\}$$

where $deg(t) = (2b_n, b_n)$.

Proof. Consider the exact sequence induced from (10.3)

$$\leftarrow H^{*,*'}(M_a^{i-1}(b_n)[2b_n];\mathbb{Z}/p) \overset{j_1}{\leftarrow} H^{*,*'}(M_a^i;\mathbb{Z}/p) \overset{j_2}{\leftarrow} H^{*,*'}(\chi_a;\mathbb{Z}/p) \leftarrow .$$

By induction we assume for $(*,*') \in \cup_{j=1}^{i-1} D_j$

$$H^{*,*'}(M_a^{i-1};\mathbb{Z}/p) \cong K[t]/(t^{i-1})\otimes Q(n-1)\{a'\}.$$

Then for $(*,*') \in \cup_{j=1}^{i} D_j$, we see

$$H^{*,*'}(M_a^{i-1}(b_n)[2b_n];\mathbb{Z}/p) \cong (K[t]/(t^{i-1})\otimes Q(n-1)\{a'\})\otimes\{t\}.$$

In particular, both sides of the above are zero if $(*,*') \in D_1$. Hence j_2 is injective for this case.

Note $|Q_n a'| = 2p^n + n$, and for $* < 2p^n + n$ we have the isomorphism $H^{*,*'}(\chi_a; \mathbb{Z}/p) \cong K \otimes Q(n-1)\{a'\}$ which is zero for $(*,*') \in \cup_{j=2}^{i} D_j$. Hence j_2 is injective and j_1 is surjective in $\cup_{j=1}^{i} D_j$. □

Question. Is the isomorphism in the above lemma is that of $Q(n-1)$-modules ?

Corollary 10.8. *Let $c_i = Q_0 ... \hat{Q}_i ... Q_{n-1}(a')$. Then there is the additive isomorphism*

$$CH^*(M_a^{p-1})/p \cong \mathbb{Z}/p\{1\} \oplus \mathbb{Z}/p[t]/(t^{p-1})\{c_0, ..., c_{p-1}\}.$$

Here we consider some easy cases such that $K_*^M(k)/p = 0$ for $* > n+1$. Then note $K_*^M(k)/(Ker(a)) = \mathbb{Z}/p$.

Proposition 10.9. *Let $0 \neq a \in K_{n+1}(k)/p$ and $K_*^M(k)/p = 0$ for $* > n+1$. Then there is the additive isomorphism*

$$H^{*,*'}(M_a^{p-1}; \mathbb{Z}/p) \cong H^{*,*}(pt; \mathbb{Z}/p)[t]/(t^p) \oplus \mathbb{Z}/p[t]/(t^{p-1}) \otimes \tilde{Q}(n-1)\{a'\}$$

where $\tilde{Q}(n-1) = Q(n-1) - \mathbb{Z}/p\{Q_0...Q_{n-1}, Q_1...Q_{n-1}\}$.

Proof. For $n+1 < *$, we know $H^{*,*'}(pt.; \mathbb{Z}.p) = 0$. Hence for $n+1 < * < |Q_n a'| = 2p^n + n$, there is the isomorphism

$$H^{*,*'}(\chi_a; \mathbb{Z}/p) \cong Q(n-1)\{a'\}.$$

In particular $H^{*,*'}(\chi_a; \mathbb{Z}/p) = 0$ for $n+1 < * < 2p^n + n$. By using arguments similar to the proof of Lemma 10.7, we get the proof. (Here we identify $t = Q_1...Q_{n-1}(a')$.) □

Remark. As examples satisfying the assumption of the above proposition, we can take the high dimensional local fields defined by Kato and Parsin. Let k be a complete discrete valuation field with residue field F. Then it is well known that

$$K_r^M(k)/p \cong K_r^M(F)/p \oplus K_{r-1}^M(F)/p.$$

Let k_0 be a finite field, and let $k_1, ..., k_n$ be the sequence of complete discrete valuation fields such that the residue field of k_i is k_{i-1} for each $1 \geq i \geq n$. Then the field $k = k_n$ obtained in this way is called an n-dimensional local field. (see [Ka]). Then

$$K_n^M(k_n)/p \cong \mathbb{Z}/p \quad \textit{and} \quad K_m^M(k_n)/p = 0 \quad \textit{for } m > n.$$

11. $ABP^{2*,*}(X)$ FOR THE NORM VARIETIES

Using the fact that $H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/p)$ is a $Q(n)$-free module, we compute AHss

$$E(ABP)_2^{*,*,*} \cong H^{*,*'}(\tilde{\chi}_a;BP^*) \Longrightarrow ABP^{*,*'}(\tilde{\chi}_a).$$

Lemma 11.1. *The E_r-term of the above AHss is computed for $r \le 2p^n$, namely, for $i \le n$, the nonzero differentials are given by $d_{2p^i-1}(x) = v_i \otimes Q_i(x)$ and we have*

$$E(ABP)_{2p^i}^{*,*,*} \cong BP^*/I_{i+1} \otimes K[\xi_a] \otimes \Lambda(Q_{i+1},...,Q_n)\{Q_0...Q_i a'\}$$

where $K = K_*^M(k)/(Ker(a))$.

Proof. For ease of notations, let us write $Q(i,n) = \Lambda(Q_i,...,Q_n)$. By induction, we assume the result for $E(ABP)_{2p^{i-1}}$, $i \ge 1$. We note that all elements in $E(ABP)_{2p^{i-1}}$ are Q_0-image and that

$$E(ABP)_{2p^{i-1}}^{*,*,0} \cong K[\xi_a] \otimes Q(i,n)\{Q_0...Q_{i-1}(a')\}$$

$$\subset H^{*,*'}(\tilde{\chi}_a) \subset H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/p).$$

Moreover we can identify

$$E_{2p^{i-1}}^{*,*,*''} \subset H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/p) \otimes \mathbb{Z}/p[v_i] \quad for\ *'' > -2p^{n+1}+2 = |v_{i+1}|.$$

Let $E(Ak(i))_r^{*,*,*}$ be the AHss converging to $Ak(i)^{*,*'}(\tilde{\chi}_a)$. The natural map $ABP \to Ak(i)$ of spectra induces the map of AHss

$$j : E(ABP)_{2p^{i-1}}^{*,*,*''} \to E(Ak(i))_{2p^{i-1}}^{*,*,*''} \cong E(Ak(i))_2^{*,*,*''} \cong H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/p) \otimes k(i)^{*''}.$$

Then we see that the map j is injective for $*'' > -2p^{i+1}+2$. The first nonzero differential for $E(Ak(i))_r$ is $d_{2p^i-1}(x) = v_i \otimes Q_i(x)$ in $E(Ak(i))$ and so in $E(ABP)$. In particular the image $(d_{2p^i-1}E(ABP)^{*,*,*})$ is equal to

$$BP^*/I_i\{v_i\} \otimes K[\xi_a] \otimes Q(i+1,n)\{Q_iQ_0...Q_{i-1}(a)\}.$$

Since the sequence $(p, v_1, ..., v_i)$ is a regular sequence in BP^*, we have the result. □

The same fact holds for $ABP\langle m\rangle^{*,*'}(-)$ theory with $m \ge n$. In particular , we have for $m = n$

Corollary 11.2. $ABP\langle n\rangle^{*,*'}(\tilde{\chi}_a) \cong K_*^M(k)/(Ker(a))[\xi_a]^+$.

Conjecture. $ABP^{*,*'}(\tilde{\chi}_a) \cong BP^*/I_{n+1} \otimes K_*^M(k)/(Ker(a))[\xi_a]^+$.

Lemma 11.3. *Let $E(\chi_\alpha)_r$ (resp. $E(\tilde{\chi}_\alpha)_r$, $E(pt.)_r$) be AHss converging to $ABP^{*,*'}(\chi_\alpha)$ (resp. $ABP^{*,*'}(\tilde{\chi}_\alpha)$, $ABP^{*,*'}(pt.)$). Then*

$$E(\chi_\alpha)_r^{*,*,*} \cong E(\tilde{\chi}_\alpha)_r^{*,*,*} \oplus E(pt.)_r^{*,*,*}.$$

Proof. We consider maps of AHss

$$E(\tilde{\chi}_\alpha)_r \leftarrow E(\chi_\alpha)_r \leftarrow E(pt.)_r.$$

We have the additive decomposition $H^{*,*'}(\chi_\alpha) \cong A \oplus B$ where A (resp. B) is the $d(x) > 0$ (resp. $d(x) \leq 0$) parts of $H^{*,*'}(\tilde{\chi}_\alpha)$. Then we know $A \cong H^{*,*'}(\tilde{\chi}_\alpha)$ and $B \cong H^{*,*'}(pt.)$.

By induction on r, suppose that $E(\chi_\alpha)_r \cong E(\tilde{\chi}_\alpha)_r \oplus E(pt.)_r$ and $E(\tilde{\chi}_\alpha)_r$ (resp. $E(pt.)_r$) is isomorphic to the $d(x) > 0$ (resp. $d(x) \leq 0$) parts of $E(\chi_\alpha)_r$.

Then there is no nonzero differential for $x \in E(\chi_\alpha)_r$ such that $d_r(x) = y$ with $d(x) \leq 0$ but $d(y) > 0$ because if $d(x) \leq 0$ then x is in the image from $E(pt.)_r$, and so is y. Note that $d(d_{2s+1}) = s$ is positive. Hence the differential is closed in both $d(x) > 0$ and $d(x) \leq 0$ parts. Therefore we have the decomposition of the spectral sequence. □

Now we consider the AHss

$$E_2^{*,*',*''} = H^{*,*'}(V_a : BP^*) \Longrightarrow ABP^{*,*'}(V_a).$$

Here recall that (Cor 10.8)

$$\mathbb{Z}_{(p)}\{1, c_0\} \oplus \mathbb{Z}/p\{c_1, ..., c_{n-1}\} \subset H^{*,*'}(V_a; \mathbb{Z}_{(p)}).$$

Lemma 11.4. *The element c_i is I_i-torsion in $E_\infty^{2*,*,0}$.*

Proof. From Lemma 11.1, the element $c_i = Q_0...\hat{Q}_i...Q_n(a')$ is I_i-torsion in $E_{2p^i}^{2*,*,0}(\tilde{\chi}_a)$ converging to $ABP^{*,*'}(\tilde{\chi}_a)$. Considering the map of spectral sequences

$$E(\tilde{\chi}_a)_r \leftarrow E(\chi_a)_r \leftarrow E(V_a)_r$$

and the above lemma, we see that c_i in E_∞ is also I_i-torsion. □

We note the following lemma.

Lemma 11.5. *We have the filtration such that*

$$grBP^*\{c_0, ...c_{n-1}\}/(v_i c_j + v_j c_i | i < j) \cong \oplus_{0 \leq i \leq n-1} BP^*/I_i\{c_i\}.$$

Proof. Let $F_i = BP^*\{c_0, ..., c_{i-1}\}/(v_j c_k = v_k c_j | 0 \leq j < k \leq i-1)$. Then we have $F_{n-1} \supset .... \supset F_0$ and

$$F_i/F_{i-1} \cong BP^*\{c_i\}/(v_j \bar{c}_i = 0 | j < i) = BP^*/I_i\{c_i\}.$$

This induces the isomorphism of graded rings. □

Lemma 11.6. *Let $\bar{c}_i \in ABP^{2*,*}(V_a)$ be a lift of $c_i \in grABP^{2*,*}(V_a)$. There is the relation*

$$v_i\bar{c}_j - v_j\bar{c}_i = 0 \ \ mod(I_\infty^2) \quad for\ all\ i < j.$$

Proof. Since c_i is a I_i-torsion in $grABP^*(V_a)$, it is a v_k-torsion for $k < i$. From Corollary 3.7, there is z such that $Q_k(z) = c_i$. Let $pr : M(V_a) \to M(V_a)$ be the projector for $M^{pr-1} = pM(V_a)$ and $M' = (1 - pr)M(V_a)$ the orthogonal summand for pr. This z is uniquely written in $H^{*,*'}(V_a; \mathbb{Z}/p)$ as

$$z = Q_0...\hat{Q}_k...\hat{Q}_i...Q_{n-1}(\alpha') + b, \quad b \in M'$$

from Theorem 10.5 (by using $w(z) = 1$). Then

$$prQ_i(z) = Q_i(Q_0...\hat{Q}_k...\hat{Q}_i...Q_{n-1}(\alpha')) + pQ_i(b) = c_k + pQ_i(b).$$

Here $prQ_i(b) = Q_i(pr(b))$ from Lemma 9.1 and so $prQ_i(b) = 0$. Moreover $prQ_j(z) = 0$ for $j \neq k$, $j \neq i$. It follows from Lemma 3.6 that we get the relation $v_k\bar{c}_i + v_i\bar{c}_k = 0\ mod(I^2)$. □

The following lemma is immediate by $v_i \mapsto c_i$.

Lemma 11.7.

$$Ideal(I_n) \cong BP^*\{c_0, ..., c_{n-1}\}/(v_ic_j - v_jc_i).$$

Let $\bar{k}$ be the algebraic closure of k and $X|_{\bar{k}} = X \otimes_k \bar{k}$. Let $i_{\bar{k}} : ABP^{2*,*}(X) \to ABP^{2*,*}(X|_{\bar{k}})$ be the induced map. Of course we have the isomorphism of BP^*-modules

$$(11.1) \quad ABP^{2*,*}(M_a^{p-1}|_{\bar{k}}) \cong BP^* \otimes CH^*(M_a^{p-1}|_{\bar{k}}) \cong BP^*[\bar{t}]/(\bar{t}^p)$$

for $deg(\bar{t}) = (2b_n, b_n)$.

Lemma 11.8. $i_{\bar{k}}(t^jc_0) = p\bar{t}^{j+1}$.

Proof. First we prove $i_{\bar{k}}(t^{p-2}c_0) = p\bar{t}^{p-1}$. Here $t^{p-2}\bar{c}_0$ (resp. $\bar{t}^{p-1}$) generates $\mathbb{Z}_{(p)} \subset ABP^{2d,d}(V_n) \cong H^{2d,d}(V_n; \mathbb{Z}_{(p)})$ (resp. $ABP^{2d,d}(V_a|_{\bar{k}}) \cong \mathbb{Z}_{(p)}$) where $d = dim(V_a) = p^n - 1$.

Since V_k has no k-rational points,

$$degH^{2d,d}(V_a : \mathbb{Z}_{(p)}) \subset pH^{0,0}(Spec(k); \mathbb{Z}_{(p)}) = p\mathbb{Z}_{(p)}.$$

On the other hand, the fact $t_{\mathbb{C}}(V_a) = v_n$ implies that

$$deg(r_{\Delta_n}(-T_{V_a})) = deg(r_{\Delta_n}(-T_{V_a|\bar{k}})) = p\ mod(p^2).$$

Hence we have $degH^{2d,d}(V_a;\mathbb{Z}_{(p)}) = p\mathbb{Z}_{(p)}$, while $degH^{2d,d}(V_a|_{\bar{k}};\mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}$. Since the following diagram is commutative

$$\begin{array}{ccc} H^{2d,d}(V_a;\mathbb{Z}_{(p)}) & \xrightarrow{deg} & H^{0,0}(Spec(k);\mathbb{Z}_{(p)}) \\ {\scriptstyle i_{\bar{k}}}\downarrow & & \cong\downarrow \\ H^{2d,d}(V_a|_{\bar{k}};\mathbb{Z}_{(p)}) & \xrightarrow{deg} & H^{0,0}(Spec(\bar{k});\mathbb{Z}_{(p)}) \end{array}$$

we see that $i_{\bar{k}}(t^{p-2}c_0) = p\bar{t}^{p-1}$.

Consider the commutative diagram

$$\begin{array}{ccccc} ABP^{2*-2b_n,*-b_n}(M_a^{i-1}|_{\bar{k}}) & \xleftarrow{j_1|\bar{k}} & ABP^{2*,*}(M_a^i|_{\bar{k}}) & \longleftarrow & ABP^{2*,*}(\chi_a|_{\bar{k}}) \\ {\scriptstyle i_{\bar{k}}}\uparrow & & {\scriptstyle i_{\bar{k}}}\uparrow & & {\scriptstyle i_{\bar{k}}}\uparrow \\ ABP^{2*-2b_n,*-b_n}(M_a^{i-1}) & \xleftarrow{j_1} & ABP^{2*,*}(M_a^i) & \longleftarrow & ABP^{2*,*}(\chi_a) \end{array}$$

When $(*,*') = (2b_ni, b_ni)$, we see that $j_1|_{\bar{k}}$ and j_1 are isomorphism since $ABP^{*,*'}(\chi_a) = ABP^{*,*'}(\chi_a|_{\bar{k}}) = 0$. Moreover $j_1|_{\bar{k}}(\bar{t}^i) = \bar{t}^{i-1}$ and $j_1(c_0t^{i-1}) = c_0t^{i-2}$. By induction starting $i_{\bar{k}}(t^{p-2}c_0) = p\bar{t}^{p-1}$, we have the desired result $i_{\bar{k}}(t^{i-2}c_0) = p\bar{t}^{i-1}$, from the above diagram. □

Lemma 11.9. *In $ABP^{2*,*}(V_a|_{\bar{k}})$, we see $i_{\bar{k}}(c_i) = v_i\bar{t}\ mod(I_\infty^2)$.*

Proof. By induction on i, we assume $i_{\bar{k}}(c_k) = v_k\bar{t}\ mod(I_\infty^2)$, for $k < i$. Since $(v_i\bar{c}_0 - p\bar{c}_i) \in I_\infty^2\{\bar{c}_0, ..., \bar{c}_{i-1}\}$, we have

$$i_{\bar{k}}(v_i\bar{c}_0 - p\bar{c}_i) = p(v_i\bar{t} - i_{\bar{k}}(\bar{c}_i)) \quad mod(I^3\{\bar{t}\}).$$

Hence $i_{\bar{k}}(\bar{c}_i) = v_i\bar{t}\ mod(I^2)$, since $\bar{t}$ generates a free BP^*-module, that is integral domain, indeed, BP^* is a polynomial algebra. □

Theorem 11.10. *The cohomology $ABP^{2*,*}(V_a)$ contains sub BP^*-module isomorphic to $I_n\{\bar{t}\}$.*

Proposition 11.11. *Suppose that the isomorphism in Lemma 10.7 is that of $Q(n-1)$-modules. Then the map $i_{\bar{k}} : ABP^{2*,*}(M_a^{p-1}) \to ABP^{2*,*}(M_a^{p-1}|_{\bar{k}})$ is injective and hence*

$$ABP^{2*,*}(M_a^{p-1}) \cong BP^*\{1\} \oplus \mathbb{Z}_{(p)}[t]/(t^{p-2}) \otimes I_n\{c\}$$

where $pc = \bar{c}_0$ and $deg(c) = (2b_n, b_n)$.

Theorem 11.12. *Suppose that the projection of the motive*

$$CH^*(V_a|_{\bar{k}}) \to CH^*(M_a^{p-1}|_{\bar{k}}) \cong \mathbb{Z}[\bar{t}]/(\bar{t}^{p-1})$$

induces the map of rings. Then

$$Im(i_{\bar{k}} : ABP^{2*,*}(M_a^{p-1})) \cong BP^*\{1\} \oplus I_n[\bar{t}]/(\bar{t}^{p-2}).$$

Proof. From the above corollary we see $Im(i_{\bar{k}}) \supset I_n\bar{t}$. By the commutative diagram

$$\begin{array}{ccc} ABP^{2*,*}(V_a) & \xrightarrow{p} & ABP^{2*,*}(M_a^{p-1}) \\ {\scriptstyle i_{\bar{k}}}\downarrow & & {\scriptstyle i_{\bar{k}}}\downarrow \\ ABP^{2*,*}(V_a|_{\bar{k}}) & \xrightarrow{p} & ABP^{2*,*}(M_a^{p-1}|_{\bar{k}}), \end{array}$$

we see $Im(p \cdot i_{\bar{k}}) \supset I_n\bar{t}$. Hence

$$Im(i_{\bar{k}} \cdot p) = Im(p \cdot i_{\bar{k}}) \supset I_n^2\bar{t}^2.$$

In particular $v_1^2\bar{t}^2 \in i_{\bar{k}}ABP^{2*,*}(M_a^{p-1})$.

Suppose that $v_1\bar{t}^2 \notin i_{\bar{k}}ABP^{2*,*}(M_a^{p-1})$. Then $v_1^2\bar{t}^2$ is a BP^*-module generator of $i_{\bar{k}}ABP^{2*,*}(M_a^{p-1})$. Hence there is a nonzero element

$$c \in \Omega^{2*,*}(M_a^{p-1}) \otimes_{\Omega^*} \mathbb{Z} \cong CH^*(M_a^{p-1})$$

such that $|c| = |v_1^2\bar{t}^2| = 2b_n - 4(p-1)$. But such element does not exist in $CH^*(M_a^{p-1})$ form Corollary 10.8. Thus we have $v_1\bar{t}^2 \in i_{\bar{k}}ABP^{2*,*}(M_a^{p-1})$.

Similarly we see that

$$v_i\bar{t}^j \in i_{\bar{k}}ABP^{2*,*}(M_a^{p-1})$$

for all $0 \le i \le n-1$ and $1 \le j \le p-1$. Then we can prove the theorem. □

12. Smith and Toda spectrum $V(n)$

The Smith and Toda spectrum $V(n)$ is defined as a (topological) spectrum such that

$$H^*(V(n);\mathbb{Z}/p) \cong Q(n), \quad BP^*(V(n)) \cong P(n+1)^* = BP^*/I_n.$$

It is known that such $V(n)$ exists for $n = 1$ if $p \ge 3$, $n = 2$ if $p \ge 5$ and $n = 3$ if $p \ge 7$. Suppose that $V(n)$ exists. Then $V(n)$ is constructed from $V(n-1)$ by the cofiber sequence

$$S^{2p^n-2}V(n-1) \xrightarrow{f} V(n-1) \to V(n)$$

where f is a map so that $f^* = v_n$ identifying $BP^*(V(n-1)) \cong BP^*/I_n$. The Greek letter element is defined as the stable map

$$G(n)_s : S^{-s|v_n|} \subset S^{-s|v_n|}V(n-1) \xrightarrow{f^s} V(n-1) \xrightarrow{q} S^{2b_n-n}$$

where q is the projection map to the biggest cell. Usually $G(n)_k$ is written as

$$G(1)_s = \alpha_s, \quad G(2)_s = \beta_s, \quad G(3)_s = \gamma_s$$

and called the Greek letter elements. It is known that if $V(n)$ exists then the stable homotopy group of sphere

$$\pi_i^s = lim_{N\to\infty}[S^{N+i}, S]$$

is multiplicative generated by the Greek letter elements when $i < |G(n+1)_1|$.

We note here there exists analogous spaces in the stable A_1-homotopy category. First we consider the space defined by the cone

$$M_a|_{\bar{k}} \to M_a \to cone = MV(n-1).$$

Lemma 12.1. *Let $p = 2$ or Suppose the assumption in Proposition 11.11. Then $ABP^{2*,*}(MV(n-1)) \cong P(n)^*[t]^+/(t^{p-1})$.*

Note that there are nonzero elements $ABP^{*,*'}(MV(n-1))$ when $* \neq 2*'$.

Another candidate for $V(n)$ is the reduced *Čeck* spaces.

Lemma 12.2. *Let k be a field such that $K_*^M(k)/p = 0$ for $* > n+1$. Then $ABP^{*,*'}(\tilde{\chi}_a) \cong P(n+1)^*[\xi_a]^+$.*

13. The case M_a^{p-1} exists as a variety

In general, we can not identify the motive M_a^{p-1} as an object in the stable homotopy category and so we can not consider its generalized cohomology theories. However we consider (hereafter of this section) the cases that

(**Assumption**) There is a space U_a and a map $U_a \to V_a$ such that this map induces the isomorphism

$$H^{*,*'}(U_a;\mathbb{Z}/p) \cong H^{*,*'}(M_a^{p-1};\mathbb{Z}/p).$$

Note when $p = 2$, this assumption holds (Theorem [Vi-Ya]).

Define a space W_a by the cofibering

$$U_a \xrightarrow{p} \chi_a \xrightarrow{i} cone = W_a \xrightarrow{\partial} U_a(1).$$

Remark. When $p = 2$, we have the isomorphism of motives $M(W_a) = M(\chi_a)(2^n-1)[2^{n+1}-1]$. However the $\mathbb{A}^1$-homotopy type of W_a is not that of $S^{2^{n+1}-1,2^n-1}\chi_a$. In fact, their $\Lambda(Q_n)$-module structures are different.

We consider some more easy theory

$$A\bar{h}^{*,*'}(X) = ABP\langle n-1\rangle^{*,*'}(X), \quad \bar{h}^* = \mathbb{Z}_{(p)}[v_1, ...v_{n-1}]$$

and give a short another proof of the $A\bar{h}^{*,*}$-version of Theorem 11.10.

By the arguments similar to the proof of Lemma 11.1 and Corollary 11.2, we get

Lemma 13.1. *$A\bar{h}^{*,*'}(\tilde{\chi}_a)\cong K_*^M(k)/(Ker(a))[\xi_a]\{\delta_a,Q_n\delta_a\}$ where $\delta_a=Q_0...Q_{n-1}a'$.*

In particular, the above $BP\langle n-1\rangle^*$-module is indeed a $BP\langle n-1\rangle^*/(I_n)=\mathbb{Z}/p$ module.

Lemma 13.2. *For $*\le 2b_n$, we see $gr\bar{A}h^{*,*',*''}(W_a)=0$ and*

$$grA\bar{h}^{2b_n+1,b_n,*''}(W_a)\cong BP\langle n-1\rangle^*\{\delta'\}\quad with\ i^*(\delta')=\delta_a.$$

Proof. Consider the exact sequence

$$\leftarrow H^{*,*'}(U_a;\mathbb{Z}/p)\overset{p^*}{\leftarrow}H^{*,*'}(\chi_a;\mathbb{Z}/p)\overset{i^*}{\leftarrow}H^{*,*'}(W_a;\mathbb{Z}/p)\leftarrow.$$

Since $H^{*,*'}(U_a;\mathbb{Z}/p)\cong H^{*,*'}(\chi_a;\mathbb{Z}/p)$ for $*\le 2b_n$ from Lemma 6.6, we have $H^{*,*'}(W_a;\mathbb{Z}/p)=0$ for these degrees. Moreover by the exact sequence

$$0\leftarrow H^{2b_n+1,b_n}(\chi_a;\mathbb{Z}_{(p)})=\mathbb{Z}/p\leftarrow H^{2b_n+1,b_n}(W_a;\mathbb{Z}_{(p)})$$

$$\leftarrow H^{2b_n,b_n}(U_a;\mathbb{Z}_{(p)})=\mathbb{Z}_{(p)}\leftarrow 0,$$

we see $H^{2b_n+1,b_n}(W_a;\mathbb{Z}_{(p)})\cong\mathbb{Z}_{(p)}$ and write its generator by δ_a'.

Consider AHss for $A\bar{h}^{*,*}(W_a)$. Since $w(\delta_a)=-1$, we see $p^*(\delta_a)=0\in A\bar{h}^{*,*'}(U_a)$. Hence there is $\delta'\in A\bar{h}^{*,*}(W_a)$ with $i^*(\delta')=\delta_a$. i.e., δ' is represented by δ_a' in AHss. By dimensional reason, there is no nonzero differential which targets in

$$BP\langle n-1\rangle^*\{\delta'\}\subset BP\langle n-1\rangle^*\otimes H^{2b_n+1,b_n}(W_a;\mathbb{Z}_{(p)})\cong E_2^{2b_n+1,b_n,*}.$$

Thus we have the lemma. □

We consider the map

$$A\bar{h}^{*,*'}(W_a)\xrightarrow{i^*}A\bar{h}^{*,*}(\chi_a),\qquad i^*(\delta')=\delta_a=Q_0...Q_{n-1}(a').$$

Here $\delta'\in A\bar{h}^{2^{n+1}-1,2^n-1}(\chi_a)$ is a free $BP\langle n-1\rangle^*$-module but δ_a is a I_n-torsion module. Thus we know

$$Ker(i^*)|BP\langle n-1\rangle^*\{\delta'\}\cong I_n\{\delta'\}$$

which must be contained in $A\bar{h}^{*,*'}(U_a)$.

Proposition 13.3.

$$ABP\langle n-1\rangle^{2*,*}(U_a)\supset I_n\{\alpha\}\quad deg(\alpha)=(2b_n,b_n),\ \partial^*(v_i\alpha)=v_i\delta'.$$

The element $v_i\alpha$ corresponds $\bar{c}_i$ in Lemma 11.6.

Remark. For the case $ABP\langle n\rangle^{*,*'}(-)$, the situation is quite different. The arguments for $n-1$ does not work for this case, e.g., $Q_n\delta_a\neq 0\in$

$H^{*,*'}(\chi_a;\mathbb{Z}/p)$ and hence δ_a does not exist in $ABP\langle n\rangle^{*,*'}(\chi_a)$. Consider AHss

$$E_2 = H^{*,*'}(W_a; BP\langle n\rangle^*) \Longrightarrow ABP\langle n\rangle^{*,*'}(W_a).$$

In $E_{2p^n-1}^{*,*,*}$, the differential is given by $d_{2p^n-1}(\delta') = v_nQ_n\delta'$. Here $Q_n\delta'$ generates I_n- torsion module but δ' generates a free $BP\langle n\rangle^*$-module. Thus we have

$$E_{2p^n}^{2b_n+1,b_n,*} \supset I_n\{\delta'\}.$$

Elements $v_i\delta'$ represent corresponding c_i in $ABP\langle n\rangle^{*,*'}(U_a)$.

14. Real case

In the last section of this paper, we restricted the case $k=\mathbb{R}$ and $p=2$. The mod 2 motivic cohomology is $H^{*,*'} = H^{*,*'}(pt;\mathbb{Z}/2) \cong \mathbb{Z}/2[\rho,\tau]$ with $deg(\rho) = (1,1)$, $deg(\tau) = (0,1)$. We want to study $ABP^{*,*'}(U_a)$ for all bidegree. Recall that

$$H^{*,*'}(\chi_a;\mathbb{Z}/2) \cong H^{*,*'}(pt.;\mathbb{Z}/2) \oplus \mathbb{Z}/2[\rho,\xi_a]\otimes Q(n)\{a'\}$$

here note $\xi_a = \delta_a^2$. We still know from [Ya]

$$H^{*,*'}(U_a;\mathbb{Z}/2) \cong H^{*,*'}(\chi_a;\mathbb{Z}/2)/(f.deg > 2^{n+1}-2).$$

We write down it more explicitly.

Theorem 14.1. *([Ya4]) Let $k=\mathbb{R}$ and $a=\rho^{n+1}$. Then*

$$H^{*,*'}(U_a;\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau,\rho]/(\rho^{2^{n+1}-1}) \oplus \oplus_{\epsilon\neq(1,..,1)}\mathbb{Z}/[\rho]\{Q^\epsilon(a')\}/(\rho^{k(\epsilon)}Q^\epsilon(a'))$$

where $k(\epsilon) = (2^{n+1}-1) - f.deg(Q^\epsilon(a')) = 2^{n+1} - \sum_{i=0}^{n-1}(\epsilon_i(2^{i+1}-1)) - n - 2$.

Now we recall some facts for cohomology operations.

Lemma 14.2. *([Ya4]) Let $t_k = \tau^{2^k}$ and let $grH^{*,*'} = \mathbb{Z}/2[\rho]\otimes\Lambda(t_0,t_1,...)$. Then Q_n acts on $H^{*,*'}$ as a derivation on $grH^{*,*'}$ with*

$$Q_n(\rho) = 0, \quad Q_n(t_n) = \rho^{2^{n+1}-1}, \quad Q_n(t_j) = 0 \ for\ n\neq j,$$

namely, $Q_n(\Pi t_i^{e_i}\rho^k) = (\Pi_{n\neq i}t_i^{e_i})(e_n\rho^{2^{n+1}-1})\rho^k$ for $e_i = 0$ or 1.

Theorem 14.3. *([Ya4]) Let us write $\Lambda(\hat{n}) = \Lambda(t_0,...,\hat{t}_n,...)$. Then we have*

$$grAK(n)^*(pt.) \cong K(n)^*[\rho]/(\rho^{2^{n+1}-1})\otimes\Lambda(\hat{n})$$

Proof. From Lemma 3.4, the first nonzero differential is $d_{2^{n+1}-1}(x) = v_n \otimes Q_n(x)$. The fact that $Q_n t_n = \rho^{2^{n+1}-1}$ implies that $E_{2^{n+1}}^{*,*'}$ is isomorphic to the righthand module in the theorem. Since $E_r^{*,*',*''} \cong 0$ if $* \geq 2^{n+1}-1$, we see that $d_r = 0$ for all $r \geq 2^{n+1}$. Thus $E_{2^{n+1}}^{*,*'}$ is isomorphic to the infinitive term of AHss. □

Theorem 14.4. *For $1 \leq i \leq n-1$, we have the isomorphism*

$$AK(i)^{*,*'}(U_a) \cong K(i)^*[\rho]/(\rho^{2^{i+1}-1}) \otimes (A \oplus B \oplus C)$$

where $A = \Lambda(\hat{i})$, $B = \Lambda(\hat{i})\{\rho^{2^{n+1}-2^{i+1}} t_i\}$, *and* $C = \oplus_{\epsilon_i=0} \mathbb{Z}/2\{\rho^{k(\epsilon)-2^{i+1}+1} Q_\epsilon(a')\}$.

Proof. Consider the motivic AHss

$$E_2^{*,*,*} = H^{*,*'}(U_a; K(i)^*) \Longrightarrow AK(i)^*(U_a).$$

First consider elements of the $difference \leq 0$. The first nonzero differential is $d_{2^{i+1}-1}(x) = v_i \otimes Q_i(x)$. From the fact $Q_i(t_j) = \delta_{i,j}\rho^{2^i-1}$ and from the above lemma, we have

$$H(E_2^{*,*,*} \cap (difference \leq 0); Q_i) \cong (A \oplus B) \otimes \mathbb{Z}/2[\rho]/(\rho^{2^{i+1}-1}).$$

Since $A \subset K(i)^*(Spec(\mathbb{R}))$, elements in A are permanent cycles. By the dimensional reason such as $f.deg > 2^{n+1}-2^{i+1}$, elements in B are also permanent cycles.

Next consider elements of $difference > 0$. We can see that

$$Q_i Q_\epsilon(\rho^k a') = 0 \quad if \quad \begin{cases} \epsilon_i = 1 \quad or \\ k + 2^{i+1} + \sum_i \epsilon_i(2^{i+1}-1) + n \geq 2^{n+1}-1. \end{cases}$$

Of course, $\epsilon_i = 1$ case, $Q_\epsilon(a')$ is in the $Im(Q_i)$, and hence we have

$$H(E_2^{*,*,*} \cap (difference > 0); Q_i) \cong C \otimes \mathbb{Z}/2[\rho]/(\rho^{2^{i+1}-1}).$$

By also dimensional reason, all elements in C are permanent cycles. □

It seems not so easy to compute $ABP^*(pt), ABP^*(U_a)$. Hence we consider the more easy case

$$Ah^{*,*'}(-) = A\bar{h}/p^{*,*'}(-) = ABP\langle n-1\rangle/(p)^{*,*'}(-),$$

namely, throughout this section, let $h^* = \mathbb{Z}/p[v_1, ..., v_{n-1}]$. Let us write for $i \leq j$

$$\Lambda_i^j = \Lambda(t_i, ..., t_j), \quad V_i^j = \Lambda(t_0, ..., t_i, t_j, ...)$$

Theorem 14.5. *([Ya4]) There is an isomorphism*

$$grAh^{*,*'}(pt.) \cong V_0^n \otimes h^*[\rho]\{1, v_i x | x \in (\Lambda_{i+1}^{n-1})^+\}/(v_1\rho^3, ..., v_{n-1}\rho^{2^n-1}).$$

Proof. Consider AHss

$$E(Ah)_2^{*,*,*} = H^{*,*'}(pt;h^*) \Longrightarrow Ah^{*,*'}(pt).$$

By induction on n, we assume

$$(*) \quad E_{2^{i+1}}^{*,*,*} = V_0^n \otimes (A_i \oplus (h^*[\rho] \otimes \Lambda_{i+1}^{n-1}))/(v_1\rho^3, ..., v_i\rho^{2^{i+1}-1})$$

where A_i is an h^*-module with generators in $E_{2^{i+1}}^{*,*',minus}, 0 \le * < 2^{i+1} - 1$. The next nonzero differential is

$$d_{2^{i+2}-1}(x) = v_{i+1}Q_{i+1}(x)\ mod(I_{i+1}) \quad for\ x \in E_{2^{i+2}-1}^{*,*',0}.$$

In particular

$$d_{2^{i+2}-1}(at_{i+1}) = v_{i+1}\rho^{2^{i+2}-1}a \quad for\ a \in \mathbb{Z}/2[\rho] \otimes \Lambda_{i+2}^{n-1}.$$

Moreover we can prove $d_{2^{i+2}-1}(a) = 0$ for $a \in A_i$ considering the map of spectral sequences induced from $Ah \to ABP\langle i+2, ...n-1\rangle/(2)$ where $BP\langle i+2, ...n-1\rangle/(2)^* = \mathbb{Z}/2[v_{i+2}, ..., v_{n-1}]$. (For details, see [Ya4].) Thus we can prove the $(i+1)$-version of $(*)$.

Moreover we see that A_{i+1} is isomorphic to

$$A_i \oplus BP^*[\rho]\{v_1t_{i+1}, ..., v_it_{i+1}\} \otimes \Lambda_{i+1}^{n-1}/(v_1\rho^3, ..., v_{n-1}\rho^{2^n-1})$$

because $d_{2^{i+2}-1}(v_jt_{i+1}) = 0$ for all $j < i+1$. Thus we get

$$A_\infty \cong BP^*[\rho]\{v_ix | x \in (\Lambda_{i+1}^{n-1})^+\}/(v_1\rho^3, ..., v_{n-1}\rho^{2^n-1}).$$

□

Theorem 14.6. *There is the isomorphism*

$$grAh^{*,*'}(U_a) \cong (A \oplus B) \otimes V_0^n \oplus C \qquad with$$

$$(1) \quad A = h^*[\rho]\{1, v_ix | x \in (\Lambda_{i+1}^{n-1})^+\}/(v_1\rho^3, ..., v_{n-1}\rho^{2^n-1}, \rho^{2^{n+1}-1}),$$

$$(2) \quad B = \oplus_{i=0}^{n-1} h^*/I_i[\rho]\{\rho^{2^{n+1}-2^{i+1}}t_i\}/(\rho^{2^{i+1}-1}) \otimes \Lambda_{i+1}^{n-1},$$

$$(3) \quad C = \oplus_{i=0}^{n-1}(h^*/I_i[\rho]\{\oplus_{\epsilon_0=1,...\epsilon_{i-1}=1,\epsilon_i=0}\rho^{k(\epsilon)-2^{i+1}+1}Q_\epsilon(a')\}/(\rho^{2^{i+1}-1})).$$

Proof. The proof is similar to the proof of Theorems 6.7 and 6.8. □

Next we also compute $AK(n)^{*,*'}(U_a)$ and $Ah^{*,*'}(U_a)$ by using the cofiber sequence

$$U_a \longrightarrow \chi_a \xrightarrow{i} W_a$$

given in Section 13. The cohomology of W_a is easily computed from the long exact sequence induced from the above cofibering and the cohomologies of M_a and χ_a. Recall again that

$$H^{*,*'}(\chi_a; \mathbb{Z}/2) \cong H^{*,*'}(pt.; \mathbb{Z}/2) \oplus \mathbb{Z}/2[\rho, \xi_a] \otimes Q(n)\{a'\}$$

$$H^{*,*'}(U_a;\mathbb{Z}/2)\cong H^{*,*'}(\chi_a;\mathbb{Z}/2)/(f.deg>2^{n+1}-2)$$

here note $\xi_a=\delta_a^2$. Hence we have

$$H^{*,*'}(W_a;\mathbb{Z}/2)\cong \{(f.deg>2^{n+1}-2)\ parts\ of\ H^{*,*'}(\chi_a;\mathbb{Z}/2)\}$$

which is isomorphic to the ideal $Ideal(Q_n(a'),\delta_a)$ of $Q(n-1)$-algebra $H^{*,*'}(\chi_a;\mathbb{Z}/2)$. So it is written explicitly

$$\mathbb{Z}/2[\rho]\otimes(\mathbb{Z}/2[\tau]\{\delta_a\}\oplus Q(n-1)\{Q_n(a')\}\oplus \mathbb{Z}/2[\delta_a^2]\otimes Q(n)\{\delta_a^2a'\}).$$

Next we consider just $Q(n-1)$-module structures ignoring Q_n-actions. Note $deg(\delta_a)=deg(Q_n)$ and let us write Q_n also by δ_a, then we have the isomorphism of $Q(n-1)$-modules

$$H^{*,*'}(W_a;\mathbb{Z}/2)\cong (H^{*,*'}(pt.;\mathbb{Z}/2)\oplus \mathbb{Z}/2[\rho,\delta_a]\otimes Q(n-1)\{a'\})\{\delta_a\}$$

$$\cong H^{*,*'}(\chi_a;\mathbb{Z}/2)\{\delta_a\}.$$

Corollary 14.7. *There is the isomorphism $H^{*,*'}(W_a;\mathbb{Z}/2)\cong H^{*,*'}(\chi_a;\mathbb{Z}/2)$ as $Q(n-1)$-modules but not as $Q(n)$-modules.*

Proof. Note $Q_n\delta_a = Q_nQ_{n-1}...Q_0(a')\neq 0$ in $H^{*,*'}(W_a;\mathbb{Z}/2)$ but of course $Q_n(1)=0$ in $H^{*,*'}(\chi_a;\mathbb{Z}/2)$. □

From Theorem 10.5, the cohomology $H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/2)$ is $Q(n)$-free. Hence

$$H(H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/2),Q_i)\cong 0 \quad for\ 0<i\leq n.$$

By the AHss converging to $AK(i)^{*,*'}(\tilde{\chi}_a)$, we have

Lemma 14.8. *For $1\leq i\leq n$, we see $AK(i)^{*,*'}(\tilde{\chi}_a)\cong 0$.*

Corollary 14.9. *For $1\leq i\leq n-1$, we have*

$$AK(i)^{*,*'}(\chi_a)\cong AK(i)^{*,*'}(pt.),\quad AK(i)^{*,*'}(W_a)\cong AK(i)^{*,*'}(pt.)\{\delta_a\}.$$

Let us write

$$AK(i)^{\flat,\flat'}(X)=AK(i)^{*-2^{n+1}+2,*-2^n+1}(X).$$

Then we can write ;

Theorem 14.10. *For $0<i\leq n-1$, we have ;*

$$grAK(i)^{*,*'}(U_a)\cong AK(i)^{*,*'}(pt.)\oplus AK(i)^{\flat,\flat'}(pt.).$$

Proof. We consider the exact sequence

$$AK(i)^{*,*'}(U_a)\leftarrow AK(i)^{*,*'}(\chi_a)\stackrel{i^*}{\leftarrow} AK(i)^{*.*'}(W_a)\stackrel{\delta_a}{\cong} AK(i)^{\flat-1,\flat'}(\chi_a)$$

Since $\delta_a(1)=Q_0...Q_{n-1}(a')\in Im(Q_i)$ in $H^*(\chi_a;\mathbb{Z}/2)$, we have $i^*\delta_a(1)=0$ in $AK(i)^{*,*'}(\chi_a)$ by AHss. □

Remark. The corresponding elements in Theorem 14.4 and Theorem 14.10 are given by the following

$$K(i)^*[\rho]/(\rho^{2^{i+1}-1})\otimes A\cong AK(i)^{*,*'}(pt.)$$

$$K(i)^*\otimes B\cong K(i)^*\otimes\Lambda(\hat{i})\{t_n,t_{n+1},...\}\subset AK(i)^{\flat,\flat'}(pt.)$$

$$by\quad \rho^{2^{n+1}-2^{i+1}}t_i\leftrightarrow v_it_n\quad (note\ t_n^2=t_{n+1})$$

$$K(i)^*\otimes C\cong K(i)^*\otimes\Lambda(t_0,...,\hat{t}_i,...,t_{n-1})\subset AK(i)^{\flat,\flat'}(pt.)$$

$$by\quad \rho^{k(\epsilon)-2^{i+1}+1}Q_\epsilon(a')\leftrightarrow v_it^{c(\epsilon)},\quad with\quad t^{c(\epsilon)}=\Pi_{i\neq j}t_j(1-\epsilon_j).$$

The above corresponding is given by checking both first degree and difference degree, by using $d(Q_j)=2^j=-d(t_j)=-d(v_j)-1$.

We also write $Q_1...Q_{n-1}(a')=\delta_a''$ *so that* $Q_0\delta_a''=\delta_a$.

Lemma 14.11. *Writing* Q_n *also by* δ_a *(see [Po2]), we have*

$$Ah^{*,*'}(\tilde{\chi}_a)\cong\mathbb{Z}/2[\rho]\otimes\Lambda(Q_0,Q_n)[\delta_a^2]\{\delta_a''\}\cong\mathbb{Z}/2[\rho]\otimes\Lambda(Q_0)[\delta_a]\{\delta_a''\}.$$

Proof. Consider AHss for $Ah^{*,*'}(\tilde{\chi}_a)$. Since $H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/2)$ is $Q(n-1)$-free, we can see that the right hand side module is isomorphic to $E_{2^i}^{*,*,*}$. It is I_n-torsion and hence $E_\infty^{*,*,*}$ by dimensional reason. We note $Ah^{*,*'}(\tilde{\chi}_a)$ is I_n-torsion by also dimensional reason. □

Lemma 14.12. $grAh^{*,*'}(\chi_a)\cong grAh^{*,*'}(pt)\oplus grAh^{*,*'}(\tilde{\chi}_a)$.

Proof. As $Q(n-1)$-modules, we have the decomposition

$$H^{*,*'}(\chi_a;\mathbb{Z}/2)\cong H^{*,*'}(pt.;\mathbb{Z}/2)\oplus H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/2).$$

We also get the decompositions of the E_r-term of AHss by induction on r, and hence of $grA^{*,*'}(\chi_a)$. □

Since $H^*(\chi_a;\mathbb{Z}/2)$ and $H^*(W_a;\mathbb{Z}/2)$ are isomorphic as $Q(n-1)$-modules, we can prove that

Lemma 14.13. $grAh^{*,*'}(W_a)\cong grAh^{*,*'}(\chi_a)\{\delta_a\}$.

Recall Theorem 14.5. If $*\geq 2^{n+1}-1=f.deg(\delta_a)$, then the fact that $v_i\rho^*x=0$ in $Ah^{*,*'}(pt.)$ implies

$$Ah^{*,*'}(pt)\cong\mathbb{Z}/2[\rho]\otimes V_0^n\quad for\ *\geq 2^{n+1}-1.$$

Let us write

$$KAh^{*,*'}(pt)=Ker(Ah^{*,*'}(pt)\to\mathbb{Z}/2[\rho]\otimes V_0^n)$$

$$\cong h^*[\rho](I_n\{1\}\oplus\mathbb{Z}/2\{v_ix|x\in(\Lambda_{i+1}^{n-1})^+\}/(v_i\rho^{2^{i+1}-1}|1\leq i\leq n-1)\otimes V_0^n$$

$$\cong h^*[\rho]\{v_ix|x\in\Lambda_{i+1}^{n-1}\}/(v_i\rho^{2^{i+1}-1}|1\leq i\leq n-1)\otimes V_0^n.$$

Theorem 14.14.

$$Ah^{*,*'}(U_a) \cong Ah^{*,*'}(pt.)/(\rho^{2^{n+1}-1}) \oplus \mathbb{Z}/2\{\delta_a''\} \oplus KAh^{\flat,\flat'}.$$

Proof. We consider the exact sequence

$$\leftarrow Ah^{*,*'}(M_a) \leftarrow Ah^{*,*'}(\chi_a) \overset{i^*}{\leftarrow} Ah^{*,*'}(W_a) \leftarrow .$$

Using the facts that $i^*(\delta_a) = \delta_a(1) = t_n^{-1}\rho^{2^{n+1}-1}$ and $\rho\delta_a'' = \tau\delta_a$, we see

$$Ah^{*,*'}(\chi_a)/(\delta_a) \cong Ah^{*,*'}(pt.)/(\rho^{2^{n+1}-1}) \oplus \mathbb{Z}/2\{\delta''\}.$$

The map $\times\delta_a : Ah^{\flat,\flat'}(\tilde{\chi}_a) \to Ah^{*,*'}(\tilde{\chi}_a)$ is injective from Lemma 14.11. The kernel $Ker(i^*)$ is isomorphic to the kernel

$$\rho^{2^{n+1}-1} : Ah^{\flat,\flat'}(pt.) \to Ah^{*,*'}(pt.) \cong \mathbb{Z}/2[\rho] \otimes V_0^n$$

for $* \geq 2^{n+1}-1$. Thus from Lemma 9.11, we can prove the theorem. □

Remark. The elements $v_i \in I_n\{1\} \subset KAh^{\flat,\flat'}$ for $i \geq 1$ correspond c_i in Section 10 and δ_a'' corresponds c_0. By isomorphisms given for $AK(i)^*(U_a)$, we also know the correspondence with Theorem 14.6 ;

$$A \otimes V_0^n \cong Ah^{*,*'}(pt.)/(\rho^{2^{n+1}-1}),$$

$$B \otimes V_0^n \cong gr(I_n\{1\}) \otimes V_0^n \quad by \quad \rho^{2^{n+1}-2^{i+1}}t_i \leftrightarrow v_i t_n,$$

$$C \cong h^*[\rho]\{v_i x | x \in (\Lambda_{i+1}^{n-1})^+\}/(v_i\rho^{2^{i+1}-1}) \otimes V_0^n$$

$$by \quad \rho^{k(\epsilon)-2^{i+1}+1}Q_\epsilon(a') \leftrightarrow v_i t^{c(\epsilon)}, \quad with \quad t^{c(\epsilon)} = \Pi_{i\neq j} t_j(1-\epsilon_j).$$

In fact, we have the following relations. Since $Q_j t_i t_j = \rho^{2^{j+1}-1}t_i$ and $Q_i t_i t_j = \rho^{2^{i+1}-1}t_j$, we see

$$v_j\rho^{2^{n+1}-2^{i+1}}t_i = v_i\rho^{2^{n+1}-2^{j+1}}t_j \quad mod(I_\infty^2) \qquad in\ Ah^{*,*'}(U_a).$$

When $\epsilon_0 = 1, ...\epsilon_{i-1} = 1, \epsilon_i = 0$, we consider operations $Q_\epsilon = Q_j Q_{\epsilon-\Delta_j}$ and $Q_i Q_{\epsilon-\Delta_j}$ for $j < i$. Then we have the relation

$$v_j\rho^{k(\epsilon)-2^{i+1}+1}Q_\epsilon(a') = v_i\rho^{k(\epsilon+\Delta_i-\Delta_j)-2^{j+1}+1}Q_{\epsilon+\Delta_i-\Delta_j}(a') \quad mod(I_\infty^2),$$

since $k(\epsilon) - 2^{i+1} = k(\epsilon+\Delta_i-\Delta_j) - 2^{j+1}$.

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Department of Mathematics, Faculty of Education, Ibaraki University, Mito, Ibaraki, Japan

E-mail address: yagita@mx.ibaraki.ac.jp