NOTE ON THE MOD $p$ MOTIVIC COHOMOLOGY OF ALGEBRAIC GROUPS

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Abstract. Let $G_k$ be a split reductive group over a field $k$ of $ch(k) = 0$ corresponding to a compact Lie group $G$. Let $H^{*,*}(G_k;\mathbb{Z}/p)$ (resp. $H^*(G;\mathbb{Z}/p)$) be the mod $p$ motivic (resp. singular (topological)) cohomology. Then we show the isomorphism

$$grH^{*,*}(G_k;\mathbb{Z}/p) \cong grH^*(G;\mathbb{Z}/p) \otimes H^{*,*}(pt.;\mathbb{Z}/p)$$

for $G = SO_n, G_2, F_4, E_6$. Here the bidegree of the right hand side cohomology is given by $deg(y) = (2*,*)$ (resp. $deg(x) = (2*-1,*)$) for even (resp. odd) dimensional ring generators $y$ (resp. $x$).

1. Introduction

Let $k$ be a subfield of $\mathbb{C}$ which contains primitive $p$-th root of the unity. Let $G$ be a compact connected Lie group. Let us denote by $G_k$ the split reductive group over $k$ which corresponds to $G$. Let $H^{*,*}(G_k;\mathbb{Z}/p)$ (resp. $H^*(G;\mathbb{Z}/p)$) be the mod $p$ motivic (resp. singular (topological)) cohomology.

Let $T_1 \subset ... \subset T_\ell = T$ be a sequence of tori of $G$ where $T_i \cong (S^1)^{x_i}$. Note that the flag variety $G_k/T_k$ is cellular. If the sequence of tori satisfies some good condition, then we can compute the motivic cohomology $H^{*,*}(G_k/(T_i)_k;\mathbb{Z}/p)$ inductively from $H^{*,*}(G/T;\mathbb{Z}/p)$. In fact, we will show if the ring $H^*(G/T;\mathbb{Z}/p)$ satisfies some condition (stated in Assumption(*) in §2), then there is the isomorphism

$$grH^{*,*}(G_k;\mathbb{Z}/p) \cong grH^*(G;\mathbb{Z}/p) \otimes H^{*,*}(pt.;\mathbb{Z}/p)$$

Here the bidegree of the right hand side cohomology is given by $deg(y) = (2*,*)$ (resp. $deg(x) = (2*-1,*)$) for even (resp. odd) dimensional ring generators $y$ (resp. $x$).

The ring structure of $H^*(G/T;\mathbb{Z})$ is completely determined by Toda, Watanabe and Nakagawa for $G = SO_n, G_2, F_4, E_6, E_7$. We easily check
that the condition is satisfied for these cases except for $E_7$ and $p = 3$. Thus we have the isomorphisms above for these cases.

If the group $G_k$ is nonsplit, then the situation is completely different. In the last section we consider the $\text{mod}(2)$ motivic cohomology of nonsplit (twisted) form of $G_2$ for $k = \mathbb{R}$.

2. COMPACT LIE GROUP $G$

Let $G$ be a compact connected Lie group. By the Borel theorem, we have the ring isomorphism for $p$ odd

\[(3.1) \quad H^*(G; \mathbb{Z}/p) \cong P(y)/(p) \otimes \Lambda(x_1, ..., x_l)\]

with $P(y) = \mathbb{Z}[y_1, ..., y_k]/(y_1^{p^1}, ..., y_k^{p^k})$

where $|y_i| = \text{even}$ and $|x_j| = \text{odd}$. When $p = 2$, for each $y_i$, there is $x_j$ with $x_j^2 = y_i$. Hence we have $grH^*(G; \mathbb{Z}/2) \cong P(y)/(2) \otimes \Lambda(x_1, ..., x_l)$.

Let $T$ be the maximal torus of $G$ and $BT$ the classifying space of $T$. We consider the fibering

\[(3.2) \quad G \xrightarrow{\pi} G/T \xrightarrow{i} BT\]

and the induced spectral sequence

\[E_2^{*,*} = H^*(BT; H^*(G; \mathbb{Z}/p)) \implies H^*(G/T; \mathbb{Z}/p).\]

The cohomology of the classifying space of the torus is given by

\[H^*(BT) \cong S(t) = \mathbb{Z}[t_1, ..., t_\ell] \quad \text{with} \quad |t_i| = 2.\]

where $\ell$ is also the number of the odd degree generators $x_i$ in $H^*(G; \mathbb{Z}/p)$.

It is known that $y_i$ are permanent cycles and that there is a regular sequence $([\text{Tod}], [\text{Mi-Ni}])$ $(b_1, ..., b_l)$ in $H^*(BT)/(p)$ such that $d_{x_i+1}(x_i) = b_i$. Thus we get

\[E_\infty^{*,*} \cong grH^*(G/T; \mathbb{Z}/p) \cong P(y) \otimes \mathbb{Z}/p[t_1, ..., t_\ell]/(b_1, ..., b_l).\]

Moreover we know that $G/T$ is a manifold of torsion free, and we get

\[(3.3) \quad H^*(G/T)_\langle p \rangle \cong \mathbb{Z}/(y_1, ..., y_k, t_1, ..., t_\ell)/(f_1, ..., f_k, b_1, ..., b_l)\]

where $b_i = b_i \text{ mod}(p)$ and $f_i = y_i^{p^i} \text{ mod}(p, t_1, ..., t_\ell)$.

Here we consider the following assumption

Assumption(*) We can take the Torus $T_1 \subset ... \subset T_\ell = T$ with $T_i \cong (S^1)^{x_i}$ and the corresponding basis $t_i$ in $S(t)$ such that for all $1 \leq i \leq \ell$

1. $b_1, ..., b_k$ is regular in $S(t)/(t_{i+1}, ..., t_\ell)$,

2. $b_i = t_i g_i$ in $S(t)/(b_1, ..., b_{i-1}, t_{i+1}, ..., t_\ell)$ for some $g_i \in S(t)$.
Remark. Note that if above assumption is satisfied, then we can take $b' \in S(t)$ such that $b_i = b'_i \mod (b_1, ..., b_{i-1})$ and $b'_i = t_i g'$ in $S(t)/(t_{i+1}, ..., t_{\ell})$ but not only $S(t)/(b_1, ..., b_{i-1}, t_{i+1}, ..., t_{\ell})$.

Lemma 2.1. Suppose the Assumption (*). Then

$$gr H^* (G/T; \mathbb{Z}/p) \cong H^* (G/T)/(t_{i+1}, ..., t_{\ell}) \otimes \Lambda (x_{i+1}, ..., x_{\ell}).$$

Proof. Compare the two spectral sequences

$$E^*_2 = S(t) \otimes P(y) \otimes \Lambda (x_1, ..., x_{\ell}) \Longrightarrow H^* (G/T; \mathbb{Z}/p).$$

$$E(i)_2^* = S(t) \otimes P(y)/(t_{i+1}, ..., t_{\ell}) \otimes \Lambda (x_1, ..., x_{\ell}) \Longrightarrow H^* (G/T; \mathbb{Z}/p)$$

and the induced map $j : E^* \to E(i)_{r'}$. Since $d_r (x_k) = b_k$ in $E_r$ for $r = |b_k|$, so is in $E(i)_r$. By the Assumption (1), $b_1, ..., b_i$ is regular. Hence we see

$$E(i)|_{b_i+1} \cong H^* (G/T; \mathbb{Z}/p)/(t_{i+1}, ..., t_{\ell}) \otimes \Lambda (x_{i+1}, ..., x_{\ell}).$$

In fact, from Assumption(2), $b_{i+1}, ..., b_{\ell}$ are all zero in $S(t)/(t_{i+1}, ..., t_{\ell})$. Hence we get $E(i)|_{b_i+1} \cong E(i)_{\infty}$ Thus we get the lemma.

Corollary 2.2.

$$gr H^* (G/T_{i-1}; \mathbb{Z}/p) \cong gr H^* (G/T_i; \mathbb{Z}/p)/(t_i) \otimes \Lambda (x_i).$$

The fibering

$$S^1 \to G/T_{i-1} \to G/T_i$$

induces the Gysin exact sequence

$$\delta : H^{* - 2} (G/T_i; \mathbb{Z}/p) \overset{j_* - x_{t_i}}{\longrightarrow} H^* (G/T_i; \mathbb{Z}/p) \to H^* (G/T_{i-1}; \mathbb{Z}/p) \overset{\delta}{\longrightarrow}.$$

Hence we see

$$gr H^* (G/T_{i-1}; \mathbb{Z}/p) \cong gr H^* (G/T_i)/(t_i) \oplus Ker (x_{t_i}).$$

From the above corollary, we get

Corollary 2.3. We have $Ker (x_{t_i}) \cong H^* (G/T_i; \mathbb{Z}/p)\{x_i\}$ and $\delta (x_i) = g_i$ in Assumption(2).

Indeed, $\times t_i H^* (G/T_i; \mathbb{Z}/p)\{g_i\} = 0$. Moreover

$$H^* (G/T_i; \mathbb{Z}/p)/(t_i)\{g_i\} \subset H^* (G/T_i; \mathbb{Z}/p).$$
3. MOTIVIC COHOMOLOGY

Let $X$ be an algebraic variety over $k$. Let $H^{*,*}(X; \mathbb{Z}/p)$ be the $mod(p)$ motivic cohomology constructed by Suslin and Voevodsky [Vo]. For nonzero element $x \in H^{m,n}(X; \mathbb{Z}/p)$, we define the weight degree and the different degree by

$$w(x) = 2n - m, \quad d(x) = m - n.$$ 

When $X$ is smooth, it is known that

$$w(x) \geq 0, \quad d(x) \leq \dim(X).$$

Moreover from the affirmative answer of the Bloch-Kato conjecture (and hence Beilinson-Lichtenbaum conjecture) implies

$$H^{*,*}(pt.; \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau] \otimes K^M_*(k)$$

where $0 \neq \tau \in H^{0,1}(pt.; \mathbb{Z}/p) \cong \mathbb{Z}/p$ and the Milnor’s K-theory is

$$K^M_*(k)/p \cong H^{*,*}(pt.; \mathbb{Z}/p).$$

Let us denote by $G_k$ the split reductive group over $k$ corresponding to the compact Lie group $G$ and $T_i = (T_i)_k$ the split torus. We give here the main theorem of this paper

**Theorem 3.1.** Suppose the Assumption(*) in the preceding section. Then

$$\text{gr} H^{*,*}(G_k/T_i; \mathbb{Z}/p) \cong \text{gr} H^*(G/T_i; \mathbb{Z}/p) \otimes H^{*,*}(pt.; \mathbb{Z}/p)$$

where the bidegree in $H^*(G/T_i; \mathbb{Z}/p)$ is given for nonzero element $x \in H^*(G/T; \mathbb{Z}/p)$ by $w(x) = 0$ and $w(x_i) = 1$

**Corollary 3.2.** Suppose the Assumption(*). Then

$$H^{*,*}(G; \mathbb{Z}/p) \cong H^{*,*}(pt.; \mathbb{Z}/p) \otimes P(y) \otimes \Lambda(x_1, ..., x_\ell)$$

where $w(P(y)) = 0$ and $w(x_i) = 1$.

For the motivic theory, there is the Thom isomorphism and hence the Gysin exact sequence

$$H^{*-2,*,*}(G/T_1; \mathbb{Z}/p) \xrightarrow{j_* \otimes \tau} H^{*,*,*}(G/T_i; \mathbb{Z}/p) \rightarrow H^{*,*,*}(G/T_{i-1}; \mathbb{Z}/p) \rightarrow.$$ 

By descending induction on $i$, we easily show

$$H^{2*,*}(G_k/T_i; \mathbb{Z}/p) \cong H^{2*}(G/T; \mathbb{Z}/p)/(t_{i+1}, ..., t_\ell)$$

and there is $x_i \in H^{2*-1,*}(G_k/T_{i-1}; \mathbb{Z}/p)$ with $\delta(x_i) = b_i$ as Corollary 3.3.
We will prove the main theorem also by descending induction on \( i \). Indeed when \( i = \ell \), the space \( G_k/T \) is cellular and there is the isomorphism

\[
H^{*,*'}(G_k/T; \mathbb{Z}/p) \cong H^*(G/T; \mathbb{Z}/p) \otimes H^{*,*'}(pt.; \mathbb{Z}/p)
\]

identifying \( w(H^*(G/T; \mathbb{Z}/p)) = 0 \).

By induction, we assume the theorem for \( i \). We will prove the theorem for \( i - 1 \), at first the case \( k = \mathbb{C} \). Recall that

\[
H^{*,*'}(pt.; \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau],
\]
in fact, \( K_*^M(\mathbb{C})/p \cong \mathbb{Z}/p \). Let us write

\[
H^{m,n}(i, \tau) = (H^{*,*'}(G/T_i; \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau])^{m,n}
\]

\[
\cong \bigoplus_{(i + 1 \leq i_1 < \ldots < i_s \leq \ell),
\begin{array}{l}
m = 2 * + |x_{i_1}| + \ldots + |x_{i_s}|, \\
n = 1/2(m + s) + t
\end{array}
\bigoplus H^{2*,*}(G/T; \mathbb{Z}/p)/(t_{i_1}, \ldots, t_{i_s}) \{x_{i_1}\ldots x_{i_s} \tau^t\}.
\]

Note here

\begin{equation}
(\ast) \quad H^{m,n}(i - 1, \tau) \cong H^{m,n}(i, \tau)/(t_i) \oplus H^{m-|x_i|, n-1/2(|x_i|+1)}(i, \tau)/(t_i) \{x_i\}.
\end{equation}

Next consider \( \text{Ker}(t_i|H^{m,n}(i, \tau)) \). From Corollary 3.3, we still know

\[
\text{Ker}(t_i|H^{2*}(G/T; \mathbb{Z}/p)/(t_i, \ldots, t_i)) \cong H^{2* - |x_i| + 1}(G/T; \mathbb{Z}/p)/(t_i, \ldots, t_i) \{g_i\}.
\]

This implies that \( \text{Ker}(t_i|H^{m,n}(i, \tau)) \) is isomorphic to

\[
\bigoplus H^{2*,*}(G/T; \mathbb{Z}/p)/(t_i, \ldots, t_i) \{x_{i_1}\ldots x_{i_s} \tau^t\} \{g_i\},
\]

where direct sum runs

\( (i + 1 \leq i_1 < \ldots < i_s \leq \ell), \quad m - |x_i| = 2 * + |x_{i_1}| + \ldots + |x_{i_s}|, \quad
\]

\( n - 1/2(|x_i| - 1) = 1/2(m + s) + t. \)

Hence this is isomorphic to

\[
H^{m-|x_i|, n-1/2(|x_i| - 1)}(i, \tau)/(t_i) \{x_i\}
\]

by the map \( x_i \mapsto g_i \) from \( (\ast) \).

Therefore we have the exact sequence for fixed \( n \)

\begin{equation}
(\ast\ast) \quad \delta : H^{*,2n-1}(i, \tau) \xrightarrow{j_* \times t_i} H^{*,n}(i, \tau) \rightarrow H^{*,n}(i - 1, \tau) \xrightarrow{\delta}.
\end{equation}

Here, by inductive assumption on \( i \),

\( H^{*,*'}(i, \tau) \cong H^{*,*'}(G_k/T_i; \mathbb{Z}/p) \).

Of course there is the natural map

\[
H^{*,*'}(j, \tau) \rightarrow H^{*,*'}(G_k/T_j; \mathbb{Z}/p) \quad \text{for all } 1 \leq j \leq \ell.
\]
Thus we get the main result for $k = \mathbb{C}$ by the five lemma and the induction on $i$.

For general field $k$ case, we consider

$$H^{m,n}(i, k) = (H^*(i, \tau) \otimes k_*(k)/p)^{m,n}$$

$$\cong \oplus_a H^{*-[a],*-[a]}(i, \tau)\{a\}$$

where $\{a\}$ is a $\mathbb{Z}/p$-base of $K_*(k)/p$. Then from the result for $k = \mathbb{C}$,

$$\delta \rightarrow H^{*-[2a],n-1-[a]}(i, \tau)\{a\}$$

By five lemma, we also have the result of the main theorem.

First, we consider the orthogonal groups $SO_k$ for the case

When $H^*(X;\mathbb{Z}/p) \cong A \otimes \mathbb{Z}/p[\tau]$, we show the following theorem, by using the preceding arguments exchanging $K_*(k)/p$ by $A \otimes K_*(k)/p$.

**Theorem 3.3.** Suppose that $G$ satisfies Assumption(*) and $H^*(X;\mathbb{Z}/p)$ is $\mathbb{Z}/p[\tau]$-free. Then

$$H^*(X \times G_k/T_i;\mathbb{Z}/p) \cong H^*(X;\mathbb{Z}/p) \otimes_{H^*(pt,\mathbb{Z})} H^*(G_k/T_i;\mathbb{Z}/p).$$

**Corollary 3.4.** The Kunneth formula holds for $H^*(G_k;\mathbb{Z}/p)$. Hence it is a Hopf algebra.

Since $w(x_i) = 1$, the coproduct is given as

$$\psi(x_i) = \sum y(1)j_j x_j \otimes y(2)j_j + y(3)j_j \otimes y(4)j_j x_j$$

Here the above coproduct is determined from that of the topological case $H^*(G;\mathbb{Z}/p)$ because $t_{G}^{*}t^\prime$ is injective for $k = \mathbb{C}$ and for each $(\ast, \ast^\prime)$.

4. **Examples**

The cohomology $H^*(G/T)$ is computed by Toda-Watanabe ([To-Wa]) for the case $G = SO(m), G_2, F_4, E_6$. The case $G = E_7$ is computed by Nakagawa [Wa], [Na].

First, we consider the orthogonal groups $G = SO(m)$ and $p = 2$. The mod 2-cohomology is written as (see for example [Ni])

$$gr H^*(SO(m);\mathbb{Z}/2) \cong \Lambda(x_1, x_2, \ldots, x_{m-1})$$

where the multiplications are given by $x^2_i = x_{2i}$ for odd $i$. We write $y_{2(odd)} = x^2_{odd}$. Hence we can write

$$H^*(SO(m);\mathbb{Z}/2) \cong \mathbb{Z}/2[y_{4i+2}|2 \leq 4i+2 \leq m-1]/(y_{2i+2}^{(i)}) \otimes \Lambda(x_1, x_3, \ldots, x_m)$$
where \( s(i) \) is the smallest number such that \( 2s(i)(4i + 2) \geq m \) and \( \tilde{m} = m - 1 \) (resp. \( \tilde{m} = m - 2 \)) if \( m \) is even (resp. odd).

By Toda-Watanabe [To-Wa], it is known that

**Theorem 4.1.**

1. \( H^*(SO(2n + 1)/T_n) \cong \mathbb{Z}[t_i, t_n, y_{2i}|1 \leq i \leq n]/(c_i - 2y_{2i}, J_i) \)
   
   where \( J_i = y_{3i} + \sum_{0 < j < 2i} (-1)^j y_{2j} y_{4i-2j}, \quad (y_{2k} = 0 \ for \ k \geq n) \)
   
   \( c_i = \sigma_i(t_1, ..., t_n) \ i-th \ elementary \ symmetric \ function. \)

2. \( H^*(SO(2n + 1)/T_n) \cong H^*(SO(2n + 1)/T_{n+1})/\langle t_n \rangle. \)

The mod 2-cohomology is

\[
H^*(SO(2n)/T; \mathbb{Z}/2) \cong P(y) \otimes S(t)/(c_1, ..., c_n).
\]

\[
H^*(SO(2n - 1)/T; \mathbb{Z}/2) \cong H^*(SO(2n); \mathbb{Z}/2)/(t_n).
\]

Of course \((c_1, ..., c_n)\) is regular in \( S(t)/2 = \mathbb{Z}/2[t_1, ..., t_n] \) and

\[
c_i = t_1...t_i \ \text{in} \ S(t)/(t_{i+1}, ..., t_n)\).
\]

Hence the Assumption(*) is satisfied. (The case \( SO(odd) \) is similar.)

Next consider the exceptional Lie group \( G = G_2 \). Also from [To-Wa]

\[
H^*(G/T; \mathbb{Z}) \cong \mathbb{Z}[t_1, t_2, y]/(t_1^2 + t_1 t_2 + t_2^2, t_1^3 - 2y, y^2)
\]

with \( |t_i| = 2 \) and \( |y| = 6 \). Since \((t_1^2 + t_1 t_2 + t_2^2, t_1^3)\) is a regular sequence in \( \mathbb{Z}/2[t_1, t_2] \) and \( t_1^2 + t_1 t_2 + t_2^2 = t_1^3 \ modem(t_2) \), this case also satisfies the Assumption(*).

The case \( F_4 \) is stated as following.

**Theorem 4.2.** ([To-Wa])

\[
H^*(F_4/T) \cong \mathbb{Z}[t_1, t_2, t_3, t_4, y_1, y_3, w]/(\rho_1, \rho_2, \rho_3, \rho_4, \rho_6, \rho_8, \rho_{12})
\]

where \( |t_1| = |y_1| = 2, |y_3| = 6, |w| = 8 \), and

\[
\rho_1 = c_1 - 2y_1, \ \rho_2 = c_2 - 2y_1^2, \ \rho_3 = c_3 - 2y_3, \ \rho_4 = c_4 - 2c_3y_1 + 2y_1^2 - 3w,
\]

\[
\rho_6 = -c_4y_1^2 + y_3^2, \ \rho_8 = 3c_4y_1^8 + 3w(w + c_3y_1), \ \rho_{12} = w^3
\]

\[
\text{for} \quad c_i = \sigma_i(t_1, t_2, t_3, t_4).
\]

Hence we can write the mod(2) cohomology as

\[
gr H^*(F_4; \mathbb{Z}/2) \cong \mathbb{Z}/2[y_3]/(y_3^2) \otimes \mathbb{Z}/2[t_1, t_2, t_3, y_1]/(\rho_2, \rho_3, \rho_{12}, \rho_8)
\]

where \( \rho_2 = c_2, \ \rho_3 = c_3, \ \rho_8 = y_1^8 + c_4y_1^2 + c_4^2, \ \rho_{12} = c_4^3 \)

\[
c_i = \sigma_i(t_1, t_2, t_3, t_1 + t_2 + t_3).
\]

We can take new generator \( t_i \) (write it by \( t_i' \)) which satisfies Assumption(*) as follows

\[
t_4' = t_1 + t_2 + t_3, \ \ t_3' = y_1, \ t_2' = t_2, \ t_1' = t_1.
\]
The mod(3) cohomology is written as
\[ H^*(F_4/T; \mathbb{Z}/3) \cong \mathbb{Z}/3[w]/(w^3) \otimes \mathbb{Z}/3[t_1, t_2, t_3, t_4]/(\rho_2, \rho_4, \rho_6, \rho_8) \]
where \( \rho_2 = c_2 + c_2^2, \ \rho_6 = c_3^2 - c_4 c_1^2, \ \rho_4 = c_4 + c_3 c_1 - c_1^4, \ \rho_8 = -c_1^8. \)

We can see the Assumption(*) is satisfied by taking
\[ t'_4 = c_1, \ t'_3 = t_4, \ t'_2 = t_3, \ t'_1 = t_2. \]

We can check Assumption(*) for the cases \( E_6, E_7 \) similarly by using the result Toda-Watanabe and Nakagawa. We only give the outline of arguments. From Toda-Watanabe, we see
\[ H^*(E_6/T; \mathbb{Z}) \cong \mathbb{Z}[t_1, ..., t_6, \gamma_1, \gamma_3, \gamma_4]/(\rho_1, \rho_2, \rho_3, \rho_5, \rho_6, \rho_8, \rho_9, \rho_{12}). \]

With mod(2), we have
\[ \rho_1 = c_1 + \gamma_1, \ \rho_2 = c_2, \ \rho_3 = c_3, \ \rho_4 = c_4 + \gamma_4, \]
\[ \rho_5 = c_1 \text{ mod}(\gamma_1), \ \rho_6 = \gamma_3^2 \text{ mod}(\gamma_1), \ \rho_8 = c_4^2 + \gamma_1^4, \ \rho_9 = w^2 t + t^9, \]
\[ \rho_{12} = c_4^3 \text{ mod}(t, \gamma_1), \ \text{(where } t = \gamma_1 - t_1). \]

Here we can take \( \rho'_{12} = \rho_12 - c_4 \rho_8 = 0 \text{ mod}(\gamma_1, t) \). Then we can take \( t' \) satisfying Assumption(*) i.e.,
\[ t'_7 = t, \ t'_6 = \gamma_1, \ t'_5 = y_2, ... \]

For mod(3) case, we see
\[ \rho_1 = c_1, \ \rho_2 = c_2 - \gamma_1^2, \ \rho_3 = c_3 - \gamma_3, \ \rho_4 = c_4 \text{ mod}(\gamma_1), \]
\[ \rho_5 = c_5 \text{ mod}(\gamma_1), \ \rho_6 = \gamma_5^2 \text{ mod}(\gamma_1), \]
\[ \rho_8 = c_6 \gamma_3^2 - \gamma_5^8, \ \rho_9 = t^9, \ \rho_{12} = \gamma_4 \text{ mod}(t, \gamma_1). \]

Hence we can take \( t'_7 = t, \ t'_6 = \gamma_1, ..... \)

The cohomology of \( H^*(E_7/T; \mathbb{Z}) \) is given by Nakagawa [Na]
\[ H^*(E_7/T; \mathbb{Z}) \cong \mathbb{Z}[t_1, ..., t_7, \gamma_3, \gamma_4, \gamma_5, \gamma_9]/(\rho_1, \rho_2, \rho_3, \rho_5, \rho_6, \rho_8, \rho_9, \rho_{10}, \rho_{12}, \rho_{14}, \rho_{18}). \]

With mod(2), we have
\[ \rho_1 = c_1 + \gamma_1, \ \rho_2 = c_2, \ \rho_3 = c_3, \ \rho_4 = c_4 + \gamma_4, \]
\[ \rho_5 = c_5 \text{ mod}(t), \ \rho_6 = \gamma_3^2 \text{ mod}(t), \ \rho_8 = c_4^2 + t^2 c_6 + t^4 c_4 + t^8, \]
\[ \rho_9 = t^2 c_7 + t^3 c_6, \ \rho_{10} = \gamma_5^2 + t^3 c_7, \ \rho_{12} = t_0^4 u^2 + u^3 v^2, \]
\[ \rho_{14} = t_0^{14} + t_0^6 u^2 + u^2 v + t_0 v^2, \ \rho_{18} = u^2 + u^3 v + t_0^6 v^2, \]

where \( t_0 = t - t_1, \ u = \gamma_4 \text{ mod}(t_0, t_1), \ v = c_6 + t_0 \gamma_5 \text{ mod}(t_0, t_1), \ w = \gamma_9 + c_6 \gamma_3 \text{ mod}(t_0, t_1). \)

From the relation \( \rho_9 \), we first take \( t'_7 = t \). Next from \( \rho_8 \), we know that \( c_4^2 = 0 \) is in the relation. Then \( \rho_{12} = 0 \text{ mod}(t_0, c_4^2) \) implies that we can
take $t'_6 = t_0$. ($t_1 = 0$ if $t = t_0 = 0$.) We know next $\rho_{12} = c_2^2 \mod(t, t_1, c_2^2)$ and take $t'_5 = y_2$, and so on.

For $mod(3)$ case, if $\rho_8 = 0 \mod(t)$, then the assumption was satisfied but the relation is given by $\rho_8 = \gamma \gamma_5 + t c_7 - t^b$. The author does not check the case $G = E_7$ and $p = 3$.

5. The non split group ($G_2$, $p = 2$)

In this section, we give a computation of the $mod(p)$ motivic cohomology of a non split group. Throughout this section, let $G$ be a non split group of type $G_2$ and $p = 2$. Moreover let $k = \mathbb{R}$ the field of real numbers. Recall that $H^*(X; \mathbb{Z}/2)$ (resp. $H^*_{et}(X; \mathbb{Z}/2)$) is the mod 2 motivic (resp. etale) cohomology.

It is well known that

$$H^*_{et}(pt.; \mathbb{Z}/2) \cong K^M_*(\mathbb{R})/2 \cong \mathbb{Z}/2[\rho]$$

where $\rho = \{-1\} \in K^M_1(\mathbb{R})/2 \cong \mathbb{R}^*/(\mathbb{R}^*)^2$. The motivic cohomology of a point is

$$H^*_{et}(pt.; \mathbb{Z}/2) \cong H^*_{et}(X; \mathbb{Z}/2)[\tau] \cong \mathbb{Z}/2[\rho, \tau]$$

with $0 \neq \tau \in H^{0,1}(pt.; \mathbb{Z}/2) \cong \mathbb{Z}/2$.

Let $X$ be a smooth variety over $\mathbb{R}$. The manifold $X(\mathbb{C})$ of $\mathbb{C}$-rational points $X(\mathbb{C})$ is a $\mathbb{Z}/2$-equivariant space by the Galois group $Gal(\mathbb{C}/\mathbb{R})$ action. By Cox [Co], it is known that there is a natural weak homotopy equivalence

$$\{X\}^\wedge \cong (X(\mathbb{C}) \times_{\mathbb{Z}/2} E\mathbb{Z}/2)^\wedge$$

where $\{-\}_{et}$ means the etale homotopy type, $\{\}^\wedge$ means the profinite completion and $E\mathbb{Z}/2$ is a contractible space with free $\mathbb{Z}/2$-action.

Then we have

$$H^*_{et}(X; \mathbb{Z}/2) \cong H^*_{et}(X(\mathbb{C}); \mathbb{Z}/2) = H^*(X(\mathbb{C}) \times_{\mathbb{Z}/2} E\mathbb{Z}/2; \mathbb{Z}/2).$$

Here the right hand side is called the Borel cohomology. Thus we have the following Borel spectral sequence

$$E^*_{2,0} = H^*(B\mathbb{Z}/2; H^*(X(\mathbb{C}); \mathbb{Z}/2)) \implies H^*_{et}(X; \mathbb{Z}/2),$$

$$d_r : E^*_{r,0} \to E^*_{r+r+1,0-r}.$$ 

Since the Borel spectral sequence is the topological one, it is multiplicative, in particular, the differential is a derivation.

For the case $X = pt.$, we see $H^*(X(\mathbb{C}); \mathbb{Z}/2) \cong \mathbb{Z}/2$ and the Borel spectral sequence is trivial

$$H^*_{et}(pt.; \mathbb{Z}/2) \cong E^*_{\infty,0} \cong H^*(B\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2[x].$$

Hence we can identify $\rho = x \in H^1_{et}(pt.; \mathbb{Z}/2) \cong \mathbb{Z}/2.$
It is well known that the usual $\text{mod}(2)$ cohomology is
\[ grH^*(G_2; \mathbb{Z}/2) \cong \Lambda(y, x_3, x_5) \]
with $x_3^2 = y$, $Sq^2x_3 = x_5$ and $Sq^1x_5 = y$. For cohomology of $G_2/T_i$, we know from Toda-Watanabe
\[ grH^*(G_2/T_1; \mathbb{Z}/2) \cong \Lambda(t_1, y, x_5), \]
\[ grH^*(G_2/T; \mathbb{Z}/2) \cong \mathbb{Z}/2[t_1, t_2, y]/(t_1^2 + t_1t_2 + t_2^2, t_2, y^2). \]

Lemma 5.1.
\[ grH_{et}^*(G; \mathbb{Z}/2) \cong \Lambda(y, x_3) \otimes \mathbb{Z}/2[\rho]/(\rho^4), \]
\[ grH_{et}^*(G/T_1; \mathbb{Z}/2) \cong \Lambda(y, t_1) \otimes \mathbb{Z}/2[\rho]/(\rho^5), \]
\[ grH_{et}^*(G/T; \mathbb{Z}/2) \cong \mathbb{Z}/2[t_1, t_2]/(t_1^2 + t_1t_2 + t_2^2, t_2^3) \otimes \mathbb{Z}/2[\rho]/(\rho^7). \]

Proof. Consider the Borel spectral sequence
\[ E_2^{s,s'} = H^*(B\mathbb{Z}/2; H^*(G_2; \mathbb{Z}/2)) \longrightarrow H_{et}^*(G; \mathbb{Z}/2). \]
Here $E_2^{s,s'} \cong \mathbb{Z}/2[x] \otimes \Lambda(y, x_3, x_5)$.

Suppose $d_4(x_3) = 0$. Then by the transgression theorem
\[ d_6(x_5) = d_6(Sq^2x_3) = Sq^2d_4(x_3) = 0. \]
Similarly $d_7(y) = 0$. Hence all differentials are zero. Therefore $E_2^{s,s'} \cong E_\infty^{s,s'}$, which is infinite dimensional. But $H_{et}^*(G; \mathbb{Z}/2)$ is finite dimensional (since $G$ has no-rational points). Thus we know $d_4(x_3) \neq 0$. By dimensional reason, $d_4(x_3) = x^4$. Hence
\[ E_5^{s,s'} \cong \Lambda(x_5, y) \otimes \mathbb{Z}/2[x]/(x^4). \]
Since $E_5^{s,s'} = 0$ for $s' \geq 4$, we know $E_5^{s,s'} \cong E_\infty^{s,s'}$.

The other cases are proved similarly by using that $t_i$ are permanent cycles in these spectral sequences. $\Box$

Corollary 5.2. In $H_{et}^*(G/T; \mathbb{Z}/2)$, it holds
\[ t_2^3 = \rho^6, \quad t_1^2 + t_1t_2 + t_2^2 = \rho^4. \]

Proof. Consider the Gysin sequence for etale cohomology
\[ \delta: H_{et}^{*-2}(G/T; \mathbb{Z}/p) \xrightarrow{j_!} H_{et}^*(G/T; \mathbb{Z}/p) \rightarrow H_{et}^*(G/T_1; \mathbb{Z}/p) \xrightarrow{\delta}. \]
By the above theorem,
\[ \text{Ker}(H_{et}^*(G/T; \mathbb{Z}/2) \rightarrow H_{et}^*(G/T_1; \mathbb{Z}/2)) \]
is generated by $\rho^6$. Hence $\rho^6 = t_2a$ for some $a \in H_{et}^*(G/T; \mathbb{Z}/2)$. Here note as $\mathbb{Z}/2[\rho]$-modules
\[ H_{et}^*(G/T; \mathbb{Z}/2) \cong \mathbb{Z}/2[\rho]/(\rho^7)\{1, t_1\} \{1, t_2, t_2^2\}. \]
Hence \( a = t_2^2 \). Otherwise for \( a = \rho^j t_1^k t_2^l \), \( j, k \leq 1 \), the equation \( \rho^6 = \rho^j t_1^k t_2^{l+1} \) contradicts to the \( \mathbb{Z}[\rho]/(\rho^7) \)-freeness of \( H^*_{et}(G/T; \mathbb{Z}/2) \).

Similarly, we have

\[
\rho^4 = t_1^2 \pmod{t_2}
\]

from the Gysin sequence for \( H^*_{et}(G/T; \mathbb{Z}/2) \) (and \( H^*_{et}(G; \mathbb{Z}/2) \)). Of course

\[
\rho = 0 \quad \text{in} \quad H^*_{et}(G/T_{|C}; \mathbb{Z}/2) \cong H^*(G/T; \mathbb{Z}/2).
\]

Hence \( \rho^4 = t_1^2 + t_1 t_2 + t_2^2 \), which is zero in \( H^*(G/T; \mathbb{Z}/2) \).

Before studying the motivic cohomology \( H^{*,*'}(G/T; \mathbb{Z}/2) \), we recall that of the Rost motive \( M_2 \) for \( \rho^3 \in K_3^M(\mathbb{R})/2 \)

\[
H^*(M_2|C; \mathbb{Z}/2) \cong \mathbb{Z}/2[y]/(y^2) = \Lambda(y).
\]

(The Rost motive can be considered as a twisted form \( 6 - \dim \text{ sphere} \).) The étale cohomology is also compute by the Borel spectral sequence

\[
H^*_{et}(M_2; \mathbb{Z}/2) \cong \mathbb{Z}/2[\rho]/(\rho^7).
\]

By using arguments of Voevodsky, its motivic cohomology is given in [Ya]

**Lemma 5.3.** The mod(2) motivic cohomology \( H^*(M_2; \mathbb{Z}/2) \) can be identified with a \( \mathbb{Z}/2[\tau] \)-subalgebra

\[
\mathbb{Z}/2[\tau]\{1, \rho, \rho^2, \rho^3 \tau^{-1}, \rho^4 \tau^{-2}, \rho^5 \tau^{-2}, \rho^6 \tau^{-3}\}
\]

of \( H^*_{et}(M_2; \mathbb{Z}/2)[\tau, \tau^{-1}] \cong \mathbb{Z}/2[\rho, \tau, \tau^{-1}]/(\rho^7) \).

**Remark.** The above \( \mathbb{Z}/2[\tau] \)-module generators are related by the Milnor operation

\[
Q_0(\rho^3 \tau^{-1}) = \rho^4 \tau^{-2}, \quad Q_1(\rho^3 \tau^{-1}) = \rho^6 \tau^{-3}.
\]

By using the arguments of general splitting fields, Petrov-Semenov-Zainoulline [Pe-Se-Za] and Bonnet [Bo] prove the following result for the motive \( M(G/T) \) of \( G/T \).

**Theorem 5.4.**

\( M(G/T) \cong M_2 \otimes H^*(G/T; \mathbb{Z}/2)/(y) \cong M_2 \otimes \mathbb{Z}/2[t_1, t_2]/(t_1^2 + t_1 t_2 + t_2^2, t_1^3). \)

**Corollary 5.5.** The mod(2) motivic cohomology \( gr H^*(G/T; \mathbb{Z}/2) \) can be identified with a \( \mathbb{Z}/2[\tau] \)-subalgebra

\[
\mathbb{Z}/2[\tau]\{1, \rho, \rho^2, \rho^3 \tau^{-1}, \rho^4 \tau^{-2}, \rho^5 \tau^{-2}, \rho^6 \tau^{-3}\} \otimes \mathbb{Z}/2[t_1, t_2]/(t_1^2 + t_1 t_2 + t_2^2, t_1^3)
\]

of \( \mathbb{Z}/2[\rho, \tau, \tau^{-1}]/(\rho^7) \otimes \mathbb{Z}/2[t_1, t_2]/(t_1^2 + t_1 t_2 + t_2^2, t_1^3) \).

The products in \( H^{*,*'}(G/T; \mathbb{Z}/2) \) is given by

\[
\rho^7 = 0, \quad t_2^3 = \rho^3 \tau^{-3}, \quad t_1^2 + t_1 t_2 + t_2^2 = \rho^4 \tau^{-2}.
\]
Proof. From the preceding corollary, we know
\[ t_2^3 = \rho^6r^* , \quad t_1^2 + t_1t_2 + t_2^2 = \rho^4r^{*'} . \]
Since \( w(t_i) = 0 \), we see \( * = -3 \) and \( *' = -2 \).

Lemma 5.6. The cycle map induces the injection
\[ H^{*,*'}(G/T_1; \mathbb{Z}/2) \subset H^\text{et}_e(G/T_1; \mathbb{Z}/2) \otimes \mathbb{Z}/2[\tau, \tau^{-1}] . \]

Proof. We want to see that
\[ (1) \quad \tau : H^{*,-1,*'}(G/T; \mathbb{Z}/2)/(t_2H^{*,-3,*'-1}(G/T; \mathbb{Z}/2)) \rightarrow H^{*,*'}(G/T; \mathbb{Z}/2)/(t_2H^{*,-2,*'}(G/T; \mathbb{Z}/2)) \]
is injective, that is, if \( t_2x \in \text{Im}(\tau) \), then \( x = \tau x' \) for some \( x' \).

The cohomology \( H^{*,*'}(G/T; \mathbb{Z}/2) \) is written
\[ (**) \quad \mathbb{Z}/2[\tau] \otimes \Lambda(t_1) \otimes \mathbb{Z}/2[1, \rho, \rho^2, \rho^3\tau^{-1}, \rho^4\tau^{-2}, \rho^5\tau^{-2}, t_2^3][1, t_2, t_2^2] . \]
Hence the multiplication \( \times t_2 \) acts on a basis of \( \mathbb{Z}/2[\tau] \)-module generators of the free \( \mathbb{Z}/2[\tau] \)-module \( H^{*,*'}(G/T; \mathbb{Z}/2) \). The \( t_2 \)-image of each \( \mathbb{Z}/2[\tau] \)-generators are also generators. Then it is immediate that the above property (1) is satisfied.

Lemma 5.7. The mod(2) motivic cohomology \( grH^*(G/T_1; \mathbb{Z}/2) \) can be identified with a \( \mathbb{Z}/2[\tau, \rho]/(\rho^6) \)-subalgebra
\[ \mathbb{Z}/2[\tau] \otimes \Lambda(t_1) \otimes (A \oplus B) \quad \text{where} \quad A = \mathbb{Z}/2[1, \rho, \rho^2, \rho^3\tau^{-1}, \rho^4\tau^{-2}, \rho^5\tau^{-2}], \]
\[ B = \mathbb{Z}/2[\tau, \theta, \rho^2, \rho^3\tau^{-1}, \rho^4\tau^{-2}, \rho^5\tau^{-2}] \{ y \} \]
of the algebra
\[ H^\text{et}_e(G/T_1)[\tau, \tau^{-1}] \cong \mathbb{Z}/2[\tau, \tau^{-1}] \otimes \mathbb{Z}/2[\rho]/(\rho^6) \otimes \Lambda(t_1, y) . \]
Here the product in \( H^{*,*'}(G/T_1; \mathbb{Z}/2) \) is given by
\[ \rho^4\tau^{-2} = t_1 \quad \text{(hence} \quad \rho^5\tau^{-2} = \rho^4t_1, \rho^4\tau^{-1}y = t_1\tau y, \rho^5\tau^{-1}y = \tau \rho t_1 y). \]

Proof. We use the Gysin exact sequence for motivic cohomology theories. By the expression (**) of \( H^{*,*'}(G/T; \mathbb{Z}/2) \) in the preceding proof, we know
\[ H^{*,*'}(G/T; \mathbb{Z}/2)/(t_2) \cong \mathbb{Z}/2[\tau] \otimes \Lambda(t_1) \otimes A, \]
\[ \text{Ker}(t_2|H^{*,*'}(G/T; \mathbb{Z}/2)) \cong \mathbb{Z}/2[\tau] \otimes \Lambda(t_1) \otimes B', \]
where \( B' = \mathbb{Z}/2[\rho, \rho^2, \rho^3\tau^{-1}, \rho^4\tau^{-2}, \rho^5\tau^{-2}, \rho^6\tau^{-3}] \).
Take \( x_6 \in H^{*,*'}(G/T_1; \mathbb{Z}/2) \) such that \( \delta(x_6) = \rho t_2^2 \). Note \( w(x_6) = 2 \)
(since \( w(\delta(x)) = w(x) + 1 \) and \( w(\rho t_2^2) = 1 \)). By the preceding lemma and by the dimensional reason, we see
\[ x_6 = \tau y_6 \quad \text{in} \quad H^*(G/T_1; \mathbb{Z}/2) \otimes \mathbb{Z}/2[\tau, \tau^{-1}] . \]
Thus we get hte expression of this lemma. \( \square \)
Here note that the $t_1$-image of each $\mathbb{Z}/2[\tau]$-module generator of $H^{*,*'}(G/T; \mathbb{Z}/2)$ is not a generator in general, e.g., $t_1(\tau \rho) = \tau \rho^5 \tau^{-2}$. Therefore we can not represent $H^{*,*'}(G; \mathbb{Z}/p)$ as a subalgebra of $H^{*,*}_{et}(G; \mathbb{Z}/2)[\tau, \tau^{-1}]$. However we get the following theorem.

**Theorem 5.8.** The mod(2) motivic cohomology $grH^{*,*'}(G; \mathbb{Z}/2)$ can be identified with a $\mathbb{Z}/2[\tau, \rho]/(\rho^4)$-subalgebra

$$
\mathbb{Z}/2[\tau] \otimes (A_1 \oplus A_2 \oplus B_1 \oplus B_2) \oplus \mathbb{Z}/2\{c\}
$$

where $A_1 = \mathbb{Z}/2\{1, \rho, \rho^2, \rho^3\tau^{-1}\}$, $A_2 = \mathbb{Z}/2\{\tau, \rho^2\tau^{-1}, \rho^3\tau^{-1}\}\{x_5\}$, $B_1 = \mathbb{Z}/2\{\tau, \tau\rho, \rho^2, \rho^3\tau^{-1}\}\{y\}$, $B_2 = \mathbb{Z}/2\{\tau, \rho^2, \rho^3\tau^{-1}\}\{x_5y\}$

of the algebra $H^{*,*}_{et}(G)[\tau, \tau^{-1}] \oplus \mathbb{Z}/2\{c\} \cong \mathbb{Z}/2[\tau, \tau^{-1} \otimes \mathbb{Z}/2[\rho]/(\rho^4) \otimes \Lambda(x_5, y) \oplus \mathbb{Z}/2\{c\}$.

Here $deg(c) = deg(\rho^5 \tau^{-2} y) = (11, 6)$.

**Proof.** We also use the Gysin exact sequence for motivic cohomology theories. Recall that

$$A \otimes \Lambda(t_1) \cong \mathbb{Z}/2\{1, \rho, \rho^2, \rho^3\tau^{-1}, \rho^4\tau^{-2} = t_1^2, \rho^5\tau^{-2} = \rho t_1^2\}\{1, t_1\}.$$ 

Hence we have

$$(A \otimes \Lambda(t_1))/(t_1) \cong \mathbb{Z}/2\{1, \rho, \rho^2, \rho^3\tau^{-1}\} = A_1.$$ 

$$Ker(t_1|A \otimes \Lambda(t_1)) \cong \mathbb{Z}/2\{\rho^2, \rho^3\tau^{-1}, \rho^4\tau^{-2}, \rho^5\tau^{-2}\}\{t_1\} \cong A'_2.$$ 

Take $x'_5 \in H^{*,*'}(G/T; \mathbb{Z}/2)$ so that $\delta(x'_5) = \rho^2 t_1$. Hence we have $x'_5 = \tau x_5$, and

$$A_2 = \mathbb{Z}/2\{\tau, \rho, \rho^2\tau^{-1}, \rho^3\tau^{-1}\} \cong A'_2.$$ 

Next consider

$$B \otimes \Lambda(t_1) = \mathbb{Z}/2\{\tau, \tau\rho, \rho^2, \rho^3\tau^{-1}, \rho^4\tau^{-1}, \rho^5\tau^{-2}\}\{y, yt_1\}.$$ 

We can compute

$$(B \otimes \Lambda(t_1))/(t_1) \cong B_1 \oplus \mathbb{Z}/2\{c\} = \mathbb{Z}/2\{\tau, \rho, \rho^2, \rho^3\tau^{-1}\}\{y\} \oplus \mathbb{Z}/2\{\rho^5\tau^{-2} y\},$$

$$Ker(t_1|B \otimes \Lambda(t_1)) = \mathbb{Z}/2\{\rho^2, \rho^3\tau^{-1}, \rho^4\tau^{-1}, \rho^5\tau^{-2}\}\{yt_1\} \cong \mathbb{Z}/2\{\tau, \rho^2, \rho^3\tau^{-1}\}\{x_5 y\}.$$ 

We can prove the theorem by using above isomorphisms. \qed
References


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