ON λ-RING STRUCTURES OVER $\mathbf{Z}[[x]]$

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Abstract. It is shown that the λ-ring structure over the power series ring $\mathbf{Z}[[x]]$ given by the K-theory of $\mathbf{CP}^\infty$ is uniquely determined by the following condition: $\psi^p(x) \equiv px \pmod{x^2}$ for each prime p, where ψ^p is the Adams operation. Applications to algebraic topology and formal group laws are given.

1. Statement of the main result

The main corollary in [11] establishes the existence of uncountably many mutually non-isomorphic λ-ring structures over the power series ring $\mathbf{Z}[[x_1, \ldots, x_n]]$ for any positive integer n. Although this is a purely algebraic statement, its proof is not. In fact, it is proved by combining three algebraic topological theorems from [8, 9, 11] about spaces in the localization genus of classifying spaces of compact Lie groups and their K-theories. In particular, the proof does not construct the λ-ring structures; it merely shows their existence. This prompts the following question: *For a fixed n, how can these uncountably many non-isomorphic λ-ring structures over $\mathbf{Z}[[x_1, \ldots, x_n]]$ be constructed algebraically?*

In this note we contribute to this question in the case $n = 1$ by ruling out certain potential candidates for such λ-ring structures. To be more precise, recall that a λ-ring R has certain mutually-commuting endomorphisms ψ^n ($n \geq 1$), called Adams operations, such that ψ^1 is the identity. Moreover, for any prime p and any element r in R, the congruence relation

$$\psi^p(r) \equiv r^p \pmod{pR}$$

holds. The commutativity of the Adams operations implies that they are determined uniquely by the ψ^p for primes p.

Specializing to the case where R is a λ-ring structure over the power series ring $\mathbf{Z}[[x]]$, the above congruence relation implies that the coefficients of the power series $\psi^p(x) = \sum_{i=0}^\infty a_{p,\,i} x^i$ are all divisible by p, except for $a_{p,p}$. In algebraic topology, when a space has a power series ring as its K-theory, the generators x_i are usually in strictly positive filtrations and, since the Adams operations preserve filtrations, the constant term $a_{p,\,0}$ is 0. A case in point is the infinite-dimensional complex projective space $\mathbf{CP}^\infty$, which has $\mathbf{Z}[[x]]$ as its K-theory ring with x in filtration exactly 2. Its Adams operations are given by

$$\psi^p(x) = 1 - (1-x)^p \equiv px \pmod{x^2}$$

for any prime p.

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It is, therefore, very tempting to ask the following variant of the question stated at the end of the first paragraph above:

> *Given the linear polynomials $f_p(x) = px$, where p runs through the primes, is it possible to extend them to a λ-ring structure over $\mathbf{Z}[[x]]$ with Adams operations satisfying $\psi^p(x) \equiv f_p(x) \pmod{x^2}$?*

There is, of course, at least one such λ-ring, which is given by the K-theory of $\mathbf{CP}^\infty$. The main result of this note is that, up to λ-ring isomorphism, this is the only one. We now record it formally in the following theorem.

Main Theorem. *Let R be a λ-ring whose underlying ring is the power series ring $\mathbf{Z}[[x]]$. Suppose that the Adams operations of R satisfy the property that $\psi^p(x) \equiv px \pmod{x^2}$ for any prime p. Then R is isomorphic as a λ-ring to the K-theory of $\mathbf{CP}^\infty$.*

It should be pointed out that the assumption on the linear coefficients of the Adams operations are necessary in order that the conclusion be true. In fact, the uncountably many non-isomorphic λ-ring structures over $\mathbf{Z}[[x]]$ discovered in [11] all have the property that $\psi^p(x) \equiv p^2 x \pmod{x^2}$. They arise as the K-theories of the spaces in the localization genus of the classifying space $B\mathrm{SU}(2)$; see [10] for a classification of these spaces. Therefore, these λ-rings cannot be distinguished or identified simply by considering the linear terms in the Adams operations, in stark contrast to the situation in the theorem above.

We now apply the Main Theorem to obtain a result about the K-theory of spaces. Let X be a space. Say that it is *even and torsionfree* if its integral cohomology is concentrated in even dimensions and is $\mathbf{Z}$-torsionfree.

Corollary A. *Let X be a space that is even and torsionfree. Suppose that the K-theory ring of X is the power series ring $\mathbf{Z}[[x]]$ with x in filtration exactly 2. Then $K(X)$ is isomorphic as a λ-ring to $K(\mathbf{CP}^\infty)$.*

Indeed, a classical result of Adams [1] implies that such a space has the property that $\psi^p(x) = px +$ terms of higher filtrations. The Corollary then follows directly from the Main Theorem.

Next we would like to discuss another consequence of the Main Theorem that has to do with (one-dimensional, commutative) formal group laws. The reader is referred to [4] for background information about formal group laws.

Recall that the multiplicative group law is defined by

$$\mathbf{G}_m \;=\; 1 - (1-x)(1-y) = x + y - xy,$$

and it is the formal group law associated to complex K-theory. For an integer n, the n-series of $\mathbf{G}_m$ is the polynomial

$$[n]_{\mathbf{G}_m}(x) \;=\; 1 - (1-x)^n,$$

which coincides with the Adams operation $\psi^n(x)$ in the K-theory of $\mathbf{CP}^\infty$. Since we would like to construct λ-ring structures over $\mathbf{Z}[[x]]$ algebraically, it is natural to ask the following question:

> *Are there any formal group laws over the integers, other than* $\mathbf{G}_m$*, whose* n*-series,* $n \geq 1$*, form the Adams operations* $\psi^n(x)$ *of a* λ*-ring structure over* $\mathbf{Z}[[x]]$*?*

Since there are uncountably many formal group laws over the integers, a positive answer to this question seems plausible. However, this is not the case, meaning that $\mathbf{G}_m$ is the only formal group law whose n-series are the Adams operations of a λ-ring structure over $\mathbf{Z}[[x]]$.

Corollary B. *Let* F *be a (one-dimensional, commutative) formal group law over* $\mathbf{Z}$*. The following statements are equivalent:*

1. F *is strictly isomorphic to* $\mathbf{G}_m$*.*
2. *There exists a* λ*-ring structure* R *over* $\mathbf{Z}[[x]]$ *with the property that, for* $n \geq 1$*,* $\psi^n(x) = [n]_F(x)$*, the* n*-series of* F*.*
3. *There exists a* λ*-ring structure* R *over* $\mathbf{Z}[[x]]$ *with the property that, for* $n \geq 1$*,* $\psi^n(x) = [n]_F(x)$*. Moreover,* R *is isomorphic, as a* λ*-ring, to* $K(\mathbf{CP}^\infty)$*.*

Indeed, it is clear that (1) implies (3) and that (3) implies (2). The n-series of a formal group law must satisfy

$$[n](x) \;\equiv\; nx \pmod{x^2}.$$

Therefore, if R is as in (2), then it follows from the Main Theorem that it is isomorphic as a λ-ring to $K(\mathbf{CP}^\infty)$. Moreover, the proof of the Main Theorem shows that, when R is as in (2), $\log_F^{-1}(\log_{\mathbf{G}_m}(x))$ is an integral power series, where $\log_F(x)$ is the logarithm of F. This implies that F and $\mathbf{G}_m$ are strictly isomorphic.

This finishes the presentation of the results in this note. In the next section, we review some basics about λ-rings and Adams operations. The proof of the Main Theorem is given in the last section.

2. λ-RINGS

The purpose of this section is to review some basics about λ-rings, which is essential to understanding this article. Some standard references for λ-rings are the article by Atiyah and Tall [2] and the lecture notes by Knutson [5].

A *λ-ring* is a commutative ring R with a multiplicative identity equipped with functions

$$\lambda^i \colon R \;\to\; R \quad (i \geq 0),$$

called λ-operations, which satisfy the following conditions. For any integers $i, j \geq 0$ and elements r and s in R:

- $\lambda^0(r) = 1$
- $\lambda^1(r) = r$

- $\lambda^i(1) = 0$ for $i > 1$
- $\lambda^i(r+s) = \sum_{k=0}^{i} \lambda^k(r)\lambda^{i-k}(s)$
- $\lambda^i(rs) = P_i(\lambda^1(r), \ldots, \lambda^i(r); \lambda^1(s), \ldots, \lambda^i(s))$
- $\lambda^i(\lambda^j(r)) = P_{i,j}(\lambda^1(r), \ldots, \lambda^{ij}(r))$.

The P_i and $P_{i,j}$ are integral polynomials defined as follows. Consider the variables $\xi_1, \ldots, \xi_i$ and $\eta_1, \ldots, \eta_i$. Denote by $s_1, \ldots, s_i$ and $\sigma_1, \ldots, \sigma_i$, respectively, the elementary symmetric functions of the ξ's and the η's. The polynomial P_i is defined by the requirement that the expression $P_i(s_1, \ldots, s_i; \sigma_1, \ldots, \sigma_i)$ be the coefficient of t^i in the finite product

$$\prod_{m,n=1}^{i} (1 + \xi_m \eta_n t).$$

Similarly, if $s_1, \ldots, s_{ij}$ are the elementary symmetric functions of $\xi_1, \ldots, \xi_{ij}$, then the polynomial $P_{i,j}$ is defined by the requirement that the expression $P_{i,j}(s_1, \ldots, s_{ij})$ be the coefficient of t^i in the finite product

$$\prod_{l_1 < \cdots < l_j} (1 + \xi_{l_1} \cdots \xi_{l_j} t).$$

Given two λ-rings R and S, a λ-ring map $f\colon R \to S$ is a ring map from R to S which is compatible with the λ-operations in the sense that $f\lambda^i = \lambda^i f$ $(i \geq 0)$.

A λ-ring has the so-called Adams operations

$$\psi^n\colon R \to R \quad (n \geq 1)$$

defined by the Newton formula:

$$\psi^n(r) - \lambda^1(r)\psi^{n-1}(r) + \cdots + (-1)^{n-1}\lambda^{n-1}(r)\psi^1(r) + (-1)^n n\lambda^n(r) = 0.$$

These operations satisfy the following properties:

- all the ψ^n are λ-ring maps,
- $\psi^1 = \mathrm{Id}$,
- $\psi^m\psi^n = \psi^{mn} = \psi^n\psi^m$,
- $\psi^p(r) \equiv r^p \pmod{pR}$ for each prime p and element r in R.

3. PROOF OF THE MAIN THEOREM

Now we begin the proof of the Main Theorem in the previous section. We need a couple of lemmas, whose proofs will be given at the end of this section.

Lemma 3.1. *Let $f(x)$ and $g(x)$ be power series over a field F. Suppose that $f(x) \equiv \alpha x \equiv g(x) \pmod{x^2}$ for some element $\alpha \in F$ which is neither 0 nor a root of unity. Then for every element $c \in F$, there exists a unique power series $h(x)$ over F satisfying the following conditions:*

- *$h(0) = 0$,*
- *$h'(0) = c$, where "$\prime$" means formal derivative, and*
- *$h(g(x)) = f(h(x))$.*

Let R be a λ-ring as in the statement of the Main Theorem with Adams operations satisfying the property that $\psi^q(x) \equiv qx \pmod{x^2}$ for each prime q. Let p be an arbitrary but fixed prime. Using Lemma 3.1 we obtain a power series $f(x) = \sum_{i=1}^{\infty} a_i x^i$ with rational coefficients such that $a_1 = 1$ and that

$$pf(x) \;=\; f(\psi^p(x)). \tag{3.1.1}$$

Note that f is dependent on the prime p (and R), and we will write f_p for f if it is necessary to mention p.

In what follows, $\mathbf{Z}_{(p)}$ denotes the ring of p-local integers, that is, the subring of the rationals consisting of the elements $\frac{m}{n}$ with m and n integers such that p does not divide n. For an integer r, the notation $\frac{1}{p^r}\mathbf{Z}_{(p)}$ denotes the set of rational numbers of the form $\frac{\alpha}{p^r}$ where $\alpha \in \mathbf{Z}_{(p)}$.

We will need the following lemma about the p-integrality of the coefficients of f. For an integer n, we define $\theta_p(n)$ to be the largest integer for which $p^{\theta_p(n)}$ divides n, so that n is the product of $p^{\theta_p(n)}$ over its prime factors. In particular, if p does not divide n, then $\theta_p(n) = 0$.

Lemma 3.2. *With $f(x) = \sum_{i=1}^{\infty} a_i x^i$ defined as above, we have that for each $n \geq 1$, $a_n \in \frac{1}{p^{\theta_p(n)}}\mathbf{Z}_{(p)}$ and, if p divides n, $\left(a_n - \frac{1}{p}a_{n/p}\right) \in \mathbf{Z}_{(p)}$.*

Now we define another power series $g(x) = \sum_{i=1}^{\infty} c_i x^i$ over the rationals by the equation

$$g(x) \;\overset{\text{def}}{=}\; f(x) - \frac{1}{p}f(x^p). \tag{3.2.1}$$

It is easy to see that the coefficients of g are given by

$$c_n \;=\; \begin{cases} a_n & \text{if } p \text{ does not divide } n, \\ a_n - \frac{1}{p}a_{n/p} & \text{if } p \text{ divides } n. \end{cases}$$

In particular, it follows that $c_1 = 1$ and that, by Lemma 3.2, $c_n \in \mathbf{Z}_{(p)}$ for $n \geq 2$. So $g(x)$ is actually a power series over the p-local integers. We will write g_p for g if it is necessary to mention the dependency of g on the prime p.

Let S be an arbitrary λ-ring structure over $\mathbf{Z}[[x]]$ with Adams operations $\bar{\psi}^q$ satisfying the property that $\bar{\psi}^q(x) \equiv qx \pmod{x^2}$ for each prime q. To prove the Main Theorem, we need to show that R and S are isomorphic as λ-rings. There are similarly defined power series f_p and g_p associated with S; we denote them by $\bar{f}_p$ and $\bar{g}_p$ to avoid confusion with those associated with R. By the Functional Equation Lemma from the theory of formal group laws [4, 2.2(ii)], it follows that the power series $\varphi_p(x)$ defined by

$$\varphi_p(x) \;\overset{\text{def}}{=}\; f_p^{-1}(\bar{f}_p(x)),$$

where $f_p^{-1}(x)$ denotes the compositional inverse of $f_p(x)$, actually has coefficients in $\mathbf{Z}_{(p)}$. (In the notation of [4, 2.2], we are using the Functional Equation Lemma in the case in which $A = \mathbf{Z}_{(p)}$, $K = \mathbf{Q}$, $\sigma = \mathrm{Id}$, $q = p$, $s_1 = p^{-1}$, $s_n = 0$ for $n \geq 2$, and $\mathcal{A} = p\mathbf{Z}_{(p)}$.)

Observe that

$$\psi^p(x) \;=\; f_p^{-1}(pf_p(x)).$$

This implies that

$$\psi^p(\varphi_p(x)) \;=\; f_p^{-1}(pf_p(\varphi_p(x))) \;=\; f_p^{-1}(p\bar{f}_p(x)) \;=\; \varphi_p(\bar{\psi}^p(x)). \tag{3.2.2}$$

Now if q is an arbitrary prime, then both $\psi^q(\varphi_p(x))$ and $\varphi_p(\bar{\psi}^q(x))$ are power series over the rationals with constant terms 0 and linear terms qx. Moreover, if h denotes either one of these two power series, then using eq. (3.2.2) and the commutativity of Adams operations we infer that

$$\psi^p(h(x)) \;=\; h(\bar{\psi}^p(x)). \tag{3.2.3}$$

But since $h(0) = 0$ and $h'(0) = q$, eq. (3.2.3) and the uniqueness statement of Lemma 3.1 implies that

$$\psi^q(\varphi_p(x)) \;=\; \varphi_p(\bar{\psi}^q(x)). \tag{3.2.4}$$

Combining eq. (3.2.2) (for the prime q) and eq. (3.2.4), using the uniqueness part of Lemma 3.1 once again, we observe that

$$\varphi_p(x) \;=\; \varphi_q(x)$$

for arbitrary primes p and q. Denote the common value of the various $\varphi_p(x)$ by $\varphi(x)$. Since $\varphi_p(x)$ has coefficients in $\mathbf{Z}_{(p)}$, we conclude that $\varphi(x)$ is a power series with integer coefficients.

The integral power series $\varphi(x)$ extends to a ring automorphism φ of $\mathbf{Z}[[x]]$ with the property that

$$\psi^p(\varphi(x)) \;=\; \varphi(\bar{\psi}^p(x))$$

for each prime p. It follows that

$$\psi^r\varphi \;=\; \varphi\bar{\psi}^r$$

as endomorphisms for each positive integer r. But since every λ-operation λ^r is a polynomial in the Adams operations ψ^k ($k \leq r$), it follows that

$$\lambda^r\varphi \;=\; \varphi\bar{\lambda}^r$$

as functions on $\mathbf{Z}[[x]]$ for each $r \geq 1$, where $\bar{\lambda}^r$ denotes the rth λ-operation for S. Therefore, φ gives a λ-ring isomorphism from R to S, thereby proving the Main Theorem.

It remains to prove Lemma 3.1 and Lemma 3.2.

Proof of Lemma 3.1. Write $f(x) = \sum_{i=1}^{\infty} a_i x^i$ and $g(x) = \sum_{i=1}^{\infty} b_i x^i$ with $a_1 = b_1 = \alpha$. We will define $h(x) = \sum_{i=1}^{\infty} c_i x^i$ inductively, starting with $c_1 = \alpha$.

Let $n > 1$ be an integer. Suppose that the coefficients c_i have been defined for $i < n$ such that

$$h(g(x)) \;\equiv\; f(h(x)) \qquad (\text{mod } x^n) \tag{3.2.5}$$

and that the c_i are unique. If c_n exists, then we must have

$$\sum_{j=1}^{n} c_j \Big(\sum_{i=1}^{\infty} b_i x^i \Big)^j \;\equiv\; \sum_{i=1}^{\infty} a_i \Big(\sum_{j=1}^{n} c_j x^j \Big)^i \qquad (\mathrm{mod}\ x^{n+1}). \tag{3.2.6}$$

Equating the coefficients of x^n on both sides of eq. (3.2.6), we see that

$$c_n \alpha^n + t \;=\; c_n \alpha + s, \tag{3.2.7}$$

where t and s are rational polynomial expressions in, respectively, $b_1, \ldots, b_n$, $c_1, \ldots, c_{n-1}$ and $a_2, \ldots, a_n$, $c_1, \ldots, c_{n-1}$. Solving for c_n in eq. (3.2.7), we conclude that

$$c_n \;=\; \frac{s-t}{\alpha^n - \alpha}, \tag{3.2.8}$$

which is a well-defined element in the field F, since $n > 1$ and α is neither 0 nor a root of unity. Therefore, if we define c_n by the expression in eq. (3.2.8), then eq. (3.2.5) holds modulo x^{n+1} instead of x^n, and c_n is unique with respect to this property.

The lemma can now be finished by an induction. □

Remark 3.2.9. Lemma 3.1 is a generalization of a similar result of Lubin [6, Proposition 1.1]. The proof above is rather standard and follows the same pattern as Lubin's.

Proof of Lemma 3.2. In this proof we write $\psi^p(x) = \sum_{i=1}^{\infty} b_i x^i \in \mathbf{Z}[[x]]$. In particular, we have that $b_1 = p$ and that for $j > 1$,

$$b_j \;\equiv\; \begin{cases} 0 \ \ (\mathrm{mod}\ p) \text{ if } & j \neq p, \\ 1 \ \ (\mathrm{mod}\ p) \text{ if } & j = p. \end{cases}$$

We now prove the Lemma by induction on n, with the initial case $a_1 = 1$ being trivial.

Let, then, $n > 1$ be an integer and suppose that the Lemma has been proved for integers less than n. Observe that by equating the coefficients of x^n on both sides of eq. (3.1.1), we obtain the equation

$$p a_n \;=\; b_n + p^n a_n + \sum_{l=2}^{n-1} \pi_l^n, \tag{3.2.10}$$

where π_l^n has the form

$$\pi_l^n \;=\; a_l l p^{l-1} b_{n-l+1} + a_l h_l^n (b_1, \ldots, b_{n-l}). \tag{3.2.11}$$

Here the term h_l^n is given by

$$h_l^n (b_1, \ldots, b_{n-l}) \;=\; \sum_{i_1 + \cdots + i_l = n} b_{i_1} \cdots b_{i_l}, \tag{3.2.12}$$

where in the sum the i_j satisfy $1 \leq i_j \leq n - l$. We are writing π_l^n in this form because it is convenient for the arguments below.

Notice that the first term in π_l^n, $a_l l p^{l-1} b_{n-l+1}$ $(2 \leq l \leq n-1)$, is always in $p\mathbf{Z}_{(p)}$ regardless of whether p divides n or not. In fact, if p does not divide

l, then by induction hypothesis $a_l \in \mathbf{Z}_{(p)}$ and $l-1 \geq 1$. If p does divide l, then by induction hypothesis $a_l \in \frac{1}{p^{\theta_p(l)}}\mathbf{Z}_{(p)}$, but lp^{l-1} is divisible by p^l and $l > \theta_p(l)$.

To prove the p-integrality statement about a_n, let's first consider the situation when p does not divide n. For $2 \leq l \leq n-1$, we can then rewrite h_l^n as

$$h_l^n \;=\; \sum_{s=0}^{l-2} \sum_{i_1+\cdots+i_{l-s}=n-ps} \binom{l}{s} b_p^s b_{i_1}\cdots b_{i_{l-s}}, \tag{3.2.13}$$

in which none of the i_j is equal to p. In particular, h_l^n is divisible by $p^{\theta_p(l)+1}$, since it is divisible by lp^{l-s} unless $s=0$, in which case it is divisible by p^l. Together with the induction hypothesis on a_l, this implies that the second term in π_l^n, $a_l h_l^n$, is always in $p\mathbf{Z}_{(p)}$. Combining this with the previous paragraph and eq. (3.2.10), we infer that $a_n p(1-p^{n-1})$ is in $p\mathbf{Z}_{(p)}$, and hence a_n is in $\mathbf{Z}_{(p)}$.

Consider now the case when p divides n. The special case $n = p$ needs to be treated separately, but the argument is very similar to the one below, and so we will omit it. We are now assuming, in addition, that $n > p$. Just like we did above, we need to analyze h_l^n. If $l \neq n/p$, then h_l^n can be written in the form (3.2.13), and an argument similar to the one in the previous paragraph shows that $a_l h_l^n$ lies in $p\mathbf{Z}_{(p)}$.

Consider the case $l = \frac{n}{p}$. We need the following result about h_l^n.

Lemma 3.3. $h^n_{\frac{n}{p}} \equiv 1 \pmod{p^{\theta_p(n)}\mathbf{Z}}$

Proof. Similar to the case when p does not divide n, we can rewrite $h^n_{n/p}$ as

$$h^n_{n/p} \;=\; b_p^{\frac{n}{p}} + \sum_{s=0}^{l-2} \sum_{i_1+\cdots+i_{\frac{n}{p}-s}=n-ps} \binom{\frac{n}{p}}{s} b_p^s b_{i_1}\cdots b_{i_{\frac{n}{p}-s}}, \tag{3.3.1}$$

in which none of the i_j is equal to p. Since $b_p \equiv 1 \pmod{p}$, we have that

$$b_p^{p^{\theta_p(\frac{n}{p})}} \;\equiv\; 1 \pmod{p^{\theta_p(n)}\mathbf{Z}},$$

and thus

$$b_p^{\frac{n}{p}} \;\equiv\; 1 \pmod{p^{\theta_p(n)}\mathbf{Z}}.$$

Therefore, it suffices to show that each term inside the double summation is divisible by $p^{\theta_p(n)}$.

To see this, consider first when $s \neq 0$. In this case, $\binom{\frac{n}{p}}{s} b_p^s b_{i_1}\cdots b_{i_{\frac{n}{p}-s}}$ is divisible by $\frac{n}{p}\cdot p^{\frac{n}{p}-s}$, and hence by n since $s \leq \frac{n}{p}-2$. In particular, it is divisible by $p^{\theta_p(n)}$. If $s=0$, then $b_{i_1}\cdots b_{i_{\frac{n}{p}}}$ is divisible by $p^{\frac{n}{p}}$, and hence by $p^{\theta_p(n)}$ as well.

This finishes the proof of the Lemma. $\square$

We can now finish the proof of Lemma 3.2. As discussed above, the number

$$\alpha \overset{\text{def}}{=} b_n + a_{\frac{n}{p}} \frac{n}{p} p^{\frac{n}{p}-1} b_{n-\frac{n}{p}+1} + \sum_{l \neq \frac{n}{p}} \pi_l^n$$

is in $p\mathbf{Z}_{(p)}$. Combining this with eq. (3.2.10) and the induction hypothesis on $a_{\frac{n}{p}}$, we can infer that

$$a_n p(1-p^{n-1}) \;=\; \alpha + a_{\frac{n}{p}} h^n_{\frac{n}{p}}$$

lies in $\frac{1}{p^{\theta_p(n)-1}}\mathbf{Z}_{(p)}$, and hence $a_n \in \frac{1}{p^{\theta_p(n)}}\mathbf{Z}_{(p)}$.

Finally, both $a_n p^{\theta_p(n)}$ and $a_{\frac{n}{p}} p^{\theta_p(n)-1}$ lie in $\mathbf{Z}_{(p)}$ (the latter by induction hypothesis). We compute

$$\begin{aligned} a_n p^{\theta_p(n)} - a_{\frac{n}{p}} p^{\theta_p(n)-1} &\equiv a_n p^{\theta_p(n)}(1-p^{n-1}) - p^{\theta_p(n)-1} a_{\frac{n}{p}} h^n_{\frac{n}{p}} \qquad \pmod{p^{\theta_p(n)}\mathbf{Z}_{(p)}} \\ &= p^{\theta_p(n)-1}\alpha \\ &\equiv 0 \qquad \pmod{p^{\theta_p(n)}\mathbf{Z}_{(p)}}. \end{aligned}$$

The first congruence is a consequence of Lemma 3.3. This finishes the induction step, and the proof of the Lemma is complete. □

The proof of the main theorem is now complete.

REFERENCES

[1] J. F. Adams, Vector fields on spheres, Ann. Math. 75 (1962), 603-632.

[2] M. F. Atiyah and D. O. Tall, Group representations, λ-rings and the J-homomorphism, Topology 8 (1969), 253-297.

[3] F. J. B. J. Clauwens, Commuting polynomials and λ-ring structures on $\mathbf{Z}[x]$, J. Pure Appl. Algebra 95 (1994), 261-269.

[4] M. Hazewinkel, Formal groups and applications, Pure and Applied Mathematics 78, Academic Press, New York-London, 1978.

[5] D. Knutson, λ-rings and the representation theory of the symmetric group. Lecture Notes in Math. 308, Springer-Verlag, Berlin-New York, 1973.

[6] J. Lubin, Nonarchimedean dynamical systems, Comp. Math. 94 (1994), 321-346.

[7] C. A. McGibbon, Which group structures on S^3 have a maximal torus?, Lecture Notes in Math. 657 (1978), 353-360.

[8] J. M. Møller, The normalizer of the Weyl group, Math. Ann. 294 (1992), 59-80.

[9] D. Notbohm, Maps between classifying spaces and applications, J. Pure Appl. Algebra 89 (1993), 273-294.

[10] D. L. Rector, Loop structures on the homotopy type of S^3, Lecture Notes in Math. 249 (1971), 99-105.

[11] D. Yau, On adic genus and lambda-rings, Trans. Amer. Math. Soc., accepted for publication.

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