NOTE ON MOTIVIC COHOMOLOGY OF ANISOTROPIC REAL QUADRICS

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ABSTRACT. In this paper, we compute the mod 2 motivic cohomology $H^{*,*'}(X;\mathbb{Z}/2)$ for the anisotropic quadric X over $\mathbb{R}$ the field of real numbers.

1. INTRODUCTION

Each anisotropic quadratic form of dimension n over $\mathbb{R}$ is written uniquely by $q_n = x_1^2 + ... + x_n^2$. Let X_d be the d-dimensional projective quadric defined by $q_{d+2} = 0$. Let $H^{*,*'}(X_d;\mathbb{Z}/2)$ (resp. $H^*_{et}(X_d;\mathbb{Z}/2)$) be the $mod(2)$ motivic cohomology (resp. etale cohomology) constructed by Suslin and Voevodsky [Vo1,2]. It is shown in [Ya1] that

$$(1.1) \quad H^{*,*'}(X_d;\mathbb{Z}/2) \subset H^*_{et}(X_d;\mathbb{Z}/2)[\tau,\tau^{-1}]$$

where τ is the nonzero elemant in $H^{0,1}(Spec(\mathbb{R});\mathbb{Z}/2) \cong \mathbb{Z}/2$.

The above fact is showed by using the decomposition of its motives, i.e., $M(X_d) \cong \oplus_j M_{i_j}$ for some $i_j > 0$. Here M_{i_j} is the Rost motive (over $\mathbb{R}$) of dimension $2^{i_j} - 1$. The cohomology $H^{*,*'}(M_{i_j};\mathbb{Z}/2)$ is computed in [Ya1] by using arguments by Voevodsky [Vo1,2]. Hence we know the additive structure of $H^{*,*'}(X_d;\mathbb{Z}/2)$ completely. Moreover, the ring structure of $CH^*(X_d)/2 = H^{2*,*}(X_d;\mathbb{Z}/2)$ is given in [Ya2] by using the algebraic cobordism theory $\Omega^*(X_d)$.

On the other hand, there is the homeomorphism $X_d(\mathbb{C})/\mathbb{Z}/2 \cong G_2(\mathbb{R}^{d+2})$. Here $X_d(\mathbb{C})/\mathbb{Z}/2$ is the quotient space of the manifold $X_d(\mathbb{C})$ of $\mathbb{C}$-rational points of X_d by the free $Gal(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2$ action, and $G_2(\mathbb{R}^{2+d})$ is the Grassmannian of 2-planes in $\mathbb{R}^{d+2}$. Hence we know as cohomology rings

$$H^*_{et}(X_d;\mathbb{Z}/2) \cong H^*(G_2(\mathbb{R}^{d+2});\mathbb{Z}/2).$$

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Let $w_i \in H^*(G_2(\mathbb{R}^\infty);\mathbb{Z}/2)$ be the i-th Stiefel-Whitney class of the canonical 2-dimensional bundle. Define $f_i \in \mathbb{Z}/2[w_1,w_2]$ inductively

$$f_0 = w_1, \quad f_1 = w_1^2 + w_2, \quad and \quad f_{n+1} = w_1 f_n + w_2 f_{n-1} \qquad for\ n \geq 0.$$

Then by Borel, Hiller [Bo], [Hi] there is the isomorphism

$$H^*(G_2(\mathbb{R}^{d+2});\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1,w_2]/(f_d, w_2 f_{d-1}).$$

Hence we can write down $H^{*,*'}(X_d;\mathbb{Z}/2)$ as a subring of

$$\mathbb{Z}/2[w_1,w_2]/(f_d, w_2 f_{d-1}) \otimes \mathbb{Z}/2[\tau,\tau^{-1}]$$

as follows.

Let us write the alternating 2-adic expansion of $d+2$ by

$$d+2 = 2^{n_0+1} - 2^{n_1+1} + ... + (-1)^r 2^{n_r+1}$$

for $n_0 > n_1 > ....n_{r-1} > n_r + 1 \geq 0$. For $0 \leq j \leq r$, let $s_j = s_j(d)$ be the number defined by

$$s_j = \begin{cases} 2^{n_0} - 2^{n_1} + ... + 2^{n_{j-2}} - 2^{n_{j-1}} & j : even \\ 2^{n_0} - 2^{n_1} + ... - 2^{n_{j-2}} + 2^{n_{j-1}} - 2^{n_j+1} + ... + (-1)^r 2^{n_r+1} & j : odd. \end{cases}$$

We identity $w_1 \in H^{1,1}(X_d;\mathbb{Z}/2)$ and $w_2 \in H^{2,2}(X_d;\mathbb{Z}/2)$. Moreover let $h = w_2\tau^{-1} \in H^{2,1}(X_d;\mathbb{Z}/2)$. Let $\alpha(n)$ be the number of 1 in the 2-adic expansion of n, e.g., $\alpha(d) = n_0 - n_1 + ...(-1)^r n_r$. Let us write

$$QT = \{\sum_n \tau^{-n} w_1^f | f \geq 2n + \alpha(n)\} \subset \mathbb{Z}/2[\tau^{-1}, w_1].$$

Theorem 1.1. *For $d \geq 1$, the motivic cohomology $H^{*,*'}(X_d;\mathbb{Z}/2)$ is isomorphic to the $\mathbb{Z}/2[h]$-subalgebra of*

$$\mathbb{Z}/2[w_1, h, \tau, \tau^{-1}]/(f_d, h f_{d-1})$$

generated by $\mathbb{Z}/2[\tau]$ and $QT\tau^{-1}w_1^{n_j+1}h^{s_j}$ for $0 \leq j \leq r$.

By using the above embedding, we will give the ring structure of $CH^*(X_d)/2$ without using the algebraic cobordism theory.

2. Chow rings of excellent quadrics

Let k be a field of $ch(k) = 0$ and X the smooth variety. We consider the Chow ring $CH^*(X)$ generated by cycles modulo rational equivalence. For a non zero symbol $a = \{a_0, ..., a_n\}$ in the mod 2 Milnor K-theory $K_{n+1}^M(k)/2$, let $\phi_a = \langle\langle a_0, ..., a_n\rangle\rangle$ be the $(n+1)$-fold Pfister form. Let X_{ϕ_a} be the projective quadric of dimension $2^{n+1} - 2$ defined by ϕ_a. The Rost motive $M_a (= M_{\phi_a})$ is a direct summand of the motive $M(X_{\phi_a})$ representing X_{ϕ_a} so that

$$(2.1) \quad M(X_{\phi_a}) \cong M_a \otimes M(\mathbb{P}^{2^n-1}).$$

The Chow ring of the Rost motive is well known. Let $\bar{k}$ be an algebraic closure of k, $X|_{\bar{k}} = X \otimes_k \bar{k}$, and $i_{\bar{k}} : CH^*(X) \to CH^*(X|_{\bar{k}})$ the restriction map.

Lemma 2.1. *(Rost [R1]) The Chow ring $CH^*(M_a)$ is only dependent on n. There are isomorphisms*

$$CH^*(M_a) \cong \mathbb{Z}\{1, c_{n,0}\} \oplus \mathbb{Z}/2\{c_{n,1}, ..., c_{n,n-1}\} \qquad |c_{n,i}| = 2^n - 2^i,$$

and $CH^*(M_a|_{\bar{k}}) \cong \mathbb{Z}\{1, \bar{\alpha}_n\}$ with $|\bar{\alpha}_n| = 2^n - 1$. The restriction map is given by $i_{\bar{k}}(c_{n,0}) = 2\bar{\alpha}_n$ and $i_{\bar{k}}(c_{n,i}) = 0$ for $i > 0$.

Here we consider the quadrics defined by a subform ξ of the Pfister form ϕ_a. Recall that $\mathbb{T}$ is the Tate motive i.e., $M(\mathbb{P}^1) = \mathbb{T}^0 \oplus \mathbb{T}$. By using results of Rost and Hoffmann, we see the following theorem.

Theorem 2.2. *(Rost, Hoffmann [Ro],[Ho],[Ka-Me],[Vi-Ya]) Let ξ be a subform of the Pfister form $\phi_a = \langle\langle a_0, ..., a_n\rangle\rangle$ of $dim(\xi) = 2^n + m, 2^n \geq m > 0$ (i.e., ξ is a neighbor of ϕ_a). Let η be a complementary form ($\phi_a = \xi \oplus \eta$). Then*

$$M(X_\xi) = M_a \otimes M(\mathbb{P}^{m-1}) \oplus M(X_\eta) \otimes \mathbb{T}^{\otimes m}.$$

The typical example of this theorem is the isomorphism (2.1) which is the case $m = 2^n$.

Now we recall the definition of excellent quadrics ([Kn],[Ka-Me]). A quadratic form ξ over k is called excellent if for every field extension K/k, the anisotropic part of ξ_K is defined over k. An anisotropic form is excellent if and only if it is a Pfister neighbor whose complementary form is excellent also (Knebusch [Kn],[Ka-Me]). It is known (see §3 bellow) that all quadrics over $\mathbb{R}$ is excellent.

Suppose that $\xi = \xi_0$ is excellent. Then we have a decreasing sequence

$$\pi_0 \supset \pi_1 \supset ... \supset \pi_r$$

of embedded Pfister forms such that the class $[\xi_k]$ in Witt ring is given by $[\xi_k] = [\pi_k] - [\pi_{k+1}] + ... + (-1)^{r-k}[\pi_r]$, namely, $\xi = \xi_0$ and $[\xi_i] + [\xi_{i+1}] = [\pi_i]$.

Let us write $dim(\pi_i) = 2^{n_i+1}$. Then $n_0 > n_1 > .... > n_r + 1 \geq 0$ and

$$(2.2) \quad dim(\xi) = 2^{n_0+1} - 2^{n_1+1} + ... + (-1)^r 2^{n_r+1}.$$

Thus n_i is the places changing zero to one (or one to zero) in the 2-adic expansion of $dim(X_\xi) + 2 = d + 2$. Let us write

$$(2.3) \quad m_j = 1/2(dim(\xi_j) - dim(\xi_{j+1})) = 2^{n_j} - 2^{n_{j+1}+1} + ... + (-1)^{r-j} 2^{n_r+1},$$

$$(2.4) \quad s_j = m_0 + ... + m_{j-1} = 1/2(dim(\xi_0) - dim(\xi_j))$$

$$= \begin{cases} 2^{n_0} - 2^{n_1} + ... + 2^{n_{j-2}} - 2^{n_{j-1}} & j : even \\ 2^{n_0} - 2^{n_1} + ... - 2^{n_{j-2}} + 2^{n_{j-1}} - 2^{n_j+1} + ... + (-1)^r 2^{n_r+1} & j : odd. \end{cases}$$

Note that $s_{r+1} = [1/2dim(\xi)]$. Iterating Theorem 2.2, we can see ;

Lemma 2.3. *(Rost [Ro], [Ka-Me]) There is an isomorphism of motives*

$$M(X_\xi) \cong \oplus_{i=0}^{r} M_{\pi_i} \otimes M(\mathbb{P}^{m_i-1}) \otimes \mathbb{T}^{\otimes s_i}.$$

When $dim(\xi) = odd$, we see $\pi_r = \langle 1 \rangle$ and $dim(X_{\pi_r}) = -1$. So the respective term should be omitted.

Corollary 2.4. *Let $r' = r$ for $d = even$ and $r' = r-1$ otherwise. There is an additive isomorphism*

$$CH^*(X_\xi) \cong \oplus_{i=0}^{r'} CH^*(M_{\pi_i})[t]/(t^{m_i})\{t^{s_i}\}.$$

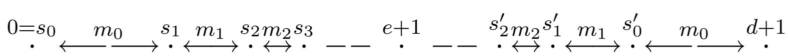
Here t^i is a generator of $CH^i(\mathbb{T}^i)$ (but it does not mean the real product of t in $CH^1(X_\xi)$.) Let $e = d/2$ for $d = even$ and $e = (d-1)/2$ for $d = odd$. Recall that $s_{r'+1} = 1/2deg(\xi) = e+1$ for $d = even$, and $s_{r'+1} = 1/2(deg(\xi)-1) = e+1$. Hence we have the additive isomorphism

$$\oplus_{i=0}^{r'} \mathbb{Z}[t]/(t^{m_i})\{t^{s_i}\} \cong \mathbb{Z}[h]/(h^{e+1}) \quad deg(h) = 1,$$
$$\oplus_{i=0}^{r'} \mathbb{Z}[t]/(t^{m_i})\{t^{s_i} c_{n,0}\} \cong \mathbb{Z}[h]/(h^{e+1})\{t^{e+\epsilon}\}$$

where $\epsilon = 0$ or 1 for $d = even$ or $d = odd$ respectively. Let us write

$$s_i' = |t^{s_i} c_{n_i,0}| = s_i + 2^{n_i} - 1.$$

We write the picture for s_i, s_i' and m_i for small $i's$.



In fact, we can compute $s_{i-1}' - s_i' = m_i$.

Let ψ be a (not assumed to be excellent) quadratic form. For each quadric X_ψ, the Chow ring of $X_\psi|_{\bar{k}} = X_\psi \otimes \bar{k}$ is given by Rost.

Let $dim(X_\psi) = d$. Let $\bar{h}$ (resp. $\bar{\alpha}$) be an element of $CH^*(X_\psi|_{\bar{k}})$ which is represented by a hyperplane section (resp. a maximal projective space) in $X_\psi|_{\bar{k}}$. So $|\bar{h}| = 1$ and $|\bar{\alpha}| = e$ if $d = even$ ($|\bar{\alpha}| = e+1$ for odd).

Lemma 2.5. *([Ro2]) There is an isomorphism of rings*

$$CH^*(X_\psi|_{\bar{k}}) \cong \mathbb{Z}\{1, \bar{h}, ..., \bar{h}^e\} \oplus \mathbb{Z}[\bar{h}]/(\bar{h}^{e+1})\{\bar{\alpha}\}.$$

The multiplication of $CH^*(X_\psi|_{\bar{k}})$ is given by

(1) $\bar{h}^{e+1} = 2\bar{h}\bar{\alpha}$ *for* $d = even$ ($\bar{h}^{e+1} = 2\bar{\alpha}$ *for* $d = odd$),

(2) $\bar{\alpha}^2 = \bar{h}^d = 2\bar{h}^e\bar{\alpha}$ *if* $d = 0\ mod(4)$ ($\bar{\alpha}^2 = 0$ *otherwise*).

Recall that X_ξ is an excellent anisotropic quadric with $2^n - 1 \le dim(X_\xi) = d \le 2^{n+1} - 2$. Let $h \in CH^1(X_\xi)$ be represented by a hyperplane section. Of course $i_{\bar{k}}(h) = \bar{h}$. It is known that $i_{\bar{k}}(c_{n_i,0}t^{s_i})$ can be written by $\bar{h}^{s_i'} = 2\bar{h}^{s_i}\bar{\alpha}$. Moeover we have ;

Theorem 2.6. *([Ya2]) There are elements $c_1', ..., c_{n-1}'$ (and c_0' when $d = 2\ mod(4)$) in $CH^*(X_\xi)$ and positive integers $d_1 \ge ... \ge d_{n-1}$ such that there is the $\mathbb{Z}[h]$-algebra isomorphism*

$$CH^*(X_\xi) \cong F \oplus \oplus_{i=1}^{n-1} \mathbb{Z}/2[h]/(h^{d_i})\{c_i'\}$$

$$where \quad F = \begin{cases} \mathbb{Z}[h]/(h^{d+1}) \oplus \mathbb{Z}\{c_0'\} & for\ d = 2\ mod(4) \\ \mathbb{Z}[h]/(h^{d+1}) & otherwise \end{cases}$$

with multiplication $c_i'c_j' = 0$ for all i, j and $hc_0' = h^{d/2+1}\ mod(c_j'|1 \le j < n)$. The degree is given as follows ; if $n_{i+1} \le j < n_i$ then $d_j = s_{i+1}$ and $|c_j'| = 2^{n_i} - 2^j + s_i = s_i' - 2^j + 1$.

3. cohomology of real Rost motives

Hereafter we always assume $k = \mathbb{R}$ the field of real numbers. Recall that $H^{*,*'}(X;\mathbb{Z}/2)$ (resp. $H_{et}^*(X;\mathbb{Z}/2)$) is the mod 2 motivic (resp. etale) cohomology defined by A.Suslin and V.Voevodsky [Vo1,2]. It is well known that

$$H_{et}^*(pt.;\mathbb{Z}/2) \cong K_*^M(\mathbb{R})/2 \cong \mathbb{Z}/2[\rho]$$

where $\rho = \{-1\} \in K_1^M(\mathbb{R})/2 \cong \mathbb{R}^*/(\mathbb{R}^*)^2$. The motivic ohomology of a point is

$$H^{*,*'}(pt.;\mathbb{Z}/2) \cong H_{et}^*(X;\mathbb{Z}/2)[\tau] \cong \mathbb{Z}/2[\rho, \tau]$$

with $0 \ne \tau \in H^{0,1}(pt.;\mathbb{Z}.2) \cong \mathbb{Z}/2$.

It is also well known that $K_*^M(\mathbb{R})/2 \cong grW^*(\mathbb{R})$; graded ring from the Witt ring $W(\mathbb{R})$ of constructed from anisotropic forms. Hence all anisotroic form is written as $q = (d+2)\langle 1\rangle$. Therefore they are excellent. Let $X_d = X_\xi = X_{(d+2)\langle 1\rangle}$ of $dim(X_d) = d$. We will study $H^{*,*'}(X_d;\mathbb{Z}/2)$.

At first we study the cohomology of the Rost motive for $a = \rho^{n+1}$. Recall that

$$Q_i : H^{*,*'}(X;\mathbb{Z}/2) \to H^{*+2^{i+1}-1,*'+2^i-1}(X;\mathbb{Z}/2)$$

is the Milnor operation in the motivic cohomology defined by Voevodsky [Vo3]. Let us write $Q(n) = \Lambda(Q_0, ..., Q_n)$.

Theorem 3.1. *([Ya1]) The cohomology* $H^{*,*'}(M_a;\mathbb{Z}/2)$ *is isomorphic to the subalgebra of* $H^*_{et}(M_a;\mathbb{Z}/2)[\tau,\tau^{-1}] \cong \mathbb{Z}/2[\rho,\tau,\tau^{-1}]/(\rho^{2^{n+1}-1})$*, that is isomorphic to*

$$(\mathbb{Z}/2[\rho,\tau] \oplus \mathbb{Z}/2[\rho] \otimes Q(n-1)\{a\tau^{-1}\})/(\rho^{2^{n+1}-1})$$

identifying with $Q^{\epsilon}(a\tau^{-1}) = Q_0^{\epsilon_0}...Q_{n-1}^{\epsilon_{n-1}}(a\tau^{-1}) = \tau^{-1-d(\epsilon)}\rho^{f(\epsilon)}a$ *where* $\epsilon \neq (1,..,1)$ *and* $f(\epsilon) = \sum_i \epsilon_i(2^{i+1}-1)$*,* $d(\epsilon) = \sum_i \epsilon_i 2^i$*.*

The sequence ϵ is the 2-adic expantion of d. So write $\epsilon = \epsilon(d)$ and $f = f(\epsilon(d)) = f(d)$. Then

$$f(\epsilon(d)) = \sum \epsilon_i(2^{i+1}-1) = 2d - \alpha(d)$$

where $\alpha(d)$ is the number of 1 in the expantion of d. We can identify the algebra of operations $Q(n)$ by the $\mathbb{Z}[\rho]$-submodule

$$\mathbb{Z}/2[\rho] \otimes Q(n) = \{ \sum_{0 \le d \le 2^n-2} \tau^{-d}\rho^f | f \ge 2d - \alpha(d)\} \subset \mathbb{Z}/2[\rho,\tau^{-1}]$$

In the above theorem $Q(n-1)\{a\tau^{-1}\} = Q(n-1)\cdot(a\tau^{-1})$ in the last ring. (The $\cdot$ in the last term means the multiplication in the ring $\mathbb{Z}/2[\rho,\tau^{-1}]$, while the first one means the actions Q^{ϵ} on $a\tau^{-1}$)

Corollary 3.2. *Let* $f(d) = f(\epsilon(d)) = 2d - \alpha(d)$ *for* $d \ge 0$ *and* $f(-1) = -n-1$*. Then there is the isomorphism*

$$H^{*,*'}(M_a;\mathbb{Z}/2)/(\tau) \cong \oplus_{0 \le d \le 2^n-1}\mathbb{Z}/2\{\tau^{-d}\rho^{m+n+1} | f(d-1) \le m < f(d)\}.$$

The element $c_{n,i} \in CH^*(M_a;\mathbb{Z}/2) \cong H^{2*,*}(M_a;\mathbb{Z}/2)$ in Lemma 2.1 is represented by

$$(3.1) \quad c_{n,i} = Q_0....\hat{Q}_i....Q_{n-1}a\tau^{-1} = (\tau^{-1}\rho^2)^{2^n-2^i}.$$

In particular, for the cycle map

$$cl : H^{*,*'}(M_a;\mathbb{Z}/2) \to H^*_{et}(M_a;\mathbb{Z}/2)$$

we have $cl(c_{n,i}) = \rho^{2^{n+1}-2^{i+1}}$.

Corollary 3.3. *There is an additive isomorphism*

$$H^{*,*'}(X_d;\mathbb{Z}/2) \cong \oplus_{i=0}^{r'} H^{*,*'}(M_{\pi_i};\mathbb{Z}/2)[h]/(h^{m_i})\{t_i\}$$

where $t_i = h^{s_i}\ mod(c_j' h^k)$ *in* $H^{2*,*}(X_d;\mathbb{Z}/2)$*.*

For ease of notation, let $2(i) = 2^{n_i+1}-1$. Since we have the similar etale motives decomposition, from Corollary 3.3, we get ;

Corollary 3.4. *There is an* $H^*_{et}(pt.;\mathbb{Z}/2) = \mathbb{Z}/2[\rho]$*-module isomorphism*

$$H^*_{et}(X_d;\mathbb{Z}/2) \cong \oplus_{i=0}^{r'}\mathbb{Z}/2[\rho,h]/(\rho^{2(i)},h^{m_i})\{t_i\}.$$

Next we recall the coniveau spectral sequence. Let X be a smooth variety over $k=\mathbb{R}$. The filtration *coniveau* is given by

$$N^cH_{et}^m(X;\mathbb{Z}/p)=\cup_Z Ker\{H_{et}^m(X;\mathbb{Z}/p)\to H_{et}^m(X-Z;\mathbb{Z}/p)\}$$

where Z runs in the set of closed subschemes of X of $codim=c$. We consider its associated (coniveau) spectral sequence. Bloch-Ogus [Bl-Og] showed that the E_2-term is given by

$$E_2^{*,*'}\cong H_{Zar}^*(X,H_{\mathbb{Z}/p}^{*'})\Longrightarrow H_{et}^{*+*'}(X;\mathbb{Z}/p)$$

where $H_{\mathbb{Z}/p}^{*'}$ is the (Zarisky) sheaf assopciated to the presheaf $U\mapsto H_{et}^{*'}(U;\mathbb{Z}/p)$. From Corollary 2.4 in [Ya1], we see the following lemma.

Lemma 3.5. *If the cycle map $cl:H^{*,*'}(X;\mathbb{Z}/p)\to H_{et}^*(X;\mathbb{Z}/p)$ is injective for each $*,*'$, then*

$$E_2^{*,*'}\cong H_{Zar}^*(X;H_{\mathbb{Z}/p}^{*'})\cong H^{*+*',*'}(X;\mathbb{Z}/p)/(\tau).$$

Moreover the spectral sequence collapses from the E_2-term.

Note $deg(Q^{\epsilon(d)})=(2d-\alpha(d),d-\alpha(d))$ in $H^{*,*'}(M_a;\mathbb{Z}/2)$ but it is represented $(d,d-\alpha(d))$ in $H_{Zar}^{*-*'}(M_a;H_{\mathbb{Z}/2}^{*'})$. From Corollary 3.2, we can write down the E_2-term for $X=M_a$ explicitly.

Lemma 3.6. *There is the $\mathbb{Z}/2[\rho]$-module isomorphism*

$$H_{Zar}^*(M_a;H_{\mathbb{Z}/2}^{*'})\cong\mathbb{Z}/2[\rho]/(\rho^{n+1})\oplus\oplus_{0\le d\le 2^n-2}\mathbb{Z}/2[\rho]/(\rho^{e_d})\{q_d\}$$

where $e_d=1+\alpha(d)-\alpha(d+1)$ and $deg(q_d)=(1+d,n+1+d-\alpha(d))$. (Here $q_d=Q^{\epsilon(d)}(a\tau^{-1})$ in $H^{*,*'}(M_a;\mathbb{Z}/2)/(\tau)$.)

Corollary 3.7. *The coniveau spectral sequence for $X=X_d$ and $p=2$ collapses from the E_2-term and*

$$H_{Zar}^*(X_d;H_{\mathbb{Z}/2}^{*'})\cong\oplus H_{Zar}^*(M_{n_i};H_{\mathbb{Z}/2}^{*'})[h]/(h^{m_i})\{t_i\}.$$

4. Borel spectral sequence

Let X be a smooth variety over $\mathbb{R}$. The manifold $X(\mathbb{C})$ of $\mathbb{C}$-rational points $X(\mathbb{C})$ is a $\mathbb{Z}/2$-equivariant space by the Galois group $Gal(\mathbb{C}/\mathbb{R})$ action. By Cox [Co], it is known that there is a natural weak homotopy equivalence

$$\{X\}_{et}^{\wedge}\cong(X(\mathbb{C})\times_{\mathbb{Z}/2}E\mathbb{Z}/2)^{\wedge}$$

where $\{-\}_{et}$ means the etale homotopy type, $\{-\}^{\wedge}$ means the profinite completion and $E\mathbb{Z}/2$ is a contractible space with free $\mathbb{Z}/2$-action. Then we have

$$H_{et}^*(X;\mathbb{Z}/2)\cong H_{\mathbb{Z}/2}^*(X(\mathbb{C});\mathbb{Z}/2)=H^*(X(\mathbb{C})\times_{\mathbb{Z}/2}E\mathbb{Z}/2;\mathbb{Z}/2).$$

Here the right hand side is called the Borel cohomology or $\mathbb{Z}/2$-equivariant cohomology. Thus we have the following Borel spectral sequence

$$E_2^{*,*'} = H^*(B\mathbb{Z}/2; H^*(X(\mathbb{C});\mathbb{Z}/2)) \Longrightarrow H^*_{et}(X;\mathbb{Z}/2),$$
$$d_r : E_r^{*,*'} \to E_r^{*+r+1,*'-r}.$$

Since the Borel spectral sequence is the topological one, it is multiplicative, in particular, the differential is a derivation. For the case $X = pt.$, we see $H^*(X(\mathbb{C});\mathbb{Z}/2) \cong \mathbb{Z}/2$ and the Borel spectral sequence is trivial

$$H^*_{et}(pt.;\mathbb{Z}/2) \cong E_\infty^{*,0} \cong H^*(B\mathbb{Z}/2;\mathbb{Z}/2) \cong \mathbb{Z}/2[x].$$

Hence we can identify $\rho = x \in H^1_{et}(pt.;\mathbb{Z}/2) \cong \mathbb{Z}/2$.

Lemma 4.1. *The E_∞-term of the Borel spectral sequence for $H^*_{et}(X_d;\mathbb{Z}/2)$ is given by*

$$E_\infty^{*,*} \cong \mathbb{Z}/2[x,h]/(x^{2(i)}h^{s_i}, h^{e+1} | 0 \le i \le r').$$

Proof. From Lemma 2.5, we have

$$H^*(X_d(\mathbb{C});\mathbb{Z}/2) \cong \mathbb{Z}/2[h]/(h^{e+1}) \otimes \Lambda(\alpha).$$

Let $0 \ne \sigma \in Gal(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2$. Then it is known $\sigma(\alpha) = \alpha + h^e$ (resp. $\sigma(\alpha) = \alpha$) if $d = 0\ mod(4)$ (resp. otherwise). Let $d = 0\ mod(4)$. (The other cases are similar but more easy.)

The E_2-term of this spectral sequence is

$$E_2^{*,*'} = H^*(\mathbb{Z}/2; H^{*'}(X(\mathbb{C});\mathbb{Z}/2))$$
$$\cong \begin{cases} H^{*'}(X(\mathbb{C});\mathbb{Z}/2)^{<\sigma>} & * = 0 \\ (Ker(1-\sigma)/Im(1+\sigma)|H^{*'}(X(\mathbb{C});\mathbb{Z}/2) & * > 0. \end{cases}$$

The nontrivial case is $*' = d = 2e$. Since $\sigma(\alpha) = \alpha + h^e$, we see

$$Ker(1-\sigma) = Im(1+\sigma) = \mathbb{Z}/2\{h^e\}.$$

Hence $E_2^{*>0,d} = 0$ and $E_2^{*,d} \cong \mathbb{Z}/2\{h^e\}$. Thus we have the isomorphism

$$E_2^{*,*'} \cong \mathbb{Z}/2[x,h]/(h^{e+1}, xh^e) \oplus \mathbb{Z}/2[x,h]/(h^e)\{\alpha h\}$$

which is additively isomorphic to

$$\mathbb{Z}/2[x,h]/(h^e) \otimes \Lambda(\alpha h) \oplus \mathbb{Z}/2\{h^e\}.$$

Note that $n_r = 0$ and $2(r) = 2^{n_r+1} - 1 = 1$. We can decompse the above term as

$$E_2^{*,*'} \cong \oplus_{i=0}^{r-1} A(i) \otimes \Lambda(\alpha h) \oplus A(r)/(x^{2(r)})$$

with $A(i) = E_2^{*,2s_i \le *' < 2s_{i+1}} \cong \mathbb{Z}/2[x]\{1,...,h^{m_i-1}\}\{h^{s_i}\}$.

The element h is a permanent cycle and the first nonzero differential must be $d_t(\alpha h)$. From Corollary 3.3 (or Corollary 3.2), we see $x^{2(r-1)}h^{s_{r-1}} = 0$ in $E_\infty^{*,*'}$. Hence $t = 2(r-1)$ and

$$d_{2(r-1)}(\alpha h) = x^{2(r-1)}h^{s_{r-1}}.$$

Note here that

$$d_{2(r-1)}(h^{m_{r-1}}\alpha) = x^{2(r-1)}h^{s_{r-1}+m_{r-1}} = x^{2(r-1)}h^{e+1} = 0.$$

Thus we have the isomorphism

$$E^{*,*}_{2(r-1)+1} \cong \oplus_{j=r-1}^{r} A(j)/(x^{2(j)}) \oplus \oplus_{i=0}^{r-2-1} A(i) \otimes \Lambda(\alpha h^{m_{r-1}}).$$

Similarly, by decsending induction on j, we can prove

$$E^{*,*}_{2(j)} \cong \oplus_{i=j+1}^{i=r'} A(i)/(x^{2(i)}) \oplus \oplus_{i=0}^{j} A(i) \otimes \Lambda(\alpha h^{s_{r'+1}-s_j}),$$

$$d_{2(j)}(\alpha h^{s_{r'+1}-s_j}) = x^{2(j)}h^{s_j}.$$

In particular , we get the desired result

$$E_\infty^{*,*} \cong \oplus_{i=0}^{r'} A(i')/(x^{2(i)}) \cong \oplus_{i=0}^{r'} \mathbb{Z}/2[x]/(x^{2(i)})\{h^{s_i}, ..., h^{s_{i+1}-1}\}$$
$$\cong \mathbb{Z}/2[x,h]/(x^{2(i)}h^{s_i}, h^{e+1})$$

□

Remark. In §3 in [Sa-Li], the integral Borel spectral sequence

$$IE_2^{*,*'} \cong H^*(\mathbb{Z}/2; H^{*'}(X_d(\mathbb{C}); \mathbb{Z})_{(2)} \Longrightarrow H^*(X; \mathbb{Z}_{(2)})$$

is studied, while $IE_\infty^{*,*'}$ seems not given there. By the filtration of multiplying 2, we have the isomorphism

$$grIE_2^{*,*'} \cong E_2^{*,*'} \oplus 2H^{*'}(X_d(\mathbb{C}); \mathbb{Z}_{(2)})^{<\sigma>}.$$

Since all elements in $2H^{*'}(X_d(\mathbb{C}); \mathbb{Z}_{(2)})$ are permanent cycles, we also get the isomorphism $grIE_\infty^{*,*'} \cong E_\infty^{*,*'} \oplus 2H^{*'}(X_d(\mathbb{C}); \mathbb{Z}_{(2)})^{<\sigma>}$.

Lemma 4.2. *Let t_i be an element in $H^*_{et}(X_d; \mathbb{Z}/2)$ such that $t_i = h^{s_i}$ mod(ρ) and $\rho^{2(i)}t_i = 0$. Then such t_i exists uniquely.*

Proof. We still know the existence. From the above corollary, if $s_j \le k < s_{j+1}$, there is the isomorphism $E_\infty^{*,2k} \cong \mathbb{Z}/2[x]/(x^{2(j)})\{h^k\}$. So the map $x^m : E_\infty^{*,2k} \to E_\infty^{*+m,2k}$ is injective if $* + m < 2(j)$.

When $* + 2k = 2s_i$ and $* > 0$ (i.e., $j < i$), we see

$$* + 2(i) = 2s_i + 2(i) - 2k = 2s_i' + 1 - 2k$$
$$< 2s_j' + 1 - 2k \le 2s_j' + 1 - 2s_j = 2(j).$$

Hence the map $x^{2(i)} : E_\infty^{*,2k} \to E_\infty^{*+2(i),2k}$ is injective when $* + 2k = 2s_i$.

Suppose t_i' is the another element which satisfies the assunption of the lemma. For $0 \neq t \in H_{et}^*(X_d;\mathbb{Z}/2)$, let us write by $0 \neq [t] \in E_\infty^{*,*'}$ the corresponding nonzero element. Then

$$0 \neq [t_i - t_i'] \in E_\infty^{*,*'} \quad with\ * < 2s_i,\ and\ \ * + *' = 2s_i$$

which must be $x^{2(i)}$-free by the above dimensional reason. However t_i, t_i' are $\rho^{2(i)}$-torsion and so is $t_i - t_i'$. This is a contradiction. □

5. Cohomology of Grassmaniann $G_2(\mathbb{R}^{2+d})$.

Let X be a smooth variety over the real number field $\mathbb{R}$. Let $X(\mathbb{R}) = \emptyset$. Then $Gal(\mathbb{C}/\mathbb{R})$ acts on $X(\mathbb{C})$ freely. Hence we have the isomorphism

$$H_{\mathbb{Z}/2}^*(X(\mathbb{C});\mathbb{Z}/2) \cong H^*(X(\mathbb{C})/\mathbb{Z}/2;\mathbb{Z}/2)$$

from the triviality of the spectral sequence induced from the cofibeing

$$E\mathbb{Z}/2 \to X(\mathbb{C}) \times_{\mathbb{Z}/2} E\mathbb{Z}/2 \to X(\mathbb{C})/\mathbb{Z}/2.$$

Now we consider anisotropic quadrics over $\mathbb{R}$. Each anisotropic quadratic form q of dimension n over $\mathbb{R}$ is written by

$$n\langle 1\rangle = q_n = x_1^2 + ... + x_n^2.$$

Reacll that X_d is the projective quadric defined by $q_{d+2} = 0$ so that $dim(X_d) = d$. Let $G_k(\mathbb{R}^{k+n})$ be the Grassmannian of k-planes in $\mathbb{R}^{k+n}$. It is well known that there is the homeomrphism

$$X_d(\mathbb{C})/\mathbb{Z}/2 \cong Gr_2(\mathbb{R}^{2+d})$$

via the map from $(z_j = x_j + iy_j) \in X_d(\mathbb{C}) \subset \mathbb{CP}^{d+1}$ to the plane generated by $(x_j), (y_j)$ in $\mathbb{R}^{2+d}$. (In fact, $||(x_i)|| = ||(y_i)||$ and $(x_i) \perp (y_i)$ in $\mathbb{R}^{d+2}$ for $(x_i + iy_i) \in X_d(\mathbb{C})$.) Thus we see that

Lemma 5.1. $H_{et}^*(X_d;\mathbb{Z}/2) \cong H^*(Gr_2(\mathbb{R}^{2+d});\mathbb{Z}/2)$.

The mod 2 cohomology of $Gr_k(\mathbb{R}^{n+k})$ is computed by Borel

$$H^*(G_k(\mathbb{R}^{n+k});\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, ..., w_k, \bar{w}_1, ..., \bar{w}_n]/I_{n,k}$$

$$with \quad I_{n,k} = ((1 + w_1 + w_2...) \cdot (1 + \bar{w}_1 + \bar{w}_2 + ...)$$

in fact, w_i(resp.$\bar{w}_i$) is the i-th Stiefel-Whitney class of the universal k-plane bundle (its complementary bundle). Hiller [Hi] write down this relation explicitly.

Theorem 5.2. *([Hi]) There is the isomorphism*

$$H^*(Gr_k(\mathbb{R}^{n+k});\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, ..., w_k]/(f_{1,n}, ..., f_{k,n})$$

$$\text{where} \quad \begin{pmatrix} f_{1,n} \\ . \\ . \\ . \\ f_{k,n} \end{pmatrix} = \begin{pmatrix} w_1 & 1 & 0 & . & . & 0 \\ w_2 & 0 & 1 & . & . & 0 \\ . & & & & & \\ . & 0 & . & . & . & 1 \\ w_k & 0 & . & . & . & 0 \end{pmatrix}^n \begin{pmatrix} w_1 \\ . \\ . \\ . \\ w_k \end{pmatrix}$$

We also note the following fact.

Lemma 5.3. *The sequence* $(f_{1,n}, ..., f_{k,n})$ *is a regular in* $\mathbb{Z}/2[w_1, ..., w_k]$.

Proof. Let $V_{n+k,k}$ be the variety of orthogonal k vectors in $\mathbb{R}^{n+k}$ and $O(k)$ be the Orthogonal group. Then we have the principal bundle

$$O(k) \to V_{n+k,k} \to G_k(\mathbb{R}^{n+k})$$

and the induced spectral sequence

$$E_2^{*,*} = H^*(BO(k); H^*(V_{n+k,k}; \mathbb{Z}/2)) \Longrightarrow H^*(G_k(\mathbb{R}^{n+k}); \mathbb{Z}/2).$$

Of course $H^*(BO(k); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, ..., w_k]$ and (by Borel)

$$grH^*(V_{n+k,k}; \mathbb{Z}/2) \cong \Lambda(e_n, ..., e_{n+k-1})$$

with $deg(e_i) = i$.

Hence the differential must be $d(e_{n+i}) = f_{n,i+1}$ from the above theorem. Moreover $H^*(Gr_k(\mathbb{R}^{n+k}); \mathbb{Z}/2)$ is multiplicatively generated by w_i. This imlies that the sequence is regular. □

Let $P(t)$ be the Poincare series $\sum_i H^*(Gr_k(\mathbb{R}^{n+k}); \mathbb{Z}/2)t^i$. Since $dim(w_i) = i$ and $dim(f_{i,n}) = n + i$, we get ;

Corollary 5.4.

$$P(t) = \frac{(1 - t^{n+1})(1 - t^{n+2})...(1 - t^{n+k})}{(1 - t)(1 - t^2)...(1 - t^k)}.$$

In particular, the case $k = 2$ of Theorem 5.2 is stated as follows by letting $f_{1,n} = f_n$ and $f_{2,k} = w_2 f_{n-1}$

Corollary 5.5. *There is the isomorphism*

$$H^*_{et}(X_d; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2]/(f_d, w_2 f_{d-1})$$

Here $f_0 = w_1$, $f_1 = w_1^2 + w_2$ *and* $f_{n+1} = w_1 f_n + w_2 f_{n-1}$ *for* $n \geq 0$, *namely,*

$$f_d = \sum \binom{d+1-b}{b} w_1^{d+1-2b} w_2^b.$$

Proof. The last equation follows from induction on d. The first isomorphism is immediate from Theorem 5.2. □

Corollary 5.6. *The elements* ρ, h *in* $H^*_{et}(X_d;\mathbb{Z}/2)$ *correspond to* w_1, w_2 *respectively in* $H^*(G_2(\mathbb{R}^{2+d});\mathbb{Z}/2)$.

Proof. We only need the proof for h. By dimensional reason, h corresponds to w_2 or $w_2+w_1^2=f_1$. But in $H^*_{et}(X_1;\mathbb{Z}/2)$ we know $\rho^2=h\neq 0$. □

Let $P=P(t,E_\infty^{*,*'})$ be the Poincare series of the infinite term of the spectral sequence in lemma 3.4. Of course from above corollaries

$$P(t)=(1-t^{d+1})(1-t^{d+2})/(1-t)(1-t^2).$$

This fact is also computed directly, infact $(1-t)(1-t^2)P$ is written

$$\sum_i(1-t^{2(i)})(1-t^{2m_i})t^{2s_i}=\sum_i(t^{2s_i}-t^{2s_i+2m_i}+t^{2s_i+2(i)}-t^{2s_i+2m_i+2(i)})$$

$$=\sum_i(t^{2s_i}-t^{2s_{i+1}})+\sum_i(t^{2s_i'+1}-t^{2s_{i-1}'+1})=1-t^{d+1}-t^{d+2}+t^{2d+3}.$$

Remark. In [Hi], Hiller computed some relations for w_1, w_2 by using above f_d in the corollary. Most of them are natural consequences from the cohomology theory of quadrics. For example, Proposition 4 in [Hi] is $w_2^d\neq 0$, which corresponds $h^d\neq 0$ in $CH^*(X_d)/2$. One of the main results of [Hi] is about the Lusternik-Schnierlmann category, that is,

$$Cat(Gr_2(\mathbb{R}^{2^n+1}))=dim(Gr_2(\mathbb{R}^{2^n+1}))=2^{n+1}-2$$

following from $w_1^{2^{n+1}-2}\neq 0$. This corresponds the fact $\rho^{2^{n+1}-2}=h^{2^n-1}\neq 0$ in the norm variety (the smallest neighbour of the Pfister form) X_{2^n-1}.

6. $\rho^{2(i)}$-torsion elements

In this section, we look for t_i in Corollary 4.5. That is $t_i\in\mathbb{Z}/2[w_1,w_2]$ such that $t_i=w_2^{s_i}\ mod(w_1)$ and $w_1^{2(i)}t_i\in(f_d,fh_{d-1})$ where

$$f_d=\sum_b\binom{d+1-b}{b}w_1^{d+1-2b}w_2^b,\quad w_2f_{d-1}=\sum_b\binom{d+b}{b}w_1^{d-2b}w_2^{b+1}.$$

We first recall the famous relation about the binary coefficient

Lemma 6.1. *([Ep-St]) If* $n=\sum n_i2^i$ *and* $m=\sum m_i2^i$ *for* $m_i,n_j=0$ *or* 1, *then*

$$\binom{m}{n}=\prod_i\binom{m_i}{n_i}\ mod(2).$$

Recall that

$$(6.1)\quad d+2 = 2^{n_0+1} - 2^{n_1+1} + ... + (-1)^r 2^{n_{r'}+1} + \epsilon.$$

where $\epsilon = 0, 1$ or -1. For $k \le r'$, let us write

$$(6.2)\quad d(k)+2 = 2^{n_0+1} - 2^{n_1} + ... + (-1)^k 2^{n_k}.$$

Lemma 6.2. *In* $\mathbb{Z}/2[w_1, w_2]$*, we can take*

$$t_k = \begin{cases} f_{d(k)}/w_1^{2(k)} & if\ d(k) \ge d \\ f_{d(k)} w_2^{d-d(k)}/w_1^{2(k)} & if\ d(k) < d. \end{cases}$$

Proof. We first prove $k = r'$ and $\epsilon = 0$. Let $d = 2d'$. Of course $\binom{d-d'}{d'} = 1$, we see $w_2 f_{d-1} = h^{d'+1}\ mod(w_1)$. Hence we can take $t_{r'+1} = w_2 f_{d-1}$. Next look for $t_{r'}$. Let $d' - b = b'$. Then

$$\binom{d+1-b}{b} = \binom{2d'+1-(d'-b')}{d'-b'} = \binom{d'+1+b'}{d'-b'} = \binom{d'+1+b'}{2b'+1}$$

Here $d'+1 = 0\ mod(2^{n_{r'}})$ from (6.1). Hence the above number is zero for each $b' < 2^{n_{r'}} - 1$ from the above lemma. (See also the proof of Lemma 9.1 bellow.) For $b' = 2^{n_{r'}} - 1$, the nonzero value of the above binomial coefficients is one and

$$f_d = w_1^{2(r')} w_2^{d'-2^{n_{r'}}+1} = w_1^{2(r')} w_2^{s_{r'}}\ mod(w_1^{2(r')+1}).$$

Therefore we can take

$$t_{r'} = f_d/(w_1^{2(r')}) \quad in\ \mathbb{Z}/2[w_1, w_2].$$

Next consider the case $k > r'$. (The case $k = r'$ and $\epsilon = \pm 1$ is also proved similarly.)

Let $k = even$ so that $d(k) \ge d$. Let $s_i(d(k))$ (resp. $r'(d(k))$, $t_i(d(k))$ be the number s_i (resp. r', t_i) for $X_{d(k)}$. Then from (2.3), we see

$$s_k(d(k)) = s_k = 2^{n_0} - 2^{n_1} + ... + 2^{n_{k-2}} - 2^{n_{k-1}}.$$

Since $r'(d(k)) = k$, we still know

$$t_k(d(k)) = f_{d(k)}/(w_1^{2(k)}) = w_2^{s_k(d(k))} + ...$$

Let $i_k : X_d \subset X_{d(k)}$ be the natural embedding. Then we can take

$$t_k = i_k^* t_k(d(k)) = (f_{d(k)})/(w_1^{2(k)}).$$

Let $k = odd$ so that $d(k) \le d$. Then

$$s_i - s_i(d(k)) = 2^{n_{k+1}+1} - ... + (-1)^r 2^{n_r+1} = d - d(k).$$

Let $i^k : X_{d(k)} \subset X_d$ be the embedding. For the Gysin map i_*^k, we see

$$i_*^k(1) = w_2^{d-d(k)}, \quad and \quad i_*^k(w_1^a w_2^b) = w_1^a w_2^{b+d-d(k)}.$$

(Note $w_2 = h$ is the hyperplane section.) Thus we can take

$$t_k = i_*^k(f_{d(k)})/w_1^{2(k)} = f_{d(k)}w_2^{d-d(k)}/w_1^{2(k)}.$$

□

From the Borel spectral sequence (Lemma 4.1), we see

$$w_1^{2(k+1)}w_2^{m_k}t_k = w_1^{2(k+1)}w_2^{m_{k+1}} = 0 \quad mod(w_1^*t_j|j \le k).$$

Moreover $w_1^{2^k}t_k$ is nonzero and it is written as $w_1^*t_{k-1}$ by the following reason by using Theorem 2.6. Recall that each element $c_it^{s_k}$ (with the notation in §2, see Corollary 2.4, Corollary 3.2,) is represented by

$$(\tau^{-1}\rho^2)^{2^{n_k}-2^i}t_k$$

In particular $c_{n_k-1}t^{s_k}$ is represent by $\tau^{-2^{n_k-1}}\rho^{2^{n_k}}t_k$. Using Theorem 2.6, we can prove that $h^{m_k}c_{n_k-1}t_k$ must be written as $\rho^*c_{n_k-1}t_{k-1}$.

However we show the following lemma by direct computation of f_d. The proof is just a computation of binomial coefficient but not so short, hence we give it in the last section.

Lemma 6.3.

$$w_1^{2^{n_k}-1}w_2^{m_k}t_k = w_1^{2(k-1)-2^{n_k}}t_{k-1}.$$

Lemma 6.4.

$$f_{d(k)} = w_1^{2(k)}(\sum_{j=n_{k+1}}^{n_{k-1}} w_1^{2^{n_{k-1}+1}-2^{j+1}}w_2^{s_k-2^{n_{k-1}}+2^j}) \quad mod(w_1^{2(k)+2^{n_{k-1}+1}}),$$

$$i.e., \quad t_k = w_2^{s_k} + w_1^{2^{n_{k-1}}}w_2^{s_k-2^{n_{k-1}-1}} + ...$$

7. MOTIVIC COHOMOLOGY

In this section, we study the motivic cohomology. Take element

$$w_1 \in H^{1,1}(X_d;\mathbb{Z}/2), \quad and \quad w_2 \in H^{2,2}(X_d;\mathbb{Z}/2)$$

such that the images of the cycle map cl are the same letter elements. (So $w_1 = \rho$ and $w_2 = \tau h$ in the notation §2 and §3.) Hence $deg(f_d) = (d+1, d+1)$. Take $\bar{t}_k \in H^{2*,*}(X_d;\mathbb{Z}/2) = CH^*(X_d)/2$ such that $cl(\tilde{t}_k) = t_k$, namely,

$$\bar{t}_k = \begin{cases} f_{d(k)}/\rho^{2(k)}\tau^{s_k} & if \ d(k) \ge d \\ f_{d(k)}h^{d-d(k)}/\rho^{2(k)}\tau^{s_k} & if \ d(k) < d. \end{cases}$$

Since we still know

$$H^{*,*'}(X_d;\mathbb{Z}/2) \subset H_{et}^*(X_d;\mathbb{Z}/2)[\tau,\tau^{-1}],$$

the following theorem is the immediate consequence of Theorem 3.1, Corollary 3.2, and Lemma 5.2

Theorem 7.1. *Given $d > 0$, the motivic cohomology $H^{*,*'}(X_d;\mathbb{Z}/2)$ is isomorphic to the $\mathbb{Z}/2[\rho,h]$-subalgebra of the algebra*

$$\mathbb{Z}/2[\rho,\tau,\tau^{-1},h]/(f_d, hf_{d-1})$$

generated by $Z/2[\tau]$ and

$$\mathbb{Z}/2[\rho]\otimes Q(n_i-1)\{\rho^{n_i+1}\tau^{-1}\bar{t}_i\} \quad 0\le i\le r'$$

where $\mathbb{Z}/2[\rho]\otimes Q(n_i-1)=\{\sum_{0\le m\le 2^{n_i-1}-2}\tau^{-m}\rho^f|f\ge 2m-\alpha(m)\}$.

Here we can take h^{s_i} instead of $\bar{t}_i$.

Corollary 7.2. *The motivic cohomology $H^{*,*'}(X_d;\mathbb{Z}/2)$ is isomorphic to the $\mathbb{Z}/2[\rho,h]$-subalgebra generated by $Z/2[\tau]$ and*

$$\mathbb{Z}/2[\rho]\otimes Q(n_i-1)\{\rho^{n_i+1}\tau^{-1}h_i^{s_i}\} \quad 0\le i\le r'.$$

Proof. By inductin on k for $k<r'$, we assume

$$\mathbb{Z}/2[\rho]\otimes Q(n_i-1)\{\tau^{-1}\rho^{n_i+1}\bar{t}_i\} \quad 0\le i<k$$

is generated by

$$(1) \quad \mathbb{Z}/2[\rho]\otimes Q(n_i-1)\{\tau^{-1}\rho^{n_i+1}h_i^{s_i}\} \quad 0\le i<k.$$

From Lemma 6.4, we see

$$\bar{t}_k = h^{s_k}+\rho^{2^{n_{k-1}}}h^{s_k-2^{n_{k-1}-1}}+...$$

Of course $2^{n_{k-1}}+n_k+1\ge n_{k-1}+1$. Hence

$$\mathbb{Z}/2[\rho]\otimes Q(n_{k-1}-1)\{(\tau^{-1}\rho^{n_k+1+2^{n_{k-1}}}h^{s_k-2^{n_{k-1}-1}}+...)\}$$

is contained in (1). ∎

Now we consider the proof of Theorem 2.6 without algebraic cobordism theory (for the field $\mathbb{R}$).

Proof of Theorem 2.6. The additive structure of $CH^*(X_d)/2$ is still known. Hence we only need the h-divisibility of element c_i. Recall that each element $c_it^{s_k}$ (with the notation in §2, see Corollary 2.4, Corollary 3.2,) is represented by

$$(\tau^{-1}\rho^2)^{2^{n_k}-2^i}\bar{t}_k.$$

Consider the element

$$h^{m_k}(c_it^{s_k}) = h^{m_k}(\tau^{-1}\rho^2)^{2^{n_k}-2^i}\bar{t}_k = (\tau^{-1}\rho)^*\bar{t}_{k-1}.$$

Here we see

$$* = 2^{n_k}-2^i+(2(k-1)-2^{n_k})/2-(2^{n_k}-1)/2$$

$$= 2^{n_k} - 2^i + 2^{n_{k-1}} + 1/2 - 2^{n_k-1} - 2^{n_k-1} - 1/2 = 2^{n_{k-1}} - 2^i.$$

Thus we get $h^{m_k} c_i t^{s_k} = c_i t^{s_{k-1}}$. $\square$

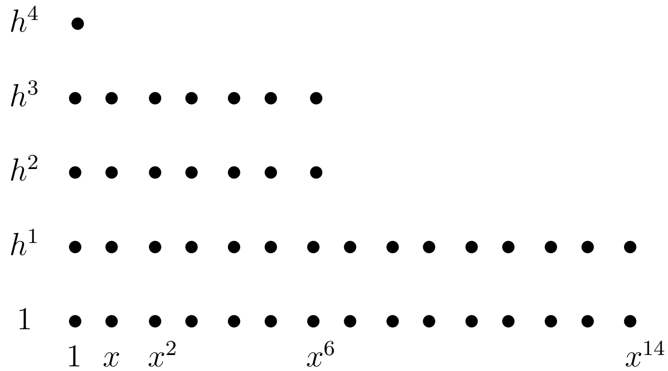
8. Example

We consider the case $d = 8$. Since $d + 2 = 2^4 - 2^3 + 2^1$, we see

$$n_0 = 3, n_1 = 2, n_2 = 0 \quad m_0 = 2, m_1 = 2, m_3 = 1.$$

Hence the infinity term $E_\infty^{*,*'}$ of the Borel spectral sequence is

$$\mathbb{Z}/2\{h^4\} \oplus \mathbb{Z}/2[x]/(x^7)\{h^2, h^3\} \oplus \mathbb{Z}/2[x]/(x^{15})\{1, h\}$$



On the other hand

$$f_8 = w_1^9 + w_1^5 w_2^2 + w_1 w_2^4, \quad w_2 f_7 = w_1^8 w_2 + w_1^6 w_2^2 + w_1^4 w_2^3 + w_2^5$$

are zero in $H_{et}^*(X_8; \mathbb{Z}/2)$. From the first equation, we have $t_2 = f_8/w_1 = w_2^4 + w_1^4 w_2^2 + w_1^8$. We also know that

$$f_6 w_2^2 = w_1^7 w_2^2 = w_2 f_8 - w_1 w_2 f_7, \quad f_{16} = w_1^{15} = w_1^6 f_8 + w_1^4 f_6 + w_2^2 w_2 f_6$$

are zero in $H_{et}^*(X_8; \mathbb{Z}/2)$. Thus we can take $t_1 = f_6 w_2^2 / w_1^7 = w_2^2$ and $t_0 = 1$ of course.

The statement of Lemma 6.3 is described

$$w_2 t_2 = w_1^6 w_2^2 = w_1^6 t_1, \quad w_1 w_2^2 t_1 = w_1^9 + w_1^5 w_2^2 = w_1^9 t_0.$$

Now we consider the motivic cohomology $H^{*,*'}(X_8; \mathbb{Z}/2)$. It is a subalgebra of

$$\mathbb{Z}/2[\rho, \tau, \tau^{-1}, h]/(f_8, w_2 f_7)$$

(identifying $w_1 = \rho, w_2 = h\tau$) generated as a $QT \otimes \mathbb{Z}/2[h]$-module by

$$1, \quad \rho^4 \tau^{-1} \quad \textit{and} \quad \rho^3 \tau^{-1} h^2.$$

(Here $\rho \tau^{-1} h^4 = \tau^{-1} \rho^9$ does not need as generators.)

Let us write $\sigma = \tau^{-1} \rho^2$. Then the Chow ring $CH^*(X_8)/2$ is described as

$$\mathbb{Z}\{1, h, ..., h^4\} \oplus \mathbb{Z}/2\{1, h\} \otimes \{\sigma^4, \sigma^6, \sigma^7, \sigma^2 h^2, \sigma^3 h^2\}.$$

Here the Q_i actions are

$$\sigma^4 = Q_0Q_1(\rho^4\tau^{-1}), \quad \sigma^6 = Q_0Q_2(\rho^4\tau^{-1}), \quad \sigma^7 = Q_1Q_2(\rho^4\tau^{-1}),$$
$$\sigma^2h^2 = Q_0(\rho^3\tau^{-1}h^2), \quad \sigma^3h^2 = Q_1(\rho^3\tau^{-1}h^2).$$

Its multiplicatin is given by

$$h^5 = \sigma^4h + \sigma^3h^2 + \sigma^2h^3, \quad h^6 = \sigma^6 + \sigma^3h^3,$$
$$h^7 = \sigma^7 + \sigma^6h + \sigma^5h^2, \quad h^8 = \sigma^7h.$$

Moreover let $c_1 = \sigma^2h^2$ and $c_2 = \sigma^4$. Then $h^2c_1 = \sigma^4h^4 = \sigma^6$. Thus we have the isomorphism

$$CH^*(X_8)/2 \cong \mathbb{Z}/2[h]/(h^9) \oplus \mathbb{Z}/2[h]/(h^4)\{c_1\} \oplus \mathbb{Z}/2[h]/(h^2)\{c_2\}.$$

9. Proofs of lemmas in §5

Proof of Lemma 6.4. Let $d = even$ and $d = d'$. Recall that

$$f_d = \binom{d'+1+b'}{2b'+1} w_1^{2b'+1}w_2^{d'-b'}$$

as described in the proof of Lemma 6.2. We will prove the case $k = even$ and the odd case is proved similarly. Let

$$d(k) + 2 = 2^{n_0+1} - 2^{n_1+1} + ... - 2^{n_{k-1}+1} + 2^{n_k+1}.$$

Here let $d(k) = 2d'$ and $0 \le b' \le d'$. We want to compute the 2-adic expansion $\epsilon(-)$ of $d'+1, b', 2b'+1$, namely,

$$\epsilon(d'+1) = (\overset{1}{0}, ..., 0, \overset{n_k}{1}, 0, ..., 0, \overset{n_{k-1}}{1}, 1, ...)$$
$$\epsilon(b') = (*, *', *'', ...), \qquad \epsilon(2b'+1) = (1, *, *', *'', ...).$$

Note that the expansion of $2b'+1$ is the shiftting of that of b' to the right and adding 1 at the first entry.

Assume $\binom{d'+1+b'}{2b'+1} \ne 0 \ mod(2)$.

Suppose $b' < 2^{n_k}$. Then

$$\epsilon(d'+1+b') = (*, *', *'', ..., \overset{n_k}{1}, 0, ..., 0, 1, 1, ...)$$

The fact $\epsilon_1(2b'+1) = 1$ imlies $\epsilon_1(d'+1+b') = 1$, which means $\epsilon_1(b') = 1$. That implies $\epsilon_2(2b'+1) = 1$, and so $\epsilon_2(b') = 1$,... Thus we see $b' = 2^{n_k} - 1$.

Next consider the case $b' \ge 2^{n_k}$. By the same argument as above we see $\epsilon_1(b') = ... = \epsilon_{n_k-1}(b') = 1$.

Suppose $\epsilon_{n_k}(b') = 1$. Since $\epsilon_{n_k}(d'+1) = 1$, we know $\epsilon_{n_k}(d'+1+b') = 0$. This implies $\epsilon_{n_k}(2b'+1) = 0$, and so $\epsilon_{n_k-1}(b') = 0$. This is a contradiction. Hence $\epsilon_{n_k}(b') = 0$. Thus we can write

$$\epsilon(b') = (1, .., 1, \overset{n_k}{0}, *, *', *'', ...), \quad \epsilon(d'+1+b') = (1, .., 1, \overset{n_k}{1}, *, *', *'', ..).$$

Suppose that there is i such that

$$\epsilon_i(b')=1,\quad but\quad \epsilon_{i+1}(b')=0\quad for\ i<n_{k-1}-1,\ i.e.,$$

$$\epsilon(b')=(1,..,1,\overset{n_k}{0},*,*',*'',\overset{i}{1},0,...\overset{n_{k-1}}{1},*''',..).$$

Then $\epsilon_{i+1}(d'+1+b')=0$ but $\epsilon_{i+1}(2b'+1)=\epsilon_i(b')=1$. This is a contradiction.

Thus if $b'<2^{n_{k-1}}$, then we get

$$\epsilon(b')=(1,...,1\ \overset{n_k}{0},0,...,0,\overset{j}{1},...,\overset{n_{k-1}-1}{1},\overset{n_{k-1}}{0},0,..)\quad i.e.,$$

$$b'=2^{n_k}-1+2^{n_{k-1}}-2^j\quad for\ n_k<j\le n_{k-1}.$$

□

Lemma 9.1.

$$f_{d+2^{k+1}}=(\sum_{j=0}^{k}w_1^{2^{k+1}-2^{j+1}}w_2^{2^j})f_d+w_1^{2^{k+1}-1}w_2f_{d-1}.$$

$$f_{d+2^{k+1}-2^{\ell+1}}=(\sum_{0\le j'<\ell<j\le k}w_1^{2^{\ell+1}-2^{j'+1}+2^{k+1}-2^{j+1}}w_2^{2^j-2^{\ell+1}+2j'})f_d+$$

$$(\sum_{j=\ell+1}^{k}w_1^{2^{\ell+1}+2^{k+1}-2^{j+1}-1}w_2^{2^j-2^{\ell+1}+1})f_{d-1}.$$

Proof. Inductively we easily see that

$$f_{d+i}=\sum\binom{i-b}{b}w_1^{i-2b}w_2^bf_d+\sum\binom{i-1-b}{b}w_1^{i-2b-1}w_2^{b+1}f_{d-1}$$

$$=f_{i-1}f_d+w_2f_{i-2}f_{i-1}.$$

First consider the case $i=2^{k+1}$. Then we note $\binom{i-1-b}{b}=0$ for $b\neq 0$. Hence we see $f_{d+2^{k+1}}=af_d+w_1^{2^{k+1}-1}w_2f_{d-1}$. We will study the coefficient a of f_d.

Let $i=2d'$ and Suppose $\binom{i-b}{b}=\binom{d'+b'}{2b'}\neq 0$.

Here we consider the 2-adic expansions.

$$\epsilon(d')=(0,...,0,\overset{k}{1}),\quad \epsilon(b')=(*,*',...,0),\quad \epsilon(2b')=(0,*,*',...).$$

Hence there is not i such that

$$\epsilon_i(b')=1,\quad but\quad \epsilon_{i+1}(b')=0\quad for\ i<k-1.$$

Indeed, if such i exists, then $\epsilon_{i+1}(d'+b')=0$. Hence $\epsilon_{i+1}(2b')=0$ and so $\epsilon_i(b')=0$, and this is a contradiction. Thus we know

$$\epsilon(b')=(0,...,0,11...,1),\quad i.e.,\quad b'=2^k-2^j.$$

This shows the first equation.

Next we show the second equation. Let $i = 2^k - 2^\ell = 2d'$ and $b'' = d' - 1 - b$. Then

$$\binom{i-1-b}{b} w_1^{i-2b-1} w_2^{b+1} = \binom{d'+b''}{2b''+1} w_1^{2b''+1} w_2^{d'-b''}.$$

Here $\epsilon(d') = (0, ..., 0, \overset{\ell}{1} \; ..., \overset{k-1}{1}, 0)$. By the arguments similar to the proof of the preceding lemma (or the above arguments), we get

$$\epsilon(b'') = (1, ..., 1, \overset{\ell}{0}, .., 0, \overset{j}{1}, .., 1, \overset{k}{0})$$

That is $b'' = 2^\ell - 1 + 2^k - 2^j$ for $\ell < j \le k$. This show the the coefficient of f_{d-1} of the second equation.

For that of f_d, we study

$$\binom{i-b}{b} w_1^{i-2b} w_2^b = \binom{d'+b'}{2b'} w_1^{2b'+1} w_2^{d'-b'}.$$

By the arguments similar to also the proof of the preceding lemma, we can prove

$$\epsilon(b') = (0, ..., 0, \overset{j'}{1}, ..., 1, \overset{\ell}{0}, 0, ..., 0, \overset{j}{1}, 1, ..., 1, \overset{k}{0}),$$

that is, $b' = 2^\ell - 2^{j'} + 2^k - 2^j$ for $0 \le j' \le \ell < j \le k$. This shows the coefficient of f_d in the second equation. □

Proof of Lemma 6.3 for the case $k = odd$. For this case

$$d + 2 = 2^{n_0+1} - ... + 2^{n_{k-1}+1} - 2^{n_k+1} + 2^{n_{k+1}+1} - ...$$

Then we see

$$d(k-1) = d(k) + 2^{n_k+1} \quad \textit{and} \quad d(k+1) = d(k) + 2^{n_{k+1}+1}.$$

From the preceding lemma, we have

$$f_{d(k-1)} = (\sum w_1^{2^{n_k+1} - 2^{j+1}} w_2^{2^j}) f_{d(k)} + w_1^{2(k)} w_2 f_{d(k)-1},$$

$$f_{d(k+1)} = (\sum w_1^{2^{n_{k+1}+1} - 2^{j+1}} w_2^{2^j}) f_{d(k)} + w_1^{2(k+1)} w_2 f_{d(k)-1}.$$

Eliminate the terms $w_1^* w_2 f_{d(k)-1}$

$$(1) \quad f_{d(k-1)} - w_1^{2(k)-2(k+1)} f_{d(k+1)} = (\sum_{j=n_{k+1}+1}^{n_k} w_1^{2^{n_k+1} - 2^{j+1}} w_2^{2^j}) f_{d(k)}$$

$$= (w_2^{2^{n_k}} + w_1^{2^{n_k}} w_2^{2^{n_k-1}} + ... + w_1^* w_2^{2^{n_{k+1}+1}}) f_{d(k)}.$$

Here note

$$2^{n_{k+1}+1} \ge 2^{n_{k+1}+1} - ...(-1)^r 2^{n_r+1} = d - d(k).$$

Also note $m_k = 2^n_k + d(k) - d$, and $t_{k-1} = w_1^{-2(k-1)} f_{d(k-1)}$, $t_k = w_1^{-2(k)} w_2^{d-d(k)}$. Hence the equation (1) is rewritten

$$w_1^{2(k-1)} t_{d(k-1)} - w_1^{2(k)-2(k+1)} f_{d(k+1)}$$

$$= (w_2^{m_k} + w_1^{2^{n_k}} w_2^{m_k - 2^{n_k - 1}} + ... + w^* w_2^{2^{n_{k+1}+1} - d + d(k)}) t_k w_1^{2(k)}.$$

Dividing by $w_1^{2^{n_k}}$ the above equation, we have

$$w_1^{2(k-1)-2^{n_k}} t_{k-1} - w_1^* f_{d(k+1)} = w_1^{2^{n_k - 1}} w_2^{m_k} t_k + w_1^{2(k)} a t_k.$$

Since $f_{d(k+1)} = 0$, $w_1^{2(k)} t_k = 0$ in $H^*_{et}(X_d; \mathbb{Z}/2)$, we get the desired formula $w_1^{2(k-1)-2^{n_k}} t_{k-1} = w_1^{2^{n_k}-1} w_2^{m_k} t_k$, for $k = odd$. □

Proof of Lemma 6.3 for the case $k = even$. For this case

$$d(k) = d(k-1) + 2^{n_k+1}, \quad d(k+1) = d(k-1) + 2^{n_k+1} - 2^{n_{k+1}+1}.$$

From the preceding lemma, we have

$$f_{d(k)} = (w_2^{2^{n_k}} + w_1 *)) f_{d(k-1)} + w_1^{2(k)} w_2 f_{d(k-1)-1},$$

$$f_{d(k+1)} = (*) f_{d(k-1)} + (\sum_{j=n_{k+1}+1}^{n_k} w_1^{2(k+1)+2(k)-2^{j+1}+1} w_2^{2j - 2^{n_{k+1}+1}+1}) f_{d(k-1)-1}.$$

Let us write the coefficient of $f_{d(k-1)-1}$ of the second equation by $w_1^{2(k+1)} w_2 b$, namely,

$$b = w_2^{2^{n_k} - 2^{n_{k+1}+1}} + w_1^{2^{n_k}} w_2^* + ... + w_1^{2^{n_k+1} - 2^{n_{k+1}+2}}.$$

Eliminate the terms of $f_{d(k-1)-1}$ and we get

$$(2) \quad w_1^{-2(k)} b f_{d(k)} + w_1^{-2(k+1)} f_{d(k+1)} = w_1^{-2(k)} (w_2^{2^{n_k+1} - 2^{n_{k+1}+1}} + w_1 c) f_{d(k-1)}.$$

Note that $f_{d(k)} = 0 \in H^*_{et}(X_d; \mathbb{Z}/2)$, and we see

$$w_1^{-2^{n_k}+1} b f_{d(k)} = w_1^{-2^{n_k}+1} w_2^{2^{n_k} - 2^{n_{k+1}+1}} f_{d(k)}.$$

Recall that

$$d - d(k+1) = d - d(k-1) - 2^{n_k+1} + 2^{n_{k+1}+1} = m_k - 2^{n_k} + 2^{n_{k+1}+1}.$$

Hence $w_1^{2^{n_k}+1} w_2^{d-d(k+1)} \times (2)$ is written by

$$w_1^{-2^{n_k}+1} w_2^{m_k} f_d + w_1^{2^{n_k} - 1 - 2(k+1)} w_2^{d - d(k+1)} f_{d(k+1)}$$

$$= w_1^{-2^{n_k}+1} (w_2^{d - d(k-1)} + w_1 c) f_{d(k-1)}.$$

This induces the almost the desired result

$$w_1^{2^{n_k}-1} w_2^{m_k} t_k = w_1^{2(k-1)-2^{n_k}} t_{k-1} + w_1^{2^{n_k}} c f_{d(k-1)}.$$

But the last term is zero because this term is also $w_1^{2^{n_k-1}}$-torsion but there is no such nonzero elemnts except for $w_1^* t_{k-1}$ by the same arguments in the proof of the preceding lemma. (The element c does not contain $w_2^{d-d(k-1)}$ by dimensional reason.) □

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