NOTE ON CHOW RINGS OF NONTRIVIAL G-TORSORS OVER A FIELD

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Abstract. Let G_k be a split reductive group over a field k corresponding to a compact Lie group G. Let E be a nontrivial G_k-torsor over a field k. In this paper we study the Chow ring of nontrivial G_k-torsors. For example when $(G,p)=(F_4,3)$, we have the isomorphism $CH^*(E)_{(3)}\cong \mathbb{Z}_{(3)}$.

1. Introduction

Let k be a subfield of $\mathbb{C}$ which contains primitive p-th root of the unity. Let G be a compact connected Lie group. Let us denote by G_k the split reductive group over k which corresponds G. By definition, a G_k-torsor E over k is a variety over k with a free G_k-action such that the quotient variety is $Spec(k)$. A G_k-torsor over k is called trivial, if it is isomorphic to G_k or equivalently it has a k-rational point.

Let H be a subgroup of G. Given a torsor E over k, we can form the twisted form of G/H by

$$(E\times G_k/H_k)/G_k\cong E/H_k.$$

In this paper, we always assume that E is a (inner) *nontrivial* torsor over k. We mainly study the cases that G are exceptional Lie groups and the (p component) torsion index $t(G)_{(p)}=p$. In particular, when $(G,p)=(G_2,2),(F_4,3)$ and $H=T$; the maximal torus, $H=P$ maximal parabolic subroups, we compute $CH^*(E/H_k)_{(p)}$ explicitly. Moreover we show $CH^*(E)_{(p)}=\mathbb{Z}_{(p)}$ for these cases. We also study the case $(G,p)=(SO_{2^{n+1}-1},2), n\ge 3$. This case $CH^*(E)_{(2)}\subset CH^*(G_k)_{(2)}$ but it is not isomorphic to $\mathbb{Z}_{(2)}$ nor $CH^*(G_k)_{(2)}$.

For these groups, Petrov, Semenov and Zainoulline [Pe-Se-Za] showed that the Chow motive of E/P_k and E/T_k are isomorphic to direct sums of the generalized Rost motive ([Vo4],[Ro2],[Su-Jo],[Vi-Za]). The algebraic cobordism $MGL^{2*,*}(-)$ of the Rost motives are given in [Vi-Ya],[Ya3]. From this, we show the multiplicative structures of

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$CH^*(E/P_k)$ and $CH^*(E/T_k)$. The algebraic cobordism $MGL^{2*,*}(G_k)$ is studied in [Ya1]. By using arguments in [Ya1], we can compute $CH^*(E)_{(p)}$.

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2. Rost motive

Let k be a field of $ch(k) = 0$ and X the smooth variety. We consider the Chow ring $CH^*(X)$ generated by cycles modulo rational equivalence. For a non zero symbol $a = \{a_0, ..., a_n\}$ in the mod 2 Milnor K-theory $K^M_{n+1}(k)/2$, let $\phi_a = \langle\langle a_0, ..., a_n\rangle\rangle$ be the $(n+1)$-fold Pfister form. Let X_{ϕ_a} be the projective quadric of dimension $2^{n+1} - 2$ defined by ϕ_a. The Rost motive $M_a (= M_{\phi_a})$ is a direct summand of the motive $M(X_{\phi_a})$ representing X_{ϕ_a} so that $M(X_{\phi_a}) \cong M_a \otimes M(\mathbb{P}^{2^n-1})$.

Moreover for an odd prime p and nonzero symbol $0 \neq a \in K^M_{n+1}/p$, we can define ([Ro],[Vo],[Su],[Vi-Za]) the generalized Rost motive M_a, which is irreducible and is split over K/k if and only if $a|_K = 0$ as the case $p = 2$.

The Chow ring of the Rost motive is well known. Let $\bar{k}$ be an algebraic closure of k, $X|_{\bar{k}} = X \otimes_k \bar{k}$, and $i_{\bar{k}} : CH^*(X) \to CH^*(X|_{\bar{k}})$ the restriction map.

Lemma 2.1. *(Rost [R1],[Vo4], [Vi-Ya], [Ya3]) The Chow ring $CH^*(M_a)$ is only dependent on n. There are isomorphisms*

$$CH^*(M_a) \cong \mathbb{Z}\{1\} \oplus (\mathbb{Z}\{c_0\} \oplus \mathbb{Z}/p\{c_1, ..., c_{n-1}\})[y]/(c_i y^{p-1})$$
$$\text{and} \quad CH^*(M_a|_{\bar{k}}) \cong \mathbb{Z}[y]/(y^p)$$

where $|y| = 2(p^{n-1} + ... + p + 1)$ and $|c_i| = |y| + 2 - 2p^i$. Here the multiplications are given by $c_i \cdot c_j = 0$ for all $0 \leq i, j \leq n-1$. Moreover the restriction map is given by $i_{\bar{k}}(c_0) = py$ and $i_{\bar{k}}(c_i) = 0$ for $i > 0$.

Remark. The element y does not exist in $CH^*(M_a)$ while $c_i y$ exists. Usually $CH^*(M_a)$ is defined only additively, however we give the ring structure as above in this paper.

Remark. In this paper the degree $|x|$ of an element $x \in CH^*(X)$ means the 2-times of the usual degree of the Chow ring so that it is compatible with the degree of the (topological) cohomology $H^*(X(\mathbb{C}))$.

Let us use notation $\Omega^*(X)$ for the motivic cobordism $MGL^{2*,*}(X)_{(p)}$ defined by Voevodsky (but not the algebraic cobordism of Levine and Morel, because Lemma 2.2 bellow is not proved for this theory). It is known that

$$\Omega^* = \Omega^*(pt.) \cong MU^{2*}(pt.)_{(p)} \cong \mathbb{Z}_{(p)}[x_1, x_2, ...]$$

where $MU^{2*}(pt.)$ is the complex cobordism ring and $|x_i| = -2i$. There is the relation ([Ya2])

$$(2.1) \quad \Omega^*(X) \otimes_{\Omega^*} \mathbb{Z}_{(p)} \cong CH^*(X)_{(p)}.$$

We can take for x_{p^i-1} the cobordism class of a $2(p^i-1)$-dimensional manifold whose characteristic numbers are divisible by p but the additive characteristic number s_{p^i-1} is not by p^2. Let us denote x_{p^i-1} as v_i. Let us denote by

$$(2.2) \quad I_n = (p = v_0, v_1, ..., v_{n-1}) \subset \Omega^*$$

the ideal of Ω^* generated by $p, ..., v_{n-1}$. Then it is well known that I_n and I_∞ are the only prime ideals stable under the Landweber-Novikov cohomology operations ([Ra]) in Ω^*.

The category of cobordism motives is defined and studied in [Vi-Ya]. In particular, we can define the algebraic cobordism of motives. The following is the main result in [Vi-Ya] (in [Ya3] for odd primes).

Lemma 2.2. *([Vi-Ya], [Ya3]) The restriction map*

$$i_{\bar{k}} : \Omega^*(M_a) \to \Omega^*(M_a|_{\bar{k}}) \cong \Omega^*[y]/(y^p)$$

is injective and there is the Ω^*-algebra isomorphism

$$\Omega^*(M_a) \cong \Omega^*\{1\} \oplus I_n\{y, ..., y^{p-1}\} \subset \Omega^*[y]/(y^p)$$

such that $v_i y = c_i$ in $\Omega^*(M_a) \otimes_{\Omega^*} \mathbb{Z}_{(p)} \cong CH^*(M_a)_{(p)}$ in (2.2).

Remark. When $n = 1$, the restriction map $i_{\bar{k}} : CH^*(M_1) \to CH^*(M_1|_{\bar{k}})$ is injective and

$$Im(i_{\bar{k}}) \cong \mathbb{Z}_{(p)}\{1\} \oplus \mathbb{Z}_{(p)}[y]/(y^{p-1})\{py\}$$

$$\subset \mathbb{Z}_{(p)}[y]/(y^p) \cong CH^*(X|_{\bar{k}})_{(p)}.$$

Remark. Let $BP\langle n\rangle^* = \mathbb{Z}_{(p)}[v_1, ..., v_n]$. Recall ([Ya2]) that

$$ABP\langle n\rangle^{2*,*}(X) = \Omega^*(X) \otimes_{\Omega^*} BP\langle n\rangle^*$$

for smooth X. Then we also see that

$$i_k : ABP\langle n-1\rangle^{2*,*}(M_a) \to ABP\langle n-1\rangle^{2*,*}(M_a|_{\bar{k}})$$

is injective.

3. compact Lie group G

Let G be a compact connected Lie group. By the Borel therorem, we have the ring isomorphism for p odd

$$(3.1)\quad H^*(G;\mathbb{Z}/p)\cong P(y)/(p)\otimes \Lambda(x_1,...,x_l)$$

$$with\ \ P(y)=\mathbb{Z}[y_1,...,y_k]/(y_1^{p^{r_1}},...y_k^{p^{r_k}})$$

where $|y_i|=even$ and $|x_j|=odd$. When p=2, for each y_i, there is x_j with $x_j^2=y_i$. Hence we have $grH^*(G;\mathbb{Z}/2)\cong P(y)/(2)\otimes\Lambda(x_1,...,x_l)$.

Let T be the maximal torus of G and BT the classifying space of T. We consider the fibering

$$(3.2)\quad G\stackrel{\pi}{\to} G/T\stackrel{i}{\to} BT$$

and the induced spectral sequence

$$E_2^{*,*}=H^*(BT;H^*(G;\mathbb{Z}/p))\Longrightarrow H^*(G/T;\mathbb{Z}/p).$$

The cohomology of the classifying space of the torus is given by

$$H^*(BT)\cong \mathbb{Z}[t_1,...,t_\ell]\quad with\ \ |t_i|=2.$$

where ℓ is also the number of the odd degree generators x_i in $H^*(G;\mathbb{Z}/p)$. It is known that y_i are permanent cycles and that there is a regular sequence ([Tod],[Mi-Ni]) $(\bar{b}_1,...,\bar{b}_\ell)$ in $H^*(BT)/(p)$ such that $d_{|x_i|+1}(x_i)=\bar{b}_i$. Thus we get

$$E_\infty^{*,*'}\cong grH^*(G/T;\mathbb{Z}/p)\cong P(y)\otimes\mathbb{Z}/p[t_1,...,t_\ell]/(\bar{b}_1,...,\bar{b}_\ell).$$

Moreover we know that G/T is a manifold of torsion free, and we get

$$(3.3)\quad H^*(G/T)_{(p)}\cong\mathbb{Z}_{(p)}[y_1,..,y_k,t_1,...t_\ell]/(f_1,...,f_k,b_1,...,b_\ell)$$

where $b_i=\bar{b}_i\ mod(p)$ and $f_i=y_i^{p^{r_i}}\ mod(t_1,...,t_\ell)$. Since G/T is cellular, we also know

$$(3.4)\quad \Omega^*(G/T)\cong\Omega^*[y_1,...,y_k,t_1,...t_\ell]/(\tilde{f}_1,...,\tilde{f}_k,\tilde{b}_1,...,\tilde{b}_\ell)$$

where $\tilde{b}_i=b_i\ mod(\Omega^{<0})$ and $\tilde{f}_i=f_i\ mod(\Omega^{<0})$.

Let G_k be the split reductive algebraic group corresponding G and T_k the split maximal Torus. Hence

$$CH^*(G_k)\cong CH^*(G_{\mathbb{C}})\cong H^*(G).$$

Similarly $CH^*(G_k/T_k)\cong H^*(G/T)$ and $CH^*(BT_k)\cong H^*(BT)$. Next we consider the relation between $CH^*(G_k)$ and $CH^*(G_k/T_k)$.

Theorem 3.1. *(Grothendieck [Gr], [Ya1]) Let E be a G_k-torsor over k. (Here we do not assume the nontiviality of E). Let $h^*(-)=CH^*(-)$ or $\Omega^*(X)$. Then*

$$h^*(E)\cong h^*(E/T_k)/(i^*h^*(BT_k))\cong h^*(E/T_k)/(t_1,...,t_\ell).$$

Proof. Let $L_i\to E/T_k$ be the line bundle corresponding the element $t_i\in h^2(E/T_k)$. Then we can embed the T_k-bundle $E\to E/T_k$ into the associated bundle $\oplus_i L_i\to E/T_k$. such that E is an open subscheme of $\oplus_i L_i$. Consider the localization exact sequence

$$\oplus_i h^*(\oplus_{j\neq i}L_j)\stackrel{\oplus s_{i*}}{\to} h^*(\oplus_i L_i)\to h^*(E)\to 0$$

where $s_i: E/T_k\to L_i$ is the zero section. Since L_i are vector bundles

$$h^*(E)\cong h^*(\oplus_{i\neq j}L_j)\cong h^*(\oplus_i L_i).$$

By the definition the first Chern class, we know $t_i=c_1(L_i)=s_i^*s_{i*}(1)$. Thus we get the desired result $h^*(E)\cong h^*(E/T_k)/(t_1,..,t_\ell)$. □

Since $CH^*(G_k)\cong CH^*(G_{\mathbb{C}})$, from the result for $H^*(G/T)$ (3.3), we know

Corollary 3.2. *([Kac [Ka]) $Ch^*(G_k)_{(p)}\cong P(y)/(py_i)$.*

The result $\Omega^*(G_{\mathbb{C}})$ is one of the main result in [Ya1]. Let Q_i be the Milnor primitive operation in $H^*(X;\mathbb{Z}/p)$ inductively defined by $Q_i=[Q_{i-1},P^{p^{i-1}}]$ and $Q_0=\beta$; the Bockstein operation where $P^{p^{p-1}}$ is the p^{p-1}-th reduced power operation. It is known that we can take generators such that $Q_i(x_{odd})\in P(y_{even})/(p)$ for all $i\ge 0$ ([Mi-Ni]).

Theorem 3.3. *([Ya1]) Take generators so that $Q_i(x_{odd})\in P(y_{even})/(p)$ for all $i\ge 0$. Then there is the Ω^*-module isomorphism*

$$\Omega^*(G_k)/I_\infty^2\cong \Omega^*\otimes P(y_{even})/(I_\infty^2,\sum_i v_iQ_i(x_{odd})).$$

Let P be a parabolic subgroup. Then the inclusinon $T\subset P$ induces the fibering

$$(3.5)\quad P/T\to G/T\stackrel{p}{\to} G/P$$

and the spectral sequence (see [Tod])

$$E(E/T)_2^{*,*'}\cong H^*(G/P)\otimes H^*(P/T)\Longrightarrow H^*(G/T).$$

Since these spaces have no torsion and even dimensionally generated, this spectral sequence is collapses, that is

$$(3.6)\quad grH^*(G/T)\cong H^*(G/P)\otimes H^*(P/T).$$

Hence $H^*(G/P)$ can be computed from $H^*(G/T)$ (while many cases $H^*(G/P)$ is more easily computed than $H^*(G/T)$).

The cohomology $H^*(P/T)$ can be compute by the fibering

$$P/T \to BT \xrightarrow{i} BP.$$

Indeed, if i^* is injective, then

$$(3.7) \quad H^*(P/T) \cong H^*(BT)/(i^*\tilde{H}^*(BP)).$$

Note that for the Borel subgroup B, we have the isomorphisms $CH^*(X/T) \cong CH^*(X/B)$ and $CH^*(BB) \cong CH^*(BT)$.

4. EXCEPTIONAL GROUPS OF TYPE (I)

Let G be a compact connected Lie group of $dim(G) = 2d$. The torsion index is defined by

$$t(G) = |H^{2d}(G/T;\mathbb{Z})/i^*H^{2d}(BT;\mathbb{Z})|.$$

By Grothendieck, it is known that any G_k-torsor E splits over any fields L over k of index dividing $t(G)$. By Totaro all $t(G)$ are recently known [To1,2]. Let us write by $t(G)_{(p)}$ the p-component of $t(G)$. In this section, we restrict the cases $t(G)_{(p)} = p$ (for ease of arguments) and G are exceptional Lie groups. We call such (G,p) is of type (I), that is

$$(G_2, 2), \quad (F_4, 2), \quad (E_6, 2)$$
$$(F_4, 3), \quad (E_6, 3), \quad (E_7, 3), \quad \textit{and } (E_8, 5).$$

Throughout this section, we assume (G,p) are type of (I).

Remark. In [Pe-Se-Za], Petrov, Semenov and Zainoulline defined the J-invariant $J_G(i_1,...,i_k)$ of G from the smallest number i_s such that $y_s^{p^{i_s}} \in Im(i_{\bar{k}})$. (More accurate definition, see 4.5 in [Pe-Se-Za].) Hence type (I) group has J-invariant $J_G(1)$.

For these cases, the ordinary $mod(p)$ cohomology is well known

$$H^*(G;\mathbb{Z}/p) \cong \mathbb{Z}/p[y]/(y^p) \otimes \Lambda(x_1,...,x_\ell)$$

where $\ell = rank(G) \geq 2$, $|y| = 2p+2, |x_1| = 3, |x_2| = 2p+1$. Moreover

$$Q_1(x_1) = y, \quad Q_0(x_2) = y.$$

Hence we have the isomorphisms as (3.3) and (3.4)

$$grH^*(G;\mathbb{Z}/p) \cong \mathbb{Z}/p[y,t_1,...,t_\ell]/(y^p,\bar{b}_1,...,\bar{b}_\ell),$$
$$H^*(G) \cong \mathbb{Z}[y,t_1,...,t_\ell]/(f_1,b_1,...,b_\ell),$$
$$\Omega^*(G) \cong \mathbb{Z}/p[y,t_1,...,t_\ell]/(\tilde{f}_1,\tilde{b}_1,...,\tilde{b}_\ell),$$

where $b_i = \bar{b}_i \ mod(p)$ and $f_1 = y_i^p \ mod(t_1,...,t_\ell)$, and where $\tilde{b}_i = b_i \ mod(\Omega^{<0})$ and $\tilde{f}_1 = f_1 \ mod(\Omega^{<0})$. From Corolary 3.2,we see

Corollary 4.1. *$CH^*(G_k)_{(p)} \cong \mathbb{Z}/p[y]/(y^p)$.*

From Theorem 3.3 and the Q_i-actions, we see

$$\Omega^*(G_k)/I_\infty^2 \cong \Omega^*[y]/(py, v_1y, y^p, I_\infty^2),$$

while we have more strong result (Theorem 5.1 in [Ya1]) than Theorem 3.3.

Corollary 4.2. *$\Omega^*(G_k) \cong \Omega^*[y]/(py, v_1y, y^p)$.*

Remark. In the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*',*''} \cong H^{*,*'}(G_k; MU^{*''}) \Longrightarrow MGL^{*,*'}(G_k)$$

we know that

$$d_{2p-1}(x_1) = v_1 \otimes Q_1(x_1) = v_1y.$$

Thus we get also $E_\infty^{2*,*,*''} \cong MU^*[y]/(py, v_1, y^p)$.

In general, torsors E are parametrized by elements $\xi \in H^1(k; Aut(G))$. A torsor is said to be inner if it is in the image of the canonical map $H^1(k; G)$ to $H^1(k; Aut(G))$. In this paper, we *assume* all torsors are *inner* type.

Petrov, Semenov and Zainoulline [Pe-Se-Za] developed the theory of generally splitting varieties. We say that L is splitting field of a variety of X if $M(X|_L)$ is isomorphic to a direct sum of twisted Tate motives $\mathbb{T}^{\otimes i}$ and the restriction map $i_L : M(X) \to M(X|_L)$ is isomorphic after tensoring $\mathbb{Q}$. A smooth scheme X is said to be generically split over k if its function field $L = k(X)$ is a splitting field. Petrov Semenov and Zainoulline showed that torsors of type G of (I), (restricting E_6 to the adjoint type E_6^{ad}), are all generally split.

Theorem 4.3. *(Theorem 4.9 in [Pe-Se-Za]) There is a mod(p) motive $R_p(G)$ such that*

$$(1) \qquad CH^*(R_p(G)|_{\bar{k}}))/p \cong \mathbb{Z}/p[y]/(y)$$

$$(2) \quad M(E/T_k; \mathbb{Z}/p) \cong \oplus_s R_p(G) \otimes \mathbb{T}^{\otimes i_s} \cong R_p(G) \otimes H^*(G/T; \mathbb{Z}/p)/(y)$$

where we identify $H^*(G/T; \mathbb{Z}/p)/(y)$ as the sum of Tate motives $\oplus T^{\otimes i_s}$.

Theorem 4.4. *(Theorem 3.8 in [Pe-Se-Za]) Let $Q_k \subset P_k$ be parabolic subgroups of G_k which are generically split over k. There is a decompostion of motive $M(E/Q_k)_{(p)} \cong M(E/P_k)_{(p)} \otimes H^*(P/Q)$.*

For $p = 2, 3$ (i.e., except for E_8, $p = 5$), from Proposition 5.5 (for $m = p$) and §7 in [Pe-Se-Za], we have the *integral* motivic decopostion which deduces the *mod(p)* decomposition in Theorem 4.3. Moreover from Corollary 6 in [Vi-Za] (see also [Se],[Bo]), we see the integral motive corresponding $R_p(G)$ is really generalized Rost motive.

Theorem 4.5. *Let G of type (I) except for E_8 and E_6. Then there is a parabolic subgroup P_k such that E/P_k is generically split and*

$$CH^*(G_k/P_k)_{(p)} \cong \mathbb{Z}[y]/(y^p) \otimes A \quad and \quad M(E/P_k)_{(p)} \cong M_2 \otimes A$$

where A is a sum of twisted Tate motives and $M_2 = M_a$ is the generalized Rost motive for some $0 \neq a \in K_3^M(k)/p$.

Now we state the main theorem of this paper.

Theorem 4.6. *Let G of type (I) except for E_8 and E_6. The chow ring $CH^*(E/T_k)_{(p)}$ is multiplicatively generated by $t_1, ..., t_\ell$. Hence $CH^*(E)_{(p)} \cong \mathbb{Z}_{(p)}$.*

Proof. From the above theorems, we have the decompostion of motive

$$M(E/T_k)_{(p)} \cong M_3 \otimes H^*(G/T)/(y).$$

We consider the restriction map

$$i_{\bar{k}} : \Omega^*(E/T_k) \to \Omega^*(E/T_k|_{\bar{k}}) \cong \Omega^*(G/T).$$

Since $i_{\bar{k}}|\Omega(M_2)$ is injective, so is $i_{\bar{k}}$ above. Let us write

$$Im(i_{\bar{k}}) = i_{\bar{k}}(\Omega^*(E/T_k)) \subset \Omega^*(G_k/T_k) = \Omega^*(G/T).$$

Of course $py^i, v_1y^i \in Im(i_{\bar{k}})$ for $i > 0$ since so in $\Omega^*(M_2|_{\bar{k}})$. Note that $t_1, .., t_\ell \in Im(i_{\bar{k}})$ because they exist in $CH^*(E/T_k)$ since so in $CH^*(BT_k)$.

Recall that each element $x \in \Omega^*(E/T_k|_{\bar{k}}) \cong \Omega^*(G/T)$ is represented as

$$(*) \quad x = \sum_{i=0}^{p-1} \sum_s v(s,i)t(s,i)y^i, \qquad v(s,i) \in \Omega^*, \ \ t(s,i) \in \mathbb{Z}_{(p)}[t_1, ..., t_\ell]$$

while if $x \in Im(i_{\bar{k}})$, then $v(s,i) \in Ideal(p, v_1) \quad for \ i > 0$.

From Corollary 4.2, we see $py = v_1y = 0$ in $\Omega^*(G_k)$. From Theorem 3.1, this means

$$py, \ v_1y \ \in (t_1, ..., t_\ell)\Omega^*(G_k/T_k).$$

(But note that this does not mean $\in (t_1, ..., t_\ell)Im(i_{\bar{k}})$.) Let us write $v_1y = \sum v(s,i)t(s,i)y^i$ as $(*)$. The above fact implies $|t(s,i)| > 0$ and hence $|v(s,i)| < 0$.

Now let us write

$$\Omega\langle 1\rangle^*(X) = \Omega^*(X) \otimes_{\Omega^*} \mathbb{Z}_{(p)}[v_1] = ABP\langle 1\rangle^{2*,*}(X).$$

In $\Omega\langle 1\rangle^*(G_k/T_k)$, the fact $|v(s,i)| < 0$ means

$$v(s,i) \in (v_1) = \mathbb{Z}_{(p)}[v_1]^{<0} = \Omega\langle 1\rangle^{<0}.$$

Hence $v_1y \in (t_1,...,t_\ell)Im(i_{\bar{k}})$ in $\Omega\langle 1\rangle^*(-)$ theory. Similarly we can prove that py^i, v_1y^i have the same property. (We note $y^p \in (p,v)\Omega^*(G_k/T_k)$ so in $Im(i_{\bar{k}})$.)

Thus we can write

$$v_1y = \sum_i \sum_s v(s,i)'t(s,i)v_1y^i \quad in\ \Omega\langle 1\rangle^*(E/T_k).$$

If $v(s,i)' \neq 0$ for $i > 1$, then apply the same equation to the right hand side v_1y in the above equation. Since $t(s,i) = 0$ when $|t(s,i)| > dim(G/T)$, we can write

$$v_1y = \sum_s v(s,0)t(s,0).$$

The same property holds for py^i and v_1y^i when $i > 1$. Hence $i_{\bar{k}}(\Omega\langle 1\rangle^*(E/T_k))$ is generated as an $\Omega\langle 1\rangle^*$-algebra by $t_1,...,t_\ell$.

Since we know the isomorphisms

$$CH^*(E/T_k)_{(p)} \cong \Omega^*(E/T_k) \otimes_{\Omega^*} \mathbb{Z}_{(p)} \cong \Omega\langle 1\rangle^*(E/T_k) \otimes_{\Omega\langle 1\rangle^*} \mathbb{Z}_{(p)},$$

the elements $t_1,...,t_\ell$ are multiplicatively generate $CH^*(E/T_k)_{(p)}$. □

5. EXCEPTIONAL LIE GROUP G_2

In this section we study the case $(G,p) = (G_2,2)$. We recall the cohomologies from Toda-Watanabe [To-Wa]

$$H^*(G/T;\mathbb{Z}) \cong \mathbb{Z}[t_1,t_2,y]/(t_1^2 + t_1t_2 + t_2^2, t_2^3 - 2y, y^2)$$

with $|t_i| = 2$ and $|y| = 6$. Let P be one of the maximal parabolic subgroups. Then from (3.6) and $H^*(P/T) \cong \mathbb{Z}\{1,t_1\}$

$$H^*(G/P;\mathbb{Z}) \cong \mathbb{Z}[t_2,y]/(t_2^3 - 2y, y^2) \cong \mathbb{Z}\{1,y\} \otimes \{1,t_2,t_2^2\}.$$

By Bonnet, we have the decomposition

Theorem 5.1. *([Bo],Corollary 5.6 in [Pe-Se-Za])*

$$M(E/P_k) \cong M_2 \oplus M_2(1) \oplus M_2(2).$$

Theorem 5.2. *There is the ring isomorphism*

$$CH^*(E/P_k)_{(2)} \cong \mathbb{Z}_{(2)}[t_2,u]/(t_2^6, 2u, t_2^3u, u^2)$$

$$\cong \mathbb{Z}_{(2)}[t_2]/(t_2^6) \oplus \mathbb{Z}/2[t_2]/(t_2^3)\{u\}$$

with $|t_2| = 2, |u| = 4$.

Proof. From Lemma 2.2, we know

$$\Omega^*(M_2) \cong \Omega^*\{1, 2y, vy\} \subset \Omega^*\{1, y\}.$$

From the preceding theorem, we have the Ω^*-algebra isomorphism

$$\Omega^*(E/P_k) \cong \Omega^*\{1, v_1 y, 2y\} \otimes \{1, t_2, t_2^2\} \subset \Omega^*(G/P).$$

Since $CH^*(X)_{(p)} \cong \Omega^*(X) \otimes_{\Omega^*} \mathbb{Z}_{(p)}$, we have the isomorphism

$$CH^*(E/P_k)_{(2)} \cong \mathbb{Z}_{(2)}\{1, 2y\}\{1, t_2, t_2^2\} \oplus \mathbb{Z}/2\{v_1 y\}\{1, t_2, t_2^2\}.$$

Here the multiplications are given as follows. Since $2y = t_2^3 \; mod(\Omega^{<0})$ in $\Omega^*(G_k/T_k)$, we can take $2y = t_2^3$ so that

$$\mathbb{Z}_{(2)}\{1, 2y\}\{1, t_2, t_2^2\} = \mathbb{Z}_{(2)}[t_2]/(t_2^6).$$

Let us write $u = v_1 y$ in $CH^*(E/T_k)_{(2)}$. Then $t_2^3 u = 2y v_1 y = 0$ and $u^2 = v_1^2 y^2 = 0$ in $\Omega^*(E/T_k) \otimes_{\Omega^*} \mathbb{Z}_{(2)}$. Hence we have the isomorphism in the theorem. □

Next consider $CH^*(E/T_k)_{(2)}$.

Theorem 5.3. *There is the ring isomorphism*

$$CH^*(E/T_k)_{(2)} \cong \mathbb{Z}_{(2)}[t_1, t_2]/(t_2^6, 2u, t_2^3 u, u^2)$$

where $u = t_1^2 + t_1 t_2 + t_2^2$.

Proof. The Chow ring is isomorphic to

$$(*) \quad CH^*(E/T_k)_{(2)} \cong CH^*(E/P_k)\{1, t_1\}$$

$$\cong (\mathbb{Z}_{(2)}\{1, 2y\} \oplus \mathbb{Z}/2\{v_1 y\})\{1, t_2, t_2^2\}\{1, t_1\}.$$

Here $2y = t_2^3$. Since $v_1 y \in (t_1, t_2)$ and $v_1 y = 0 \in CH^*(G/T)$, we see

$$v_1 y = \lambda(t_1^2 + t_1 t_2 + t_2^2) \quad mod((t_1, t_2)\Omega^{<0}\Omega^*(G/T))$$

for $\lambda \in \mathbb{Z}_{(2)}$. We can take $\lambda = 1 \; mod(2)$. Otherwise $v_1 y = 0 \in \Omega^*(G/T)/2$, which is a $\Omega^*/2$-free, and this is a contradiction. Hence we can take $t_1^2 + t_1 t_2 + t_2^2$ as $v_1 y$. (This is also proved by Lemma 4.3 in [Ya1], since $Q_1(x_1) = y$ and $d_3(x_1) = t_1^2 + t_1 t_2 + t_2^2$.) Hence in $CH^*(E/T_k)$ we have the relation

$$(t_2^3)^2 = 0, \; (t_2^3)u = 0, \; u^2 = 0, \; 2u = 0.$$

We consider the mod 2 Poincare polynomial

$$\sum_i rank_{\mathbb{Z}/2}(CH^i(E/T_k)/2)t^i = (1 + t^2 + t^4)(1 + t + t^2)(1 + t)$$

$$= 1 + 2t + 3t^2 + 4t^3 + 4t^5 + 3t^5 + t^6 = \frac{(1 - t^6)(1 - t^4)}{(1 - t)(1 - t)} - t^5(1 + t)^2$$

which is the $(mod(2))$ Poincare series of the right hand side ring of the theorem. (Note (t_2^6, u^2) is a regular sequence in $\mathbb{Z}/2[t_1, t_2]$ but $(t_2^6, u^2, (t_2^3)u)$ is not.) □

The author learned the following remarks by Zainoulline.

Remark. It is well known that there is the bijection between $H^1(k; G_2)$ and the class of Cayley algebras C from the fact $G_2 = Aut(C|_{\bar{k}})$. Hence each torsor E over k corresponds a Cayley algebra. Moreover E/T_k and E/P_k correspond the following varieties [Ca-Pe-Se-Za]. By an i-space $(i = 1, 2)$, we mean i-dimensional subspace V_i of C such that $u \cdot v = 0$ for every $u, v \in V_i$. The flag variety corresponding E/T_k is the full flag variety

$$X(1,2) = \{V_1 \subset V_2 | V_i; i - subspaces \subset C\}$$

and that corresponding E/P_k is $X(2) = \{V_2 | V_2; 2 - subspace \subset C\}$. Let

$$g : H^1(k; G_2) \to H^3(k; \mathbb{Z}/2) \cong K_3^M(k)/2$$

be the Arason invariant (which is know to be isomorphic). The symbol of the Rost motive in Theorem 5.1 is $g(E)$ i.e., $M_2 = M_{g(E)}$.

Remark. Similar facts hold for $(G, p) = (F_4, 3)$. This case, the corresponding algebras are exceptional Jordan algebras of dimension 27 over k, and the symbol of the generalizedf motive is the Rost-Serre invariant.

6. EXCEPTIONAL GROUP F_4 FOR $p = 3$

Let $(G, p) = (F_4, 3)$ throughout this section. Let E be a nontrivial G_k-torsor as previous sections. Let P_k be a maximal parabolic subgroup of G_k given by the the last vertex of the Dynkin diagram.

Theorem 6.1. *([Pa-Se-Za]) Let M_2 be the generalized Rost motive. Then there is an isomorphism $M(E/P_k) \cong \oplus_{i=0}^{7} M_2(i)$.*

We first recall the ordinary cohomology of G/P ([Is-To], [Du-Za]).

$$H^*(G/P)_{(3)} \cong \mathbb{Z}[t, y]/(r_8, r_{12}), \quad |t| = 2,\ |y| = 8$$

where $r_8 = 3y^2 - t^8$ and $r_{12} = 26y^3 - 5t^{12}$. Hence we can rewrite

$$H^*(G/P)_{(3)} \cong \mathbb{Z}_{(3)}\{1, t, ..., t^7\} \otimes \{1, y, y^2\}.$$

Recall the Rost motive

$$CH^*(M_2|_{\bar{k}}) \cong \mathbb{Z}[y]/(y^3),$$

$$CH^*(M_2) \cong \mathbb{Z}\{1\} \oplus \mathbb{Z}\{3y, 3y^2\} \oplus \mathbb{Z}/3\{v_1 y, v_1 y^2\}.$$

Of course, the above y is that in $H^*(G/P)_{(3)}$ by the dimensional reason. From the above theorem, we have the decomposition

$(*)\ CH^*(E/P_k) \cong \mathbb{Z}_{(3)}\{1, t, ..., t^7\} \otimes (\mathbb{Z}_{(3)}\{1, 3y, 3y^2\} \oplus \mathbb{Z}/3\{v_1y, v_1y^2\})$.

The ring structure is given as follows.

Theorem 6.2.

$$CH^*(E/P_k)_{(3)} \cong \mathbb{Z}_{(3)}[t, b, a_1, a_2]/(t^{16}, t^8b, b^2 = 3t^8, ba_i, 3a_i, t^8a_i, a_1a_2)$$

$$\cong \mathbb{Z}_{(3)}\{1, t, ..., t^7\} \otimes (\mathbb{Z}_{(3)}\{1, \sqrt{3}t^4, t^8\} \oplus \mathbb{Z}/3\{a_1, a_2\})$$

where $|b| = 8$ *and* $|a_1| = 4, |a_2| = 12$.

Proof. From the relation r_8 in $CH^*(G/P)$, we have

$$3y^2 = t^8 + vx \in \Omega^*(G/P) \quad for\ v \in \Omega^{<0}.$$

Hence we can take t^8 instead of $3y^2$ in $(*)$. Of course

$$(3y)^2 = 3t^8 + 3vx \ \in \Omega^*(G_k/P_k).$$

Hence we write by $b = \sqrt{3}t^4$ the element $3y$. Write by a_1, a_2 the elements v_1y, v_1y^2 respectively. Elements in $I_\infty\Omega^{<0} \subset \Omega(G_k/P_k)$ reduces to zero in $CH^*(E/T_k)$. Therefore we have the desired multiplicative results. □

The cohomology $H^*(G/T)$ is given by Toda-Watanabe [To-Wa]

$$H^*(G/T)_{(3)} \cong \mathbb{Z}_3[t_1, t_2, t_3, t_4, y]/(\rho_2, \rho_4, \rho_6, \rho_8, \rho_{12}).$$

Here relations ρ_i is written by the elementary symmetric function $c_i = \sigma_i(t_1, t_2, t_3, t_4)$, that is,

$$\rho_2 = c_2 - (1/2)c_1^2, \quad \rho_4 = c_4 - c_3c_1 + (1/2)^3c_1^4 - 3y, \quad \rho_6 = -c_4c_1^2 + c_3^2,$$

$$\rho_8 = 3c_4c_1^4 - (1/2)^4c_1^8 + 3y(2^4y + 2^3c_3c_1), \quad \rho_{12} = y^3$$

By the arguments similar to the proof of Theorem 5.3 (or Lemma 4.3 in [Ya1]).

Theorem 6.3. *Let* $\pi : E/T_k \to E/P_k$. *Then*

$$\pi^*(t) = c_1, \quad \pi^*(a_1) = \rho_2, \quad \pi^*(a_2) = \rho_6, \pi^*(b) = c_4 - c_3c_1 - (2)^{-3}c_1^4.$$

Hence there is the epimorphism

$$\mathbb{Z}_{(3)}[t_1, t_2, t_3, t_4]/(c_1^{16}, c_1^8\pi^*(b), \pi^*(b)^2 - 3c_1^8, \pi^*(b)\rho_j, 3\rho_j, c_1^8\rho_j, \rho_2\rho_6)$$

$$\to CH^*(E/T_k)_{(3)}$$

where $j = 2, 6$.

7. The orthogonal group $SO(m)$ and $p = 2$

We consider the orthogonal groups $G = SO(m)$ and $p = 2$. The mod 2-cohomology is written as (see for example [Ni])

$$grH^*(SO(m);\mathbb{Z}/2) \cong \Lambda(x_1, x_2, ..., x_{m-1})$$

where the multiplications are given by $x_s^2 = x_{2s}$. We write $y_{2(odd)} = x_{odd}^2$. Hence we can write

$$H^*(SO(m);\mathbb{Z}/2) \cong \mathbb{Z}/2[y_{4i+2}|2 \le 4i+2 \le m-1]/(y_{4i+2}^{s(i)})\otimes\Lambda(x_1, x_3, ...x_{\bar{m}})$$

where $s(i)$ is the smallest number such that $2^{s(i)}(4i + 2) \ge m$ and $\bar{m} = m - 1$ (resp.$\bar{m} = m - 2$) if m is even (resp. odd).

The Q_i-operations are given by Nishimoto [Ni]

$$Q_n x_{odd} = x_{odd+|Q_n|}, \qquad Q_n x_{even} = Q_n y_{even} = 0$$

Relations in $\Omega^*(SO(m))$ are given by

$$\sum_n v_n Q_n(x_{odd}) = \sum_n v_n x_{odd+|Q_n|} = 0 \qquad mod(I_\infty^2).$$

For example, the relations in $\Omega^*(SO(m))/I_\infty^2$ starting with $2y_6$ are given by

$$2y_6 + v_1 y_2^4 + v_2 y_6^2 + v_3 y_{10}^2 + ... = 0,$$

Theorem 7.1. *([Ya1]) There are $\Omega^*_{(2)}$-algebra isomorphisms*

$$(1) \quad \Omega^*(SO(m))/I_\infty^2 \cong \Omega^*[y_{4i+2}|2 \le 4i + 2 \le m - 1]/(R, I_\infty^2)$$

where $\quad R = \{$*relations starting with* $\quad y_{4i+2}^{2^{s(i)}},\ 2y_{4i+2},\ v_1 y_{4i'+2}\ i' \ne 0.\}$

For ease of arguments, we only consider the case $G = SO(odd)$. Let $G = SO(2m' + 1)$ and $P = SO(2m' - 1) \times SO(2)$. Then it is well known [To-Wa]

Lemma 7.2. *$H^*(G/P) \cong \mathbb{Z}[t, y]/(t^{m'} - 2y, y^2) \quad |y| = 2m'$.*

By Toda-Watanabe [To-Wa], we also know

Theorem 7.3. *([To-Wa])*

$$H^*(G/T) \cong \mathbb{Z}[t_i, y_{2i}, t_{m'}, y]/(c_i - 2y_{2i}, J_{2i}, t_{m'}^{m'} - 2y, y^2)$$

where $1 \le i \le m' - 1$, $c_i = \sigma(t_1, ..., t_{m'})$ and

$$J_{2i} = 1/4(\sum_{j=0}^{2i}(-1)^j c_j c_{2i-j}) = y_{4i} - \sum_{0<j<2i}(-1)^j y_{2j} y_{4i-2j}.$$

Hence we can write

$$grH^*(G/T) \cong H^*(G/P) \otimes A \quad A = \mathbb{Z}[t_i, y_i]/(c_i' - 2y_i, J_{2i})$$

where $c_i' = \sigma(t_1, ..., t_{m'-1})$. More precisely, we can write

$$grA = P(y)' \otimes P(t)'$$

where $P(y)' = \otimes_{i<2^{n-1}-1}\mathbb{Z}[y_{4i+2}]/(y^{2^{s_i}})$ so that $P(y) = P(y)' \otimes \mathbb{Z}[y]/(y^2)$. Moreover we see

$$P(t)' = H^*(BT_{m'-1})/(H^*(BU(m'-1))) \cong Z[t_1, ..., t_{m'-1}]/(c_1', ..., c_{m'-1}')$$

Indeed, it is also known that

$$grH^*(G/(U(m'-1) \times SO(2))) \cong P(y)' \otimes H^*(G/P).$$

Now we recall an argument of quadrics. Let $m = 2m' + 1$. and let us write the quadratic form $q(x)$ defined by

$$q(x_1, ..., x_m) = x_1x_2 + ... + x_{m-2}x_{m-1} + x_m^2$$

and the projective quadric X_q defined by the quadratic form q. Then it is well known that (in fact $SO(m)$ acts on the affine quadric in $\mathbb{A}^m - 0$)

$$X_q \cong SO(m)/(SO(m-2) \times SO(2)).$$

Let $G = SO(m)$ and $P = SO(m-2) \times SO(2)$. Then the quadric q is always split over k and we know $CH^*(G_k/P_k) \cong CH^*(X_q)$.

In particular we consider the case $m = 2^{n+1} - 1$. Let $\rho = \{-1\} \in K_1^M(k) = k^*$. We consider the field k such that

$$(**) \qquad 0 \neq \rho^{n+1} \in K_{n+1}^M(k)/2.$$

Throughout this section, we assume the condition $(**)$. (If k is a real number field, then $(**)$ is satisfied.) Define the quadratic form q' by

$$q'(x_1, ..., x_m) = x_1^2 + ...x_m^2.$$

Then this q is a subform of

$$\langle\langle -1, ..., -1\rangle\rangle = \phi_{\rho^{n+1}}$$

the $(n+1)$-th Pfister form associated to ρ^{n+1}. (That is, q' is the maximal neighbor of the $(n+1)$-th Pfister form.) Of course $q|_{\bar{k}} = q'|_{\bar{k}}$ and we can identify $E/P_k \cong X_{q'}$. From Lemma 7.4 (or Rost's result) , we know

$$CH^*(X_{q'}|_{\bar{k}}) \cong \mathbb{Z}[t, y]/(t^{2^n-1} - 2y, y^2).$$

The multiplicative structure of Chow ring of the anisotropic quadric is easily compute from Lemma 2.2 [Ya4]

Lemma 7.4. *([Ya4])*

$$CH^*(E/P_k) \cong \mathbb{Z}[t]/(t^{2^{n+1}-2}) \oplus \mathbb{Z}/2[t]/(t^{2^n-1})\{u_1, ..., u_{n-1}\}$$

where $u_i = v_i y \in \Omega^*(E/p) \otimes_{\Omega^*} Z_{(2)}$ *so* $u_i u_j = 0$.

By the projection $E/T_k \to E/B_k$, Petrov, Semenov and Zainoulline also show the following (the example of the last page in [Pe-Se-Za], indeed, they show the J-invariant $J_2(G) = (0, ..., 0, 1)$).

Theorem 7.5. *When* $m = 2^{n+1}-1$. *The restriction map* $i_{\bar{k}} : \Omega^*(E/T_k) \to \Omega^*(E/T_k|_{\bar{k}}) = \Omega^*(G_k/T_k)$ *is injective and*

$$grCH^*(E/T_k) = grCH^*(E/P_k) \otimes A,$$
$$gr\Omega^*(E/T_k) = gr\Omega^*(E/P_k) \otimes A$$

where $A = \mathbb{Z}[t_i, y_{2i}]/(c_i' - 2y_i, J_{2i})$.

As a corollary, we see that t_i, y_{2i} are all in $CH^*(E/T_k)$. Hence $CH^*(E/T_k)$ is multiplicatively generated by t_i, y_i, t and $u_1, ..., u_{n-1}$.

Theorem 7.6. *Assume* $(**)$ *and* $m = 2^{n+1} - 1$. *Then*

$$CH^*(E)_{(2)} \cong P(y)'/(2) \subset P(y)' \otimes \mathbb{Z}/2[y]/(y^2) \cong CH^*(G_k)_{(2)}.$$

Proof. The proof is quite similar to that of Theorem 3.4. Let us write

$$\Omega\langle n-1\rangle^*(X) = \Omega^*(X) \otimes_{\Omega^*} \mathbb{Z}_{(2)}[v_1, ..., v_{n-1}] \cong ABP\langle n-1\rangle^{2*,*}(X).$$

By Theorem 3.1, We want to prove

$$(1) \quad u_1, ..., u_{n-1} \in (t_1, ..., t_{m'})CH^*(E/T_k).$$

This means

$$u_1, ..., u_{n-1} \in ((t_1, ..., t_{m'}) + \Omega^{<0})\Omega\langle n-1\rangle^*(E/T_k).$$

Let us write

$$Im(i_{\bar{k}}) = i_{\bar{k}}^*(\Omega\langle n-1\rangle^*(E/T_k)) \subset \Omega\langle n-1\rangle^*(G_k/T_k),$$
$$I(t, \Omega^{<0}) = ((t_1, ..., t_{m'}) + \Omega^{<0}))Im(i_{\bar{k}}).$$

(Note $I_\infty^2 \subset \Omega^{<0}Im(i_{\bar{k}})$.) Thus it is sufficient for (1) to prove

$$(2) \quad 2y, ..., v_{n-1}y \in I(t, \Omega^{<0}).$$

At first we will show $v_{n-1}y \in I(t, \Omega^{<0})$. Recall $y = y_{2^{n+1}-2} = x_{2^{n+1}-2}$. From Theorem 3.3 and Nishimoto's result, we see

$$(3) \quad x = 2x_{2^n} + v_1 x_{2^n+2} + ... + v_{n-2}x_{2^n+2^{n-1}-2} + v_{n-1}x_{2^{n+1}-2}$$
$$= 0 \quad in \quad \Omega\langle n-1\rangle^*(G_k)/(I_\infty^2).$$

So $x \in ((t_1, .., t_{m'}) + I_\infty^2)\Omega\langle n-1\rangle^*(G_k/T_k)$.

Each element $z \in \Omega\langle n-1\rangle^*(G_k/T_k)$ is written (not uniquely) by

$$(4) \quad z = \sum v_I t_J y_K + \sum v_{I'} t_{J'} y_{K'} y$$

with $v_I, v_{I'} \in \Omega\langle n-1\rangle^*$, $t_J, t_{J'} \in \mathbb{Z}_{(2)}[t_1,..,t_{m'}]$ and $y_K, y_{K'} \in P(y)'$. Note that if $z \in (t_1,..,t_{m'})\Omega\langle n-1\rangle^*(G_k/T_k)$, then we can take $|t_J| > 0$ and $|t_{J'}| > 0$.

Consider the case $z = x$ in (3). Since $y_K \in Im(i_{\bar{k}})$, we know

$$v_I t_J y_K \in (t_1,...,t_{m'})Im(i_{\bar{k}}).$$

Since $|y| < |t_{J'} y_{K'} y|$, we know $|v_{I'}| < 0$, i.e., $v_{I'} y \in Im(i_{\bar{k}})$ because $v_{I'} \in \Omega\langle n-1\rangle^* = \mathbb{Z}_{(2)}[v_1,...,v_{n-1}]$. Thus we know $v_{I'} t_{J'} y_{K'} y \in (t_1,...,t_{m'})Im(i_{\bar{k}})$. Thereore we see

$$(5) \quad x \in I(t, \Omega^{<0}).$$

In (3), $x_{2^n+2} = y_{2^n+2}, ..., x_{2^n+2^{n-1}-2}$ are in $Im(i_{\bar{k}})$. So we see

$$v_1 x_{2^n+2} + ... + v_{n-2} x_{2^n+2^{n-1}-2} \in \Omega^{<0} Imk(i_k).$$

Hence we see

$$(6) \quad 2x_{2^n} + v_{n-1} y \in I(t, \Omega^{<0}).$$

Next we will see

$$(7) \quad 2y_2, ..., 2y_{2^n-2} \in I(t, \Omega^{<0}).$$

Then in particular, $(y_2)^{2^{n-1}} = x_{2^n}$ implies $v_{n-1} y \in I(t, \Omega^{<0})$ from (6). Similarly we can prove $v_{n-2}y, ..., 2y \in I(t, \Omega^{<0})$ by using the arguments $(3) - (7)$. Thus we see (2) and so (1).

We prove (7) for $2y_2$ and the other cases are similar. By also using Nishimoto's result and Theorem 3.3, we have the relation

$$x' = 2x_2 + v_1 x_4 + ... v_{n-1} x_{2^n} = 0 \in \Omega\langle n-1\rangle^*(G_k)/I_\infty^2.$$

By using arguments similar to (3)-(5), we have $x' \in I(t, \Omega^{<0})$. Of course $v_1 x_4 + ... v_{n-1} x_{2^n} \in \Omega^{<0} Im(i_{\bar{k}})$. Thus we see $2y_2 \in I(t, \Omega^{<0})$. □

Remark. Note that $i_{\bar{k}}(CH^*(E))_{(2)} = CH^*(G_k)_{(2)} - \{y\}$. The similar fact happens for $\Omega\langle n-1\rangle$-theory

$$Im(i_{\bar{k}}) = \Omega\langle n-1\rangle^*(G_k) - \{y\}.$$

Indeed $2y, ..., v_{n-1}y \in \Omega\langle n-1\rangle^*(G_k)$ are in the $\Omega\langle n-1\rangle^*$-subalgebra generated by y_i.

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